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Kähler-Einstein metrics with positive scalar curvature

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Abstract. In this paper, we prove that the existence of Kähler-Einstein metrics implies the stability of the underlying Kähler manifold in a suitable sense. In particular, this disproves a long-standing conjecture that a compact Kähler manifold admits Kähler-Einstein metrics if it has positive first Chern class and no nontrivial holomorphic vector fields. We will also establish an analytic criterion for the existence of Kähler-Einstein metrics. Our arguments also yield that the analytic criterion is satisfied on stable Kähler manifolds, provided that the partial C^0-estimate posed in [T6] is true.

1 Introduction

More than forty years ago, E. Calabi asked if a compact Kähler manifold M admits any Kähler-Einstein metrics. A metric is Kähler-Einstein if it is Kähler and its Ricci curvature form is a constant multiple of its Kähler form. Such a metric provides a special solution of the Einstein equation on Riemannian manifolds.

Since the Ricci form represents the first Chern class $c_1(M)$, a necessary condition for the existence of Kähler-Einstein metrics is that $c_1(M)$ is definite. In fact, Calabi conjectured that any (1,1)-form representing $c_1(M)$ is the Ricci form of some Kähler metric on M (the Calabi conjecture). In particular, the conjecture implies the existence of Ricci-flat Kähler metrics in case $c_1(M) = 0$. The Calabi conjecture was solved by Yau in 1977 [Y]. Around the same time, Aubin and Yau proved independently the existence of Kähler-Einstein metrics on Kähler manifolds with negative first Chern class [Au1], [Y]. Therefore, it had been known by the middle of 70's that $c_1(M)$ being zero or negative is also sufficient for the existence of Kähler-Einstein metrics on the underlying manifold.

Back to early 50's, using the maximum principle, Calabi had proved the uniqueness of Kähler-Einstein metrics within a fixed Kähler class for Kähler manifolds with nonpositive first Chern class. In 1986, Bando and Mabuchi proved the uniqueness of Kähler-Einstein metrics on compact Kähler manifolds with positive first Chern class.

What about the remaining case where $c_1(M) > 0$? In this case, the Kähler-Einstein metric, if it exists, must have positive scalar curvature. Note that a Kähler manifold M with $c_1(M) > 0$ is called a Fano manifold in algebraic geometry.

New difficulties and phenomena arise in this remaining case. In 1957, Matsushima proved that there is a Kähler-Einstein metric with positive scalar curvature on M only if the Lie algebra $\eta(M)$ of holomorphic fields is reductive [Mat]. This immediately implies that the positivity of the first Chern class is not enough for the existence of Kähler-Einstein metrics. For instance, if M is the blow-up of $\mathbf{C}P^2$ at one or two points, then $\eta(M)$ is not reductive, consequently, such an M does not have any Kähler-Einstein metrics. In 1983, Futaki introduced another analytic invariant–the Futaki invariant f_M [Fu1]. This f_M is a character of the Lie algebra $\eta(M)$. He proved that f_M is zero if M has a Kähler-Einstein metric. This invariant plays a more important role in our study here.

Since 1987, inhomogeneous Kähler-Einstein metrics are constructed on some M with $c_1(M) > 0$ by solving the corresponding complex Monge-Ampere equation (cf. [T1], [TY], [Di], [Si], [Na]). Previously, only known Kähler-Einstein metrics are either homogeneous or of cohomogeneity one. In later case, the problem can be reduced to solving an ODE equation ([Sa]).

Despite of those, the original problem of Calabi had been essentially untouched in the case of positive first Chern class, i.e., when is there a Kähler-Einstein metric on a compact manifold M with $c_1(M) > 0$? In 1989, I solved this problem for complex surfaces. The author proved that any complex surface M with $c_1(M) > 0$ has a Kähler-Einstein metric if and only if $\eta(M)$ is reductive. The solution also yields new insight into geometric aspects of the Calabi problem, though there are new technical difficulties in higher dimensions.

A folklore conjecture claims that there is a Kähler-Einstein metric on any compact Kähler manifold M with $c_1(M) > 0$ and without any holomorphic vector fields. One evidence for this was that all known obstructions came from holomorphic vector fields. Indeed, the conjecture was verified for complex surfaces in [T2].

However, the conjecture can not be generalized to Kähler orbifolds, since there is a two-dimensional Kähler orbifold S, such that $c_1(S) > 0$, $\eta(S) = \{0\}$ and it does not admit any Kähler-Einstein orbifold metrics (cf. [T6]). This orbifold has an isolated singularity, which may be responsible for nonexistence of Kähler-Einstein orbifold metrics on such an orbifold, like in the case of the Yamabe problem (cf. [Sc]).

Ding and Tian took a further step in [DT] (cf. Sect. 7). They defined the generalized Futaki invariant f_Y for any almost Fano variety (possibly singular). Using this invariant, we can now introduce a new notion of stability.

We recall that a degeneration of M is an algebraic fibration $\pi : W \mapsto \Delta$ without multiple fibers, such that M is biholomorphic to a fiber $W_z = \pi^{-1}(z)$ for some $z \in \Delta$, where Δ is the unit disk in $\mathbf{C}^1$. We say that this degeneration is special if its central fiber $W_0 = \pi^{-1}(0)$ is a normal variety, the determinant of its relative tangent bundle extends to be an ample bundle over W and the dilations $z \mapsto \lambda z$ on $\Delta\,(\lambda \leq 1)$ can be lifted to be a family of automorphisms $\sigma(\lambda)$ of W. Usually, we denote by v_W the holomorphic vector field on Y induced by those automorphisms. More precisely, $v_W = -\sigma'(1)$.

A particular case of such degenerations is the trivial fibration $M \times \Delta \mapsto \Delta$ for any M with nontrivial holomorphic vector fields. In this case, we say that W is trivial.

For any special degeneration $\pi : W \mapsto \Delta$, the central fiber $W_0 = \pi^{-1}(0)$ is an almost Fano variety (cf. Sect. 6). Clearly, the associated vector field v_W is tangent to Y_0 along the central fiber Y_0. Therefore, we can assign a number to each of such special degenerations, namely, the generalized Futaki invariant $f_{W_0}(v_W)$. Note that $f_{W_0}(v_W)$ can be calculated by using a residue formula of Atiyah-Bott-Lefschetz type, at least in the case that the singularities of W_0 are not too bad (cf. [Fu2], [DT1], [T4]). Such a residue formula depends only on the zeroes of v_W.

Definition 1.1. *We say that M is K-stable (resp. K-semistable), if M has no nontrivial holomorphic vector fields, and for any special degeneration W of M, the invariant* $f_{W_0}(v_W)$ *has positive (resp. nonnegative) real part. We say that M is weakly K-stable if* $\mathrm{Re}(f_{W_0}(v_W)) \geq 0$ *for any special degeneration W, and the equality holds if and only if W is trivial.*

More details on the K-stability can be found in Sect. 6.

Theorem 1.2. *If M admits a Kähler-Einstein metric with positive scalar curvature, then M is weakly K-stable. In particular, if M has no nonzero holomorphic vector fields, M is K-stable.*

Theorem 1.2 can be used to disprove the long-standing conjecture in the case of complex dimensions higher than two. A counterexample can be briefly described as follows (see Sect. 6 for details): let $G(4,7)$ be the complex Grassmannian manifold consisting of all 4-dimensional subspaces in $\mathbf{C}^7$, for any 3-dimensional subspace $P \subset \wedge^2\mathbf{C}^7$, one can define a subvariety X_P in $G(4,7)$ by

$$X_P = \{U \in G(4,7) | P \text{ projects to zero in } \wedge^2 (C^7/U)\}$$

For a generic P, X_P is a smooth 3-fold with $c_1(X_P) > 0$. I learned these manifolds from Mukai at Tokyo Metropolitan University, 1990. They were first constructed by Iskovskih (cf. [Is], [Muk]).

Take P_a to be the subspace spanned by bi-vectors

$$3e_1 \wedge e_6 - 5e_2 \wedge e_5 + 6e_3 \wedge e_4 + \sum_{j+k\geq 8} a_{1jk} e_j,$$

$$3e_1 \wedge e_7 - 2e_2 \wedge e_6 + e_3 \wedge e_5 + \sum_{j+k\geq 9} a_{2jk} e_j,$$

$$e_2 \wedge e_7 - e_3 \wedge e_6 + e_4 \wedge e_5 + \sum_{j+k\geq 10} a_{3jk} e_j,$$

where e_i are euclidean basis of $\mathbf{C}^7$ and $a = \{a_{ijk}\}$.

Then we can deduce from Theorem 1.2 that

Corollary 1.3. *For generic a, X_{P_a} has neither nontrivial holomorphic vector fields nor Kähler-Einstein metrics. In particular, the folklore conjecture is false for dimensions higher than two.*

Previously, in [DT1] Ding and the author proved that a cubic surface has a Kähler-Einstein orbifold metric only if the surface is K-semistable. There are many K-semistable, nonstable cubic surfaces, but they all have quotient singularities. Corollary 1.3 provides the first smooth example.

We still need to understand when Kähler-Einstein metrics exist on manifolds with positive first Chern class. The following conjecture provides the right answer to this problem.

Conjecture 1.4. *Let M be a compact Kähler manifold with positive first Chern class. Then M has a Kähler-Einstein metric if and only if M is weakly K-stable in the sense of Definition* 1.1.

This is the fully nonlinear version of the Hitchin-Kobayashi conjecture, which relates the existence of Hermitian-Yang-Mills metrics to the stability of holomorphic vector bundles. The Donaldson-Uhlenbeck-Yau theorem provides a solution for the Hitchin-Kobayashi conjecture [Do], [UY]. Theorem 1.2 proves the necessary part of Conjecture 1.4. As we will see, the proof is much harder than the counterpart for Hermitian-Yang-Mills metrics. So is the sufficient part of Conjecture 1.4. In a subsequent paper, we will consider the sufficient part. We will reduce the sufficient part to a partial C^0-estimate. Such a C^0-estimate was stated in [T6] and established in [T2] for complex surfaces.

Fix any $m > 0$ such that K_M^{-m} is very ample. Then we can embed M as a submanifold in the projective space $P(V)$, where $V = H^0(M, K_M^{-m})^*$. Then considering M as a point in the Hilbert scheme, on which $G = SL(V)$ acts naturally, we can introduce the CM-stability of M with respect to K_M^{-m}. We refer the readers to Sect. 8 (also [T4]) for details on the CM-stability.

Theorem 1.5 (Theorem 8.1). *If M admits a Kähler-Einstein metric with positive scalar curvature, then M is weakly CM-stable with respect to those plurianticanonical bundles which are very ample. Furthermore, if M has no nonzero holomorphic vector fields, then it is CM-stable.*

In fact, we expect that the converse to this theorem is also true (cf. Sect. 8).

In [Mum], Mumford gives an numerical criterion for his notion of stability. It is very interesting to compare this numerical criterion with the generalized Futaki invariant. It is not unreasonable to expect that they are equivalent, because of Theorem 1.2 and 1.5.

Now let us discuss analytic aspects of the above theorems. Let us fix a Kähler metric with its Kähler class ω representing $c_1(M)$. Let $P(M,\omega)$ be the set of smooth functions φ satisfying: $\omega + \partial\overline{\partial}\varphi > 0$. We define a functional F_ω on $P(M,\omega)(cf.\ [D_i])$ by

$$F_\omega(\varphi) = J_\omega(\varphi) - \frac{1}{V}\int_M \varphi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega - \varphi}\omega^n\right)$$

$$J_\omega(\varphi) = \sum_{i=0}^{n-1} \frac{n-i}{n+1}\frac{1}{V}\int_M \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i \wedge (\omega + \partial\overline{\partial}\varphi)^{n-i-1},$$

where $V = \int_M \omega^n = c_1(M)^n$, and h_ω is uniquely determined by

$$\mathrm{Ric}(\omega) - \omega = \partial\overline{\partial}h_\omega \quad \text{and} \quad \int_M (e^{h_\omega} - 1)\omega^n = 0.$$

One can regard $J_\omega(\varphi)$ as a generalized "energy" of φ. An easy computation shows that the critical points of F_ω are Kähler-Einstein metrics.

We say that F_ω is proper on $P(M,\omega)$, if for any $C > 0$ and $\{\varphi_i\} \subset P(M,\omega)_C$,

$$\overline{\lim}_{i\to\infty} F_\omega(\varphi_i) = \infty,$$

whenever $\lim_{i\to\infty} J(\varphi_i) = \infty$, where $P(M,\omega)_C$ consists of all φ in $P(M,\omega)$ satisfying:

$$\mathrm{osc}_M \varphi = \sup_M \varphi - \inf_M \varphi \le C(1 + J(\varphi)).$$

The properness of F_ω is independent of particular choices of Kähler metrics in $c_1(M)$.

Theorem 1.6. *Let M be a compact Kähler manifold with positive Chern class and without any nontrivial holomorphic fields. Then M has a Kähler-Einstein metric if and only if F_ω is proper.*

The sufficient part of this theorem is essentially proved in [DT2], by using the estimates in [T1]. We will sketch a proof of this in Sect. 2 for the reader's convenience. Clearly, Conjecture 1.4 follows from Theorem 1.6, if one can deduce the properness of F_ω from the K-stability of M. However, this is a highly nontrivial problem.

The organization of this paper is as follows: in Sect. 2, we discuss a few basic properties of F_ω and outline the proof of the sufficient part of Theorem 1.6. Sect. 3 contains a technical lemma needed for proving Theorem 1.6. In Sect. 4, we prove Theorem 1.6. In Sect. 5, based on the arguments in Sect. 4, we derive new inequalities involving F_ω and the K-energy of T. Mabuchi. In Sect. 6, we discuss the K-stability in details and prove Theorem 1.2. In Sect. 7, applying Theorem 1.2, we construct a counterexample to the long-standing conjecture described above. We will also introduce a class of Fano manifolds with additional structures, so called obstruction triples. We will show that those are obstructions to the existence of Kähler-Einstein metrics with positive scalar curvature. In Sect. 8, we prove the CM-stability of Fano manifolds which admit Kähler-Einstein metrics. The basic idea of the proof was already in [T4] and can be applied to other general cases (cf. [T5]). A conjecture will be stated concerning the Kähler-Einstein metrics and the CM-stability. In last section, we discuss the relation between obstruction triples and Ricci solitons. Two problems will be proposed.

