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The Kähler–Ricci flow on surfaces of positive Kodaira dimension*

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1 Introduction

The existence of Kähler–Einstein metrics on a compact Kähler manifold has been the subject of intensive study over the last few decades, following Yau's solution to the Calabi conjecture (see [Ya2,Au,Ti2,Ti3]). The Ricci flow, introduced by Richard Hamilton in [Ha1,Ha2], has become one of the most powerful tools in geometric analysis. The Ricci flow preserves the Kählerian property, so it provides a natural flow in Kähler geometry, referred to as the Kähler–Ricci flow. Using the Kähler–Ricci flow, Cao [Ca] gave an alternative proof of the existence of Kähler–Einstein metrics on a compact Kähler manifold with trivial or negative first Chern class. In the early 90's, Hamilton and Chow also used the Ricci flow to give another proof of the classical uniformization for Riemann surfaces (see [Ha2,Ch,ChLuTi]). Recently Perelman [Pe1] has made a major breakthrough in studying the Ricci

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flow. The convergence of the Kähler–Ricci flow on Kähler–Einstein Fano manifolds was claimed by Perelman [Pe2] and a proof of this convergence and its generalization to any Kähler manifolds admitting a Kähler–Ricci soliton was given by the second named author and Zhu in [TiZhu]. Previously, in [ChTi], Chen and the second named author proved that the Kähler–Ricci flow converges to a Kähler–Einstein metric if the bisectional curvature of the initial metric is non-negative and positive at least at one point.

However, most algebraic manifolds do not have a definite or trivial first Chern class. It is a natural question to ask if there exist any well-defined canonical metrics on these manifolds or on varieties canonically associated to them, i.e. their canonical models. Tsuji [Ts] used the Kähler–Ricci flow to prove the existence of a canonical singular Kähler–Einstein metric on a minimal algebraic manifold of general type. In this paper, we propose a program of finding canonical metrics on canonical models of algebraic varieties of positive Kodaira dimension. We also carry out this program for minimal Kähler surfaces. To do it, we will study the Kähler–Ricci flow starting from any Kähler metric and describe its limiting behavior as time goes to infinity.

Let X be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X. In local coordinates $z_1, ..., z_n$, we can write ω as

$$\omega = \sqrt{-1}\sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge dz_{\bar{j}},$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. Consider the Kähler–Ricci flow

$$\begin{cases} \frac{\partial}{\partial t}\omega(t,\cdot) = -Ric(\omega(t,\cdot)) - \omega(t,\cdot), \\ \omega(0,\cdot) = \omega_0, \end{cases} \tag{1.1}$$

where $\omega(t,\cdot)$ is a family of Kähler metrics on X and $Ric(\omega(t,\cdot))$ denotes the Ricci curvature of $\omega(t,\cdot)$ and ω_0 is a given Kähler metric. If the canonical line bundle K_X of X is ample and ω_0 represents $[K_X]$, Cao proved in [Ca] that (1.1) has a global solution $\omega(t,\cdot)$ for all $t > 0$ and $\omega(t,\cdot)$ converges to a Kähler–Einstein metric on X. If K_X is semi-positive, Tsuji proved in [Ts] under the assumption $[\omega_0] > [K_X]$ that (1.1) has a global solution $\omega(t,\cdot)$. This additional assumption was removed in [TiZha], moreover, if K_X is also big, $\omega(t,\cdot)$ converges to a singular Kähler–Einstein metric with locally bounded Kähler potential as t tends to ∞ (see [TiZha]).

If X is a minimal Kähler surface of non-negative Kodaira dimension, then K_X is numerically effective. The Kodaira dimension $\mathrm{kod}(X)$ of X is equal to $0, 1, 2$. If $\mathrm{kod}(X) = 0$, then a finite cover of X is either a K3 surface or a complex torus, and so after an appropriate scaling, $\omega(t,\cdot)$ converges to the unique Ricci-flat metric in the Kähler class $[\omega_0]$ (cf. [Ca]).

If $\mathrm{kod}(X) = 2$, i.e., X is of general type, then $\omega(t, \cdot)$ converges to the unique Kähler–Einstein orbifold metric on its canonical model as t tends to ∞ (see [TiZha]). If $\mathrm{kod}(X) = 1$, then X is a minimal elliptic surface and does not admit any Kähler–Einstein current in $-c_1(X)$, which has bounded local potential and is smooth outside a subvariety. Hence, one does not expect that $\omega(t, \cdot)$ converges to a smooth metric outside a subvariety of X.

In this paper, we study the limiting behavior of $\omega(t, \cdot)$ as t tends to ∞ in the case that X is a minimal elliptic surface. In its sequel, we will extend our results here to higher dimensional manifolds, that is, we will study the limiting behavior of (1.1) when X is an n-dimensional algebraic variety of Kodaira dimension in $(0, n)$ and with numerically positive K_X. Hence, our first goal is to identify limiting candidates. If X is a minimal elliptic surface of $\mathrm{kod}(X) = 1$, there exist an algebraic curve Σ and a holomorphic map $f : X \to \Sigma$ such that $K_X = f^*L$ for some ample line bundle L over Σ. The general fibre of the holomorphic fibration induced by f is a non-singular elliptic curve. Let Σ_{reg} consist of all $s \in \Sigma$ such that $f^{-1}(s)$ is a nonsingular fibre and let $X_{reg} = f^{-1}(\Sigma_{reg})$. For any $s \in \Sigma_{reg}$, $f^{-1}(s)$ is an elliptic curve, so the L^2-metric on the moduli space of elliptic curves induces a semi-positive $(1, 1)$-form ω_{WP} on Σ_{reg}. A metric ω is called a generalized Kähler–Einstein metric if it is smooth on Σ_{reg}, and extends appropriately to Σ and satisfies

$$Ric(\omega) = -\omega + \omega_{\mathrm{WP}}, \quad \text{on } \Sigma_{reg}.$$

Such a metric exists and is unique in a suitable sense.[1] Here is our main result of this paper.

Theorem 1.1 *Let $f : X \to \Sigma$ be a minimal elliptic surface of $\mathrm{kod}(X) = 1$ with singular fibres $X_{s_1} = m_1 F_1, \ldots, X_{s_k} = m_k F_k$ of multiplicity $m_i \in \mathbf{N}$, $i = 1, \ldots, k$. Then for any initial Kähler metric, the Kähler–Ricci flow (1.1) has a global solution $\omega(t, \cdot)$ for all time $t \in [0, \infty)$ satisfying the following.*

1. *$\omega(t, \cdot)$ converges to $f^*\omega_\infty \in -2\pi c_1(X)$ as currents for a positive current ω_∞ on Σ.*
2. *ω_∞ is smooth on Σ_{reg} and $Ric(\omega_\infty) = -\sqrt{-1}\partial\bar{\partial} \log \omega_\infty$ is a well-defined current on Σ satisfying*

$$Ric(\omega_\infty) = -\omega_\infty + \omega_{\mathrm{WP}} + 2\pi \sum_{i=1}^{k} \frac{m_k - 1}{m_k}[s_i], \tag{1.2}$$

where ω_{WP} is the induced Weil–Petersson metric and $[s_i]$ is the current of integration associated to the divisor s_i on Σ. ω_∞ is called a generalized Kähler–Einstein metric on Σ.

[1] Such canonical metrics can be also defined for higher dimensional manifolds. We refer the readers to Sect. 3 for more details.

