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Kähler–Einstein metrics and the generalized Futaki invariant

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1 Introduction

In 1983, Futaki introduced his famous invariant. This invariant generalizes the obstruction of Kazdan–Warner to prescribing Gauss curvature on S^2 (cf. [Fu1]). The Futaki invariant is defined for any compact Kähler manifold with positive first Chern class that has nontrivial holomorphic vector fields. It is a Lie algebraic character from the Lie algebra of holomorphic vector fields into $\mathbb{C}$, and its vanishing is a necessary condition for the existence of a Kähler–Einstein metric on the underlying manifold. Therefore, it can be used to test the existence of Kähler–Einstein metrics on a given compact Kähler manifold with positive first Chern class. An excellent reference on the Futaki invariant is Futaki's book [Fu2].

Until now, all known nontrivial obstructions to Kähler–Einstein metrics come from holomorphic vector fields. This suggests the following conjecture.

Conjecture. *If a compact Kähler manifold with positive first Chern class has no nontrivial holomorphic vector fields, then it admits a Kähler–Einstein metric.*

One can also formulate a parallel conjecture for Kähler orbifolds.

In this paper, we will use the jumping of complex structures to produce new obstructions to the existence of Kähler–Einstein (or orbifold) metrics. Our obstructions do not assume that the underlying Kähler manifold (or orbifold) has nontrivial holomorphic vector fields, hence, they could lead to counterexamples to the above conjecture. Indeed, we will see that the conjecture is false for Kähler orbifolds. Our results also indicate that there might be a connection between the existence of a Kähler–Einstein metric and Mumford's stability of the point in Chow variety corresponding to the underlying Kähler manifold.

Let X be a compact Kähler manifold with positive first Chern class $C_1(X)$. Kodaira's embedding Theorem implies that the pluri-anticanonical line bundle K_X^{-m} is very ample for sufficiently large $m > 0$, namely, any basis of $H^0(X, K_X^{-m})$

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gives an embedding of X into some projective space $\mathbb{C}P^N$. Therefore, we may assume that X is a submanifold in $\mathbb{C}P^N$ and the hyperplane line bundle on $\mathbb{C}P^N$ restricts to a multiple of the anticanonical bundle K_X^{-1}. Let σ_t be a one-parameter subgroup in $\mathrm{Aut}(\mathbb{C}P^N) = \mathrm{SL}(N+1, \mathbb{C})$, which is generated by the real part of a holomorphic vector field v.

Denote by X_t the submanifold $\sigma_t(X)$ $(t \in \mathbb{R})$. It is well known that X_t converges to a subvariety X_∞ in $\mathbb{C}P^N$ as t tends to $+\infty$. Although this is not necessary, for simplicity, we assume here that X_∞ is irreducible and nondegenerate, i.e., X_∞ is not contained in any hyperplane of $\mathbb{C}P^N$. Then σ_t preserves X_∞ for all $t \in \mathbb{R}$. Consequently, the induced holomorphic vector field v is tangent to X_∞. For convenience, later on, we say that X jumps to X_∞. Let $\eta(X_\infty)$ be the Lie algebra of the holomorphic vector fields on $\mathbb{C}P^N$ which are tangent to X_∞ along it. We will first generalize the arguments in [Fu1] to introduce a character F_{X_∞} from $\eta(X_\infty)$ into $\mathbb{C}$ (cf. Sect. 1 for the details). As in the smooth case, we call F_{X_∞} the Futaki invariant of X_∞.

Theorem 0.1 *Let X, σ_t, v, X_∞ be as above. Assume that X_∞ is normal and X admits a Kähler–Einstein metric. Then the real part of the generalized Futaki invariant $F_{X_\infty}(v)$ is nonnegative.*

The following can be easily proved by applying Theorem 0.1 to both v and $-v$.

Corollary 0.1 (Futaki, 83, [Fu1]) *If σ_t preserves X, then $F_X(v) = 0$.*

Theorem 0.1 can be generalized to the case of Kähler orbifolds and yields obstructions to the existence of Kähler–Einstein orbifold metrics. A Kähler–Einstein orbifold metric is just a Kähler orbifold metric such that the Ricci curvature is proportional to the metric. The analysis on Kähler orbifolds is identical to that on smooth Kähler manifolds. In particular, if X is a normal Kähler orbifold, the first Chern class $C_1(X)$ is defined to be the cohomology class represented by the Ricci curvature of any Kähler orbifold metric. We say that $C_1(X)$ is positive if it is represented by a Kähler orbifold metric. By a theorem of Baily [Ba], if $C_1(X)$ is positive, we can still embed X into some projective space $\mathbb{C}P^N$ by pluri-anticanonical sections as in the case of smooth Kähler manifolds.

Theorem 0.2 *Let X be a Kähler orbifold embedded in $\mathbb{C}P^N$ such that the hyperplane bundle restricts to a multiple of the anticanonical bundle K_X^{-1}, and let σ_t be a one-parameter subgroup in* $\mathrm{Aut}(\mathbb{C}P^N) = \mathrm{SL}(N+1, \mathbb{C})$, *which is induced by the real part of a holomorphic vector field v. Assume that $X_t = \sigma_t(X)$ converges to an irreducible, normal and nondegenerate variety X_∞ and X admits a Kähler–Einstein orbifold metric. Then the real part of the generalized Futaki invariant $F_{X_\infty}(v)$ is nonnegative.*

Although we have not been able to construct a counterexample to the above conjecture using Theorem 0.1, we will show that Theorem 0.2 does provide many counterexamples to the conjecture in the case of Kähler orbifolds. In fact, some of these Kähler orbifolds are just cubic hypersurfaces in $\mathbb{C}P^3$. Of course, by the main theorem in [Ti], these cubic surfaces can not be smooth, but they have only mild quotient singularities, i.e., simple surface singularities (cf. [BPV, pp. 86–87]). A generic one has only one quotient singular point of type A_3 (cf. Sect. 4 for details). Recall that a singularity of type A_k $(k \geqq 1)$ is a simple surface singularity, and is isomorphic to the quotient space of $\mathbb{C}^2$ by the cyclic subgroup of order $k+1$ in

SU(2). In the applications of Theorem 0.2 to cubic hypersurfaces, the limit variety X_∞ is still diffeomorphic to X in some cases, while X_∞ is more singular than X in other ones. Curiously, our discussions yield that a cubic surface in $\mathbb{C}P^3$ has a Kähler–Einstein orbifold metric only if it is semistable in the sense of Mumford [Md] (Theorem 4.1). This makes us wonder if this is true in general.

The organization of this paper is as follows. In Sect. 1, we introduce the generalized Futaki invariant for a class of singular varieties. We also give a method of computing this generalized invariant for nondegenerate vector fields on Kähler orbifolds, following the argument of Futaki in [Fu2] in the smooth case. In Sect. 2, we briefly discuss the K-energy and state a result of Bando and Mabuchi in the setting of orbifolds. This result will be used in the proof of our main theorems. Section 3 contains the proof of Theorem 0.1. The proof for Theorem 0.2 is analogous and we omit it. In Sect. 4, we apply Theorem 0.2 to some cubic hypersurfaces in $\mathbb{C}P^3$. We find many interesting examples of Kähler orbifolds with positive first Chern class and without any holomorphic vector fields, which do not have Kähler–Einstein orbifold metrics, either. In the final section, we give two problems and discuss some possible generalizations to arbitrary Kähler manifolds with extremal metrics.

1 Futaki invariants for singular Fano varieties

In this section, we define and study a generalized Futaki invariant for a class of normal projective varieties.

Let Y be an irreducible, normal projective variety. In particular, the singular locus $\mathrm{Sing}(Y)$ of Y has complex dimension less than $\dim_{\mathbb{C}} Y - 1$. A line bundle L in Y is said to be ample if the tensor power L^m of L is very ample for some $m > 0$, i.e., any basis of the sections of L^m gives a Kodaira embedding of Y into some projective space $\mathbb{C}P^N$. An admissible Kähler metric ω on Y is defined as follows. Assume that L^m is very ample and let $\phi_m: Y \to \mathbb{C}P^N$ be the corresponding Kodaira embedding. Then any metric of the form

$$\omega = \frac{\alpha}{m}\phi_m^*\left(\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\psi\right) \quad (\alpha > 0) \tag{1.1}$$

where ω_{FS} is the Fubini–Study metric and ψ is a smooth function on $\mathbb{C}P^N$, is said to be admissible.