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Contents

2 The functional F_ω

In this section, we collect a few facts about the functional F_ω and prove the sufficient part of Theorem 1.6.

Recall that for any $\varphi \in P(M,\omega)$, we have

$$F_\omega(\varphi) = \sum_{i=0}^{n-1} \frac{n-i}{n+1} \frac{1}{V} \int_M \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i \wedge (\omega + \partial\overline{\partial}\varphi)^{n-i-1}$$
$$- \frac{1}{V} \int_M \varphi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega - \varphi}\omega^n\right)$$

where $V = \int_M \omega^n$, and h_ω is determined by $\mathrm{Ric}(\omega) - \omega = \partial\overline{\partial} h_\omega$ and $\int_M (e^{h_\omega} - 1)\omega^n = 0$. Clearly, $F_\omega(\varphi + c) = F_\omega(\varphi)$.

The following identities can be proved (cf. [DT2]):
(1) if $\omega' = \omega + \partial\overline{\partial}\varphi$ is another Kähler metric, then $F_\omega(\varphi) = -F_{\omega'}(-\varphi)$;
(2) If ω' is as above and $\omega'' = \omega' + \partial\overline{\partial}\psi$ is a Kähler metric, then

$$F_\omega(\varphi) + F_{\omega'}(\psi) = F_\omega(\varphi + \psi),$$

in particular, it follows that the properness of F_ω is independent of the choice of the initial metric ω;
(3) The critical points of F_ω correspond to Kähler-Einstein metrics ω with $\mathrm{Ric}(\omega) = \omega$. In particular, the critical points are independent of the initial metric ω.

Proposition 2.1. *If F_ω is proper, then its minimum can be attained, in particular, there is a Kähler-Einstein metric on M.*

Proof. As we said before, this proposition is essentially known (cf. [DT2]). For reader's convenience, we sketch a proof here.

Consider the complex Monge-Ampere equations:

$$(\omega + \partial\overline{\partial}\varphi)^n = e^{h_\omega - t\varphi}\omega^n, \quad \omega + \partial\overline{\partial}\varphi > 0 \qquad (2.1)_t$$

The solution for $t = 1$ gives rise to a Kähler-Einstein metric on M. To find a solution for $t = 1$, we only need to prove that $J_\omega(\varphi)$ is uniformly bounded for any solution of $(2.1)_t$ (cf. [BM], [T1], [T2]). By the implicit function theorem, one can show that the solution of $(2.1)_t$ varies smoothly with $t < 1$.

Let $\{\varphi_t\}$ be a smooth family of solutions such that φ_t solves $(2.1)_t$. By a direct computation, we have (cf. [DT2])

$$\frac{d}{dt}\left(t(J_\omega(\varphi_t) - \frac{1}{V}\int_M \varphi_t\omega^n)\right) = -(I_\omega(\varphi_t) - J_\omega(\varphi_t)),$$

where

$$I_\omega(\varphi) = \frac{1}{V}\int_M \varphi(\omega^n - (\omega + \partial\overline{\partial}\varphi)^n),$$

In fact, $J_\omega(\varphi) = \int_0^1 \frac{I_\omega(s\varphi)ds}{s}$. It is known (cf. [Au2], [BM], [T1]) that $I_\omega(\varphi_t) - J_\omega(\varphi_t) \geq 0$, consequently,

$$F_\omega(\varphi_t) \leq -\log\left(\frac{1}{V}\int_M e^{h_\omega - \varphi_t}\omega^n\right)$$

Using $(2.1)_t$ and the concavity of the logrithmic function, one can deduce

$$-\log\left(\frac{1}{V}\int_M e^{h_\omega-\varphi_t}\omega^n\right) \le \frac{1-t}{V}\int_M \varphi_t(\omega+\partial\overline{\partial}\varphi_t)^n$$

By the Moser iteration, one can show (cf. [T1]) that for some uniform constant $c > 0$,

$$\frac{1}{V}\int_M \varphi_t(\omega+\partial\overline{\partial}\varphi_t)^n \le c\left(1+\inf_M \varphi_t\right),$$

therefore, we have $F_\omega(\varphi_t) \le (1-t)c \le c$. On the other hand, by the Green formula and the fact that $\Delta_\omega\varphi_t \ge -n$, one has

$$\sup_M \varphi_t \le c\left(1+\frac{1}{V}\int_M \varphi_t\omega^n\right)$$

It follows that $\varphi_t \in P(M,\omega)_C$ for some constant C. Then the properness of F_ω implies that $J_\omega(\varphi_t)$, and consequently, $\| \varphi_t \|_{C^0}$ is uniformly bounded. Therefore, there is a Kähler-Einstein metric on M.

3 A smoothing lemma

Let M be a compact Kähler manifold with positive first Chern class as before. For any Kähler metric ω in $c_1(M)$, we denote by $\lambda_{1,\omega}$ and σ_ω the first nonzero eigenvalue and the Sobolev constant of (M,ω). Note that for any smooth function u, one has

$$\left(\frac{1}{V}\int_M |u|^{\frac{2n}{n-1}}\omega^n\right)^{\frac{n-1}{n}} \le \frac{\sigma_\omega}{V}\left(\int_M \partial u\wedge\overline{\partial}u\wedge\omega^{n-1}+\int_M |u|^2\omega^n\right). \tag{3.1}$$

This section is devoted to the proof of the following proposition.

Proposition 3.1. *Let ω be any Kähler metric in $c_1(M) > 0$ with $\mathrm{Ric}(\omega) \ge (1-\epsilon)\omega$. Then there is another Kähler metric of the form $\omega' = \omega + \partial\overline{\partial}\varphi$ satisfying: (1) $\| \varphi \|_{C^0} \le e \| h_\omega \|_{C^0}$; (2) $\|h_{\omega'}\|_{C^{\frac{1}{2}}} \le C(1+\|h_\omega\|^2_{C^0})^{n+1}\epsilon^\beta$, where $\beta = \frac{1}{4}e^{-n}$, $C = C(n,\lambda_{1,\omega'},\sigma_{\omega'})$ is a constant depending only on the dimension n, the Poincare constant $\lambda_{1,\omega'}$ and the Sobolev constant $\sigma_{\omega'}$* of ω'.

Consider the heat flow

$$\frac{\partial u}{\partial s} = \log\left(\frac{(\omega+\partial\overline{\partial}u)^n}{\omega^n}\right)+u-h_\omega,\ u|_{s=0}=0 \tag{3.2}$$

This is in fact Hamilton's Ricci flow. This has been used before in [Ca], [Ba], etc.. We will denote by u_s and ω_s the function $u(s,\cdot)$ and the Kähler form $\omega+\partial\overline{\partial}u_s$.

Differentiating (3.2), we obtain

$$\partial\overline{\partial}\left(\frac{\partial u}{\partial s}\right) = -\mathrm{Ric}(\omega_s) + \omega_s$$

This implies that $h_{\omega_s} = -\frac{\partial u}{\partial s} + c_s$, where c_s are constants. Since $u_0 = 0$, we have $c_0 = 0$.

First we collect a few basic estimates for the solutions of (3.2). They are simple corollaries of the maximum principle for heat equations.

Lemma 3.2. *We have* $\|u_s\|_{C^0} \leq e^s \|h_\omega\|_{C^0}$, *and* $\|\frac{\partial u}{\partial s}\|_{C^0} \leq e^s \|h_\omega\|_{C^0}$.

It follows from the maximum principle for heat equations. Here we need to use the equation

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) = \Delta\left(\frac{\partial u}{\partial s}\right) + \frac{\partial u}{\partial s} \tag{3.3}$$

Lemma 3.3. *We have* $\inf_M \omega_s \geq e^s \inf_M \Delta h_\omega$.

Proof. It follows from (3.3)

$$\begin{aligned}\frac{\partial}{\partial s}\left(\Delta\left(\frac{\partial u}{\partial s}\right)\right) &= \Delta^2\left(\frac{\partial u}{\partial s}\right) + \Delta\left(\frac{\partial u}{\partial s}\right) - \left|\nabla\overline{\nabla}\left(\frac{\partial u}{\partial s}\right)\right|^2 \\ &\leq \Delta^2\left(\frac{\partial u}{\partial s}\right) + \Delta\left(\frac{\partial u}{\partial s}\right)\end{aligned} \tag{3.4}$$

Then the lemma follows from the maximum principle and $h_{\omega_s} = -\frac{\partial u}{\partial s} + \mathrm{const}$.

The proof of next two lemmata are due to S. Bando [Ba]. They are more tricky than previous ones.

Lemma 3.4. *We have*

$$\left\|\left|\frac{\partial u}{\partial s}\right|_s\right\|_{C^0}^2 + s\|\nabla\left(\left.\frac{\partial u}{\partial s}\right|_s\right)\|_{C^0}^2 \leq e^{2s}\|h_\omega\|_{C^0}^2,$$

in particular, $\|\nabla h_{\omega_1}\|_{C^0} \leq e^2 \|h_\omega\|_{C^0}$.

Proof. By straightforward computations, we can deduce

$$\begin{aligned}\frac{\partial}{\partial s}\left(\left|\frac{\partial u}{\partial s}\right|^2\right) &= \Delta\left|\frac{\partial u}{\partial s}\right|^2 - 2\left|\nabla\frac{\partial u}{\partial s}\right|^2 + 2\left|\frac{\partial u}{\partial s}\right|^2 \\ \frac{\partial}{\partial s}\left(\left|\nabla\frac{\partial u}{\partial s}\right|^2\right) &= \Delta\left|\nabla\frac{\partial u}{\partial s}\right|^2 - \left|\nabla\nabla\frac{\partial u}{\partial s}\right|^2 - \left|\overline{\nabla}\nabla\frac{\partial u}{\partial s}\right|^2 + \left|\nabla\frac{\partial u}{\partial s}\right|^2\end{aligned} \tag{3.5}$$

Since $s \geq 0$, we have

$$\begin{aligned} &\frac{\partial}{\partial s}\left(\left|\frac{\partial u}{\partial s}\right|^2 + s\left|\nabla\frac{\partial u}{\partial s}\right|^2\right) \\ \leq\ &\Delta\left(\left|\frac{\partial u}{\partial s}\right|^2 + s\left|\nabla\frac{\partial u}{\partial s}\right|^2\right) + 2\left(\left|\frac{\partial u}{\partial s}\right|^2 + s\left|\nabla\frac{\partial u}{\partial s}\right|^2\right) \end{aligned}$$

Then the lemma follows from the maximum principle for heat equations.

Finally, we need to bound $\Delta(\frac{\partial u}{\partial s})$. From (3.4) and (3.5), one deduces

$$\begin{aligned} &\frac{\partial}{\partial s}\left(\left|\nabla\frac{\partial u}{\partial s}\right|^2 - \alpha s\Delta\left(\frac{\partial u}{\partial s}\right)\right) \\ \leq\ &\Delta\left(\left|\nabla\frac{\partial u}{\partial s}\right|^2 - \alpha s\Delta\left(\frac{\partial u}{\partial s}\right)\right) + \left(\left|\nabla\frac{\partial u}{\partial s}\right|^2 - \alpha s\Delta\left(\frac{\partial u}{\partial s}\right)\right) \\ &- \alpha\Delta\left(\frac{\partial u}{\partial s}\right) - (1-\alpha s)\left|\nabla\overline{\nabla}\left(\frac{\partial u}{\partial s}\right)\right|^2 \end{aligned} \tag{3.6}$$

The Cauchy-Schwartz inequality implies

$$n\left|\nabla\overline{\nabla}\frac{\partial u}{\partial s}\right|^2 \geq \left|\Delta\frac{\partial u}{\partial s}\right|^2 \tag{3.7}$$

Combining this with (3.6), we obtain

$$\left(\frac{\partial}{\partial s} - \Delta\right)\left(e^{-s}\left(\left|\nabla\frac{\partial u}{\partial s}\right|^2 - \alpha s\Delta\left(\frac{\partial u}{\partial s}\right)\right)\right) \leq -\frac{1-\alpha s}{n}\left|\Delta\frac{\partial u}{\partial s}\right|^2 - \alpha\Delta\left(\frac{\partial u}{\partial s}\right) \tag{3.8}$$

Together with Lemma 3.4 and the maximum principle, this implies that for $\alpha s < 1$,

$$\Delta\left(-\frac{\partial u}{\partial s}\right) \leq \left(\frac{e^{2s}}{\alpha s}\|h_\omega\|^2_{C^0} + \frac{n\alpha}{1-\alpha s}\right)e^s$$

Choosing $\alpha = 1/2s$ and using Lemma 3.3, 3.4, one obtains

Lemma 3.6. *We have*

$$-n\epsilon e^{2s} \leq \Delta\left(-\frac{\partial u}{\partial s}\right) \leq \left(\frac{n}{s} + 2e^{2s}\|h_\omega\|^2_{C^0}\right)e^s, \tag{3.9}$$

in particular, if $s = 1$, *we have*

$$\left|\Delta\frac{\partial u}{\partial s}\right| \leq \max\{n\epsilon e^2, ne + 2e^3\|h_\omega\|^2_{C^0}\}, \tag{3.10}$$

where ϵ *is given in* Proposition 3.1.

We put v to be

$$h_{\omega_1} - \frac{1}{V}\int_M h_{\omega_1}\omega_1^n \tag{3.11}$$

For simplicity, we denote by λ_1 and σ_1 the Poincare and Sobolev constant of the metric ω_1, respectively.

Lemma 3.7. *We have*

$$\|v\|_{C^0} \le \left(\frac{n}{n-1}\right)^{\frac{n(n-1)}{2}}(2\sigma_1 b)^{\frac{n}{2}}\left(1+\frac{b}{\lambda_1}\right), \tag{3.12}$$

where $b = \max\{n\epsilon e^2, ne + 2e^3\|h_\omega\|_{C^0}^2\}$.