3. *For any compact subset $K \subset X_{reg}$, there is a constant C_K such that for all $t \in [0, \infty)$*

$$\|\omega(t,\cdot) - f^*\omega_\infty(\cdot)\|_{L^\infty(K)} + e^t \sup_{s\in f(K)} \|\omega(t,\cdot)|_{f^{-1}(s)}\|_{L^\infty(f^{-1}(s))} \le C_K. \tag{1.3}$$

Moreover, the scalar curvature of $\omega(t,\cdot)$ is uniformly bounded on any compact subset of X_{reg}.

Remark 1.1 We conjecture that $\omega(t,\cdot)$ converges to $f^*\omega_\infty$ in the Gromov–Hausdorff topology and in the C^∞ topology outside singular fibres.

An elliptic surface $f : X \to \Sigma$ is an elliptic fibre bundle if it does not admit any singular fibre. Such X is isotrivial as an etale cover of a product of two curves.

Corollary 1.1 *Let $f : X \to \Sigma$ be an elliptic fibre bundle over a curve Σ of genus greater one. Then the Kähler–Ricci flow* (1.1) *has a global solution with any initial Kähler metric. Furthermore, $\omega(t,\cdot)$ converges weakly as currents to the pullback of the Kähler–Einstein metric on Σ with the scalar curvature and $\|\omega(t,\cdot)\|_{L^\infty(X)}$ being uniformly bounded.*

Theorem 1.1 seems to be the first general convergence result on collapsing of the Kähler–Ricci flow. Combining the results in [Ca,Ts,TiZha], we give a metric classification in Sect. 8.1 for Kähler surfaces with a nef canonical line bundle by the Kähler–Ricci flow.

2 Preliminaries

Let X be an n-dimensional compact Kähler manifold and

$$\omega = \sqrt{-1}\sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge dz_{\bar{j}}$$

be a Kähler form associated to the Kähler metric $\{g_{i\bar{j}}\}$ in local coordinates $z_1, ..., z_n$. The curvature tensor for g is locally given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial z_{\bar{l}}} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial z_{\bar{l}}}, \qquad i, j, k, l = 1, 2, ..., n.$$

The Ricci curvature is given by

$$R_{i\bar{j}} = -\frac{\partial^2 \log\det(g_{k\bar{l}})}{\partial z_i \partial z_{\bar{j}}}, \qquad i, j = 1, 2, ..., n.$$

So its Ricci curvature form is given by

$$Ric(\omega) = \sqrt{-1}\sum_{i,j=1}^{n} R_{i\bar{j}} dz_i \wedge dz_{\bar{j}} = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{k\bar{l}}).$$

2.1 Reduction of the Kähler–Ricci flow In this section, we will reduce the Kähler–Ricci flow (1.1) to a parabolic equation for the Kähler potential on any compact Kähler manifold X with semi-ample canonical line bundle K_X.

Definition 2.1 *Let L be a holomorphic line bundle L over a compact Kähler manifold X.*

1. *L is called nef, i.e. numerically effective, if for every curve $C \subset X$*

$$L \cdot C = \int_C c_1(L) \geq 0.$$

2. *L is called semi-positive if there exists a smooth Hermitian metric h on L such that $Ric(h) \geq 0$.*
3. *L is called semi-ample if a sufficiently large power of L is globally generated.*

It always holds that $3 \Rightarrow 2 \Rightarrow 1$ and the abundance conjecture in algebraic geometry predicts that $3 \Leftrightarrow 2 \Leftrightarrow 1$ for K_X. If K_X is semi-ample, it is a semi-positive line bundle so that $c_1(X) \leq 0$. Also by the semi-ampleness of K_X, there is a sufficiently large integer m such that any basis of $H^0(X, K_X^m)$ gives rise to a holomorphic map f from X into a projective space. Recall the Kodaira dimension $\mathrm{kod}(X)$ of X is defined to be the dimension of the image by the holomorphic map f and it is in fact a birational invariant of X.

Let $Ka(X)$ denote the Kähler cone of X, that is,

$$Ka(X) = \{[\omega] \in H^{1,1}(X, \mathbf{R}) \mid [\omega] > 0\}.$$

Suppose that $\omega(t, \cdot)$ is a solution of (1.1) on $[0, T)$. Then its induced equation for the Kähler class in $Ka(X)$ is given by the following ordinary differential equation

$$\begin{cases} \frac{\partial[\omega]}{\partial t} = -2\pi c_1(X) - [\omega], \\ [\omega]|_{t=0} = [\omega_0]. \end{cases} \tag{2.1}$$

It follows that

$$[\omega(t, \cdot)] = -2\pi c_1(X) + e^{-t}([\omega_0] + 2\pi c_1(X)).$$

When the canonical bundle K_X is semi-ample, $\mathrm{kod}(X) \geq 0$ and $c_1(X) \leq 0$. We can choose a smooth closed semi-positive $(1,1)$-form

$\chi \in -2\pi c_1(X)$ and define the following reference Kähler metric along the Kähler–Ricci flow

$$\omega_t = \chi + e^{-t}(\omega_0 - \chi). \tag{2.2}$$

In particular, $\omega_t \geq e^{-t}\omega_0$ is a Kähler form for all $t \in [0, \infty)$ and the solution of (1.1) can be written as

$$\omega = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi.$$

Let Ω be a smooth volume form on X such that $Ric(\Omega) = -\sqrt{-1}\partial\overline{\partial}\log\Omega = -\chi$. Then the evolution for the Kähler potential φ is given by the following initial value problem

$$\begin{cases} \frac{\partial\varphi}{\partial t} = \log \frac{e^{(n-\mathrm{kod}(X))t}(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega} - \varphi, \\ \varphi|_{t=0} = 0. \end{cases} \tag{2.3}$$

The following existence result for the Kähler–Ricci flow (1.1) (or equivalently (2.3)) was proved in [TiZha]. It was previously proved in [Ts] in the special case when K_X is semi-positive with the initial class condition that $[\omega_0] > -2\pi c_1(X)$. This was also studied in [CaLa] under a stronger technical assumption.

Theorem 2.1 *Given any compact Kähler manifold X and any Kähler metric ω_0, the Kähler–Ricci flow* (1.1) *has a solution for time $t \in [0, T)$, where* $T = \sup\{t \geq 0 \mid [\omega_t]$ is a Kähler class$\}$. *In particular,* (1.1) *has a global solution for all $t \in [0, \infty)$ if K_X is nef. Moreover, in the case when χ is semi-positive,* (2.3)*, and consequently* (1.1)*, has a global solution for all $t \in [0, \infty)$.*

By straightforward calculation (cf. [Ha1,ChTi]), the evolution equation for the scalar curvature R is given by

$$\frac{\partial R}{\partial t} = \Delta R + |Ric|^2 + R, \tag{2.4}$$

where Δ is the Laplace operator associated to the Kähler form ω. Then the following proposition is an immediate conclusion from the maximum principle for the parabolic equation (2.4).

Proposition 2.1 *The scalar curvature along the Kähler–Ricci flow* (1.1) *is uniformly bounded from below if K_X is nef.*

Proof For any $T > 0$, suppose $\inf_{[0,T]\times X} R = R(t_0, z_0)$ for some $(t_0, z_0) \in [0, T] \times X$. Applying the maximum principle for the parabolic equation (2.4), we have

$$|Ric|^2(t_0, z_0) + R(t_0, z_0) \leq 0.$$

It is easy to see by diagonalizing $Ric(t_0, z_0)$ under the normal coordinates with respect to $\omega(t, \cdot)$ at z_0 that there exists a uniform constant $C > 0$ independent of T such that $|Ric|^2(t_0, z_0) \leq C$. Therefore $R(t_0, z_0)$ is uniformly bounded from below. This proves the proposition. □

2.2 Minimal surfaces with positive Kodaira dimension An elliptic fibration of a surface X is a proper holomorphic map $f : X \to \Sigma$ from X to a curve Σ such that the general fibre is a non-singular elliptic curve. An elliptic surface is a surface admitting an elliptic fibration. Any surface X of $\mathrm{kod}(X) = 1$ must be an elliptic surface. Such an elliptic surface is sometimes called a properly elliptic surface. Since we assume that X is minimal, all fibres are free of (-1)-curves. A very simple example is the product of two curves, one elliptic and the other of genus greater than one.