One can check that the admissibility of ω is independent of the choice of an m such that L^m is very ample. Since $\dim_{\mathbb{C}}\mathrm{Sing}(Y) \leqq n-2$, where $n = \dim_{\mathbb{C}} Y$, the Kähler metric ω defines a cohomology class $[\omega]$ of type $(1,1)$ on Y, i.e., $[\omega] = \alpha C_1(L)$. It follows from the definition that, given two admissible Kähler metrics ω_1 and ω_2 in $[\omega]$, there is a bounded, continuous function φ that is smooth away from $\mathrm{Sing}(Y)$ such that

$$\omega_2 = \omega_1 + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi \quad \text{on } Y\backslash\mathrm{Sing}(Y). \tag{1.2}$$

Definition 1.1 We say that Y is a Q-Fano variety if there is an ample line bundle which restricts to the pluri-anticanonical line bundle K_Y^{-m} on the regular part of Y for some m.

Obviously, if Y is smooth, then Y is Q-Fano if and only if $C_1(Y) > 0$. There are two important special cases of Q-Fano varieties: (1) if Y has a resolution $\tilde{Y}$ with the anticanonical line bundle $K_{\tilde{Y}}^{-1}$ being nef and ample outside the exceptional divisor. The line bundle in Definition 2.1 is just the pushdown of the anticanonical line bundle $K_{\tilde{Y}}^{-1}$; (2) If Y is a nondegenerate, normal subvariety in some projective space $\mathbb{C}P^N$ with the property that there is a sequence of compact Kähler manifolds $\{X_i\}_{i \geqq 1}$ in $\mathbb{C}P^N$ with $C_1(X_i) > 0$ such that $\lim_{i\to\infty} X_i = Y$ and the hyperplane bundle restricts to the pluri-anticanonical bundle $K_{X_i}^{-m}$ for a fixed m and all i. In particular, the subvariety X_∞ in Theorem 0.1 is a Q-Fano variety, as is the one in Theorem 0.2 for a similar reason.

For convenience, in the following, we will denote by $C_1(Y)$ the cohomology class $\frac{1}{m} C_1(L)$ if L is the line bundle in Definition 1.1 on the Q-Fano variety Y. Let $\omega \in C_1(Y)$ be an admissible Kähler metric. Then there is a smooth function f on the regular part Y_{reg} of Y such that

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f \quad \text{on } Y_{\mathrm{reg}}. \tag{1.3}$$

Lemma 1.1 *The function f can be extended to Y and is L^p-integrable with respect to the volume form ω^n on Y for any given $p > 1$.*

Proof. Let $\tilde{Y}$ be a smooth resolution of Y, and $\tilde{\omega}$ be a Kähler metric on $\tilde{Y}$ such that $\pi^*\omega \leqq \tilde{\omega}$, where $\pi: \tilde{Y} \to Y$ is the natural projection. By the choice of ω, the cohomology class of $\mathrm{Ric}(\tilde{\omega}) - \pi^*\omega$ has its support in the exceptional divisors $E_1, \ldots, E_\ell$ of $\tilde{Y}$ over Y, say

$$\mathrm{Ric}(\tilde{\omega}) - \pi^*\omega = \sum_{i=1}^{\ell} \alpha_i C_1([E_i]),$$

where $C_1([E_i])$ are the Poincaré duals to E_i in $\tilde{Y}$. For each i, let $\|\cdot\|_i$ be a hermitian metric on the line bundle $[E_i]$ and S_i be a section of $[E_i]$ whose zero locus is E_i. Then we have in the sense of distribution,

$$\mathrm{Ric}(\tilde{\omega}) - \pi^*\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\left(- \sum_{i=1}^{\ell} \alpha_i \log \|S_i\|_i^2 + \psi \right),$$

where ψ is a smooth function on $\tilde{Y}$. Consequently,

$$\begin{aligned}\pi^*\mathrm{Ric}(\omega) - \pi^*\omega &= \mathrm{Ric}(\pi^*\omega) - \pi^*\omega \\ &= \mathrm{Ric}(\tilde{\omega}) - \pi^*\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(\frac{\omega^n}{(\pi^*\omega)^n} \right) \\ &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\left(- \sum_{i=1}^{\ell} \alpha_i \log \|S_i\|_i^2 + \psi + \log\left(\frac{\tilde{\omega}^n}{\pi^*\omega^n} \right) \right). \qquad (1.4)\end{aligned}$$

Since the singular locus $\mathrm{Sing}(Y)$ is of complex codimension at least two, it follows

$$f = - \sum_{i=1}^{\ell} \alpha_i \log \|S_i\|_i^2 + \psi + \log\left(\frac{\tilde{\omega}^n}{\pi^*\omega^n} \right) + c \quad \text{on } Y_{\mathrm{reg}}, \tag{1.5}$$

where c is a constant. Therefore, f can be extended to Y and is L^p-integrable with respect to ω^n for any $p > 1$.

Remark. If Y admits a smooth resolution $\tilde{Y}$ such that

$$C_1(\tilde{Y}) - \pi^* C_1(Y) = \sum_{i=1}^{\ell} \alpha_i C_1([E_i]) \tag{1.6}$$

for some $\alpha_i < 1$ ($i = 1, 2, \ldots, \ell$), where E_i are the exceptional divisors in $\tilde{Y}$, then e^f is L^1-bounded with respect to ω. This follows from (1.5) in the proof of the lemma above.

Definition 1.2 A holomorphic vector field v on Y_{reg} is *admissible* if there is an embedding of Y into some projective space $\mathbb{C}P^N$ such that the hyperplane bundle $\mathcal{O}_{\mathbb{C}P^N}(1)$ restricts to a pluri-anticanonical line bundle K_Y^{-m} on Y_{reg} and v is the restriction of some holomorphic vector field on $\mathbb{C}P^N$ to Y.

Let ω be an admissible Kähler metric as given in the definition (1.1). Then by Lemma 1.1, there is a L^4-function f on Y such that in the sense of distribution,

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f \quad \text{on } Y, \tag{1.7}$$

that is, for any smooth $(n-1, n-1)$-form ψ on $\mathbb{C}P^N$,

$$\int_Y (\mathrm{Ric}(\omega) - \omega) \wedge \psi = \frac{\sqrt{-1}}{2\pi} \int_Y f \partial\bar{\partial}\psi.$$

Since ω is the restriction of a Kähler metric on some projective space $\mathbb{C}P^N$, the scalar curvature of ω is L^1-integrable. Therefore, the above integral on the left-hand side is well-defined.

Since the Ricci curvature $\mathrm{Ric}(\omega)$ is bounded from above,

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f \leqq C\omega \tag{1.8}$$

for some constant $C \geqq 0$. In particular, it implies that f is bounded from below. This can be seen by pulling (1.8) back to a smooth resolution of Y and using Green's formula on the smooth resolution. Notice that by taking the trace of (1.8) with respect to ω, we get

$$\Delta_\omega f \leqq Cn, \tag{1.9}$$

where Δ_ω denotes the Laplacian operator of the admissible Kähler metric ω.

Multiplying both sides of (1.9) by $\dfrac{1}{1 + (f - \inf_Y f)}$ and integrating by parts, we obtain

$$\begin{aligned} \int_Y \frac{|\nabla f|^2}{(1 + (f - \inf_Y f))^2} \omega^n &\leqq Cn \int_Y \omega^n \\ &= CnV. \end{aligned} \tag{1.10}$$

Therefore

$$\int_Y |\nabla f|^{3/2}\omega^n \leqq \left(\int_Y \frac{|\nabla f|^2}{(1+(f-\inf_Y f))^2}\omega^n\right)^{3/4}\left(\int_Y (1+(f-\inf_Y f))^6\omega^n\right)^{1/4},$$
$$< \infty$$

i.e., $|\nabla f|$ is $L^{3/2}$-bounded. It follows that the integral $\int_Y v(f)\omega^n$ is well-defined for any admissible holomorphic vector field v.

Lemma 1.2 *Let v be an admissible holomorphic vector field, ω_1, ω_2 be two admissible Kähler metrics in $C_1(Y)$ on a Q-Fano variety Y. If f_1, f_2 are the functions defined in* (1.7) *for $\omega = \omega_1, \omega_2$, respectively, then*

$$\int_Y v(f_1)\omega_1^n = \int_Y v(f_2)\omega_2^n. \tag{1.11}$$

Proof. Without loss of generality, we may assume that Y is a subvariety in $\mathbb{C}P^N$, both ω_1, ω_2 are Kähler metrics on $\mathbb{C}P^N$ and v is a holomorphic vector field on $\mathbb{C}P^N$. Then there is a smooth function φ on $\mathbb{C}P^N$ such that

$$\omega_2 = \omega_1 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$$

(cf. (1.2)).