Proof. We put $v_\pm = \max\{0, \pm v\}$. By (3.9), we have

$$\frac{1}{V}\int_M |\nabla v_+|^2\omega_1^n \le \frac{\epsilon e^2}{V}\int_M v_+\omega_1^n, \quad \frac{1}{V}\int_M |\nabla v_-|^2\omega_1^n \le \frac{ne + 2e^3\|h_\omega\|_{C^0}^2}{V}\int_M v_-\omega_1^n,$$

therefore,

$$\frac{\lambda_1}{V}\int_M |v|^2\omega_1^n \le \frac{b}{V}\int_M |v|\omega_1^n,$$

consequently,

$$\frac{1}{V}\int_M |v|^2\omega_1^n \le \frac{b^2}{\lambda_1^2}. \tag{3.13}$$

Next we put $v_1 = \max\{1, v\}$. Then for any $p \ge 1$,

$$\frac{1}{V}\int_M \left|\nabla v_1^{\frac{p+1}{2}}\right|^2\omega_1^n \le \frac{(p+1)^2 b}{4pV}\int_M v_1^p\omega_1^n \tag{3.14}$$

Since $b \ge 1$ and $v_1 \ge 1$, by the Sobolev inequality, one can deduce from (3.14),

$$\left(\frac{1}{V}\int_M v_1^{\frac{n(p+1)}{n-1}}\omega_1^n\right)^{\frac{n-1}{n(p+1)}} \le (\sigma_1 b(p+1))^{\frac{1}{p+1}}\left(\frac{1}{V}\int_M v_1^{p+1}\omega_1^n\right)^{\frac{1}{p+1}}$$

Then the standard Moser iteration yields

$$\sup_M v \le \sup_M v_1 \le \left(\frac{n}{n-1}\right)^{\frac{n(n-1)}{2}}(2b\sigma_1)^{\frac{n}{2}}(1+\|v\|_{L^2}), \tag{3.15}$$

where $\|v\|_{L^2}$ denotes the L^2-norm of v with respect to ω_1.

Similarly, $\inf_M v \geq -(\frac{n}{n-1})^{\frac{n(n-1)}{2}}(2b\sigma_1)^{\frac{n}{2}}(1+\|v\|_{L^2})$. Then the lemma follows from this, (3.15) and (3.13).

Lemma 3.8. *We have*

$$\|v\|_{L^2}^2 \leq \left(\frac{n}{n-1}\right)^{\frac{n(n-1)}{2}} \frac{2ne\epsilon(2\sigma_1 b)^{\frac{n}{2}}}{\lambda_1}\left(1+\frac{b}{\lambda_1}\right) \tag{3.16}$$

where $ne\epsilon$ is the lower bound of Δh_ω.

Proof. By Lemma 3.3, we have $\Delta v + ne\epsilon \geq 0$, therefore,

$$\int_M |\Delta v + ne\epsilon|\omega_1^n = ne\epsilon V$$

It follows that

$$\frac{\lambda_1}{V}\int_M |v|^2\omega_1^n \leq \frac{1}{V}\int_M |\nabla v|^2\omega_1^n \leq 2ne\epsilon\|v\|_{C^0}$$

Then (3.16) follows from (3.12).

Proof of Proposition 3.1. First we improve the estimate of v in (3.12). We may assume that $\epsilon \leq 1$ and $\sigma_1 \geq 1$.

Using (3.9) and (3.12), we have

$$\frac{1}{V}\int_M \left|\nabla v^{\frac{p+1}{2}}\right|^2 \omega_1^n \leq \frac{(p+1)^2 b}{4pV}\int_M |v|^p\omega_1^n,$$

where $p \geq 1$. Using the Solobev inequality, we deduce from this

$$\left(\frac{1}{V}\int_M |v|^{\frac{n(p+1)}{n-1}}\omega_1^n\right)^{\frac{n-1}{n(p+1)}} \leq \left(\sigma_1\left(\|v\|_{C^0}+\frac{b(p+1)}{2}\right)\right)^{\frac{1}{p+1}}\left(\frac{1}{V}\int_M |v|^p\omega_1^n\right)^{\frac{1}{p+1}} \tag{3.17}$$

Substituting p in (3.17) by $p_i = \frac{n}{n-1}(p_{i-1}+1)$ inductively, where $p_0 = 1$ and $i \geq 0$, we obtain

$$\sup_M |v| \leq \prod_{i=0}^{\infty}\left(\sigma_1\left(\|v\|_{C^0}+\frac{b(p_i+1)}{2}\right)\right)^{\frac{1}{p_i+1}} \|v\|_{L^1}^{\prod_{i=0}^{\infty}\frac{p_i}{p_i+1}} \tag{3.18}$$

Simple computations show

$$\prod_{i=0}^{\infty}\frac{p_i}{p_i+1} \geq \frac{1}{\prod_{i=0}^{\infty}\left(1+(\frac{n-1}{n})^i\right)} \geq e^{-n},$$

$$\prod_{i=0}^{\infty}\left(\sigma_1\left(\|v\|_{C^0}+\frac{b(p_i+1)}{2}\right)\right)^{\frac{1}{p_i+1}} \leq \left(\frac{2n}{n-1}\right)^{\frac{n(n-1)}{2}}\left(\frac{b(1+\|v\|_{C^0})}{2}\right)^{\frac{n}{2}}$$

Therefore, by Lemma 3.6, we have

$$\|v\|_{C^0} \leq C(n, \lambda_1, \sigma_1)(1 + \|h_\omega\|_{C^0}^2)^{n+1} \epsilon^{\frac{1}{2}e^{-n}} \tag{3.19}$$

We will always denote by $C(n, \lambda_1, \sigma_1)$ a constant depending only on n, λ_1, σ_1. In fact, the constant in (3.19) can be given explicitely.

Since $\frac{1}{V}\int_M e^{h_{\omega_1}} \omega_1^n = 1$, by (3.11) and (3.19), we have

$$\|h_{\omega_1}\|_{C^0} \leq C(n, \lambda_1, \sigma_1)(1 + \|h_\omega\|_{C^0}^2)^{n+1} \epsilon^{\frac{1}{2}e^{-n}} \tag{3.20}$$

Let x, y be any two points in M, and let $d(x, y)$ be the distance between them with respect to the metric ω_1. If $d(x, y) \geq 2\epsilon^{\frac{1}{2}e^{-n}}$, then it follows from (3.20),

$$\frac{|h_{\omega_1}(x) - h_{\omega_1}(y)|}{\sqrt{d(x, y)}} \leq C(n, \lambda_1, \sigma_1)(1 + \|h_\omega\|_{C^0}^2)^{n+1} \epsilon^{\frac{1}{4}e^{-n}}.$$

On the other hand, if $d(x, y) < 2\epsilon^{\frac{1}{2}e^{-n}}$, then by Lemma 3.4, $\|\nabla h_{\omega_1}\|_{C^0} \leq e^2 \|h_\omega\|_{C^0}$ and consequently,

$$\frac{|h_{\omega_1}(x) - h_{\omega_1}(y)|}{\sqrt{d(x, y)}} \leq 2e^2 \|h_\omega\|_{C^0} \epsilon^{\frac{1}{4}e^{-n}}.$$

Then the proposition follows.

Remark. The estimate in Proposition 3.1 can be improved as follows: let G be the Green function of the metric ω_1. Suppose that $0 \leq G(x, y) \leq c/d(x, y)^{2n-2}$ and $\mathrm{vol}\,(B_r(x)) \leq cr^n$ for any $x, y \in M$ and $r > 0$, where c is some uniform constant. Then by the Green formula,

$$\begin{aligned}
\|v\|_{C^0} &\leq \sup_{x \in M} \left| \frac{1}{V} \int_M \Delta v(y) G(x, y) \omega_1^n(y) \right| \\
&\leq \sup_M \left(\frac{e\epsilon}{V} \int_M G(x, y) \omega_1^n(y) + \frac{1}{V} \int_M |\Delta v + e\epsilon|(y) G(x, y) \omega_1^n(y) \right) \\
&\leq \frac{C}{r^{2n-2}} \left(\epsilon + \frac{1}{V} \int_M |\Delta v + e\epsilon| \omega_1^n \right) + \sup_M \left(\frac{b}{V} \int_{B_r(x)} G(x, y) \omega_1^n(y) \right) \\
&\leq \frac{C\epsilon}{r^{2n-2}} + Cbr^2
\end{aligned} \tag{3.21}$$

where C always denotes a constant depending only on c.

Choosing $r = \left(\frac{\epsilon}{b}\right)^{\frac{1}{2n}}$ in (3.21), we deduce

$$\|h_{\omega_1}\|_{C^0} \leq C b^{\frac{n-1}{n}} \epsilon^{\frac{1}{n}} \tag{3.22}$$

Therefore,

$$\|h_{\omega_1}\|_{C^{\frac{1}{2}}} \leq C(1+\|h_\omega\|^2_{C^0})^{\frac{n-1}{n}}\epsilon^{\frac{1}{2n}} \tag{3.23}$$

The conditions on G and $B_r(x)$ are often satisfied.

4 Proof of Theorem 1.6

By Proposition 2.1, we only need to prove the necessary part of Theorem 1.6, namely, if M has a Kähler-Einstein metric ω_{KE} with $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$ and no nontrivial holomorphic fields, then $F_{\omega_{KE}}$ is proper.

Let ω be any Kähler metric in $c_1(M)$. Using the fact that $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$, we can find a function φ_1, such that $\omega_{KE} = \omega + \partial\overline{\partial}\varphi_1$, and $\omega^n_{KE} = e^{h_\omega - \varphi_1}\omega^n$.

Lemma 4.1. *The first nonzero eigenvalue $\lambda_{1,\omega_{KE}}$ is strictly greater than* 1.

This is well-known and follows from the standard Bochner identity.

Consider complex Monge-Ampere equations

$$(\omega+\partial\overline{\partial}\varphi)^n = e^{h_\omega - t\varphi}\omega^n, \quad \omega+\partial\overline{\partial}\varphi > 0, \tag*{$(4.1)_t$}$$

where $0 \leq t \leq 1$. Clearly, φ_1 is a solution of $(4.1)_1$. By Lemma 4.1 and the implicit function theorem, $(4.1)_t$ has a solution φ_t for t sufficiently close to 1. In fact, it is known (cf. [BM]) that $(4.1)_t$ has a unique solution φ_t for any $t \in [0,1]$. This is because $I_\omega(\varphi_t) - J_\omega(\varphi_t)$ is nondecreasing with t, and consequently, the C^3-norm of φ_t can be uniformly bounded.

Put $\omega_t = \omega + \partial\overline{\partial}\varphi_t$. Then $\omega_1 = \omega_{KE}$. Simple computations show that

$$\begin{aligned} h_{\omega_t} &= -(1-t)\varphi_t + c_t, \\ \mathrm{Ric}(\omega_t) &= t\omega_t + (1-t)\omega, \end{aligned}$$

where c_t is determined by

$$\int_M \left(e^{-(1-t)\varphi_t + c_t} - 1\right)\omega_t^n = 0.$$

In particular, we have $|c_t| \leq (1-t)\|\varphi_t\|_{C^0}$. Also it follows that

$$\Delta h_{\omega_t} + n(1-t) > 0 \tag{4.2}$$

We apply Proposition 3.1 to each ω_t and obtain a Kähler metric $\omega'_t = \omega_t + \partial\overline{\partial}u_t$ satisfying:

$$\begin{aligned} &\|u_t\|_{C^0} \leq e(1-t)\|\varphi_t\|_{C^0}, \\ &\|h_{\omega'_t}\|_{C^{\frac{1}{2}}} \leq C(n, \lambda_{1,\omega'_t}, \sigma_{\omega'_t})(1+(1-t)^2\|\varphi_t\|^2_{C^0})^{n+1}(1-t)^\beta, \end{aligned} \tag{4.3}$$

where $\beta = \frac{1}{4}e^{-n}$. In particular, $u_1 = 0$ and $\omega'_1 = \omega_{KE}$.

We choose μ_t by

$$\int_M e^{h_{\omega_t}-u_t+\mu_t}\omega_t^n = V \tag{4.4}$$

Then by (4.3), $|\mu_t| \le e(1-t)\|\varphi_t\|_{C^0}$.

As before, there are ψ_t such that $\omega_{KE} = \omega'_t + \partial\bar{\partial}\psi_t$ and

$$\omega_{KE}^n = e^{h_{\omega'_t}-\psi_t}\omega_t'^n \tag{4.5}$$

It follows from the maximum principle that

$$\varphi_t = \varphi_1 - \psi_t - u_t + \mu_t + c_t \tag{4.6}$$

Hence, φ_t is uniformly equivalent to φ_1 as long as ψ_t is uniformly bounded.

Consider the operator $\Phi_t : C^{2,\frac{1}{2}}(M) \mapsto C^{0,\frac{1}{2}}$:

$$\Phi_t(\psi) = \log\left(\frac{(\omega_1 - \partial\bar{\partial}\psi)^n}{\omega_1^n}\right) - h_{\omega'_t} - \psi \tag{4.7}$$

Its linearization at $\psi = 0$ is $-\Delta - 1$, so it is invertible by Lemma 4.1. Then by the implicit function theorem, there is a $\delta > 0$, which depends only on the lower bound of $\lambda_{1,\omega_{KE}} - 1$, satisfying: if the Hölder norm $\|h_{\omega'_t}\|_{C^{\frac{1}{2}}(\omega_1)}$ with respect to ω_1 is less than δ, then there is a unique ψ such that $\Phi_t(\psi) = 0$ and $\|\psi\|_{C^{2,\frac{1}{2}}(\omega_1)} \le C\delta$. Note that C always denotes a uniform constant.

We observe that

$$\lambda_{1,\omega'} \ge 2^{-n-1}\lambda_{1,\omega_{KE}}, \quad \sigma_{\omega'} \le 2^{n+1}\sigma_{\omega_{KE}},$$

whenever $\frac{1}{2}\omega_{KE} \le \omega' \le 2\omega_{KE}$.