Let $f : X \to \Sigma$ be an elliptic surface. The differential df can be viewed as an injection of sheaves $f^*(K_\Sigma) \to \Omega_X^1$. Its cokernel $\Omega_{X/\Sigma}$ is called the sheaf of relative differentials. In general, $\Omega_{X/\Sigma}$ is far from being locally free. If some fibre has a multiple component, then df vanishes along this component and $\Omega_{X/\Sigma}$ contains a torsion subsheaf with one-dimensional support. Away from the singularities of f we have the following exact sequence

$$0 \to f^*(K_\Sigma) \to \Omega_X^1 \to \Omega_{X/\Sigma} \to 0$$

inducing an isomorphism between $\Omega_{X/\Sigma}$ and $K_X \otimes f^*(K_\Sigma^\vee)$. We also call the line bundle $\Omega_{X/\Sigma}$ the dualizing sheaf of f on X. The following Kodaira canonical bundle formula is well-known (cf. [BaHuPeVa,Ko,Mi]).

Theorem 2.2 *Let $f : X \to \Sigma$ be a minimal elliptic surface such that its multiple fibres are $X_{s_1} = m_1 F_1, \ldots, X_{s_k} = m_k F_k$. Then*

$$K_X = f^*(K_\Sigma \otimes (f_{*1}\mathcal{O}_X)^\vee) \otimes \mathcal{O}_X\Big(\sum (m_i - 1)F_i\Big), \tag{2.5}$$

or

$$K_X = f^*\Big(L \otimes \mathcal{O}_X\Big(\sum (m_i - 1)F_i\Big)\Big),$$

where L is a line bundle of degree $\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_\Sigma)$ on Σ.

Note that $\deg(f_{*1}\mathcal{O}_X)^\vee = \deg(f_*\Omega_{X/\Sigma}) \geq 0$ and the equality holds if and only if f is locally trivial. The following invariant

$$\delta(f) = \chi(\mathcal{O}_X) + \left(2g(\Sigma) - 2 + \sum_{i=1}^k \left(1 - \frac{1}{m_i}\right)\right)$$

determines the Kodaira dimension of X.

Proposition 2.2 (cf. [BaHuPeVa]) *Let $f : X \to \Sigma$ be a minimal elliptic surface. Then* $\mathrm{kod}(X) = 1$ *if and only if* $\delta(f) > 0$.

Let $\mathcal{S}_1/\Gamma_1 \cong \mathbf{C}$ be the period domain, where $\mathcal{S}_1 = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ is the upper half plane and $\Gamma_1 = \mathrm{SL}(2, \mathbf{Z})/\{\pm 1\}$ is the modular group acting by $z \to \frac{az+b}{cz+d}$. The j-function gives an isomorphism $\mathcal{S}_1/\Gamma_1 \to \mathbf{C}$ with

1. $j(z) = 0$ if $z = e^{\frac{\pi}{3}\sqrt{-1}}$ modulo Γ_1,
2. $j(z) = 1$ if $z = \sqrt{-1}$ modulo Γ_1.

Thus any elliptic surface $f : X \to \Sigma$ gives a period map $p : \Sigma_{reg} \to \mathcal{S}_1/\Gamma_1$. Set $J : \Sigma_{reg} \mapsto \mathbf{C}$ by $J(s) = j(p(s))$.

If we choose a semi-positive $(1, 1)$-form $\chi \in -2\pi c_1(X)$ and apply the Kähler–Ricci flow (1.1) on a minimal elliptic surface X of $\mathrm{kod}(X) = 1$. Theorem 1.1 shows that the Kähler–Ricci flow (1.1) provides a canonical way of deforming any given Kähler metric to a canonical metric. This canonical metric on Σ satisfies the curvature equation

$$Ric(g_\infty) = -g_\infty + g_{\mathrm{WP}} + 2\pi \sum_{i=1}^{k} \frac{m_i - 1}{m_i}[s_i].$$

This can be regarded as the local version of Kodaira's canonical bundle formula (2.5), where the pullback of the Weil–Petersson metric g_{WP} by the period map p is the curvature of the dualizing sheaf $f_*\Omega_{X/\Sigma}$ and the current $2\pi \sum_{i=1}^{k} \frac{m_i-1}{m_i}[s_i]$ corresponds to the residues from the multiple fibres.

3 Generalized Kähler–Einstein metrics and the Kähler–Ricci flow

3.1 Limiting metrics on canonical models and Weil–Petersson metrics

In this subsection, we introduce a class of canonical metrics which we call generalized Kähler–Einstein metrics.

Let X be an n-dimensional smooth algebraic manifold. Suppose that K_X is semi-ample so that K_X^m is base point free for m sufficiently large and $\mathrm{kod}(X) = \kappa$ with $0 \le \kappa \le n$. Then the pluricanonical map

$$\left|K_X^m\right| : X \to X_m \subset \mathbf{CP}^{N_m}$$

is a holomorphic map. Fix a sufficiently large m, $|K_X^m|$ induces a holomorphic fibration $f : X \to X_{\mathrm{can}}$ such that $K_X^m = f^*\mathcal{O}(1)$, where X_{can} is the image of the pluricanonical map and is called the canonical model of X. X_{can} is unique and isomorphic to X_m for m sufficiently large since K_X is semi-ample and so the canonical ring of X is finitely generated. If K_X is nef and big, $\kappa = n$ and K_X is semi-ample by Kawamata's result (cf. [ClKoMo]). Such X is called a minimal model of general type and the Kähler–Ricci flow deforms any Kähler metric to a unique singular Kähler–Einstein metric on X (see [Ts,TiZha]). If $0 < \kappa < n$, for a general fibre X_s, K_{X_s} is numerically trivial and X_s is a Calabi–Yau manifold. We can choose χ to be a multiple of the Fubini–Study metric of $\mathbf{CP}^{N_m}$ restricted on X_{can} such that $f^*\chi \in -2\pi c_1(X)$. Notice that $f^*\chi$ is a smooth semi-positive $(1, 1)$-form on X. For simplicity, we sometimes denote it by χ. Denote by X°_{can} the set of all smooth points s of X_{can} such that $X_s = f^{-1}(s)$ is a nonsingular fibre. Put $X_{reg} = f^{-1}(X^{\circ}_{\mathrm{can}})$.