Define $\omega_s = \omega_1 + \frac{\sqrt{-1}}{2\pi}(s-1)\partial\bar{\partial}\varphi$, $(1 \leqq s \leqq 2)$, then all ω_s are admissible Kähler metrics in $C_1(Y)$. Let f_s be the function such that in the sense of distribution,

$$\operatorname{Ric}(\omega_s) - \omega_s = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f_s\,. \tag{1.12}$$

A direct computation shows: we can take f_s to be of the form

$$f_s = -\log\left(\frac{\omega_s^n}{\omega_1^n}\right) - (s-1)\varphi + f\,. \tag{1.13}$$

Consequently, if $\dot{f}_s$ denotes the derivative $\left.\frac{\partial f_t}{\partial t}\right|_{t=s}$, we have

$$\dot{f}_s = -\Delta_s\varphi - \varphi \quad \text{on } Y\,, \tag{1.14}$$

where Δ_s denotes the Laplacian of the metric ω_s. In particular, $\dot{f}_s$ is continuous on Y. Therefore

$$\begin{aligned}\frac{d}{ds}\left(\int_Y v(f_s)\omega_s^n\right) &= \int_Y (v(\dot{f}_s) + v(f_s)\Delta_s\varphi)\omega_s^n \\ &= \int_Y (v(-\Delta_s\varphi - \varphi) + v(f_s)\Delta_s\varphi)\omega_s^n \\ &= \int_Y ((\operatorname{div}_s(v) + v(f_s))\Delta_s\varphi - v(\varphi_s))\omega_s^n\,,\end{aligned} \tag{1.15}$$

where $\operatorname{div}_s(v)$ denotes the divergence of v with respect to ω_s. Since v is holomorphic and ω_s is a Kähler metric on $\mathbb{C}P^N$, the interior product $v \lrcorner\, \omega_s$ is a $\bar{\partial}$-closed

(0, 1)-form. Therefore, there is a smooth function ψ_s such that $\bar{\partial}\psi_s = v \rfloor \omega_s$. Since v is tangent to Y, $\mathrm{div}_s(v) = \Delta_s \psi_s$, so by (1.15),

$$\frac{d}{ds}\left(\int_Y v(f_s)\omega_s^n\right) = \int_Y (\Delta_s \psi_s + \psi_s + v(f_s))\Delta_s \varphi_s \omega_s^n . \tag{1.16}$$

It is an easy computation to show that $\bar{\partial}(\Delta_s \psi_s + \psi_s + v(f_s))$ is identically zero. Here one needs to use the Eq. (1.12) and the fact that v is holomorphic. Using the arguments in the proof of Lemma 1.1, we see that df_s can only have simple poles along the normal directions of the singular set $\mathrm{Sing}(Y)$ of Y. However, since v is tangent to $\mathrm{Sing}(Y)$ along $\mathrm{Sing}(Y)$, the normal part of v is zero. Consequently, $v(f_s)$ is bounded on Y and so is $\Delta_s \psi_s + \psi_s + v(f_s)$. Therefore, the holomorphic function $\Delta_s \psi_s + \psi_s + v(f_s)$ has to be a constant. It follows from (1.16) that

$$\frac{d}{ds}\left(\int_Y v(f_s)\omega_s^n\right) \equiv 0 , \quad 1 \leqq s \leqq 2 .$$

The lemma follows.

Now we can define the generalized Futaki invariant $F_Y(v)$ for an admissible holomorphic vector field v on a Q-Fano variety Y by

$$F_Y(v) = \int_Y v(f)\omega^n , \tag{1.17}$$

where ω is any admissible metric and f is defined by (1.7). Lemma 1.2 tells us that $F_Y(v)$ is independent of the choice of the admissible metric ω and so is a holomorphic invariant of Y.

If Y is smooth, this $F_Y(v)$ is just the Futaki invariant [Fu1]. In fact, the proof of Lemma 1.2 is essentially due to Futaki. Our contribution is only to make sure that the singularities of Y do not cause any serious trouble.

Finally, we discuss briefly how to compute this generalized Futaki invariant in case Y is a normal Kähler orbifold with $C_1(Y) > 0$. The reader can find some descriptions of Kähler orbifolds in the two paragraphs before Theorem 2.2 in Sect. 2. If we take ω to be in $C_1(Y)$, there is a smooth function f satisfying:

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f \quad \text{on } Y . \tag{1.18}$$

Here f is smooth in the following sense: for any local uniformization (U, Γ, ϕ), $\pi^* f$ is smooth in U, where $\pi : U \to U/\Gamma \subset Y$ is the projection. A Kähler orbifold Y with $C_1(Y) > 0$ is a Q-Fano variety. This follows from Kodaira's embedding Theorem in the case of Kähler orbifolds (cf. [Ba]).

Proposition 1.1 *Assume that v is an admissible holomorphic vector field on a Kähler orbifold Y with $C_1(Y) > 0$. Then for any Kähler orbifold metric ω in $C_1(Y)$ and f defined in* (1.18),

$$F_Y(v) = \int_Y v(f)\omega^n . \tag{1.19}$$

Proof. Recall that the generalized Futaki invariant $F_Y(v)$ was defined using admissible Kähler metrics, which are degenerate Kähler orbifold metrics: they degenerate along the normal directions of $\mathrm{Sing}(Y)$. We are asserting that orbifold metrics can be used instead. The point is that one can approximate an admissible Kähler metric ω_0 by Kähler orbifold metrics. Moreover, a direct computation shows that there is a function φ, smooth in the sense of orbifolds, such that $\omega_\varepsilon = \omega_0 + \varepsilon \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ are Kähler orbifold metrics for any sufficiently small $\varepsilon > 0$. Obviously, all these ω_ε are in $C_1(Y)$.

By the analogue of Lemma 1.2 for Kähler orbifold metrics on Y, we have,

$$\begin{aligned}\int_Y v(f)\omega^n &= \int_Y v(f_\varepsilon)\omega_\varepsilon^n \\ &= \lim_{\varepsilon\to 0^+}\int_Y v(f_\varepsilon)\omega_\varepsilon^n ,\end{aligned}$$

where the f_ε are the functions satisfying:

$$\mathrm{Ric}(\omega_\varepsilon) - \omega_\varepsilon = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f_\varepsilon \quad \text{on } Y .$$

Then the f_ε can be taken to be

$$-\log\left(\frac{\omega_\varepsilon^n}{\omega_0^n}\right) - \varepsilon\varphi + f_0 ,$$

where $f = f_0$ satisfies (1.7) with $\omega = \omega_0$. Therefore

$$v(f_\varepsilon) = -v\left(\frac{\omega_\varepsilon^n}{\omega_0^n}\right)\frac{\omega_0^n}{\omega_\varepsilon^n} - \varepsilon v(\varphi) + v(f_0) ,$$

consequently,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_Y v(f_\varepsilon)\omega_\varepsilon^n &= \lim_{\varepsilon\to 0^+}\left(-\int_Y v\left(\frac{\omega_\varepsilon^n}{\omega_0^n}\right)\omega_0^n + \int_Y v(f_0)\omega_\varepsilon^n\right) \\ &= F_Y(v) + \lim_{\varepsilon\to 0^+}\int_Y \mathrm{div}_{\omega_0}(v)\omega_\varepsilon^n .\end{aligned}$$

By the admissibility of v, $\mathrm{div}_{\omega_0}(v)$ is bounded, so

$$\lim_{\varepsilon\to 0^+}\int_Y \mathrm{div}_{\omega_0}(v)\omega_\varepsilon^n = \int_Y \mathrm{div}_{\omega_0}(v)\omega_0^n = 0 .$$

The proposition is proved.

Let Y be a Kähler orbifold with $C_1(Y) > 0$. A holomorphic vector field v is said to be nondegenerate if the zero set of v consists of disjoint subsets $\{Z_\lambda\}_{\lambda\in\Lambda}$ satisfying the following conditions: for each local uniformization $\pi: U \to U/\Gamma \subset Y$ with $\pi(U)\cap Z_\lambda \neq \emptyset$, $\pi^{-1}(Z_\lambda)$ is smooth in U, $\pi^* v$ vanishes along $\pi^{-1}(Z_\lambda)$ and is nondegenerate in the normal directions of $\pi^{-1}(Z_\lambda)$. Now if ω is a Kähler orbifold metric in $C_1(Y)$, then we define an endomorphism $L(v)$ of $T^{1,0}Y$ by

$$L(v)(u) = \nabla_v u - L_v u = \nabla_u v , \tag{1.20}$$

where L_v denotes the Lie derivative, and u is a vector field of type (1,0). We should point out that at a singular point of Y, (1.20) is understood to be defined via local uniformization. Let Z_λ be one of the components in Zero(v), then $L(v)$ induces an endomorphism $L_\lambda(v)$ of the normal bundle of $\pi^{-1}(Z_\lambda)$ in any local uniformization $\pi: U \to Y$. This $L_\lambda(v)$ is of full rank at any point of Z_λ by the nondegeneracy condition. The following proposition is the analogue of Theorem 5.2.8 in [Fu2] for Kähler orbifolds and can be proved by giving additional care at the singular set.