From now on, we will further assume that

$$C \ge C(n, 2^{-n-1}\lambda_{1,\omega_{KE}}, 2^{n+1}\sigma_{\omega_{KE}})$$

We choose t_0 such that

$$\begin{aligned} &(1-t_0)^\beta(1+(1-t_0)^2\|\varphi_{t_0}\|_{C^0}^2)^{n+1} \\ &= \sup_{t_0\le t\le 1}(1-t)^\beta(1+(1-t)^2\|\varphi_t\|_{C^0}^2)^{n+1} \\ &= \frac{\delta}{4C(n, 2^{-n-1}\lambda_{1,\omega_{KE}}, 2^{n+1}\sigma_{\omega_{KE}})} \end{aligned} \tag{4.8}$$

We may assume that $C\delta < \frac{1}{4}$. We claim that for any $t \in [t_0, 1]$, $\|\psi_t\|_{C^{2,\frac{1}{2}}(\omega_1)} < \frac{1}{4}$. Assume that this is not true. Since $\psi_1 = 0$, there is a t in

$[t_0, 1]$ such that $\|\psi_t\|_{C^{2,\frac{1}{2}}(\omega_1)} = \frac{1}{4}$. It follows that $\frac{1}{2}\omega_{KE} \le \omega_t' \le 2\omega_{KE}$, so the above arguments show that $\|\psi_t\|_{C^{2,\frac{1}{2}}(\omega_1)} \le C\delta < \frac{1}{4}$, a contradiction. Thus the claim is established. It follows from (4.6) that for all $t \ge \max\{t_0, 1 - \frac{1}{4e}\}$,

$$\|\varphi_t\|_{C^0} \ge (1 - 3e(1-t))\|\varphi_1\|_{C^0} - 1 \tag{4.9}$$

Since $I_\omega(\varphi_t) - J_\omega(\varphi_t)$ is nondecreasing (cf. [T1]), we have

$$\begin{aligned} F_{\omega_{KE}}(-\varphi_1) &= -F_\omega(\varphi_1) \\ &= \int_0^1 (I_\omega(\varphi_t) - J_\omega(\varphi_t))dt \\ &\ge \min\left\{1 - t_0, \frac{1}{4e}\right\}\left(I_\omega(\varphi_{t_0}) - J_\omega(\varphi_{t_0})\right) \\ &\ge \min\left\{1 - t_0, \frac{1}{4e}\right\}(I_\omega(\varphi_1) - J_\omega(\varphi_1)) - 20(1-t_0)^2\|\varphi_{t_0}\|_{C^0} - 2 \end{aligned} \tag{4.10}$$

By the choice of t_0 in (4.8), one can deduce from (4.10) that

$$F_{\omega_{KE}}(-\varphi_1) \ge \min\left\{1 - t_0, \frac{1}{4e}\right\} J_{\omega_{KE}}(-\varphi_1) - C \tag{4.11}$$

If $1 - t_0 \ge \frac{1}{4e}$, then by (4.11),

$$F_{\omega_{KE}}(-\varphi_1) \ge \frac{1}{4ne} J_{\omega_{KE}}(-\varphi_1) - C,$$

if $1 - t_0 \le \frac{1}{4e}$, then by (4.8), (4.11) and the fact that $\|\varphi_1\|_{C^0} \le \mathrm{osc}_M \varphi_1$,

$$F_{\omega_{KE}}(-\varphi_1) \ge \frac{1}{2}\left(\frac{\delta}{4C(n)}\right)^{\frac{1}{2n+2+\beta}} \frac{J_{\omega_{KE}}(-\varphi_1)}{(1 + \mathrm{osc}_M \varphi_1)^{\frac{2n+2}{2n+2+\beta}}} - C \tag{4.12}$$

Thus we have proved

Theorem 4.3. *Let (M, ω_{KE}) be a Kähler-Einstein manifold without any holomorphic vector fields. Then $F_{\omega_{KE}}$ is proper.*

Clearly, Theorem 1.6 follows from Proposition 2.1 and Theorem 4.3.

Theorem 1.6 also has the following generalization. Let G be a maximal compact subgroup in the identity component of $\mathrm{Aut}(M)$. Let ω be a G-invariant Kähler metric in $c_1(M) > 0$. Define $P_G(M, \omega)$ to be the set of smooth, G-invariant functions φ satisfying: $\omega + \partial\bar{\partial}\varphi > 0$. Then we have

Theorem 4.4. *Let (M, ω) be given as above. Then M has a G-invariant Kähler-Einstein metric if and only if F_ω is proper on $P_G(M, \omega)$.*

Its proof is exactly the same as that of Theorem 1.6, so we omit it.

5 New inequalities

In this section, we derive some nonlinear inequalities on Kähler-Einstein manifolds. They generalize the Moser-Trudinger inequality on S^2.

First let us give a corollary of (4.12). We define $P(M,\omega,\epsilon)$ to be the set of all φ in $P(M,\omega)$ such that $\sigma_{\omega_\varphi} \leq 1/\epsilon$, where $\omega_\varphi = \omega + \partial\overline{\partial}\varphi$.

Theorem 5.1. *Let (M,ω_{KE}) be a Kähler-Einstein manifold without holomorphic vector fields. Then for any φ* in $P(M,\omega_{KE},\epsilon)$,

$$F_{\omega_{KE}}(\varphi) \geq a_{1,\epsilon} J_{\omega_{KE}}(\varphi)^{\frac{\beta}{2n+2+\beta}} - a_{2,\epsilon}, \tag{5.1}$$

where $a_{1,\epsilon}$, $a_{2,\epsilon}$ are constants, which depend only on n, ϵ and the lower bound of $\lambda_{1,\omega_{KE}} - 1$ from 0.

Proof. We will adopt the notations in (4.1)–(4.12), and use C to denote a constant depending only on n and ϵ.

By (4.12), we need to prove only that

$$\mathrm{osc}_M \varphi_1 \leq C(1 + I_\omega(\varphi_1) - J_\omega(\varphi_1)), \tag{5.2}$$

where $\omega = \omega_{KE} + \partial\overline{\partial}\varphi$. We note that $\varphi = -\varphi_1$.

Notice that $-\Delta_\omega(\varphi_1 - \inf_M \varphi_1) \leq n$. By the Moser iteration and the fact that $\sigma_\omega \leq \frac{1}{\epsilon}$, one can derive

$$\begin{aligned}\mathrm{osc}_M \varphi_1 &\leq C\left(1 + \left(\frac{1}{V}\int_M \left(\varphi_1 - \inf_M \varphi_1\right)^2 \omega^n\right)^{\frac{1}{2}}\right) \\ &\leq C\left(1 + (\mathrm{osc}_M \varphi_1)^{\frac{1}{2}}\left(\frac{1}{V}\int_M \left(\varphi_1 - \inf_M \varphi_1\right)\omega^n\right)^{\frac{1}{2}}\right)\end{aligned} \tag{5.3}$$

Since $\Delta\varphi_1 \leq n$, where Δ is the laplacian of ω_{KE}, we have

$$\frac{1}{V}\int_M \left(\varphi_1 - \inf_M \varphi_1\right)\omega_{KE}^n \leq \sup_x \left(\frac{n}{V}\int_M \Delta\varphi_1(y) G(x,y)\omega_{KE}^n(y)\right),$$

where $G(\cdot,\cdot)$ denotes the Green function of ω_{KE} with $\inf G(x,y) = 0$. Hence,

$$\frac{1}{V}\int_M \left(\varphi_1 - \inf_M \varphi_1\right)\omega_{KE}^n \leq I_\omega(\varphi_1) + C \tag{5.4}$$

Though we do not know yet if C is independent of M, it implies (5.1) with constants possibly depending on M.

To make sure that $a_{1,\epsilon}$, $a_{2,\epsilon}$ are independent of M, we need a uniform bound on $G(\cdot,\cdot)$. This uniform bound follows from Cheng and Li's heat kernel estimates in [CL] (also see [BM]).

Clearly, (5.2) follows from (5.3) and (5.4), so the theorem is proved.

Example. Since $c_1(M) > 0$, one can embed M into $\mathbf{C}P^N$ by a basis $H^0(M, K_M^{-m})$ for m sufficiently large. Then there is a natural family of metrics $\frac{1}{m}\sigma^*\omega_{FS}$, where ω_{FS} is the Fubini-Study metric on $\mathbf{C}P^N$ and $\sigma \in SL(N+1, \mathbf{C})$. The Sobolev inequality holds uniformly for these metrics, since $\sigma(M)$ is a complex submanifold (cf. [Sim]). Let $P^L(M, \omega_{KE})$, where $L = K_M^{-m}$, be the set of φ satisfying:

$$\frac{1}{m}\sigma^*\omega_{FS} = \omega_{KE} + \partial\bar{\partial}\varphi, \text{ for some } \sigma \in SL(N+1, \mathbf{C}).$$

Then this set is contained in $P(M, \omega_{KE}, \epsilon)$ for some $\epsilon = \epsilon(m)$.

In general, if (M, ω) is a Kähler-Einstein manifold, nonzero holomorphic fields X correspond in one-to-one to the eigenfunctions ψ with eigenvalue 1, namely, $\Delta\psi = -\psi$, and $g_{KE}(X, Y) = Y(\psi)$ for any vector field Y, where g_{KE} is the Kähler-Einstein metric.

Theorem 5.2. *Let (M, ω_{KE}) be as above, and Λ_1 be the space of the eigenfunctions of ω_{KE} with eigenvalue 1. Then for any function $\varphi \in P(M, \omega_{KE}, \epsilon)$ $(\epsilon > 0)$ perpendicular to Λ_1, i.e., $\int_M \varphi \cdot \psi\omega_{KE}^n = 0$ for any $\psi \in \Lambda_1$, we have*

$$F_{\omega_{KE}}(\varphi) \geq a_{1,\epsilon}\, J_{\omega_{KE}}(\varphi)^{\frac{\beta}{2n+2+\beta}} - a_{2,\epsilon}, \tag{5.5}$$

where $a_{1,\epsilon}$, $a_{2,\epsilon}$ are constants. They may depend only on n, ϵ and the lower bound of $\lambda_{1,\omega_{KE}} - 1$ from 0. Here $\lambda_{1,\omega_{KE}}$ denotes the first nonzero eigenvalue of ω_{KE}, which is greater than one.

Proof. Since φ is perpendicular to Λ_1, by the following proposition, there is a smooth family of φ_t such that $\varphi_1 = \varphi$ and φ_t solves $(4.1)_t$. Then Theorem 5.2 follows from the same arguments as in the proof of (4.12).

Proposition 5.3. *For any $\varphi \in P(M, \omega_{KE})$ perpendicular to Λ_1, there is a unique family $\{\varphi_t\}_{0 \leq t \leq 1}$ such that $\varphi_1 = -\varphi + c$, where c is some constant, and φ_t solves*

$$(\omega + \partial\bar{\partial}\varphi)^n = e^{h_\omega - t\varphi}\omega^n, \quad \omega + \partial\bar{\partial}\varphi > 0, \tag{5.6}_t$$

where $\omega = \omega_{KE} + \partial\bar{\partial}\varphi$.

Proof. In [BM], Bando and Mabuchi proved this under slightly stronger conditions on φ. Their result is good enough for proving uniqueness of Kähler-Einstein metrics, but not sufficient in our case. Nevertheless, their arguments can be modified to prove this proposition. For the reader's convenience, we outline a proof here.

Put $\varphi_1 = -\varphi + c$, where c is chosen such that $\int_M e^{-\varphi+c}\omega_{KE}^n = V$. Then φ_1 solves $(5.6)_1$. It is known that any solution φ_t of $(5.6)_t$ for $t < 1$ can be deformed into a family of solutions $\{\varphi_s\}_{s \leq t}$, where φ_s solves $(5.6)_s$. This is because $I(\varphi_s) - J(\varphi_s)$ is nondecreasing in s and dominates $\|\varphi_s\|_{C^0}$ (cf. [BM], [T1]). Therefore, we only need to show that $(5.6)_t$ has a solution near φ_1 for $1-t$ sufficiently small. For this purpose, naturally, we apply the implicit function theorem.

Write $\psi = \varphi - \varphi_1$. Then $(5.6)_t$ becomes

$$\log\left(\frac{(\omega_{KE} + \partial\bar{\partial}\psi)^n}{\omega_{KE}^n}\right) = (1-t)\varphi_1 - t\psi. \tag{5.7$_t$}$$

Let P_0 be the orthogonal projection from $L^2(M, \omega_{KE})$ onto Λ_1, and Λ' be the orthogonal complement of Λ_1. By the assumption, $\varphi_1 \in \Lambda'$.

Consider the equation

$$(1-P_0)\left(\log\left(\frac{(\omega_{KE} + \partial\bar{\partial}(\theta+\psi'))^n}{\omega_{KE}^n}\right)\right) = (1-t)\varphi_1 - t\psi', \tag{5.8$_t$}$$

where $\theta \in \Lambda_1$ and $\psi' \in \Lambda'$. Its linearization at $t=1$ and $\theta = 0$ is $(1-P_0)(\Delta+1)$, which is invertible on Λ'. Therefore, by the implicit function theorem, for any θ small and t close to 1, there is a unique $\psi'_{t,\theta}$ such that $\theta + \psi'_{t,\theta}$ solves $(5.8)_t$.

For any automorphism σ of M, $\sigma^*\omega_{KE}$ is a Kähler-Einstein metric, so there is a unique function u_σ solving $(5.7)_1$. One can easily show that any θ in Λ_1 is of the form $P_0(u_\sigma)$ for some σ, in particular, we may write $u_\sigma = \theta + \psi'_{1,\theta}$.

Following [BM], we put $\psi'_{t,\theta} = \psi'_{1,\theta} + (1-t)\xi_{t,\theta}$. Then $(5.7)_t$ is equivalent to

$$\frac{1}{1-t}P_0\left(\log\left(\frac{(\omega_{KE} + \partial\bar{\partial}(\theta + \psi'_{1,\theta} + (1-t)\xi_{t,\theta}))^n}{(\omega_{KE} + \partial\bar{\partial}(\theta+\psi'_{1,\theta}))^n}\right)\right) - \theta = 0. \tag{5.9$_t$}$$

Let us denote by $\Gamma(t,\theta)$ the term on the left side of $(5.9)_t$. Then $\Gamma(1,0) = 0$ and

$$\Gamma(1,\theta) = P_0(\Delta_\theta \xi_{1,\theta}) - \theta, \tag{5.10}$$

where Δ_θ is the Laplacian of $\omega_{KE} + \partial\bar{\partial}(\theta + \psi'_{1,\theta})$.