Lemma 3.1 *For any Kähler class* $[\omega]$ *on* X*, there is a smooth function* ψ *on* X_{reg} *such that* $\omega_{\mathrm{SF}} := \omega + \sqrt{-1}\partial\overline{\partial}\psi$ *is a closed semi-flat* $(1,1)$*-form in the following sense: the restriction of* ω_{SF} *to each smooth* $X_s \subset X_{reg}$ *is a Ricci flat Kähler metric.*

Proof On each nonsingular fibre X_s, let ω_s be the restriction of ω to X_s and ∂_V and $\overline{\partial}_V$ be the restriction of ∂ and $\overline{\partial}$ to X_s. Then by Hodge theory, there is a unique function h_s on X_s defined by

$$\begin{cases} \partial_V\overline{\partial}_V h_s = -\partial_V\overline{\partial}_V \log \omega_s^{n-\kappa}, \\ \int_{X_s} e^{h_s}\omega_s^{n-\kappa} = \int_{X_s} \omega_s^{n-\kappa}. \end{cases} \tag{3.1}$$

By Yau's solution to the Calabi conjecture, there is a unique ψ_s solving the following Monge–Ampère equation

$$\begin{cases} \frac{(\omega_s+\sqrt{-1}\partial_V\overline{\partial}_V\psi_s)^{n-\kappa}}{\omega_s^{n-\kappa}} = e^{h_s} \\ \int_{X_s} \psi_s\omega_s^{n-\kappa} = 0. \end{cases} \tag{3.2}$$

Since f is holomorphic, $\psi(z,s) = \psi_s(z)$ is well-defined as a smooth function on X_{reg}. □

By Hodge theory, there exists a volume form Ω on X such that $\sqrt{-1}\partial\overline{\partial}\log\Omega = \chi$. Define

$$F = \frac{\Omega}{\binom{n}{\kappa}\omega_{\mathrm{SF}}^{n-\kappa}\wedge\chi^{\kappa}}. \tag{3.3}$$

Lemma 3.2 *F is the pullback of a function on* X°_{can}.

Proof Since χ is the pullback from X_{can}, we have

$$\sqrt{-1}\partial_V\overline{\partial}_V\log\Omega = \sqrt{-1}\partial_V\overline{\partial}_V\log\omega_{\mathrm{SF}}^{n-\kappa}\wedge\chi^{\kappa} = 0$$

on each nonsingular fibre X_s. Thus F is constant along each nonsingular fibre X_s and so it is the pullback of a function from X°_{can}. □

There is a canonical Hermitian metric on the dualizing sheaf $f_*(\Omega^{n-\kappa}_{X/X_{\mathrm{can}}}) = (f_{*1}\mathcal{O}_X)^{\vee}$ over X°_{can}.

Definition 3.1 *Let X be an n-dimensional algebraic manifold. Suppose that its canonical line bundle K_X is semi-positive and $0 < \kappa = \mathrm{kod}(X) < n$. Let X_{can} be the canonical model of X. We define a canonical Hermitian metric h_{can} on $f_*(\Omega^{n-\kappa}_{X/X_{\mathrm{can}}})$ in the way that for any smooth $(n-\kappa, 0)$-form η on a nonsingular fibre X_s,*

$$|\eta|^2_{h_{\mathrm{can}}} = \frac{\eta \wedge \bar{\eta} \wedge \chi^\kappa}{\omega_{\mathrm{SF}}^{n-\kappa} \wedge \chi^\kappa}. \tag{3.4}$$

Now let us recall some facts on the Weil–Petersson metric on the moduli space $\mathcal{M}$ of polarized Calabi–Yau manifolds of dimension $n-\kappa$ (cf. [FaLu]). Let $\mathcal{X} \to \mathcal{M}$ be a universal family of Calabi–Yau manifolds. Let $(U; t_1, ..., t_\kappa)$ be a local holomorphic coordinate chart of $\mathcal{M}$, where $\kappa = \dim \mathcal{M}$. Then each $\frac{\partial}{\partial t_i}$ corresponds to an element $\iota(\frac{\partial}{\partial t_i}) \in H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ through the Kodaira–Spencer map ι. The Weil–Petersson metric is defined by the L^2-inner product of harmonic forms representing classes in $H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$. In the case of Calabi–Yau manifolds, it was shown in [Ti1] that the metric can be expressed as follows: Let Ψ be a nonzero holomorphic $(n-\kappa, 0)$-form on the fibre $\mathcal{X}_t$ and $\Psi \lrcorner \iota(\frac{\partial}{\partial t_i})$ be the contraction of Ψ and $\frac{\partial}{\partial t_i}$, then the Weil–Petersson metric is given by

$$\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar{t}_j}\right)_{\omega_{\mathrm{WP}}} = \frac{\int_{\mathcal{X}_t} \Psi \lrcorner \iota\left(\frac{\partial}{\partial t_i}\right) \wedge \overline{\Psi \lrcorner \iota\left(\frac{\partial}{\partial t_i}\right)}}{\int_{\mathcal{X}_t} \Psi \wedge \overline{\Psi}}. \tag{3.5}$$

One can also represent ω_{WP} as the curvature form of the first Hodge bundle $f_*\Omega^{n-\kappa}_{\mathcal{X}/\mathcal{M}}$. Let Ψ be a nonzero local holomorphic section of $f_*\Omega^{n-\kappa}_{\mathcal{X}/\mathcal{M}}$ and one can define the Hermitian metric h_{WP} on $f_*\Omega^{n-\kappa}_{\mathcal{X}/\mathcal{M}}$ by

$$|\Psi_t|^2_{h_{\mathrm{WP}}} = \int_{\mathcal{X}_t} \Psi_t \wedge \overline{\Psi_t}. \tag{3.6}$$

Then the Weil–Petersson metric is given by

$$\omega_{\mathrm{WP}} = Ric(h_{\mathrm{WP}}). \tag{3.7}$$

Lemma 3.3

$$Ric(h_{\mathrm{can}}) = \omega_{\mathrm{WP}}. \tag{3.8}$$

Proof Let $u = \frac{\Psi \wedge \overline{\Psi}}{\omega_{\mathrm{SF}}^{n-\kappa}}$. Notice that Ψ restricted on each fibre $\mathcal{X}_t$ is a holomorphic $(n-\kappa, 0)$-form and $\Psi \wedge \overline{\Psi}$ is a Calabi–Yau volume form, therefore u is constant along each fibre and can be considered as the pullback of a function on $\mathcal{M}$. Then by definition

$$\omega_{\mathrm{WP}} = -\sqrt{-1}\partial\overline{\partial} \log \int_{\mathcal{X}_t} u\omega_{\mathrm{SF}}^{n-\kappa} = -\sqrt{-1}\partial\overline{\partial} \log u,$$

where the last equality makes use of the fact that $\int_{X_t} \omega_{SF}^{n-\kappa} = constant$. At the same time

$$Ric(h_{\mathrm{can}}) = -\sqrt{-1}\partial\overline{\partial}\log \frac{\Psi \wedge \overline{\Psi} \wedge \chi^{\kappa}}{\omega_{SF}^{n-\kappa} \wedge \chi^{\kappa}} = -\sqrt{-1}\partial\overline{\partial}\log u.$$

This proves the lemma. □

A singular Kähler metric on an algebraic variety X is a closed positive $(1, 1)$-current smooth outside a subvariety of X.

Definition 3.2 *Let ω be a possibly singular Kähler metric on X_{can} such that $f^*\omega \in -2\pi c_1(X)$. Then ω is called a generalized Kähler–Einstein metric if on X_{can}°*

$$Ric(\omega) = -\omega + \omega_{WP}. \tag{3.9}$$

In general, let $f : X \to \Sigma$ be a holomorphic Calabi–Yau fibration over an algebraic variety Σ. If X is nonsingular and Σ° is the set of all nonsingular points s of Σ with $f^{-1}(s)$ being nonsingular. Then a possibly singular Kähler metric ω on Σ is called a generalized Kähler–Einstein metric if on Σ°

$$Ric(\omega) = \lambda\omega + \omega_{WP}, \tag{3.10}$$

where $\lambda = -1, 0, 1$.

The following theorem is the main result of this section and its proof is based on the work of Kolodziej [Kol1,Kol2].