Proposition 1.2 *Let Y, ω, v be as above, Ω be the curvature form of ω and K_λ be the curvature form of the induced metric on the normal bundle $N_{Y|Z_\lambda}$ of Z_λ, which is defined to be $N_{U|\pi^{-1}(Z_\lambda)}$ in any local uniformization $\pi: U \to Y$. Then*

$$F_Y(v) = \sum_{\lambda \in \Lambda} \frac{1}{|\Gamma_\lambda|} \int_{Z_\lambda} \varphi(L(v) + \Omega)/\det(L_\lambda(v) + K_\lambda)\,, \tag{1.21}$$

where $|\Gamma_\lambda|$ is the order of the local uniformization group at a generic point of Z_λ, and φ denotes the $(n+1)$-power of the invariant Chern–Weil polynomial defining the first Chern class.

There are two special cases of this proposition in which $F_Y(v)$ has a very simple formula.

Corollary 1.1 *Let Y, ω, v be as above. If all Z_λ are of dimension* 0, *then*

$$F_Y(v) = \sum_{\lambda \in \Lambda} \frac{1}{|\Gamma_\lambda|} \frac{(\mathrm{div}_{Z_\lambda}(v))^{n+1}}{\det(L(v)|_{Z_\lambda})}\,. \tag{1.22}$$

Corollary 1.2 *Let Y, ω, v be as above, $n = 2$, $\Lambda = \Lambda_1 \cup \Lambda_2$. Assume that $\dim_{\mathbb{C}} Z_\lambda = 0$ for $\lambda \in \Lambda_1$ and $\dim_{\mathbb{C}} Z_1 = 1$ for $\lambda \in \Lambda_2$. Then*

$$\begin{aligned} F_Y(v) = {} & \sum_{\lambda \in \Lambda_1} \frac{1}{|\Gamma_\lambda|} \frac{(\mathrm{div}_{Z_\lambda}(v))^3}{\det(L(v)|_{Z_\lambda})} \\ & + \sum_{\lambda \in \Lambda_2} \mathrm{div}_{Z_\lambda}(v)\left(2\deg(Z_\lambda) + \frac{1}{|\Gamma_\lambda|}\left(2 - 2g(Z_\lambda) - \sum_{x \in Z_\lambda} \frac{|\Gamma_x| - 1}{|\Gamma_x|}\right)\right), \end{aligned} \tag{1.23}$$

where $g(Z_\lambda)$ is the genus of Z_λ for each $\lambda \in \Lambda_2$ and Γ_x is the local uniformization group of Y at x, in particular, if x is a smooth point, $|\Gamma_x| = 1$.

Examples. (1) Let $X_f \subset \mathbb{C}P^3$ be the zero locus of a cubic polynomial f. Put $f = z_0 z_1^2 + z_2 z_3(z_2 - z_3)$, where z_0, z_1, z_2, z_3 are homogeneous coordinates of $\mathbb{C}P^3$. Then X_f has a unique quotient singularity at $p_0 = [1, 0, 0, 0]$. This singularity is of the form C^2/Γ, where Γ is the dihedral subgroup in SU(2) of type D_4. One can check that X_f is a Kähler orbifold with $C_1(X) > 0$. In fact, the minimal resolution of X_f is an almost Del Pezzo surface. Let v be the holomorphic vector field whose real part generates the one-parameter subgroup $\{\mathrm{diag}(1, e^{3t}, e^{2t}, e^{2t})\}_{t \in R}$ in SL(4, $\mathbb{C}$). Then v restricts to a holomorphic vector field on X_f and has five zeros $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$, $[0, 0, 1, 1]$. A computation shows

$$\begin{aligned} F_{X_f}(v) &= \frac{1}{8} \cdot \frac{(\frac{1}{2})^3}{\frac{1}{16}} + \frac{(-2)^3}{1} + 3\frac{(-1)^3}{-2} \\ &= \frac{1}{4} - 8 + \frac{3}{2} = \frac{7}{4} - 8 = -\frac{25}{4}\,. \end{aligned}$$

(2) Let $X_f = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}P^3 \mid f(z_0, z_1, z_2, z_3) = 0\}$ be defined by the cubic polynomial $f = z_0 z_1^2 + z_1 z_2^2 + z_3^3$. Then X_f is a Kähler orbifold with $C_1(X) > 0$ and a unique quotient singular point of type E_6, i.e., C^2/Γ, where $\Gamma \subset \mathrm{SU}(2)$ is a finite group of type E_6. Let v be the holomorphic vector field on X whose real part generates the one-parameter subgroup $\{\mathrm{diag}(1, e^{6t}, e^{3t}, e^{4t})\}_{t \in \mathbb{R}}$ in $\mathrm{SL}(4, \mathbb{C})$. Then v has zeros $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, and by Corollary 1.2,

$$F_{X_f}(v) = \frac{1}{60} \frac{(\mathrm{div}(v))^3}{\det(L(v))}\bigg|_{[1,0,0,0]} - \frac{125}{6} + \frac{8}{3}\,.$$

$$< 0\,.$$

(3) Let $f = z_0(z_1^2 + z_2^2) + z_3^2 z_1$. Then $\mathrm{Sing}(X_f) = \{[1, 0, 0, 0], [0, 1, \pm\sqrt{-1}, 0]\}$. The first of them is of type A_3, i.e., the quotient singularity of the form C^2/Γ_4, where $\Gamma_4 \subset \mathrm{SU}(2)$ is generated by

$$\begin{pmatrix} e^{2\pi\sqrt{-1}/4} & 0 \\ 0 & e^{-2\pi\sqrt{-1}/4} \end{pmatrix}.$$

The others are of type A_1. Let v be the holomorphic vector field whose real part generates $\{\mathrm{diag}(1, e^{2t}, e^{2t}, e^{t})\}$. Then $\mathrm{Zero}(v)$ consists of $[1, 0, 0, 0]$ and the line $\{z_0 = z_3 = 0\}$. This line has the degree 1 and contains two singularities $[0, 1, \pm\sqrt{-1}, 0]$. Also, the generic point of this line is smooth. Therefore, by Corollary 1.2,

$$F_{X_f}(v) = \frac{1}{4} \frac{1^3}{1/4} - 3 = -2 < 0\,.$$

(4) Let $f = z_0(z_1^2 + z_2^2) + z_3^3$. Then $\mathrm{Sing}(X_5)$ consists of three points $[1, 0, 0, 0]$ and $[0, 1, \pm\sqrt{-1}, 0]$. Let v be the holomorphic vector field whose real part generates the group $\{\mathrm{diag}(1, e^{3t}, e^{3t}, e^{2t})\}$. The zero set of v consists of a single point $[1, 0, 0, 0]$ and a line $\{z_0 = z_3 = 0\}$. By Corollary 1.2, one obtains

$$F_{X_f}(v) = \frac{1}{3} \cdot \frac{2^3}{1} - \left(2 + 2 - \frac{4}{3}\right) = 0\,.$$

In fact, this orbifold is the quotient of $\mathbb{C}P^2$ by the cyclic group Γ_3 in $\mathrm{SU}(2)$. Therefore, it has a Kähler–Einstein orbifold metric and has vanishing Futaki invariant.

2 The lower boundedness of the K-energy

The K-energy was first introduced by Mabuchi in [Ma] on any compact Kähler manifold with positive first Chern class. It was used in [BM] to prove the uniqueness of Kähler–Einstein metrics. Its definition was inspired by the work of Donaldson on Yang–Mills connections and stable bundles. Bando and Mabuchi also proved in [BM] that if there is a Kähler–Einstein metric on the underlying manifold, then the K-energy is bounded from below. In this section, we give a brief account of the K-energy.