Differentiating $(5.8)_t$ on t at $t=1$ and $\theta=0$, we have

$$\Delta\xi_{1,0} + \xi_{1,0} = \varphi_1. \tag{5.11}$$

Using (5.11) as in [BM], one can compute

$$D_2\Gamma(1,0)(\theta)\theta' = -\frac{1}{V}\int_M \left(1 + \frac{1}{2}\Delta\varphi_1\right)\theta\theta'\omega_{KE}^n \tag{5.12}$$

If this derivative $D_2\Gamma(1,0)$ is invertible, then $(5.9)_t$ is solvable for t sufficiently close to 1. Therefore, there is a family of solutions φ_t of $(5.6)_t$ such that $\varphi_1 = -\varphi + c$ for some constant c.

In general, one can use a trick in [BM]. For any small δ, define $\omega_\delta = (1-\delta)\omega + \delta\omega_{KE}$. Then $\omega_\delta = \omega_{KE} + (1-\delta)\partial\overline{\partial}\varphi$. Let $\Gamma_\delta(t,\theta)$ be the left side of $(5.9)_t$ with φ replaced by $(1-\delta)\varphi$. Then

$$D_2\Gamma_\delta(1,0) = (1-\delta)D_2\Gamma(1,0) - \mathrm{Id}$$

This implies that $D_2\Gamma_\delta(1,0)$ is invertible on Λ_1 for $\delta \neq 0$ sufficiently small. Therefore, there are φ_t^δ solving $(5.6)_t$ with ω replaced by ω_δ and satisfying $\varphi_t^\delta = -(1-\delta)\varphi + c_\delta$ for some constant c_δ.

On the other hand, because of the monotonicity of $I_{\omega_\delta} - J_{\omega_\delta}$, we have

$$\begin{aligned} &I_{\omega_\delta}(\varphi_t^\delta) - J_{\omega_\delta}(\varphi_t^\delta) \\ &\quad \leq I_{\omega_\delta}(\varphi_1^\delta) - J_{\omega_\delta}(\varphi_1^\delta) \\ &\quad = J_{\omega_{KE}}((1-\delta)\varphi) \leq C. \end{aligned}$$

Note that C always denotes a uniform constant. Hence, as mentioned at the beginning of Sect. 4, the C^3-norm of φ_t^δ can be uniformly bounded. It follows that φ_t^δ converges to a solution of $(5.6)_t$ as δ goes to zero. The proposition is proved.

Corollary 5.4. *Let (M, ω_{KE}), P_0 and Λ_1 be as above. For any $\varphi \in P(M, \omega_{KE})$, there is a unique $\sigma^*\varphi + u_\sigma$, where $\sigma \in \mathrm{Aut}_0(M)$, such that $\sigma^*\omega_{KE} = \omega_{KE} + \partial\overline{\partial}u_\sigma$, $\int_M (\sigma^*\varphi + u_\sigma)\omega_{KE}^n = 0$, and $P_0(\sigma^*\varphi + u_\sigma) = 0$.*

This follows from Proposition 5.3 and the fact that the solution of $(5.6)_t$ is unique for any $t < 1$.

The inequality (5.5) is not sharp. In fact, we expect

Conjecture 5.5. *If M has a Kähler-Einstein metric ω_{KE} with positive scalar curvature, then there is an $\delta > 0$ such that*

$$F_{\omega_{KE}}(\varphi) \geq \delta \int_M \varphi(\omega_{KE}^n - (\omega_{KE} + \partial\overline{\partial}\varphi)^n) - C_\delta \tag{5.13}$$

for any $\varphi \in P(M, \omega_{KE})$ perpendicular to Λ_1, where C_δ is a constant, which may depend on δ.

This can be regarded as a fully nonlinear generalization of the Moser-Trudinger inequality. In the case $M = S^2$, if we choose ω to be the canonical metric, the inequality becomes

$$\int_{S^2} e^{-\varphi}\omega^n \leq C'_\delta e^{(\frac{1}{2}-\delta)\int_{S^2}\partial\varphi\wedge\overline{\partial}\varphi - \int_M \varphi\omega^n}.$$

This was proved by Aubin (cf. [Au2], [OPS], [CY]).

A weaker form of Conjecture 5.5 will follow, if one can show that there is a $t_0 \in (0,1)$ such that for any $t \geq t_0$, $\|\psi_t\|_{C^0} \leq \frac{1}{2}\|\varphi\|_{C^0} + C$ for any solution ψ_t of (4.5).

Finally, we derive an inequality involving the K-energy. We will use this inequality in the following sections.

Let us first recall the definition of Mabuchi's K-energy v_ω: let $\{\varphi_t\}_{0\leq t\leq 1} \subset P(M,\omega)$ be any path with $\varphi_0 = 0$ and $\varphi_1 = \varphi$, then

$$v_\omega(\varphi) = -\frac{1}{V}\int_0^1 \int_M \dot{\varphi}_t(\mathrm{Ric}(\omega_t) - \omega_t) \wedge \omega_t^{n-1} \wedge dt$$

where $\omega_t = \omega_{\varphi_t} = \omega + \partial\overline{\partial}\varphi_t$. Mabuchi proved [Ma] that the above integral depends only on φ. In fact, it was observed in [T3] that

$$v_\omega(\varphi) = \frac{1}{V}\int_M \left(\log\left(\frac{\omega_1^n}{\omega^n}\right)\omega_1^n + h_\omega\left(\omega^n - \omega_1^n\right)\right) - \frac{1}{n}(I_\omega(\varphi) - J_\omega(\varphi)) \tag{5.14}$$

A simple computation shows (cf. [DT2]):

$$F_\omega(\varphi) = v_\omega(\varphi) + \frac{1}{V}\int_M h_{\omega_\varphi}\omega_\varphi^n - \frac{1}{V}\int_M h_\omega \omega^n \tag{5.15}$$

Since $\int_M e^{h_{\omega_\varphi}}\omega_\varphi^n = V$, by the concavity of the logrithmic function, we have $\frac{1}{V}\int_M h_{\omega_\varphi}\omega_\varphi^n \leq 0$. Hence, we deduce from Theorem 5.3,

Theorem 5.5. *Let (M, ω_{KE}) be as above, and Λ_1 be the space of the first nonzero eigenfunctions of ω_{KE}. Then for any function $\varphi \in P(M, \omega_{KE}, \epsilon)$ perpendicular to Λ_1, we have*

$$v_{\omega_{KE}}(\varphi) \geq a_{1,\epsilon} J_{\omega_{KE}}(\varphi)^{\frac{\beta}{2n+2+\beta}} - a_{2,\epsilon}, \tag{5.16}$$

where $a_{1,\epsilon}$, $a_{2,\epsilon}$ are as above.

Similarly, there is an analogue of Theorem 1.6 by replacing F_ω by the K-energy of Mabuchi.

6 Proof of Theorem 1.2

In this section, we introduce the notion of K-stability, which appeared in Definition 1.1. We will then prove Theorem 1.2. The proof here is the refinement of that for the main theorem in [DT1].

An almost Fano variety Y is an irreducible, normal variety, such that for some m, the pluri-anticanonical bundle $K_{Y_{\mathrm{reg}}}^{-m}$ extends to be an ample line bundle over Y, where Y_{reg} is the regular part of Y.

Obviously, if Y is smooth, then Y is almost Fano if and only if $c_1(Y) > 0$. There are two important cases of almost Fano varieties: (1) Y has a resolution $\tilde{Y}$, such that the anticanonical bundle $K_{\tilde{Y}}^{-1}$ is nef and ample outside the exceptional divisor; (2) Y is an irreducible, normal subvariety in some CP^N, which is the limit of a sequence of compact Kähler manifolds Y_i in CP^N with $c_1(Y_i) > 0$.

Let Y be an almost Fano variety. We recall the definition of the generalized Futaki invariant f_Y as follows: let L be the ample line bundle over Y, which extends $K_{\mathrm{Reg}\,(\mathrm{Y})}^{-m}$ for some m, where Reg (Y) is the regular part of Y, then for l sufficiently large, any basis of $H^0(Y, L^l)$ gives rise to an embedding ϕ of Y into some $\mathbf{C}P^N$, and consequently, the Fubini-Study metric ω_{FS} on $\mathbf{C}P^N$ induces an admissible metric ω (cf. [DT1], p317), i.e., $\omega = \frac{1}{ml}\phi^*\omega_{FS}$.

If $\pi : \tilde{Y} \mapsto Y$ is any smooth resolution, then $\det(\pi)^m$ induces a section of $\pi^*L \otimes K_{\tilde{Y}}^m$, which does not vanish on $\pi^{-1}(\mathrm{Reg(Y)})$. It follows that

$$K_{\tilde{Y}}^{-m} = \pi^*L + E,$$

where E supports in exceptional divisors of $\tilde{Y}$. Therefore, there is a function h_ω such that in the weak sense,

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}h_\omega, \quad \text{on } Y \tag{6.1}$$

We denote by $\eta(Y)$ the Lie algebra of all admissible holomorphic vector fields on Y. A vector field v is called admissible, if it generates a family of automorphisms $\phi_v(t)$ of Y such that $\phi_v(t)^*L = L$.

It is shown in [DT1] that the integral

$$f_Y(v) = \int_Y v(h_\omega)\omega^n,$$

is well defined, where $v \in \eta(Y)$ is any admissible holomorphic field on Y. Following Futaki's arguments in [Fu1], they prove in [DT1] that f_Y is independent of particular choices of ω and a character of the Lie algebra $\eta(Y)$ of holomorphic vector fields.

In fact, $f_Y(v)$ can be calculated by using a residue formula of Atiyah-Bott-Lefschetz type, at least in the case that singularities of Y are not too bad. Such a residue formula depends only on the fixed-point set of v (cf. [Fu2], Theorem 5.2.8 and [DT1], Proposition 1.2).

Now let $\pi : W \mapsto \Delta$ be a special degeneration of a Fano manifold M. Then $Y = W_0$ is an almost Fano variety with an admissible holomorphic field v_W. We need to study $f_Y(v_W)$.

Since W is special, there is an embedding of W into $\mathbf{C}P^N \times \Delta$, such that π_1^*H extends $\det(\mathscr{T}_{W|\Delta})$ over the regular part of W, where $\mathscr{T}_{W|\Delta}$ is the relative tangent bundle of $\mathrm{Reg}(W)$ over Δ, H is the hyperplane bundle over $\mathbf{C}P^N$ and π_i is the projection onto the i^{th}-factor. For simplicity, we may assume that $W \subset P^N \times \Delta$. The vector field v_W induces a one-parameter algebraic subgroup $G = \{\sigma(t)\}_{t\in\mathbf{C}^*} \subset SL(N+1, \mathbf{C})$, such that $\sigma(t)(M) = W_t$ for $0 < |t| < 1$ and $\sigma(t)(Y) = Y$.

Let v be the holomorphic vector field associated to G, i.e., $v = -\sigma'(1)$. Then $v \in \eta(\mathbf{C}P^N)$ and $v|_Y = v_W$. Since $\overline{\partial}(i(v)\omega_{FS}) = 0$, there is a smooth function θ_v on $\mathbf{C}P^N$ such that $\overline{\partial}\theta_v = \frac{1}{m} i(v)\omega_{FS}$.

We put $\omega_t = \frac{1}{m}\omega_{FS}|_{W_t}$. Then ω_t induces a Kähler metric g_t on W_t. Let ∇_t be the (1,0)-gradient of ω_t, i.e., $g_t(\nabla_t \psi, u) = \overline{\partial}\psi(u)$ for any function ψ and tangent vector u on X_t.

Now we can start to prove Theorem 1.2. Let ω_{KE} be the metric on M with $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$, there are φ_t and automorphisms $\tau(t)$ of M, such that

$$\tau(t)^*\sigma(t)^*\omega_t = \omega_{KE} + \partial\overline{\partial}\varphi_t, \quad \omega_{KS}^n = e^{h_{\tau(t)^*\sigma(t)^*\omega_t} - \varphi_t}\tau(t)^*\sigma(t)^*\omega_t^n, \quad \varphi_t \perp \Lambda_1,$$

where Λ_1 is the eigenspace of ω_{KE} with eigenvalue one.

Lemma 6.1. *Assume that W is non-trivial. Then $\|\varphi_t\|_{C^0}$ diverges to the infinity as t tends to zero.*

Proof. We prove it by contradiction. If $\|\varphi_t\|_{C^0}$ are uniformly bounded, then $\sigma(t) \cdot \tau(t) : M \mapsto W_t$ converges to a holomorphic map $\Psi : M \mapsto W_0$. For any $w \in W_0$, if $\Psi^{-1}(w)$ is of complex dimension greater than zero, then for any small neighborhood U of w, $\sigma(t)(\Psi^{-1}(w)) \subset U$ so long as t is sufficiently small. This is impossible, since W is projective. Similarly, using the fact that W_0 is a fiber of W of simple multiplicity, one can show that $\Psi^{-1}(w)$ has at most one component. Therefore, Ψ is a biholomorphism, and consequently, W is trivial, a contradiction! The lemma is proved.

Remark. In the above proof, we do not assume that W_0 is irreducible, but I need each component of W_0 has multiplicity one.

As in last section, let $v_{\omega_{KE}}$ be the K-energy of Mabuchi. First we observe that all these φ_t are in $P(M, \omega_{KE}, \epsilon)$ for some small $\epsilon > 0$ (cf. the example after Theorem 5.1). This implies that $J_{\omega_{KE}}(\varphi_t)$ dominates $\|\varphi_t\|_{C^0}$. Therefore, by Theorem 5.5 and Lemma 6.1, $v_{\omega_{KE}}(\varphi_t)$ diverges to the infinity when t tends to zero.

Put $t = e^{-s}$, then $t = 1$ when $s = 0$, and $t \mapsto 0$ as $s \mapsto +\infty$. We define ψ_s by

$$\sigma(t)^*\omega_t = \omega_{KE} + \partial\overline{\partial}\psi_s, \quad \omega_{KE}^n = e^{h_{\sigma(t)^*\omega_t} - \psi_s}\sigma(t)^*\omega_t^n.$$

Then, using the invariance of the K-energy under automorphisms (cf. [BM]), we have

$$v_{\omega_{KE}}(\varphi_t) = v_{\omega_{KE}}(\psi_s) = -\frac{1}{V}\int_0^s du \int_M \dot{\psi}(\mathrm{Ric}(\tilde{\omega}_u) - \tilde{\omega}_u) \wedge \tilde{\omega}_u^{n-1},$$

where $t = e^{-s}$, $\tilde{\omega}_u = \sigma(e^{-u})^*\omega_{e^{-u}}$ and $\dot{\psi}$ denotes the derivative $\frac{\partial\psi_s}{\partial s}$.