Theorem 3.1 *Suppose that X_{can} is smooth (or has orbifold singularities) and $F \in L^{1+\epsilon}(X_{\mathrm{can}})$ for some $\epsilon > 0$, then there is a unique solution $\varphi_{\infty} \in \mathrm{PSH}(\chi) \cap C^0(X_{\mathrm{can}})$ solving the following equation on X_{can}*

$$(\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^{\kappa} = F e^{\varphi}\chi^{\kappa}. \tag{3.11}$$

Furthermore, $\omega = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_{\infty}$ is a closed positive current on X_{can}. If ω is smooth on X_{can}°, then the Ricci curvature of ω on X_{can}° is given by

$$Ric(\omega) = -\omega + \omega_{WP}. \tag{3.12}$$

In fact, the assumption $F \in L^{1+\epsilon}(X_{\mathrm{can}})$ for some $\epsilon > 0$ is always satisfied (cf. [SoTi]). In Sect. 3.3, we will show that $F \in L^{1+\epsilon}(X_{\mathrm{can}})$ when X is a minimal elliptic surface.

Such canonical metrics also belong to a class of Kähler metrics which generalize Calabi's extremal metrics. Let Y be an n-dimensional compact Kähler manifold together with a fixed closed (1,1)-form θ. Fix a Kähler class $[\omega]$, denote by $\mathcal{K}_{[\omega]}$ the space of Kähler metrics within the same Kähler class, that is, all Kähler metrics of the form $\omega_{\varphi} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$.

One may consider the following equation

$$\bar{\partial} V_\varphi = 0, \tag{3.13}$$

where V_φ is defined by

$$\omega_\varphi(V_\varphi, \cdot) = \bar{\partial}(S(\omega_\varphi) - \mathrm{tr}_{\omega_\varphi}(\theta)). \tag{3.14}$$

Clearly, when $\theta = 0$, (3.13) is exactly the equation for Calabi's extremal metrics. For this reason, we call a solution of (3.13) a generalized extremal metric. If Y does not admit any nontrivial holomorphic vector fields, then any generalized extremal metric ω_φ satisfies

$$S(\omega_\varphi) - \mathrm{tr}_{\omega_\varphi}(\theta) = \mu,$$

where μ is the constant given by

$$\mu = \frac{n(2\pi c_1(Y) - [\theta]) \cdot [\omega]^{n-1}}{[\omega]^n}.$$

Moreover, if $2\pi c_1(Y) - [\theta] = \lambda[\omega]$, then such a metric satisfies

$$Ric(\omega_\varphi) = \lambda\omega_\varphi + \theta,$$

that is, ω_φ is a generalized Kähler–Einstein metric. More interestingly, if we take θ to be the pull-back of ω_{WP} by $f : X^\circ_{\mathrm{can}} \to \mathcal{M}$ from the moduli space of polarized Calabi–Yau manifolds, we return to those generalized Kähler–Einstein metrics on the canonical models of algebraic manifolds with semi-ample canonical bundle. Such generalized Kähler–Einstein metrics arise naturally from the collapsing limits of the Kähler–Ricci flow.

3.2 Minimal surfaces of general type Let X_{can} be the canonical model of a minimal surface of general type from the contraction map $f : X \to X_{\mathrm{can}}$. X_{can} has possibly rational singularities of A-D-E-type by contracting the (-2)-curves on X. Since K_{can} is ample and $f^* K_{\mathrm{can}} = K_X$, we can assume the smooth closed (1, 1)-form $\chi = f^*\chi \in -2\pi c_1(X)$ and χ is a Kähler from on X_{can}. It is shown in [TiZha] that the Kähler–Ricci flow (1.1) converges to the canonical metric g_{KE} on X, which is the pullback of the smooth orbifold Kähler–Einstein metric on the canonical model X_{can}, although g_{KE} is degenerate along those (-2)-curves.

3.3 Minimal elliptic surfaces of Kodaira dimension one Now consider minimal elliptic surfaces. From Lemma 3.1, we know that there exists a closed semi-flat (1, 1)-form ω_{SF} in $[\omega_0]$.

Lemma 3.4 *Let F be the function on Σ defined by $F = \frac{\Omega}{2\omega_{\mathrm{SF}} \wedge \chi}$ as* (3.3). *Let $B \subset \Sigma$ be a small disk with center* 0 *such that all fibres X_s, $s \neq 0$, are nonsingular. Then there exists a constant $C > 0$ such that*

1. *if X_0 is of type mI_0, then*

$$\frac{1}{C}|s|^{-\frac{2(m-1)}{m}} < F|_B \le C|s|^{-\frac{2(m-1)}{m}}; \tag{3.15}$$

2. *if X_0 is of type mI_b or I_b^*, $b > 0$, then*

$$-\frac{1}{C}|s|^{-\frac{2(m-1)}{m}} \log|s|^2 \le F|_B \le -C|s|^{-\frac{2(m-1)}{m}} \log|s|^2; \tag{3.16}$$

3. *if X_0 is of any other singular fibre type, then*

$$\frac{1}{C} \le F|_B \le C. \tag{3.17}$$

Proof Let Y be the fibration of f over B.

1. If X_0 is of type mI_0, we start with a fibration $\tilde{Y} = \mathbf{C} \times \tilde{B}/L$, where $L = \mathbf{Z} + \mathbf{Z} \cdot z(w)$ is a holomorphic family of lattices with z being a holomorphic function on $\tilde{B}$ satisfying: $z(w) = z(0) + const \cdot w^{mh}$, w is the coordinate on $\tilde{B}$, $h \in \mathbf{N}$. The automorphism of $\mathbf{C} \times B$ given by $(c, w) \to (c + \frac{1}{m}, e^{\frac{2\pi\sqrt{-1}}{m}} w)$ descends to $\tilde{Y}$ and generates a group action without fixed points. We can assume that Y is the quotient of $\tilde{Y}$ by the group action. Therefore ω_{SF} is a smooth family of Ricci-flat metrics over B. Choose a local coordinate s on B centered around 0, and a covering $\{U_\alpha\}$ of a neighborhood U of X_0 in X by small polydiscs. Since the function f^*s vanishes to order m along X_0, we can in each U_α choose a holomorphic function w_α on U_α as the mth root of f^*s, with

$$w_\alpha^m = f^*s$$

and on $U_\alpha \cap U_\beta$

$$w_\alpha = e^{\frac{2\pi\sqrt{-1}k_{\alpha\beta}}{m}} w_\beta$$

for $k_{\alpha\beta} \in \{0, 1, ..., m-1\}$. On each U_α, $ds \wedge d\overline{s} = m^2|s|^{\frac{2(m-1)}{m}} dw_\alpha \wedge d\overline{w}_\alpha$. Then $|s|^{\frac{2(m-1)}{m}} F$ is smooth and bounded away from zero on Y. Thus (3.15) is proved.
2. If X_0 is of type I_b, $b > 0$, we can assume $Y = \mathbf{C} \times B/L$, where

$$L = \mathbf{Z} + \mathbf{Z}\frac{b}{2\pi\sqrt{-1}} \log s.$$

Let γ_0 be an arc passing through 0 in B and γ be an arc on X transverse to X_0 with $f \circ \gamma = \gamma_0$. We also assume that γ does not pass through any double point of X_0. $\Omega = 2F\omega_{\mathrm{SF}} \wedge \chi$ is smooth and non-degenerate and so is χ along γ. Since $F = \frac{1}{2}\left(\frac{\Omega}{\omega_0\wedge\chi}\right)\left(\frac{\omega_0\wedge\chi}{\omega_{\mathrm{SF}}\wedge\chi}\right)$, it suffices to estimate the function $\frac{\omega_0}{\omega_{\mathrm{SF}}}\Big|_{X_s}$ restricted to γ near X_0. Let $\omega_{\mathbf{C}}$ be the standard flat metric on $\mathbf{C}$. Along γ, $\omega_0|_{X_s}$ is uniformly equivalent to $\omega_{\mathbf{C}}$, so it suffices to estimate $\frac{\omega_{\mathbf{C}}}{\omega_{\mathrm{SF}}}\Big|_{X_s}$. But