Let X be a compact Kähler manifold with positive first Chern class $C_1(X)$. By an abuse of notation, we will also denote by $C_1(X)$ the set of Kähler metrics whose Kähler form represents $C_1(X)$, and we will identify a Kähler metric with its Kähler form. For any two Kähler metrics ω_0 and ω_1 in $C_1(X)$, there is a smooth function φ, unique up to the addition of constants, satisfying:

$$\omega_1 = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi . \tag{2.1}$$

We put $\omega_s = \omega_0 + \frac{s\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ and define

$$M(\omega_0, \omega_1) = -\frac{1}{V}\int_0^1\left(\int_X \varphi(R(\omega_s) - n)\omega_s^n\right)ds , \tag{2.2}$$

where $R(\omega_s)$ denotes the scalar curvature of the metric, n is the complex dimension of X and V is the volume of X with respect to ω_0. The functional M has the properties:

(1) $M(\omega_0, \omega_1) = -M(\omega_1, \omega_0)$
(2) $M(\omega_0, \omega_1) + M(\omega_1, \omega_2) = M(\omega_0, \omega_2)$.

These identities can be proved by a direct computation (cf. Proposition 6.3.1, [Fu2]). Fix any Kähler metric ω in $C_1(X)$, we define the K-energy $M_\omega(\cdot)$ on the set $C_1(X)$ of Kähler metrics by

$$M_\omega(\omega') = M(\omega, \omega') . \tag{2.3}$$

Lemma 2.1 *Let ω_t be a family of Kähler metrics in $C_1(X)$. Write $\omega_t = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$. Then*

$$\frac{d}{dt}M_\omega(\omega_t) = \frac{1}{V}\int_X \nabla_t\dot{\varphi}_t(f_t)\omega_t^n , \tag{2.4}$$

where the functions f_t are determined by

$$\operatorname{Ric}(\omega_t) - \omega_t = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f_t \tag{2.5}$$

$$\int_X f_t\omega_t^n = 0 . \tag{2.6}$$

This follows from a direct computation. We omit the proof.

Theorem 2.1 (Bando and Mabuchi [BM]) *If X admits a Kähler–Einstein metric, then the* K-*energy M_ω is bounded from below.*

We refer the reader to [BM] for the proof of this theorem.

A local uniformizing chart is a triple (U, Γ, ϕ) satisfying the following conditions: (1) $\phi: U \to \mathbb{C}^n$ is a holomorphic embedding; (2) Γ is a finite group acting on U by biholomorphisms and $\phi\circ\Gamma\circ\phi^{-1}$ is contained in $U(n)$. A complex orbifold is a Hausdorff topological space covered by the open subsets $\{U_\alpha/\Gamma_\alpha\}$, where the

U_α/Γ_α are the quotients of the U_α by the Γ_α, and the $\{(U_\alpha, \Gamma_\alpha, \phi_\alpha)\}$ are local uniformizing charts. A Kähler orbifold metric on the complex orbifold is a Kähler metric ω on the regular part of the orbifold such that if (U, Γ, ϕ) is a local uniformizing chart and $\pi: U \to U/\Gamma$ is the projection, then $\pi^*\omega$ extends smoothly on U. A complex orbifold is a Kähler orbifold if it has a Kähler orbifold metric. A Kähler–Einstein orbifold metric is just a Kähler orbifold metric whose Ricci curvature is proportional to the metric. The first Chern class of a compact Kähler orbifold is defined to be the cohomology class represented by the Ricci curvature form of any Kähler orbifold metric.

Let X be a compact Kähler orbifold. We assume that the first Chern class $C_1(X)$ can be represented by a d-closed positive (1,1)-form, namely, $C_1(X) > 0$. It is known that $C_1(X) > 0$ is equivalent to the existence of a Kähler orbifold metric ω such that $\mathrm{Ric}(\omega)$ is a positive-definite (1,1)-form. As before, we denote by $C_1(X)$ the set of Kähler orbifold metrics whose Kähler form represents $C_1(X)$. Then we can define a functional $M(\cdot,\cdot)$ by (2.1) and (2.2) and the K-energy in this general case. This $M(\cdot,\cdot)$ has the properties (1), (2) on Kähler orbifolds as well as on smooth Kähler manifolds. In particular, we have the variation formula (2.4) for the K-energy on the Kähler orbifold X and the following

Theorem 2.2 *Let X be a compact Kähler orbifold with positive first Chern class. If X admits a Kähler–Einstein orbifold metric, then the* K-*energy M_ω is bounded from below.*

There is another functional F_ω introduced by the first author in [Di]. This functional is cohomological to the K-energy. It bounds the K-energy from below. Moreover, this F_ω has a nice expression in terms of the solutions of the following complex Monge–Ampere equations:

$$(2.7)_t \qquad \begin{cases} \left(\omega + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n = e^{f - t\varphi} \\ \left(\omega + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0 \ , \quad \text{in } X \ . \end{cases}$$

We refer readers to [DT] for this functional.

3 The proof of the main theorems

This section is devoted to the proof of Theorem 0.1. The proof of Theorem 0.2 is analogous and we omit it. Let X be a compact Kähler manifold in $\mathbb{C}P^N$ with $C_1(X) > 0$ and σ_t be a one-parameter subgroup in $\mathrm{Aut}(\mathbb{C}P^N) = \mathrm{SL}(N+1, \mathbb{C})$ as in the introduction. Let v be the holomorphic vector field induced by σ_t.

Denote by X_t the submanifold $\sigma_t(X)(t \in \mathbb{R})$. Then X_t converge to a subvariety X_∞ in $\mathbb{C}P^N$ as t tends to $+\infty$. By assumption, this X_∞ is irreducible, normal and nondegenerate.

Let ω_{FS} be the Fubini–Study metric on $\mathbb{C}P^N$. Then $\frac{1}{m}\sigma_t^*\omega_{\mathrm{FS}}$ restricts to a Kähler metric ω_t on X with its Kähler form in $C_1(X)$ for any $t \in \mathbb{R}$. By Theorem 2.1, the existence of a Kähler–Einstein metric on X implies that the K-energy $M_\omega(\omega_t)$ is bounded from below for $t \in \mathbb{R}$, where $\omega = \omega_0$.

By changing the coordinates $[z_0, \ldots, z_N]$ of $\mathbb{C}P^N$, we may assume that v is represented by an $(N+1)\times(N+1)$-matrix $(a_{ij})_{0 \leqq i,j \leqq N}$ satisfying the following conditions: there are $0 = k_0 < k_1 < \ldots < k_\mu = N$, such that

$$a_{ii} = \lambda_\alpha \quad \text{for } k_{\alpha-1} \leqq i < k_\alpha\,, \quad \alpha = 1, 2, \ldots, \mu\,,$$

$$a_{ij} = 0 \quad \text{if } i > j \quad \text{or} \quad k_{\alpha-1} \leqq i < k_\alpha \leqq j\,,$$

$$a_{ii+1} = 1 \quad \text{if } k_{\alpha-1} \leqq i < j < k_\alpha\,,$$

namely, v is in its Jordan form. Consequently, σ_t can be represented by a family of $(N+1)\times(N+1)$ matrices

$$\sigma_t = (\sigma_{ij}(t))_{0 \leqq i,j \leqq N}\,, \tag{3.1}$$

where

$$\sigma_{ij}(t) = \begin{cases} 0 & \text{if } i > j \text{ or } k_{\alpha-1} \leqq i < k_\alpha \leqq j, \alpha = 1, 2, \ldots, \mu\,, \\ e^{\lambda_\alpha t} & \text{if } k_{\alpha-1} \leqq i = j < k_\alpha, \alpha = 1, \ldots, \mu\,, \\ \dfrac{t^{j-i} e^{\lambda_\alpha t}}{(j-i)!} & \text{if } k_{\alpha-1} \leqq i < j < k_\alpha, \alpha = 1, 2, \ldots, \mu\,. \end{cases} \tag{3.2}$$

Recall that in the homogeneous coordinates of $\mathbb{C}P^N$,

$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(\sum_{i=0}^N |z_i|^2 \right) \tag{3.3}$$

it follows

$$\sigma_t^* \omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(\sum_{i=0}^N \left| \sum_{j=0}^N \sigma_{ij}(t) z_j \right|^2 \right). \tag{3.4}$$

Let us introduce the functions

$$\varphi_t = \frac{1}{m} \log\left(\frac{\sum_{i=0}^N |\sum_{j=0}^N \sigma_{ij}(t) z_j|^2}{\sum_{i=0}^N |z_i|^2} \right). \tag{3.5}$$

Then by (3.3) and (3.4), we have

$$\begin{aligned} \omega_t - \omega_0 &= \frac{1}{m} (\sigma_t^* \omega_{\mathrm{FS}} - \omega_{\mathrm{FS}})|_X \\ &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \varphi_t\,. \end{aligned} \tag{3.6}$$