By the definition of ψ_s, one can deduce that $\dot{\psi} = \sigma(t)^*\mathrm{Re}(\theta_v) + c$ for some constant c. It follows that

$$\frac{d}{ds}(v_{\omega_{KE}}(\psi_s)) = \mathrm{Re}\left(\frac{1}{V}\int_{W_t} \nabla_t \theta_v(h_{\omega_t})\omega_t^n\right). \tag{6.2}$$

It is proved in [DT1] that

$$\lim_{t\to\infty} \frac{1}{V}\int_{W_t} (\nabla_t \theta_v)(h_{\omega_t})\omega_t^n = f_{W_0}(v_W), \tag{6.3}$$

consequently, $\mathrm{Re}(f_{W_0}(v_W)) \geq 0$ (cf. [DT1]).

The following proposition provides an estimate on the convergence rate in (6.3).

Proposition 6.2. *Let W, W_t and $\sigma(t)$ be as above. Then there are positive numbers C, γ, which may depend on W, such that*

$$\left|\lim_{t\to\infty} \frac{1}{V}\int_{W_t} (\nabla_t \theta_v)(h_{\omega_t})\omega_t^n - f_{W_0}(v_W)\right| \leq C|t|^\gamma \tag{6.4}$$

Proof. Let $p : \tilde{W} \mapsto W$ be a smooth resolution. Then $p^{-1}(W_t)$ is boholomorphic to W_t for any $t \neq 0$. Let $\mathscr{L}$ be the extension of the relative pluri-anticanonical bundle $\mathscr{K}_{W|\Delta}^{-m}$. Then

$$K_{\tilde{W}}^{-m} = p^*\mathscr{L} + E,$$

where E supports in the exceptional divisors of the resolution $p : \tilde{W} \mapsto W$.

As above, we may assume that $W \subset \mathbf{C}P^N \times \Delta$ and $\pi_1^* H = \mathscr{L}$, where H is the hyperplane bundle over $\mathbf{C}P^N$. Then for any fixed Kähler metric $\tilde{\omega}$ on $\tilde{W}$, there is a hermitian norm $\|\cdot\|$ on E, such that

$$\mathrm{Ric}(\tilde{\omega}) = \frac{1}{m}\left(p^*\pi_1^*\omega_{FS} - \partial\overline{\partial}\log\|S\|^2\right), \tag{6.5}$$

where S is the defining section of E.

We define

$$h = \log\left(\frac{\|S\|^{-\frac{2}{m}}\tilde{\omega}^{n+1}}{p^*\pi_1^*\omega_{FS}^n \wedge dt \wedge d\bar{t}}\right). \tag{6.6}$$

Then by (6.5), we have that $h|_{W_t} = h_{\omega_t} + c_t$, where c_t is some constant.

Clearly, h descends to be a smooth function on the regular part of W. Moreover, one can easily show that $h|_{W_t}$ is uniformly L^2-bounded with respect to ω_t.

It follows from (6.6) that there is a $\delta > 0$ such that

$$|\nabla h|(x) \leq |t|^{-\frac{1}{2}}, \quad \text{for any } x \in \left(\bigcup_{|t'|\leq|t|} W_{t'}\right)\backslash B_{|t|^\delta}(\mathrm{Sing}(W_0)),$$

where $B_r(F)$ denotes the r-neighborhood of the subset F in W with respect to the metric ω_{FS}. Therefore, we have

$$\begin{aligned}
&\left|\frac{1}{V}\int_{W_t}\nabla_t\theta_v(h_{\omega_t})\omega_t^n - \mathrm{Re}\left(\frac{1}{V}\int_{W_0} v_W(h_{\omega_0})\omega_0^n\right)\right| \\
= \; &\left|\frac{1}{V}\int_{W_t} h\Big|_{W_t}\Delta_t\theta_v\omega_t^n - \frac{1}{V}\int_{W_0} h|_{W_0}\Delta_0\theta_v\omega_0^n\right| \\
\leq \; & C\left(|t|^{\frac{1}{2}} + \mathrm{vol}(B_{|t|^\delta}(\mathrm{Sing}(W_0)))^{\frac{1}{2}}\right)
\end{aligned}$$

Then the proposition follows.

Now we can finish the proof of Theorem 1.2: if $\mathrm{Re}(f_{W_0}(v_W)) = 0$, then by Proposition 6.2,

$$\left|\frac{d}{ds}\nu_{\omega_{KE}}(\psi_s)\right| \leq Ce^{-\gamma s},$$

it follows that $\nu_{\omega_{KE}}(\varphi_{e^{-s}}) = \nu_{\omega_{KE}}(\psi_s)$ is bounded as s goes to $+\infty$. This contradicts to what we have shown before, i.e., $\nu_{\omega_{KE}}(\varphi_{e^{-s}})$ diverges to the infinity as s goes to $+\infty$. Therefore, $\mathrm{Re}(f_{W_0}(v_W)) > 0$, Theorem 1.2 is proved.

Remark 6.3. If M admits continuous families of automorphisms, the lifting holomorphic vector field v_W is not unique. However, the generalized Futaki invariant $f_{W_0}(v_W)$ is independent of liftings, whenever M has a Kähler-Einstein metric.

Remark 6.4. In fact, the above arguments show that $\nu_{\omega_{KE}}$ is proper on the family $\{\varphi_t\}_{0<|t|\leq 1}$ if and only if $\mathrm{Re}(f_{W_0}(v_W)) > 0$. Similar statement can be made for $F_{\omega_{KE}}$. This indicates that Conjecture 1.4 should be true.

Remark 6.5. I expect that the K-stability can be also defined in terms of subsheaves of K_M^{-m}. Choose m such that K_M^{-m} is very ample. Given any basis $s_0, \cdots, s_N$ of $H^0(M, K_M^{-m})$, we have an embedding of M into $PH^0(M, K_M^{-m})^*$. Let $\{\sigma(t)\}$ be the one-parameter subgroup of $SL(N+1, \mathbf{C})$ defined by $\sigma(t)(s_i) = t^{\lambda_i}s_i$, where $\lambda_0 \leq \cdots \leq \lambda_N$. Assume that the limit M_∞ of $\sigma(t)(M)$ is normal. Then, in principle, $f_{M_\infty}(-\sigma'(1))$ can be calculated in terms of $\mathscr{F}_i$ and λ_i $(0 \leq i \leq N)$, where $\mathscr{F}_i$ is the subsheaf of K_M^{-m} generated by $s_0, \cdots, s_i$. But I do not have an explicit formula yet.

7 Counter examples

In this section, we study a family of special Fano 3-folds and construct a counterexample to the long-standing conjecture stated in Sect. 1. I learned from Mukai the construction of those 3-folds during a conference on algebraic geometry in Tokyo, 1990. They were first constructed by Iskovskih.

Those 3-folds have the same cohomology groups as $\mathbf{C}P^3$ does. One of those 3-folds was studied by Mukai and Umemura in [MU], this particularly interesting manifold is a compactification of $SL(2,\mathbf{C})/\Gamma$, where Γ is the icosahedral group.

Let us recall Mukai's construction of those manifolds. The complex Grassmannian $G(4,7)$ consists of all 4-subspaces in $\mathbf{C}^7$. For any 3-dimensional subspace P in $\wedge^2\mathbf{C}^7$, we define a subvariety X_P as follows:

$$X_P = \{E \in G(4,7) | \pi_E(P) = 0\}, \tag{7.1}$$

where π_E denotes the orthogonal projection from $\wedge^2\mathbf{C}^7$ onto $\wedge^2 E'$, and E' is the orthogonal complement of E. We say that X_P is non-degenerate if no sections of $H^0(G(4,7),Q)$ vanish identically on X_P. This is true for generic X_P.

When X_P is smooth and non-degenerate, the normal bundle of X_P in $G(4,7)$ is the restriction of $\wedge^2 Q \oplus \wedge^2 Q \oplus \wedge^2 Q$ to X_P, where Q denotes the universal quotient bundle over $G(4,7)$. It follows that $c_1(X_P) = c_1(Q)|_{X_P}$, consequently, $c_1(X_P) > 0$.

Again we assume that X_P is smooth and non-degenerate. Then one can show that $H^i(X_P, \wedge^j Q) = 0$ for any $i \geq 1$ and $j = 1,2$. Hence, by the Riemann-Roch Theorem, $h^0(X_P, Q) = 7$ and $h^0(X_P, \wedge^2 Q) = 18$. Now we can identify the Lie algebra $\eta(X_P)$ of holomorphic vector fields on X_P. Any v in $\eta(X_P)$ induces a one-parameter family of automorphisms Φ_t of X_P. Using non-degeneracy of X_P, one can show that $H^0(X_P, Q) = H^0(G(4,7),Q) = \mathbf{C}^7$. Therefore, Φ_t corresponds to a linear transformation on $\mathbf{C}^7$. Clearly, this linear transformation induces an action on $\wedge^2\mathbf{C}^7$ which preserves P. Therefore, we can identify $\eta(X_P)$ with the set of matrices in $sl(7,\mathbf{C})$ whose induced action on $\wedge^2\mathbf{C}^7$ preserves P. Note that for any A in $sl(7,\mathbf{C})$, the induced action on $\wedge^2\mathbf{C}^7$ is given by

$$A(e_i \wedge e_j) = A(e_i) \wedge e_j + e_i \wedge A(e_j), \quad i \neq j,$$

where e_i are euclidean basis of $\mathbf{C}^7$.

Take P_0 to be the subspace spanned by bi-vectors

$$\begin{aligned} u_1 =& 3e_1 \wedge e_6 - 5e_2 \wedge e_5 + 6e_3 \wedge e_4, \\ u_2 =& 3e_1 \wedge e_7 - 2e_2 \wedge e_6 + e_3 \wedge e_5, \\ u_3 =& e_2 \wedge e_7 - e_3 \wedge e_6 + e_4 \wedge e_5. \end{aligned}$$

It is easy to check that P_0 is invariant under the group, whose Lie algebra is generated by matrices

$A_1 = \mathrm{diag}(3,2,1,0,-1,-2,-3)$,

$A_2 = \{b_{2ij}\}_{1\leq i,j\leq 7}$, where $b_{2ij} = 1$ for $j - i = 1$ and 0 otherwise ,

$A_3 = \{b_{3ij}\}_{1\leq i,j\leq 7}$, where $b_{321} = b_{376} = 3, b_{332} = b_{365} = 5, b_{343} = b_{354} = 6$, all other $b_{3ij} = 0$, These three matrices generate a Lie algebra which is isomorphic to $sl(2,\mathbf{C})$.

To see that X_{P_0} is smooth, we notice that for any $E \in X_{P_0}$, $\sigma(t)(E)$ converges to one of the following 4-subspaces:

$$\{e_1, e_2, e_3, e_4\}, \quad \{e_1, e_3, e_4, e_5\},$$
$$\{e_3, e_5, e_6, e_7\}, \quad \{e_4, e_5, e_6, e_7\},$$

where $\sigma(t) = \mathrm{diag}\,(t^{-3}, t^{-2}, t^{-1}, 1, t, t^2, t^3)$ is the one-parameter subgroup generated by A_1. It is straightforward to check that X_{P_0} is smooth at these points. It follows that X_{P_0} is smooth. Similarly, one can show that X_{P_0} is non-degenerate. Hence, $\eta(X_{P_0}) = sl(2,\mathbf{C})$. This Lie algebra is semi-simple, so the Futaki invariant is identically zero.

Now we take P_a be the subspace generated by $u_i + \sum_{j+k\geq 7+i} a_{ijk} e_j \wedge e_k$, where $i = 1, 2, 3$, and $a = \{a_{ijk}\}$. Clearly, $\lim_{t\to 0} \sigma(t)(P_a) = P_0$.

For any a, we have a special degeneration $W_a = \bigcup_{|t|\leq 1} \sigma(t)(X_{P_a})$ of X_{P_a}. In particular, each X_a is smooth and non-degenerate.

Obviously, for a generic a, $\eta(X_{P_a}) = \{0\}$. However, by Theorem 1.2, X_{P_a} does not admit any Kähler-Einstein metric. This disproves the long-standing conjecture (Corollary 1.3).

Let us discuss more on $X = X_{P_0}$. Recall that $\sigma(t)$ acts naturally on the space $H^1(X, T_X)$ of infinitesimal deformations. For each small a, X_{P_a} corresponds to a point, say ξ_a, in $H^1(X, T_X)$. Then $X_{\sigma(t)(P_a)}$ corresponds to $\sigma(t)(\xi_a)$, which converges to 0 as t goes to zero. Let v be the holomorphic vector field on X associated to $\sigma(t)$, i.e., $v = -\sigma'(1)$, then the Lie bracket $[\xi_a, v]$ lies in the positive part $E_+(ad(v))$ of $ad(v)$ in $H^1(X, T_X)$, where $ad(v)(\xi) = [\xi, v]$ for any ξ in $H^1(X, T_X)$. Note that for any linear automorphism T of a vector space V, the positive part $E_+(T)$ is the subspace generated by those vectors ξ in V satisfying: for some $i > 0$, $T^i\xi \neq 0$ and $(T - \lambda)T^i\xi = 0$ for some λ with $\mathrm{Re}(\lambda) > 0$.

Thus we have a triple (X, v, ξ_a) such that $v \in H^0(X, T_X)$, $\xi_a \in E_+(ad(v)) \subset H^1(X, T_X)$ and the Futaki invariant $\mathrm{Re}(f_X(v)) \leq 0$.

Inspired by this, we give a general definition.

Definition 7.1. *An obstruction triple (X, v, ξ) consists of a manifold X with $c_1(X) > 0$, a holomorphic vector field v on X with $\mathrm{Re}(f_X(v)) \leq 0$ and an infinitesimal deformation ξ in $E_+(ad(v)) \subset H^1(X, T_X)$.*

One can also define more general obstruction triples by only assuming that X is an almost Fano variety.