$$\frac{\omega_{\mathbf{C}}}{\omega_{\mathrm{SF}}}\Big|_{X_s} = \frac{\int_{X_s}\omega_{\mathbf{C}}}{\int_{X_s}\omega_{\mathrm{SF}}} = \frac{-b\log|s|}{2\pi\int_{X_s}\omega_{\mathrm{SF}}}$$

and $\mathrm{Vol}(X_s) = \int_{X_s}\omega_{\mathrm{SF}}$ is a constant independent of s. Therefore there exists a constant $C > 0$ such that

$$-\frac{1}{C}\log|s|^2 \le F \le -C\log|s|^2.$$

If X_0 is of type mI_b, $b > 0$, we start with a fibration $f : \tilde{Y} \to \tilde{B}$, where $\tilde{Y} = \mathbf{C}\times\tilde{B}/L$ and $L = \mathbf{Z} + \mathbf{Z}\frac{mb}{2\pi\sqrt{-1}}\log w$ and w is the coordinate function of B. So $\tilde{Y}_0 = C_1 + C_2 + \ldots + C_{mb}$ is of type I_{mb}. The automorphism $(c, w) \to (c, e^{\frac{2\pi\sqrt{-1}}{m}}w)$ of $\mathbf{C}\times\tilde{B}$ induces a fibre-preserving automorphism of order m on $\tilde{Y}$. Such an automorphism generates a group action on $\tilde{Y}$ without fixed points and the quotient of $\tilde{Y}$ has a singular fibre of type mI_b. Then by using the same arguments for singular fibres of type mI_0, we can prove (3.16). A fibration of type I_b^* ($b > 0$) is obtained by taking a quotient of a fibration of type I_{2b} after resolving the A_1-singularities. The lattices can be locally written as $L = s^{\frac{1}{2}}\mathbf{Z} + \mathbf{Z}s^{\frac{1}{2}}\frac{b}{2\pi\sqrt{-1}}\log s$. Then the above argument gives the required estimate for F.

3. If X_0 is not of type mI_b, $b \ge 0$ or I_b^*, $b > 0$, it must be of type I_0^*, II, III, IV, IV^*, III^* or II^*. Such a singular fibre is not a stable fibre. By the table of Kodaira (cf. [Ko]), the functional invariant $J(s)$ is bounded near 0 and $J(0) = 0$ or 1. One can write down the table of local lattices of periods and the periods are bounded near the singular fibre. For example, if X_0 is of type II, then X_0 is a cuspidal rational curve with $J(s) = s^{3m+1}$, $m \in \mathbf{N}\cup\{0\}$ in the local normal representation. On each fibre X_s the above fixed flat metric $\omega_{\mathbf{C}}$ on X_s has uniformly bounded area, therefore

$$0 < \frac{1}{C} \le \frac{\omega_{\mathbf{C}}}{\omega_{\mathrm{SF}}}\Big|_{X_s} = \frac{\int_{X_s}\omega_{\mathbf{C}}}{\int_{X_s}\omega_{\mathrm{SF}}} \le C.$$

The estimate is then proved by the same argument as that in the previous case. □

Immediately we have the following corollary.

Corollary 3.1 *There exists* $\epsilon > 0$ *such that* $F \in L^{1+\epsilon}(\Sigma)$.

Proof Calculate

$$\int_\Sigma F^{1+\epsilon}\chi^\kappa = \frac{1}{\int_{X_s}\omega_{SF}^{n-\kappa}}\int_\Sigma\left(\int_{X_s}F^{1+\epsilon}\omega_{SF}^{n-\kappa}\right)\chi^\kappa$$
$$= \frac{1}{\int_{X_s}\omega_{SF}^{n-\kappa}}\int_X F^{1+\epsilon}\chi^\kappa\wedge\omega_{SF}^{n-\kappa}$$
$$= \frac{1}{\binom{n}{\kappa}\int_{X_s}\omega_{SF}^{n-\kappa}}\int_X F^\epsilon\Omega \le C.$$

The last inequality holds for sufficiently small $\epsilon > 0$ because F has at worst pole singularities by Lemma 3.4 . □

Proposition 3.1 *There is a unique solution* $\varphi_\infty \in C^0(\Sigma)\cap C^\infty(\Sigma_{reg})$ *solving the following equation on* Σ

$$\chi + \sqrt{-1}\partial\overline{\partial}\varphi = Fe^\varphi\chi. \tag{3.18}$$

Proof This is a corollary of Theorem 3.1, but still we give an elementary proof for the sake of completeness. Rewrite (3.18) as

$$\Delta\varphi = Fe^\varphi - 1, \tag{3.19}$$

where Δ is the Laplacian operator associated to χ. Notice that F is strictly positive on Σ and uniformly bounded away from 0. Also by Lemma 3.4, $F \in L^p(\Sigma)$ for some $p > 1$. Therefore we can choose a family of functions $\{F_t\}$ for $t \in (0, 1]$ such that $F_t > 0$ is uniformly bounded below away from 0, $F_t \in C^\infty(\Sigma)$ and $\lim_{t\to 0}\|F_t - F\|_{L^p(X)} = 0$. Let $F_0 = F$. We will apply the method of continuity to find the solutions of the following equation parameterized by $t \in [0, 1]$

$$\Delta\varphi_t = F_t e^{\varphi_t} - 1. \tag{3.20}$$

Obviously (3.20) is solvable for all $t \in (0, 1]$. To solve for $t = 0$ we need to derive the uniform C^0-estimate for φ_t. By the maximum principle, there exists a constant $C_1 > 0$ such that for all $t \in (0, 1]$

$$\sup_{\Sigma\times(0,1]} e^{\varphi_t} \le \frac{1}{\inf_{\Sigma\times(0,1]} F_t} \le C_1.$$

The standard L^p estimate gives

$$\|\varphi_t\|_{L^p_2} \le C_2(\|F_t\|_{L^p} + 1) \le C_3.$$

The Sobolev embedding theorem implies

$$\|\varphi_t\|_{L^\infty} \le C_4$$

for $t \in (0, 1]$. With the C^0 estimate, we can derive the uniform C^k-estimate for φ_t by the local estimates of the standard theory of linear elliptic PDE due to the fact that Δ has uniformly bounded coefficients. Therefore there exists $\varphi_\infty \in C^0(\Sigma) \cap C^\infty(\Sigma_{reg})$ satisfying (3.18).