A direct computation shows

$$\dot{\varphi}_t = \frac{1}{m} \frac{\sum_{i=0}^N 2\,\mathrm{Re}((\sum_{j=0}^N \sigma'_{ij}(t) z_j)(\overline{\sum_{k=0}^N \sigma_{ik}(t) Z_k}))}{\sum_{i=0}^N |\sum_{j=0}^N \sigma_{ij}(t) z_j|^2}$$

$$= \frac{1}{m}\frac{\sum_{\alpha=1}^{\mu}\sum_{i=k_{\alpha-1}}^{k_\alpha-1} 2\,\mathrm{Re}\left(\sum_{j=0}^{N}\overline{\sigma_{ij}(t)z_j}\cdot\sum_{j=i}^{k_\alpha-1}\left(\frac{\lambda_\alpha t}{j-i}+1\right)\frac{t^{j-i-1}e^{\lambda_\alpha t}}{(j-i-1)!}z_j\right)}{\sum_{i=0}^{N}|\sum_{j=0}^{N}\sigma_{ij}(t)z_j|^2}$$

$$= \frac{1}{m}\frac{2\,\mathrm{Re}\{\sum_{\alpha=1}^{\mu}\sum_{i=k_{\alpha-1}}^{k_\alpha-1}\sum_{j=0}^{N}\overline{\sigma_{ij}(t)z_j}\sum_{j=0}^{N}(\lambda_\alpha\sigma_{ij}(t)+\delta_{i+1,k}\sigma_{i+1,j}(t))z_j\}}{\sum_{i=0}^{N}|\sum_{j=0}^{N}\sigma_{ij}(t)z_j|^2}. \tag{3.7}$$

Define a smooth function θ_v on $\mathbb{C}P^N$ by

$$\theta_v(z_0,\ldots,z_N) = \frac{\sum_{\alpha=1}^{\mu}(\sum_{i=k_{\alpha-1}}^{k_\alpha-1}\lambda_\alpha|z_i|^2 + 2\,\mathrm{Re}(\sum_{i=k_{\alpha-1}}^{k_\alpha-1}\bar{z}_i z_{i+1}))}{\sum_{i=0}^{N}|z_i|^2} \tag{3.8}$$

then (3.7) implies

$$\dot{\varphi}_t = \frac{2}{m}\mathrm{Re}(\theta_v\circ\sigma_t)\,. \tag{3.9}$$

By Lemma 2.1, we have

$$\frac{d}{dt}M_\omega(\omega_t) = \frac{1}{V}\int_X \nabla_t\dot{\varphi}_t(\tilde{f}_t)\omega_t^n\,, \tag{3.10}$$

where ∇_t denotes the gradient with respect to the metric ω_t and $\tilde{f}_t$ is the function defined by the equations

$$\mathrm{Ric}(\omega_t) - \omega_t = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\tilde{f}_t\,, \tag{3.11}$$

$$\inf_X \tilde{f}_t = 0\,. \tag{3.12}$$

Denote by f_t the composition $\tilde{f}_t\circ\sigma_t^{-1}$. Then f_t is smooth on X_t and

$$\mathrm{Ric}\left(\frac{1}{m}\omega_{\mathrm{FS}}|_{X_t}\right) - \frac{1}{m}\omega_{\mathrm{FS}}|_{X_t} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f_t \tag{3.13}$$

$$\inf_{X_t} f_t = 0\,. \tag{3.14}$$

Then (3.10) becomes

$$\frac{d}{dt}M_\omega(\omega_t) = \frac{2}{V}\mathrm{Re}\left(\int_{X_t}(\nabla_t\theta_v)(f_t)\left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n\right). \tag{3.15}$$

Claim. $\lim_{t\to+\infty}\int_{X_t}(\nabla_t\theta_v)(f_t)\left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n = F_{X_\infty}(v)$.

Obviously, Theorem 0.1 follows from this claim, (3.15) and Theorem 2.1. Now let us prove this claim.

Lemma 3.1 *There is a uniform constant C, independent of t, such that for any smooth function ψ on X_t, we have the following Sobolev inequality,*

$$(3.16)\qquad \left(\int_{X_t} |\psi|^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}} \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \leqq C \int_{X_t} (|\nabla_t \psi|^2 + |\psi|^2)\left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n .$$

Any complex submanifold is a minimal submanifold, so this lemma follows from the Sobolev inequality in (Theorem 18.6, p. 93 [Si]).

Lemma 3.2 *Let Δ_t be the Laplacian operator of the metric $\frac{1}{m}\omega_{\mathrm{FS}}$ restricted to X_t. Then there is a positive constant $c > 0$, independent of t, such that the first eigenvalue $\lambda_1(\Delta_t) \geqq c > 0$.*

Proof. We prove it by contradiction. Suppose that the lemma is false. Then there is a sequence $\{t_i\}$ satisfying: (1) $\lim_{i\to\infty} t_i = \infty$; (2) the first nonzero eigenvalue $\lambda(t_i)$ of Δ_{t_i} converges to zero. For each i, let ψ_i be the eigenfunction of Δ_{t_i} with the eigenvalue $\lambda(t_i)$ such that

$$\int_{X_{t_i}} \psi_i^2 \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n = 1 .$$

Using Lemma 3.1 and a standard Moser iteration argument, one can show

$$(3.17)\qquad |\psi_i|_{C^0} \leqq C ,$$

where C is a uniform constant. Therefore, it follows from the elliptic theory that a subsequence of $\{\psi_i\}$ converges to a bounded nonzero function ψ_∞ on X_∞, which is actually smooth away from $\mathrm{Sing}(X_\infty)$. Furthermore, by the assumption on $\lambda(t_i)$, we have $\Delta_\infty \psi_\infty = 0$ on $X_\infty \backslash \mathrm{Sing}(X_\infty)$ and

$$\int_{X_\infty} \psi_\infty \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n = 0 .$$

This implies that X_∞ is reducible, a contradiction. The lemma is proved.

Since X_t is a complex submanifold, by the Gauss equation, the scalar curvature $R\left(\frac{1}{m}\omega_{\mathrm{FS}}|_{X_t}\right)$ of the metric $\frac{1}{m}\omega_{\mathrm{FS}}|_{X_t}$ is bounded from above by $mn(n+1)$. Therefore, taking the trace of (3.13) against $\frac{1}{m}\omega_{\mathrm{FS}}|_{X_t}$, we obtain

$$(3.18)\qquad \Delta_t f_t \leqq mn(n+1) .$$

For any $\delta > 0$, multiplying both sides of (3.17) by $(1+f_t)^{-\delta}$ and integrating by parts,

$$(3.19)\qquad \frac{1}{V}\int_{X_t} |\nabla(1+f_t)^{(1-\delta)/2}|^2 \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \leqq C_\delta ,$$

where C_δ is a constant depending only on δ, m, n.

In the following, we always use C to denote a constant independent of t. There is a uniform constant $C > 0$ and a small tubular neighborhood $B_\varepsilon(\mathrm{Sing}(X_\infty))$ in $\mathbb{C}P^N$ ($\varepsilon > 0$) such that

$$\mathrm{Vol}_{\frac{1}{m}\omega_{\mathrm{FS}}|X_t}(X_t \setminus B_{4\varepsilon}(\mathrm{Sing}(X_\infty))) \geqq \tfrac{3}{4} V \tag{3.20}$$

and

$$\left| R\left(\frac{1}{m}\omega_{\mathrm{FS}}|_{X_t}\right)\right| \leqq C \quad \text{on } X_t \setminus B_\varepsilon(\mathrm{Sing}(X_\infty))\,. \tag{3.21}$$

Outside $B_\varepsilon(\mathrm{Sing}(X_\infty))$, the submanifolds X_t are uniformly smooth. Then Lemma 3.2 and the standard elliptic estimates imply

$$\sup_{X_t \setminus B_{4\varepsilon}(\mathrm{Sing}(X_\infty))} \left| f_t - \frac{1}{V}\int_{X_t} f_t \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \right| \leqq C\,. \tag{3.22}$$

Lemma 3.3 *There is a uniform constant $C' > 0$ such that for any $k > 0$,*

$$k \leqq C' + \frac{3}{2V}\int_{X_t} (k - f_t)_+ \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n, \tag{3.23}$$

where $(k - f_t)_+ = \max\{0, k - f_t\}$.