It follows from Theorem 1.2 that

Theorem 7.2. *Let $W \mapsto \Delta$ be a special degeneration of M. Let ξ be the corresponding infinitesimal deformation at W_0. If (W_0, v_W, ξ) is an obstruction triple, then M has no Kähler-Einstein metrics.*

Therefore, in order to find M without any Kähler-Einstein metrics, we only need to look for obstruction triples.

It is easy to check that there are no 2-dimensional obstruction triples. Here the dimension of (X, v, ξ) is defined to be $\dim_{\mathbf{C}} X$. The above X_{P_0} gives rise to an example of 3-dimensional obstruction triples. In general, let X be any Kähler-Einstein manifold with a holomorphic vector field v, if the action $ad(v)$ on $H^1(X, T_X)$ is nontrivial, then we can construct an obstruction triple of the form $(X, \pm v, \xi)$, consequently, the deformation of X along ξ has no Kähler-Einstein metrics.

8 CM-stability

In this section, we show the connection between the existence of Kähler-Einstein metrics and the CM-stability on algebraic manifolds. The basic ideas have been presented in [T3], [T4]. Here we confine ourselves to the case of Fano manifolds.

Let $\pi : \mathscr{X} \mapsto Z$ be a $SL(N+1, \mathbf{C})$-equivariant holomorphic fibration between smooth varieties, satisfying:

(1) $\mathscr{X} \subset Z \times CP^N$ is a family of subvarieties of dimension n, moreover, the action on $\mathscr{X}$ is induced by the canonical action of $G = SL(N+1, \mathbf{C})$ on CP^N;

(2) Let L be the hyperplane bundle over CP^N, then $K^{-1}_{\pi^{-1}(z)} = \mu L|_{\pi^{-1}(z)}$ for some rational number $\mu > 0$, and each $z \in Z_0$, where Z_0 is the subvariety of Z consisting of those smooth fibers with positive first Chern class. Clearly, Z_0 is G-invariant;

Consider the virtual bundle

$$\mathscr{E} = (n+1)(\mathscr{K}^{-1} - \mathscr{K}) \otimes (\pi_2^* L - \pi_2^* L^{-1})^n - n\mu(\pi_2^* L - \pi_2^* L^{-1})^{n+1}, \qquad (8.1)$$

where $\mathscr{K} = K_{\mathscr{X}} \otimes K_Z^{-1}$ is the relative canonical bundle, and π_i is the restriction to $\mathscr{X}$ of the projection from $Z \times CP^N$ onto its i^{th}-factor.

We define L_Z to be the inverse of the determinant line bundle $\det(\mathscr{E}, \pi)$. A straightforward computation shows:

$$\begin{aligned} &ch_{n+1}(n+1)(\mathscr{K}^{-1} - \mathscr{K}) \otimes (\pi_2^* L - \pi_2^* L^{-1})^n - n\mu(\pi_2^* L - \pi_2^* L^{-1})^{n+1} \\ &\quad = 2^{n+1}((n+1)c_1(\mathscr{K}^{-1})\pi_2^* c_1(L)^n - n\mu\pi_2^* c_1(L)^{n+1}) \end{aligned} \qquad (8.2)$$

Therefore, by the Grothendick-Riemann-Roch Theorem,

$$c_1(L_Z) = 2^{n+1}\pi_{1*}\Big((n+1)c_1(\mathscr{K})\pi_2^* c_1(L)^n + n\mu\pi_2^* c_1(L)^{n+1}\Big) \qquad (8.3)$$

We also denote by $\mathscr{L}_Z^{-1}$ the total space of the line bundle L_Z^{-1} over Z. Then $G = SL(N+1, \mathbf{C})$ acts naturally on $\mathscr{L}_Z^{-1}$. Recall that $X_z = \pi^{-1}(z)$ $(z \in Z_0)$ is weakly CM-stable with respect to L, if the orbit $G \cdot \tilde{z}$ in $\mathscr{L}_Z^{-1}$ is closed, where $\tilde{z}$ is any nonzero vector in the fiber of L_Z^{-1} over z; If, in addition, the stablizer G_z of z is finite, then X_z is CM-stable. We also recall that X_z is CM-semistable, if the 0-section is not in the closure of $G \cdot \tilde{z}$. Clearly, this G-stability (resp. G-semistability) is independent of choices of $\tilde{z}$.

Theorem 8.1. *Let $\pi : \mathscr{X} \mapsto Z$ be as above. Assume that X_z has a Kähler-Einstein metric, where $z \in Z_0$. Then X_z is weakly CM-stable. If X_z has no nontrivial holomorphic vector fields, it is actually CM-stable with respect to L.*

Remarks. 1) Similar results hold for manifolds with nef canonical bundle (cf. [T5]); 2) If L_Z is ample over Z, then Z can be embedded into $PH^0(Z, L_Z^m)^*$ for some $m > 0$. The group G acts naturally on this projective space. Then the CM-stability is the same as the stability of z in $PH^0(Z, L_Z^m)^*$ with respect to G (cf. [Mum]); 3) One can generalize Theorem 8.1 slightly. More precisely, if $\pi : \mathscr{X} \mapsto Z$ is a $SL(N+1, \mathbf{C})$-equivaraint fibration as above, except that some fibers may have dimension higher than n. Define $\mathscr{L}_Z^{-1}$ and G as in Theorem 8.1. Put Z_h to be the subvariety of Z consisting of all fibers with dimension $> n$. Clearly, Z_h has codimension at least 2. By the same arguments as in the proof of Theorem 8.1, one can show that the orbit $G \cdot \tilde{z}$ is closed in $Z \backslash Z_h$ if X_z has a Kähler-Einstein metrics. Under further assumptions on Z, one can deduce the CM-stability of X_z in this more general situation. This generalization is not substantial, but often useful (see the following example).

Example 8.2. Let us apply Theorem 8.1 to giving an alternative proof of Corollary 1.3. We will adopt the notations in Sect. 7.

Recall that $W = G(4, 7)$ consists of all 4-subspaces in $\mathbf{C}^7$. Let Q be its universal quotient bundle.

Let π_i $(i = 1, 2)$ be the projection from $W \times G(3, H^0(W, \wedge^2 Q))$ onto its i^{th}-factor, and let S be the universal bundle over $G(3, H^0(W, \wedge^2 Q))$. Then there is a natural endomorphism over $W \times G(3, H^0(W, \wedge^2 Q))$

$$\Phi : \pi_2^* S \mapsto \pi_1^* \wedge^2 Q, \quad \Phi(v)|_{(x,P)} = v_x \in \wedge^2 Q. \tag{5.4}$$

Naturally, one can regard Φ as a section in $\pi_2^* S^* \otimes \pi_1^*(\wedge^l Q)$.

We define

$$\mathscr{X} = \{(x, P) \in W \times G(3, H^0(W, \wedge^2 Q)) \ |\Phi(x, P) = 0\}.$$

One can show that $\mathscr{X}$ is smooth.

If $L = \det(Q)$, then $c_1(L)$ is the positive generator of $H^2(W, \mathbf{Z})$.

Consider the fibration $\pi = \pi_2|_{\mathscr{X}} : \mathscr{X} \mapsto Z$, where $Z = G(3, H^0(W, \wedge^2 Q))$. Its generic fibers are smooth and of dimension 3. Then Z_0 parametrizes all Fano 3-folds X_P (cf. Sect. 7).

Using the Adjunction Formula, one can show

$$c_1(\mathscr{K}) = -\pi_1^* c_1(L) - 3\pi_2^* c_1(S).$$

Therefore, it follows that

$$c_1(L_Z) = 16\pi_*\left(12\pi_2^*c_1(S^*)\pi_1^*c_1(L)^3 - \pi_1^*c_1(L)^4\right).$$

One can show that L_Z is ample.

By the definition of P_a, one can show that none of $G \cdot P_a$ is closed in $Z \backslash Z_h$. Therefore, by the above Remark 3), any generic X_{P_a} admits no Kähler-Einstein metrics, so Corollary 1.3 is reproved.

Now we prove Theorem 8.1. For simplicity, we assume that X_z has no nontrivial holomorphic vector fields. The general case can be proved similarly without difficulities. We will start with an analytic criterion for stability.

Lemma 8.3. *Let $\|\cdot\|_Z$ be any fixed hermitian metric on L_Z^{-1}. Given any z in Z_0. We define a function on G by*

$$F_0(\sigma) = \log(\|\sigma(\tilde{z})\|_Z), \quad \sigma \in G \tag{8.4}$$

where $\tilde{z}$ is any lifting of z in $\mathscr{L}_Z^{-1}$. Then X_z is CM-stable if and only if F_0 is proper on G.

The proof of this lemma is simple and is left to readers (cf. [T4]).

Let ω be the Kähler form of the Kähler-Einstein metric on X_z. Then the K-energy ν_ω (cf. (5.15)) induces a functional D_ω on G: let ω_{FS} be the Fubini-Study metric on CP^N, for any $z' \in Z_0$, we put $\omega_{z'}$ to be the restriction of ω_{FS} to $X_{z'}$, then $D_\omega(\sigma) = \nu_\omega(\sigma^*(\omega_{\sigma(z)}))$.

Let h be the pull-back metric on π_2^*L over $\mathscr{X}$ from the standard hermitian metric on the hyperplane bundle over CP^N. Then the curvature $\pi_2^*\omega_{FS}$ of h restricts to the Kähler metric $\omega_{z'}$ on each $X_{z'}$, and consequently, a hermitian metric $k_{\mathscr{X}}$ on the relative canonical bundle $\mathscr{K}$ over $\pi^{-1}(Z_0)$. We denote by $R_{\mathscr{X}|Z}$ the curvature form of $k_{\mathscr{X}}$.

Lemma 8.4. [T4] *Define GX to be the variety $\{(\sigma, x) | x \in \sigma(X)\}$ in $G \times CP^N$. Then for any smooth $2(\dim_C G - 1)$-form ϕ with compact support in G,*

$$-\int_G D_\omega(\sigma)\partial\bar{\partial}\phi \int_{GX}\left(-p_z^*R_{\mathscr{X}|z} - \frac{n\mu}{n+1}\pi_2^*\omega_{FS}\right) \wedge \pi_2^*\omega_{FS}^n \wedge \pi_1^*\phi, \tag{8.5}$$

where p_z is the map: $GX \mapsto \mathscr{X}$, assigning (σ, x) to x in $\sigma(X) \subset \mathscr{X}$.

Proof. For the reader's convenience, we outline its proof here.

Define $\Psi : G \times X \mapsto GX$ by assigning (σ, x) to $(\sigma, \sigma(x))$. We have hermitian metrics $\tilde{h} = \Psi^*h$ on $\Psi^*\pi_2^*L$ and $\tilde{k} = \Psi^*k_{\mathscr{X}}$ on $\pi_2^*K_X$. Then (8.5) becomes

$$-\int_G D_\omega(\sigma)\partial\bar{\partial}\phi = \int_{G\times X}\left(-R(\tilde{k}) - \frac{n\mu}{n+1}R(\tilde{h})\right) \wedge R(\tilde{h})^n \wedge \pi_1^*\phi \tag{8.6}$$

where $R(\cdot)$ denotes the curvature form.

Let h_0 be a hermitian metric on $L|_X$ with $\omega = R(h_0)$. Define functions φ_t on $G \times X$ by

$$\varphi_t(\sigma, x) = t \log\left(\frac{\tilde{h}(\sigma, x)}{h_0(x)}\right) \quad \textit{and} \quad \tilde{h}_t = e^{\varphi_t} h_0$$

Then $\tilde{h}_1 = \tilde{h}$ and $\tilde{h}_0 = \pi_2^* h_0$. Notice that the curvature $R(\tilde{h}_t)$ restricts to a Kähler metric on $\sigma \times X$ for each σ. Therefore, $\tilde{h}_t$ induces a hermitian metric $\tilde{k}_t$ on $\pi_2^* K_X$ such that $\tilde{k}_1 = \tilde{k}$ and $\tilde{k}_0$ is independent of σ.

We have

$$\begin{aligned}
&- \int_G D_\omega(\sigma) \partial\bar{\partial}\phi \\
&= \int_G \partial\bar{\partial}\phi(\sigma) \wedge \int_0^1 \int_X \frac{\partial}{\partial t}\varphi_t(\sigma, x)\big(-nR(\tilde{k}_t) - n\mu R(\tilde{h}_t)\big) \wedge R(\tilde{h}_t)^{n-1} \wedge dt \\
&= \int_0^1 \int_{G\times X} \frac{\partial}{\partial t}\varphi_t\big(-nR(\tilde{k}_t) - n\mu R(\tilde{h}_t)\big) \wedge R(\tilde{h}_t)^{n-1} \wedge \partial\bar{\partial}\pi_1^*\phi \wedge dt \\
&= \int_0^1 \int_{G\times X} \pi_1^*\phi \wedge \partial\bar{\partial}\left(\frac{\partial}{\partial t}\varphi_t\right) \wedge \big(-nR(\tilde{k}_t) - n\mu R(\tilde{h}_t)\big) \wedge R(\tilde{h}_t)^{n-1} \wedge dt \\
&= \int_0^1 \int_{G\times X} \pi_1^*\phi \wedge \left(-R(\tilde{k}_t) \wedge \frac{\partial}{\partial t} R(\tilde{h}_t)^n - \frac{n\mu}{n+1}\frac{\partial}{\partial t} R(\tilde{h}_t)^{n+1}\right) \wedge dt \\
&= \int_{G\times X} \pi_1^*\phi \wedge \left(-R(\tilde{k}) - \frac{n\mu}{n+1} R(\tilde{h})\right) \wedge R(\tilde{h})^n \\
&\quad + \int_0^1 \int_{G\times X} \pi_1^*\phi \wedge \frac{\partial}{\partial t} R(\tilde{k}_t) \wedge R(\tilde{h}_t)^n \wedge dt
\end{aligned} \tag{8.7}$$

However,

$$\frac{\partial}{\partial t} R(\tilde{k}_t)(\sigma, x) = \partial\bar{\partial}\big(\Delta_{\sigma,t}\varphi_t\big)(\sigma, x),$$

where $\Delta_{\sigma,t}$ denotes the Laplacian of the Kähler metric induced by $R(\tilde{h}_t)|_{\sigma\times X}$, so the last integral in (8.7) vanishes by integration by parts. Then (8.5) follows from (8.6), (8.7).