Now we prove the uniqueness. Suppose that there is another solution $\varphi' \in C^0(\Sigma) \cap C^\infty(\Sigma_{reg})$ solving (3.11). Let $\psi = \varphi' - \varphi$. Then by the comparison principle for plurisubharmonic functions (cf. [Kol1]), we have

$$\int_{\psi\le 0} Fe^{\varphi}\chi = \int_{\psi\le 0}(\chi + \sqrt{-1}\partial\overline{\partial}\varphi) \le \int_{\psi\le 0}(\chi + \sqrt{-1}\partial\overline{\partial}\varphi + \sqrt{-1}\partial\overline{\partial}\psi)$$
$$= \int_{\psi\le 0} Fe^{\varphi'}\chi.$$

This gives

$$\int_{\psi\le 0}(1 - e^{\psi})Fe^{\varphi}\chi \le 0.$$

Therefore $\psi \ge 0$ on Σ and by the same argument we can show $\psi \le 0$ so that $\psi = 0$ everywhere on Σ. This completes the proof of the proposition. □

Corollary 3.2 *Let $f : X \to \Sigma$ be a minimal elliptic surface of* $\mathrm{kod}(X) = 1$ *with singular fibres $X_{s_1} = m_1F_1, \ldots, X_{s_k} = m_kF_k$ of multiplicity $m_i \in \mathbf{N}$, $i = 1, ..., k$. If φ_∞ is the solution in Proposition 3.1, $\omega_\infty = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$ is a closed positive* $(1, 1)$*-form on Σ and smooth on Σ_{reg}. The Ricci curvature $Ric(\omega_\infty) = -\sqrt{-1}\partial\overline{\partial}\log\omega_\infty$ is a well-defined closed* $(1, 1)$*-current on Σ and smooth on Σ_{reg}. It also satisfies the following generalized Kähler–Einstein equation as currents*

$$Ric(\omega_\infty) = -\omega_\infty + \omega_{\mathrm{WP}} + 2\pi\sum_{i=1}^{k}\frac{m_k - 1}{m_k}[s_i], \tag{3.21}$$

where ω_{WP} is the induced Weil–Petersson metric and $[s_i]$ is the current of integration associated to the divisor s_i on Σ. In particular, if $f : X \to \Sigma$ has only singular fibres of type mI_0, ω_∞ is a hyperbolic cone metric on Σ_{reg} given by

$$Ric(\omega_\infty) = -\omega_\infty. \tag{3.22}$$

Corollary 3.2 shows that ω_∞ satisifies a generalized hyperbolic metric equation with a correction term $\omega_{\mathrm{WP}} + 2\pi\sum_{i=1}^{k}\frac{m_k-1}{m_k}[s_i]$ inherited from the elliptic fibration structure of X. Also we notice that the residues only come from multiple fibres.

4 A parabolic Schwarz lemma

In this section we will establish a parabolic Schwarz lemma for compact Kähler manifolds. It is a parabolic analog of the classical Schwarz lemma in [Ya1] and will lead us to identify and estimate the collapsing on the vertical direction for properly minimal elliptic surfaces and in general certain fibre spaces. It also plays a key role in estimating the scalar curvature along the Kähler–Ricci flow.

Let $f : X \to Y$ be a non-constant holomorphic mapping between two compact Kähler manifolds. Suppose that $\dim X = n$ and the Kähler metric $\omega(t, \cdot)$ on X is deformed by the Kähler–Ricci flow (1.1). Then we have the following parabolic Schwarz lemma.

Theorem 4.1 *If the holomorphic bisectional curvature of Y with respect to a fixed Kähler metric $h_{\alpha\overline{\beta}}$ is bounded from above by a negative constant $-K$ and the Kähler–Ricci flow* (1.1) *exists for all $t \in [0, T)$, then*

$$f^*h \leq \frac{C_K(t)}{K}\omega(t, \cdot), \tag{4.1}$$

where $C_K(t)$ is a bounded positive function in t dependent on the initial metric ω_0 and $\lim_{t\to\infty} C_K(t) = 1$ *if $T = \infty$.*

Proof Choose normal coordinate systems for $g = \omega(t, \cdot)$ on X and h on Y respectively. Let $u = \mathrm{tr}_g(h) = g^{i\overline{j}} f_i^{\alpha} f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}}$ and we will calculate the evolution of u. Standard calculation (cf. [Lu,Ya1]) shows that

$$\begin{aligned}\Delta u &= g^{k\overline{l}}\partial_k\partial_{\overline{l}}\big(g^{i\overline{j}} f_i^{\alpha} f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}}\big)\\ &= g^{i\overline{l}} g^{k\overline{j}} R_{k\overline{l}} f_i^{\alpha} f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}} + g^{i\overline{j}} g^{k\overline{l}} f_{i,k}^{\alpha} f_{\overline{j},\overline{l}}^{\overline{\beta}} h_{\alpha\overline{\beta}} - g^{i\overline{j}} g^{k\overline{l}} S_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^{\alpha} f_{\overline{j}}^{\overline{\beta}} f_k^{\gamma} f_{\overline{l}}^{\overline{\delta}},\end{aligned}$$

where $S_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is the curvature tensor of $h_{\alpha\bar{\beta}}$ and the Laplacian Δ acts on functions ϕ by

$$\Delta\phi = g^{i\overline{j}}\partial_i\partial_{\overline{j}}\phi.$$

By the definition of u we have

$$\Delta u \geq g^{i\overline{l}} g^{k\overline{j}} R_{k\overline{l}} f_i^{\alpha} f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}} + Ku^2.$$

Now

$$\begin{aligned}\frac{\partial u}{\partial t} &= -g^{i\overline{l}} g^{k\overline{j}} \frac{\partial g_{k\overline{l}}}{\partial t} f_i^{\alpha} f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}}\\ &= g^{i\overline{l}} g^{k\overline{j}} (R_{k\overline{l}} + g_{k\overline{l}}) f_i^{\alpha} f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}}\\ &= g^{i\overline{l}} g^{k\overline{j}} R_{k\overline{l}} f_i^{\alpha} f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}} + u,\end{aligned}$$

therefore

$$\left(\frac{\partial}{\partial t} - \Delta\right)u \leq u - Ku^2. \tag{4.2}$$

Let $u_{\max}(t) = \max_X u(t, \cdot) = u(t, z_t)$ for some $z_t \in X$. By the maximum principle, $\Delta u(t, z_t) \leq 0$ so that we have

$$\frac{d}{dt}u_{\max} \leq u_{\max} - Ku_{\max}^2.$$

Thus $u_{\max}(t) \leq \frac{1}{K}$ if $u_{\max}(0) \leq \frac{1}{K}$ and

$$u_{\max}(t) \leq \frac{1}{K - Ce^{-t}}$$

for some $C < K$ if $u_{\max}(0) > \frac{1}{K}$. This proves the theorem. □

By a similar argument as in the proof of Theorem 4.1 one can also derive the following Schwarz lemma for volume forms with weaker curvature bounds on the target manifold.

Theorem 4.2 *Suppose that* $\dim X = n \geq \dim Y = \kappa$. *Let* χ *be the Kähler form on* Y *with respect to the Kähler metric* $h_{\alpha\overline{\beta}}$. *If* $Ric(h) \leq -Kh$ *for some* $K > 0$ *and the Kähler–Ricci flow* (1.1) *exists for all* $t \in [0, T)$, *then there exists a constant* $C > 0$ *dependent on the initial metric* ω_0 *such that*

$$\frac{\omega^{n-\kappa} \wedge f^*\chi^\kappa}{\omega^n} \leq C. \tag{4.3}$$

Suppose $2\pi c_1(X) = -[f^*\chi]$ for a Kähler form χ on Y, i.e. K_X is a semi-positive line bundle pulled back from an ample line bundle from Y. From now on, we will write $f^*\chi$ as χ for convenience. Since $c_1(X) \leq 0$, by Theorem 2.1, the Kähler–Ricci flow has long time existence.

Theorem 4.3 *Suppose that* $\dim X = n \geq \dim Y = \kappa$ *and* $f : X \to Y$ *is a holomorphic fibration such that* $2\pi c_1(X) = -[f^*\chi]$ *for some Kähler form* χ *on* Y. *Then the Kähler–Ricci flow* (1.1) *exists for all* $t \in [0, \infty)$ *and there exist constants* A, $C > 0$ *such that for all* (t, z),

$$f^*\chi(z) \leq C \max_{(s,w)\in[0,t]\times X}\left\{2\log\frac{\Omega}{e^{(n-\kappa)s}\omega(s,w)^n}e^{-A\varphi(s,w)} + 3ne^{-A\varphi(s,w)}, 1\right\}e^{A\varphi(t,z)}\omega(t,z), \tag{4.4}$$

where Ω *is a smooth volume form on* X *such that* $Ric(\Omega) = -f^*\chi$.