Proof. Put ψ to be

$$(k - f_t)_+ - \frac{1}{V}\int_{X_t} (k - f_t)_+ \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n,$$

then by (3.18), we have

$$-\Delta_t \psi \leqq mn(n+1)\,. \tag{3.24}$$

For any $p > 0$, multiplying both sides of (3.24) by ψ_+^p and integrating by parts, we obtain

$$\int_{X_t} \left| \nabla_t \psi_+^{\frac{p+1}{2}} \right|^2 \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \leqq mn(n+1)\frac{(p+1)^2}{4p} \int_{X_t} \psi_+^p \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n. \tag{3.25}$$

With help of the Sobolev inequality in Lemma 3.1, the standard Moser iteration argument yields

$$\sup_{X_t} \psi \leqq C\left(1 + \left(\int_{X_t} \psi^2 \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n\right)^{\frac{1}{2}}\right), \tag{3.26}$$

where C is a uniform constant.

On the other hand, multiplying (3.18) by $(k - f_t)_+$ and integrating by parts, we have

$$\int_{X_t} |\nabla_t \psi|^2 \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \leqq mn(n+1) \int_{X_t} (k - f_t)_+ \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n. \tag{3.27}$$

The Lemma 3.2 implies

$$\int_{X_t} \psi^2 \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \leqq \frac{mn(n+1)}{c} \int_{X_t} (k - f_t)_+ \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n, \tag{3.28}$$

where c is the lower bound of the first eigenvalues in Lemma 3.2. Then (3.23) follows from (3.26), (3.28) and the Schwartz inequality.

Corollary 3.1 *Let E_t be the subset $\{x \in X_t \mid f_t(x) \leqq 8C'\}$. Denote by* $\mathrm{mes}(\cdot)$ *the measure on X_t induced by the metric $\frac{1}{m}\omega_{\mathrm{FS}}$. Then we have*

$$\mathrm{mes}(E_t) > \tfrac{1}{4} V . \tag{3.29}$$

Proof. By (3.23), we have

$$\begin{aligned} 16C' &\leqq C' + \frac{3}{2V} \int_{X_t} (16C' - f_t)_+ \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \\ &< C' + \frac{24C'}{V}\mathrm{mes}(E_t) + \frac{12C'}{V}(V - \mathrm{mes}(E_t)) \\ &= 13C' + \frac{12C'}{V}\mathrm{mes}(E_t) . \end{aligned}$$

Then (3.29) follows.

By this corollary, there is at least one point in the set $E_t \backslash B_{4\varepsilon}(\mathrm{Sing}(X_\infty))$, and consequently (3.22) implies

$$\frac{1}{V}\int_{X_t} f_t \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \leqq C . \tag{3.30}$$

since $f_t \geqq 0$, we have

$$\frac{1}{V}\int_{X_t} |1 + f_t|^1 \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \leqq 1 + C . \tag{3.31}$$

Next, we fix δ to be $\frac{1}{2n}$, then applying Lemma 3.1 to the inequality (3.19) and using (3.31), we have

$$\int_{X_t} |1 + f_t|^{\frac{2n-1}{2n-2}} \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \leqq C . \tag{3.32}$$

Consequently

$$\begin{aligned} &\int_{X_t} |\nabla f_t|^{\frac{4n+1}{4n}} \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \\ &= \int_{X_t} \frac{|\nabla f_t|^{\frac{4n+1}{4n}}}{(1 + f_t)^{\frac{8n^2-6n+1}{16n(n-1)}}} (1 + f_t)^{\frac{8n^2-6n+1}{16n(n-1)}} \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n \\ &\leqq \left(\int_{X_t} \frac{|\nabla f_t|^2}{(1 + f_t)^{\frac{8n^2-6n+1}{8n^2-6n-2}}} \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n\right)^{\frac{4n+1}{8n}} \left(\int_{X_t} (1 + f_t)^{\frac{2n-1}{2n-2}} \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n\right)^{\frac{4n-1}{8n}} \\ &\leqq \left(\left(\frac{16n^2 - 12n - 4}{8n^2 - 6n - 5}\right)^2 \int_{X_t} \frac{|\nabla(1 + f_t)^{\frac{1}{2} - \frac{3}{4(4n^2-3n-1)}}|^2}{(1 + f_t)^{\frac{8n^2-6n+1}{8n^2-6n-2}}} \left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n\right)^{\frac{4n+1}{8n}} \end{aligned}$$

$$\cdot\left(\int_{X_t}(1+f_t)^{\frac{2n-1}{2n-2}}\left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n\right)^{\frac{4n-1}{8n}}$$

$$\leqq C\,. \tag{3.33}$$

On the other hand, the $\Delta_t f_t$ are uniformly bounded in any compact subset away from $\mathrm{Sing}(X_\infty)$. It follows that the f_t converge to f_∞ in C^∞-norms on any compact subset away from $\mathrm{Sing}(X_\infty)$. Therefore, using (3.33) and the fact that $\mathrm{Sing}(X_\infty)$ is of complex dimension smaller than n, we have

$$\lim_{t\to+\infty}\int_{X_t}(\nabla\theta_v)(f_t)\left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n=\int_{X_\infty}\nabla\theta_v(f_\infty)\left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n. \tag{3.34}$$

Furthermore, in the sense of distribution,

$$\mathrm{Ric}\left(\frac{1}{m}\omega_{\mathrm{FS}}|_{X_\infty}\right)-\frac{1}{m}\omega_{\mathrm{FS}}|_{X_\infty}=\frac{\sqrt{-1}}{2\pi}\partial\bar\partial f_\infty\,. \tag{3.35}$$

In fact, the above equation is valid in C^∞-sense away from the singular set of X_∞. Note that the solution of (3.35) is unique up to an additive constant and θ_v is a smooth function on $\mathbb{C}P^N$. On X_∞ $\nabla\theta_v$ is nothing else but v. Therefore, according to Sect. 1, we have

$$\lim_{t\to+\infty}\int_{X_t}(\nabla\theta_v)(f_t)\left(\frac{1}{m}\omega_{\mathrm{FS}}\right)^n=F_{X_\infty}(v)\,. \tag{3.36}$$

The claim, and consequently Theorem 0.1, have been proved.

4 Examples

In this section, we will apply Theorem 0.2 to study the existence of Kähler–Einstein orbifold metrics on a cubic hypersurface X_f in $\mathbb{C}P^3$. The cubic hypersurface X_f is the zero locus in $\mathbb{C}P^3$ of a cubic homogeneous polynomial f on C^4. By the main theorem in [Ti], if X_f is smooth, it has a Kähler–Einstein metric. Therefore, we may assume that X_f is a Kähler orbifold with isolated singular points. Note that $C_1(X_f)>0$. By changing the coordinates z_0, z_1, z_2, z_3, we may further assume that $[1,0,0,0]$ is a singular point of X_f and that the order of its local uniformization group is maximal. This implies that the polynomial f is of the form

$$f(z_0,z_1,z_2,z_3)=z_0f_2(z_1,z_2,z_3)+f_3(z_1,z_2,z_3)\,, \tag{4.1}$$

where f_j are homogeneous polynomials of degree j on z_1, z_2, z_3.

By a linear transformation on z_1, z_2, z_3, we may assume f_2 takes one of the following forms.

$$(1)\ f_2=z_1^2\,,$$

$$(2)\ f_2=z_1^2+z_2^2\,,$$

$$(3)\ f_2=z_1z_2-z_3^2\,. \tag{4.2}$$

Since $[1, 0, 0, 0]$ is the singular point of X_f with maximal order of the local uniformization group, the singularities of X_f can only be ordinary double points in the case 3).

Proposition 4.1 *Let X_f be a cubic surface in $\mathbb{C}P^3$ defined as above. Assume that f_2 is of form* (1) *or* (2) *in* (4.2) *and $f_3(0, 0, z_3) \equiv 0$. Then X_f has no Kähler–Einstein orbifold metric. Moreover, the generic X_f in these two cases does not have any nontrivial holomorphic vector fields and has only one singular point of type A_3.*

Proof. Let us first assume that $f_2 = z_1^2$. There are three subcases according to the number of different linear factors in $f_3(0, z_2, z_3)$. (i) If $f_3(0, z_2, z_3)$ has three different linear factors, then by a linear transformation, we may assume that $f_3(0, z_2, z_3) = z_2 z_3(z_2 - z_3)$. Let $\{\sigma_t\} = \{\mathrm{diag}(1, e^{3t}, e^{2t}, e^{2t})\}_{t \in \mathbb{R}}$ and $X_t = \sigma_t(X_f)$. Then X_t is defined by a cubic polynomial of the form $z_0 z_1^2 + z_2 z_3(z_2 - z_3) + O(e^{-t})$, where $O(e^{-t})$ stands for the terms with coefficients $\leqq ce^{-t}$ for some constant c. Therefore, X_t converges to a Kähler orbifold X_∞ defined by the cubic polynomial $z_0 z_1^2 + z_2 z_3(z_2 - z_3)$. The automorphisms σ_t preserve X_∞. By Example (1) at the end of section 1, if v is the holomorphic vector field whose real part generates σ_t, then $F_{X_\infty}(v) < 0$. Therefore, by Theorem 0.2, there is no Kähler–Einstein orbifold metric on X_f. Note that X_f is diffeomorphic to X_∞ and generically such an X_f has no nontrivial holomorphic vector fields.