Lemma 8.5. *Let $g_{\mathscr{X}}$ and g_Z be hermitian metrics on canonical bundles K and K_Z, respectively. Then*

$$R_{\mathscr{X}|Z} = R(g_{\mathscr{X}}) - R(g_Z) + \partial\bar{\partial}\psi \quad \text{on} \quad \mathscr{X}\backslash\{x | \mathrm{Im}(d\pi(x)) \neq T_{\pi(x)}Z\} \tag{8.8}$$

where ψ is a smooth function on $\mathscr{X}\backslash\{x | \mathrm{Im}(d\pi(x)) \neq T_{\pi(x)}Z\}$ satisfying $\sup\psi < \infty$.

Proof. For any x such that $\mathrm{Im}(d\pi(x)) = T_{\pi(x)}Z$, we will define $\psi(x)$ as follows: choose $s_{\mathscr{X}}$ in $K_{\mathscr{X}}|_x$ and s_Z in $K_Z|_{\pi(x)}$, such that $g_{\mathscr{X}}(s_{\mathscr{X}}, s_{\mathscr{X}})(x) = 1$ and $g_Z(s_Z, s_Z)(\pi(x)) = 1$. Since $d\pi|_x : T_x\mathscr{X} \mapsto T_{\pi(x)}Z$ is surjective, there is a unique vector s of $\mathscr{K}_x$ such that $s \cdot (d\pi)^*(s_Z) = s_{\mathscr{X}}$. We simply define $\psi(x) = -\log k_{\mathscr{X}}(s,s)(x)$. Clearly, $\sup\psi < \infty$ and $\psi(x)$ diverges to $-\infty$ as x tends to any point where $d\pi$ is not surjective.

We can also write

$$\psi(x) = -\log\left(\frac{k_{\mathscr{X}}\pi^* g_Z}{g_{\mathscr{X}}}\right) \tag{8.9}$$

Then (8.8) follows easily.

Corollary 8.6. *For any smooth* $2(\dim_C G - 1)$*-form* ϕ *with compact support in* G, *we have*

$$\begin{aligned} &-\int_G D_\omega \partial\bar{\partial}\phi + \int_G \psi_Z \partial\bar{\partial}\phi \\ &= \int_{GX}\left(-R(g_{\mathscr{X}}) + R(g_Z) - \frac{n\mu}{n+1}\pi_2^*\omega_{FS}\right)\pi_2^*\omega_{FS}^n \wedge \pi_1^*\phi \end{aligned} \tag{8.10}$$

where ψ_Z *is a smooth function on* Z_0, *moreover,* ψ_Z *is bounded from above and for any* $z \in Z_0$,

$$\psi_Z(z) = \int_{X_z} \psi\omega_{FS}^n,$$

consequently, ψ *extends to a continuous function outside the set of points where* X_z *contains a component of multiplicity* > 1. *Furthermore, if* X_z *contains a component of multiplicity* > 1, $\psi_Z(z')$ *diverges to infinity as* $z' \mapsto z$.

This follows directly from Lemma 8.4, 8.5.

Let Φ be the push-forward current

$$\pi_*\left(\left(-R(g_{\mathscr{X}}) + R(g_Z) - \frac{n\mu}{n+1}\pi_2^*\omega_{FS}\right) \wedge \pi_2^*\omega_{FS}^n\right)$$

Then $2^{n+1}(n+1)\Phi$ represents the Chern class of L_Z^{-1}. Therefore, there is a function θ_Z, which is smooth in Z_0, such that in the weak sense,

$$\Phi = \frac{1}{(n+1)2^{n+1}} R(\|\cdot\|_Z) + \partial\bar{\partial}\theta_Z, \quad \text{on} \quad Z \tag{8.11}$$

Lemma 8.7. *The function* θ_Z *is Hölder continuous.*

Proof. First we prove the following: there are uniform constants $\delta, C > 0$, such that for any point $z' \in Z$, and $r \leq 1$,

$$r^{2-2b}\int_{B_r(z')}|\mathrm{tr}_{\omega_Z}\Phi|\left(\frac{\sqrt{-1}}{2\pi}\omega_Z\right)^b \leq Cr^\delta \tag{8.12}$$

where $b = \dim_{\mathbf{C}} Z$, ω_Z is a fixed Kähler metric on Z, and $B_r(z')$ is the ball of ω_Z with radius r and center at z'. Using the definition of Φ, one can deduce (8.12) from the following: for any $r \leq 1$ and $z' \in Z\backslash Z_h$,

$$r^{2-2b}\int_{\pi^{-1}(B_r(z'))}\left(\frac{\sqrt{-1}}{2\pi}\omega_{FS}\right)^{n+1}\wedge\left(\frac{\sqrt{-1}}{2\pi}\pi^*\omega_Z\right)^{b-1} \leq Cr^\delta \tag{8.13}$$

To prove this, we choose local coordinates $t_1, \cdots, t_b$ such that $z' = (0, \cdots, 0)$. Without loss of the generality, we may further assume that ω_Z is just the euclidean metric on $B_r(z') \subset \mathbf{C}^b$. We denote by $B_r^i(z')$ the intersection of $B_r(z')$ with the hyperplane $\{t_i = 0\}$ $(i = 1, \cdots, b)$. Then

$$\begin{aligned} & r^{2-2b}\int_{\pi^{-1}(B_r(z'))}\left(\frac{\sqrt{-1}}{2\pi}\omega_{FS}\right)^{n+1}\wedge\left(\frac{\sqrt{-1}}{2\pi}\pi^*\omega_Z\right)^{b-1} \\ \leq \quad & C\sum_{i=1}^b \sup\nolimits_{|a_j|\leq r, j\neq i}\int_{\pi^{-1}(\{t_j=a_j\})}\left(\frac{\sqrt{-1}}{2\pi}\omega_{FS}\right)^{n+1}, \end{aligned}$$

where C is a uniform constant. Hence, it suffices to show (8.13) in the case of $b = 1$.

When $b = 1$, π is a holomorphic function on $\pi^{-1}(B_r(z'))$. Since each fiber $\pi^{-1}(z)$ is a compact variety of $\mathbf{C}P^N$ of degree d, there is a uniform $\delta < 1$ such that

$$\pi^{-1}(B_r(z')) \subset B_{r^\delta}(X_{z'}). \tag{8.14}$$

Then (8.13) follows.

Now the lemma can be easily deduced from the standard Green formula and (8.12).

By Corollary 8.6 and Lemma 8.7, we have

$$\partial\bar\partial\left((D_\omega - \psi_Z + \theta_Z)(\sigma) - \frac{1}{(n+1)2^{n+1}}\log\left(\frac{\|\cdot\|_Z(\sigma(z))}{\|\cdot\|_Z(z)}\right)\right) = 0$$

namely, the function $(D_\omega - \psi + \theta_Z)(\sigma) - \frac{1}{(n+1)2^{n+1}}\log\left(\frac{\|\cdot\|_Z(\sigma(z))}{\|\cdot\|_Z(z)}\right)$ is of the form $\log|f|^2$ for some holomorphic function f on $SL(N+1,\mathbf{C})$.

Let us denote by $\{z_{ij}\}_{0\leq i,j\leq N}$, w the homogeneous coordinates of $CP^{(N+1)^2}$, then $SL(N+1,)$ can be naturally identified with the affine subvariety $W \cap \{w \neq 0\}$, where

$$W = \{[\{z_{ij}\}_{0\leq i,j\leq N}, w] | \det(z_{ij}) = w^{N+1}\}$$

Then by using the definition of D_ω and straightforward computations, one can show

Lemma 8.8. *The function f has at most polynomial growth near $W \backslash SL(N+1, \mathbf{C})$, i.e., there are constants $\ell > 0$, $C > 0$, such that $f(\sigma) \leq Cd(\sigma, W \backslash SL(N+1, \mathbf{C}))^\ell$, where $d(\sigma, W \backslash SL(N+1, \mathbf{C}))$ denotes the distance from σ to $W \backslash SL(N+1, \mathbf{C})$ with respect to the standard metric on $CP^{(N+1)^2}$.*

Therefore, f extends to be a meromorphic function on W. Notice that W is normal and $W \backslash SL(N+1, \mathbf{C})$ is irreducible. It follows that f has to be an nonzero constant c, and consequently, we have

$$\left(\frac{\| \cdot \|_Z(\sigma(z))}{\| \cdot \|_Z(z)} \right)^{\frac{1}{2^{n+1}(n+1)}} = |c|^2 e^{-\psi_Z(\sigma) + \theta_Z(\sigma)} e^{D_{\omega,m}(\sigma)}, \tag{8.15}$$

or equivalently,

$$\frac{2^{-(n+1)}}{n+1} F_0(\sigma) = D_\omega(\sigma) + \theta_Z(\sigma) - \psi_Z(\sigma) - C. \tag{8.16}$$

Note that $C > 0$ always denotes some uniform constant.

By Lemma 6.1, Corollary 8.6 and Theorem 5.5, $\psi_Z(\sigma) + D_\omega(\sigma)$ diverges to infinity as σ goes to $W \backslash G$. It follows from Lemma 8.7 that $F_0(\cdot)$ is proper on G, then Theorem 8.1 follows from Lemma 8.3.

Remark. Theorem 8.1 can be also proved by using the functional F_ω directly. In that case, we start with $\mathscr{E} = (\mathscr{K}^{-1} - \mathscr{K})^{n+1}$ instead. However, its determinant line bundle $\det(\mathscr{E}, \pi)$ coincides with L_Z^{-1} considered above. We used the K-energy in the above proof because it works for more general cases where the Kähler class may not be canonical.

For any $\sigma \in G$, we can define a function φ_σ by

$$\mu \sigma^* \omega_{FS} = \omega + \partial\bar{\partial}\varphi_\sigma, \qquad \int_{X_z} \varphi_\sigma \omega^n = 0.$$

Let $P^L(X_z, \omega)$ be the set of all these φ_σ. The proof of Theorem 8.1 also yields

Theorem 8.9. *Assume that X_z has no nonzero holomorphic vector fields. Then v_ω is proper on $P^L(X_z, \omega)$ if and only if X_z is CM-stable with respect to L.*

Similarly, one may have

Theorem 8.10. *Assume that X_z has no nonzero holomorphic vector fields. Then the functional F_ω is proper on $P^L(X_z, \omega)$ if and only if X_z is CM-stable with respect to L.*

The general case where X_z has nontrivial holomorphic vector fields can be treated identically, involving weakly CM-stability in place of the CM-stability.

Finally, these results motivate us to propose

Conjecture 8.11. *Let M be a compact Kähler manifold with $c_1(M) > 0$. Then M admits a Kähler-Einstein metric if and only if M is weakly CM-stable with respect to K_M^{-m} for m sufficiently large.*

The necessary part of this conjecture follows from Theorem 8.1. The other part follows from Theorem 8.9 or 8.10 if one can establish the partial C^0-estimates posed in [T6]. We plan to discuss these in details in a subsequent paper.

If Conjecture 8.11 is affirmed, then moduli spaces of Kähler-Einstein manifolds with positive scalar curvature are quasi-projective.

9 Further remarks

Let (M, ω) be a compact Kähler manifold with the Kähler class $c_1(M) > 0$. In order to construct a Kähler-Einstein metric on M, we need to solve the complex Monge-Ampere equations in $(2.1)_t$ for $0 \le t \le 1$. By the continuity method (cf. [Au], [BM], [T1]), one can show that either M admits Kähler-Einstein metrics, i.e., $(2.1)_1$ is solvable, or for some $t_0 \in (0, 1]$, $(2.1)_t$ is solvable if and only if $t < t_0$. The main issue is to understand when the second case occurs. In previous sections, we approach this by using the K-stability or the CM-stability.

Here we want to make a few remarks on the differential geometric approach. More precisely, when the second case occurs, we would like to characterize the limit (M, ω_t), where $\omega_t = \omega + \partial\overline{\partial}\varphi_t$ and φ_t solves $(2.1)_t$.

Conjecture 9.1. *By taking subsequences if necessary, one should have that (M, ω_t) converges to a space $(M_\infty, \omega_\infty)$, which is smooth outside a subset of real Hausdorff codimension at least 4, in the Cheeger-Gromov-Hausdorff topology. Furthermore, $(M_\infty, \omega_\infty)$ can be expanded to be an obstruction triple (M_∞, v, ξ) (possibly singular) satisfying:*

$$\mathrm{Ric}(\omega_\infty) - \omega_\infty = -L_v(\omega_\infty), \quad \text{on the regular part of } M_\infty, \tag{9.1}$$

where L_v denotes the Lie derivative in the direction of v. In particular, $(M_\infty, \omega_\infty)$ is a Ricci soliton if ω_∞ is not Kähler-Einstein.

Here by the Cheeger-Gromov-Hausdorff topology, we mean that 1) (M, ω_t) converges to $(M_\infty, \omega_\infty)$ in the Gromov-Hausdorff topology; 2) For any $\{x_t\} \subset M$ with $\lim_{t \to t_0} x_t = x_\infty$ in M_∞, there are C^1 diffeomorphisms Φ_t from $B_r(x_\infty)$ onto $B_r(x_t)$ for some small $r > 0$, such that $\Phi_t^* \omega_t$ converges to ω_∞ in the C^0-topology on $B_r(x_\infty)$.

The singularities of M_∞ should be very mild. One may even guess that M_∞ is actually smooth, but there are no convincing evidences for this.

Similar things can be said for the Ricci flow (3.2), i.e.,

$$\frac{\partial\varphi}{\partial t} = \log\left(\frac{(\omega + \partial\overline{\partial}\varphi)^n}{\omega^n}\right) + \varphi - h_\omega, \quad \varphi|_{t=0} = 0. \tag{9.2}$$

It is known (cf. [Ca]) that (9.2) is solvable for all $t > 0$. To prove the existence of Kähler-Einstein metrics on M, we need to show that φ has a limit in the C^0-topology as t goes to infinity. We believe that $(M, \omega + \partial\overline{\partial}(\varphi|_t))$ converges to $(M_\infty, \omega_\infty)$ described in Conjecture 9.1. Previously, R. Hamilton thought that the limit $(M_\infty, \omega_\infty)$ should be a Ricci soliton. Our new observation here is that ω_∞ may be Kähler-Einstein, and otherwise, it is a special Ricci soliton.

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