(ii) If $f_3(0, z_2, z_3)$ has two different factors, then we may assume $f_3(0, z_2, z_3) = z_2 z_3^2$. By changing coordinates, we can take f to be of the form $z_0 z_1^2 + \alpha z_1 z_2^2 + z_2 z_3^2$. Since X_f has only isolated singular points, $\alpha \neq 0$, then X_f has a holomorphic vector field v whose real part generates the one-parameter subgroup $\{\mathrm{diag}(1, e^{4t}, e^{2t}, e^{3t})\}_{t \in \mathbb{R}}$. One can easily compute that $F_{X_f}(v) \neq 0$. Therefore, X_f has no Kähler–Einstein orbifold metric, either.

(iii) If $f_3(0, z_2, z_3)$ has only one factor, then by a transformation, we may take f to be $z_0 z_1^2 + z_1 z_2^2 + z_3^3$. By Example (2) in Sect. 1, such an X_f has a holomorphic vector field v with $F_{X_f}(v) \neq 0$. So this X_f has no Kähler–Einstein orbifold metric.

Next, we assume that $f_2 = z_1^2 + z_2^2$ and $f_3(0, 0, z_3) \equiv 0$. Since X_f has only isolated singular points, f_3 contains either $z_3^2 z_1$ or $z_3^2 z_2$. Define $\sigma_t = \mathrm{diag}(1, e^{2t}, e^{2t}, e^t)$. Then $X_t = \sigma(X_f)$ converge to X_∞ defined by the cubic polynomial $z_0(z_1^2 + z_2^2) + z_3^2(\alpha z_1 + \beta z_2)$ as t goes to ∞, where $|\alpha|^2 + |\beta|^2 > 0$. By a transformation, we may assume that $\alpha = 1$ and $\beta = 0$. Obviously, σ_t preserves X_∞, so the corresponding holomorphic vector field v is tangent to X_∞. By Example (3) in Sect. 1, $F_{X_\infty}(v) < 0$. Therefore, Theorem 0.2 implies that there is no Kähler–Einstein orbifold metric on X_f.

Claim. For generic f in this case, X_f has no nontrivial holomorphic vector fields and has a unique singular point of the A_3-type at $[1, 0, 0, 0]$.

Let τ_t be any family of automorphisms in $\mathbb{C}P^3$ preserving X_f. Then $\tau_t([1, 0, 0, 0]) = [1, 0, 0, 0]$. If $\tau_t = (\tau_{tij})_{0 \leqq i,j \leqq 3}$, then $\tau_{ti0} = 0$ for $i \geqq 1$. We may assume $\tau_{t00} = 1$. But $\tau_t^* f = \lambda_t f$, where λ_t are complex numbers, it follows that $\tau_t^*(z_1^2 + z_2^2) = \lambda_t(z_1^2 + z_2^2)$. Therefore, by making a linear transformation on z_1, z_2, we may assume $\tau_{tij} = \delta_{ij}\lambda_t$ for $i = 1, 2$, consequently, $\tau_{ti3} = 0$ for $i = 1, 2$. Let us take f to be of the form

$$z_0(z_1^2 + z_2^2) + z_3^2 z_1 + z_1^2 z_2 + z_2^2 z_3 .$$

Then $\tau_{tii}=1$ for $i=0,\ldots,3$. Since f contains no terms like $z_3z_1z_2$ or z_2^3, $\tau_{t32}=\tau_{t02}=0$. Then using $\tau_t^*f=f$, one can see that τ_t has to be the identity for each t, i.e., X_f has no nontrivial holomorphic vector fields. A simple computation shows that X_f has a unique singular point of A_3-type at $[1, 0, 0, 0]$. So the claim is proved.

The proof of Proposition 4.1 is complete.

Let us discuss the rest of the cases. First we recall the definition of Mumford's stability for hypersurfaces in a projective space $\mathbb{C}P^{n+1}$. Let $\mathfrak{R}_{n,m}$ be the space of all homogeneous polynomials of degree m on $\mathbb{C}P^{n+1}$. Then the linear group $\mathrm{SL}(n+2)$ acts on this space. A polynomial f in $\mathfrak{R}_{n,m}$ is said to be stable if the orbit $\mathrm{SL}(n+2)f$ in $\mathfrak{R}_{n,m}$ is closed and the stabilizer of f in $\mathrm{SL}(n+2)$ is finite, and it is semistable if 0 is not in the closure of the orbit $\mathrm{SL}(n+2)f$. Then the hypersurface $X_f=\{x\in\mathbb{C}P^{n+1}\mid f(x)=0\}$ is stable (resp. semistable) in the sense of Mumford if f is stable (resp. semistable). The following proposition can be found in [Md, p. 51] or proved by a straightforward computation.

Proposition 4.2 *Let X_f be a cubic surface in $\mathbb{C}P^3$ defined by the polynomial f in $\mathfrak{R}_{2,3}$. Then X_f is stable if and only if it is either smooth or an orbifold with only isolated ordinary double points, and X_f is semistable and not stable if and only if it is a singular orbifold with only isolated simple surface singularities of type A_2.*

Putting the above two propositions together, we obtain

Theorem 4.1 *Let X_f be a cubic surface as above. Then X_f admits a Kähler–Einstein orbifold metric only if it is semistable in the sense of Mumford.*

It is very likely that the method in [Ti] can be applied to prove the existence of a Kähler–Einstein orbifold metric on a stable (possibly even semistable) cubic surface. However, extra effort is needed in computing those analytic invariants in [Ti] because of the presence of singularities. We have not carried this out.

5 Some remarks

In this last section, we discuss some open problems. Mukai and Umemura constructed in [MU] a Fano 3-fold X which can jump to another Fano 3-fold X_∞, moreover, X has no nontrivial holomorphic vector fields and the Futaki invariant on X_∞ vanishes.

Problem 1 *Is there a Fano n-fold X which has no nontrivial holomorhic vector fields and which jumps to another Fano n-fold X_∞ with nonvanishing Futaki invariant?*

If the answer to this problem is affirmative, then one can further ask

Problem 2 *Let X_∞ be a Fano n-fold with a holomorphic vector field v. Assume that the Futaki invariant* $\mathrm{Re}(F_{X_\infty}(v))\neq 0$. *We normalize v such that* $\mathrm{Re}(F_{X_\infty}(v))<0$. *This holomorphic vector field v induces an adjoint action on the infinitesimal deformation group $H^1(X_\infty, T_{X_\infty})$. Is it possible that this adjoint representation of* $\mathrm{Re}(v)$ *has negative eigenvalues in $H^1(X_\infty, T_{X_\infty})$?*

By our main theorem, if there is a Kähler–Einstein manifold X which jumps to X_∞, then the adjoint representation of $\mathrm{Re}(v)$ in $H^1(X_\infty, T_{X_\infty})$ has no negative eigenvalues. In particular, it follows from [Ti] that the answer to Problem 2 is negative in the case of complex surfaces. One can also show that the answer to Problem 1 is negative in the case of complex surfaces. On the other hand, the examples of last section show that if X_∞ is a Kähler orbifold with positive first Chern class, the possibility in Problem 2 certainly exists. This shows the subtlety of Problem 2 in the smooth case.

The K-energy in Sect. 2 can be defined for Kähler metrics with Kähler class other than $C_1(X)$ and the manifold X need not have positive first Chern class, either. In this general case, Calabi and Futaki have introduced the analogous Futaki invariant and proved that it is an obstruction to the existence of Kähler metrics with constant scalar curvature. The Kähler metrics with constant scalar curvature are the extremal metrics in the sense of Calabi (cf. [Fu2]). All the discussions in Sects. 1, 3 are still valid for the general case. The problem is that we do not know if the K-energy is bounded from below under the assumption that there exists a Kähler metric of constant scalar curvature in the given Kähler class. In other words, for the general case we have not gotten a result similar to Theorem 2.1.

Finally, as one can see from the discussions in Sect. 2, the generalized Futaki invariant can be defined for any (even reducible) variety. An interesting problem is to compute it in terms of the zeroes of the admissible holomorphic vector field and the singularities of the variety. Is there a formula analogous to (1.21)? Even an estimate of the upper bound of the generalized Futaki invariant is of interest. Such an upper bound may be related to the numerical criterion in [Md] for Mumford's stability.

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