Proof Let $u = g^{i\overline{j}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} \chi_{\alpha\overline{\beta}}$ and choose normal coordinates for g and χ. We will calculate the evolution for $\log u$. Note that $\Delta \log u = \frac{\Delta u}{u} - \frac{|\nabla u|_g^2}{u^2}$

and

$$\Delta u = g^{i\bar{l}} g^{k\bar{j}} R_{k\bar{l}} f_i^{\alpha} f_{\bar{j}}^{\overline{\beta}} \chi_{\alpha\overline{\beta}} + g^{i\bar{j}} g^{k\bar{l}} f_{i,k}^{\alpha} f_{\bar{j},\bar{l}}^{\overline{\beta}} \chi_{\alpha\overline{\beta}} - g^{i\bar{j}} g^{k\bar{l}} S_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^{\alpha} f_{\bar{j}}^{\overline{\beta}} f_k^{\gamma} f_{\bar{l}}^{\overline{\delta}}. \tag{4.5}$$

Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|\nabla u|_g^2 &= \sum_{i,j,k,\alpha,\beta} f_i^{\alpha} f_{\bar{j}}^{\overline{\beta}} f_{i,k}^{\alpha} f_{\bar{j},\bar{k}}^{\overline{\beta}} \\
&\le \sum_{i,j,\alpha,\beta} |f_i^{\alpha} f_{\bar{j}}^{\overline{\beta}}| \Big(\sum_k |f_{i,k}^{\alpha}|^2\Big)^{\frac{1}{2}} \Big(\sum_l |f_{\bar{j},\bar{l}}^{\overline{\beta}}|^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{i,\alpha} |f_i^{\alpha}| \Big(\sum_k |f_{i,k}^{\alpha}|^2\Big)^{\frac{1}{2}}\Big)^2 \\
&\le \Big(\sum_{j,\beta} |f_j^{\beta}|^2\Big)\Big(\sum_{i,k,\alpha} |f_{i,k}^{\alpha}|^2\Big).
\end{aligned}$$

There exists $C_1 > 0$ such that $|S_{\alpha\overline{\beta}\gamma\overline{\delta}} V^{\alpha} V^{\overline{\beta}} W^{\gamma} W^{\overline{\delta}}| \le C_1 |V|_{\chi}^2 |W|_{\chi}^2$. Then we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right) \log u \\
&= \frac{1}{u}\left(- g^{k\bar{l}} g^{i\bar{j}} f_{i,k}^{\alpha} f_{\bar{j},\bar{l}}^{\overline{\beta}} \chi_{\alpha\overline{\beta}} + g^{i\bar{j}} g^{k\bar{l}} S_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^{\alpha} f_{\bar{j}}^{\overline{\beta}} f_k^{\gamma} f_{\bar{l}}^{\overline{\delta}} + \frac{|\nabla u|_g^2}{u}\right) + 1 \\
&\le \frac{1}{u} g^{i\bar{j}} g^{k\bar{l}} S_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^{\alpha} f_{\bar{j}}^{\overline{\beta}} f_k^{\gamma} f_{\bar{l}}^{\overline{\delta}} + 1 \\
&\le C_2 u + 1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\varphi &= -\mathrm{tr}_{\omega}(\sqrt{-1}\partial\overline{\partial}\varphi) + \frac{\partial\varphi}{\partial t} \\
&= -\mathrm{tr}_{\omega}(\omega - \omega_t) + \frac{\partial\varphi}{\partial t} \\
&= \mathrm{tr}_{\omega}(\omega_t) + \frac{\partial\varphi}{\partial t} - n.
\end{aligned}$$

Combining the above estimates we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)(\log u - 2A\varphi) \\
&\le C_2 u - 2A\mathrm{tr}_{\omega}(\omega_t) - 2A \log \frac{e^{(n-\kappa)t}\omega^n}{\Omega} + 2A\varphi + 2nA + 1 \\
&\le -A' u + 2A \log \frac{\Omega}{e^{(n-\kappa)t}\omega^n} + 2A\varphi + 3nA
\end{aligned}$$

for some constant $A' > 0$ if we choose A sufficiently large. The last inequality holds because $\omega_t \geq C_3\chi$ for some constant $C_3 > 0$.

Suppose on each time interval $[0, t]$, the maximum of $\log u - 2A\varphi$ is achieved at (t_0, z_0), by the maximum principle we have

$$u(t_0, z_0) \leq \frac{A}{A'}\left(2\log\frac{\Omega}{e^{(n-\kappa)t_0}\omega^n}(t_0, z_0) + 2\varphi(t_0, z_0) + 3n\right)$$

and

$$\begin{aligned}
&u(t, z)e^{-2A\varphi(t,z)}\\
&\leq u(t_0, z_0)e^{-2A\varphi(t_0,z_0)}\\
&\leq \frac{A}{A'}\left(2\left(\log\frac{\Omega}{e^{(n-\kappa)t_0}\omega^n}(t_0, z_0)\right)e^{-2A\varphi(t_0,z_0)} + 2\varphi(t_0, z_0)e^{-2A\varphi(t_0,z_0)}\right.\\
&\qquad\left. + 3ne^{-2A\varphi(t_0,z_0)}\right)\\
&\leq \frac{2A}{A'}\left(\log\frac{\Omega}{e^{(n-\kappa)t_0}\omega^n}(t_0, z_0)\right)e^{-2A\varphi(t_0,z_0)} + C_4 + \frac{3nA}{A'}e^{-2A\varphi(t_0,z_0)}.
\end{aligned}$$

This completes the proof. □

5 Estimates

In this section, we prove the uniform zeroth order and second order estimate of the Kähler potential φ along the Kähler–Ricci flow. A gradient estimate is also derived and it gives a uniform bound of the scalar curvature. We assume that $f : X \to \Sigma$ is a minimal elliptic surface of $\mathrm{kod}(X) = 1$ over a curve Σ with singular fibres over $\Delta = \{s_1, ..., s_k\} \subset \Sigma$. Let $X_{s_i} = f^{-1}(s_i)$ be the corresponding singular fibres for $i = 1, ..., k$ and S be the defining section of the divisor

$$\sum_{i=1}^{k}[X_{s_i}] = f^*\Big(\sum_{i=1}^{k}[s_i]\Big)$$

vanishing exactly on all the singular fibres. We can always find a Hermitian metric h on the line bundle induced by the divisor $\sum_{i=1}^{k}[X_{s_i}]$ such that $Ric(h)$ is a multiple of χ and

$$|S|_h^2 \leq 1.$$

We also write $|S|_h^{2\lambda}$ for $(|S|_h^2)^\lambda$ for simplicity.

5.1 Zeroth order and volume estimates We will first derive the zeroth order estimates for φ and $\frac{d\varphi}{dt}$.

Lemma 5.1 *Let φ be a solution of the Kähler–Ricci flow* (2.3)*. There exists a constant $C > 0$ such that $\varphi \leq C$.*

Proof This is a straightforward application of the maximum principle. Let $\varphi_{\max}(t) = \max_X \varphi(t, \cdot)$. Applying the maximum principle, we have

$$\begin{aligned}\frac{\partial \varphi_{\max}}{\partial t} &\leq \log \frac{e^t \omega_t^2}{\Omega} - \varphi_{\max} \\ &\leq \log \frac{2\chi \wedge (\omega_0 - \chi) + e^{-t}(\omega_0 - \chi)^2}{\Omega} - \varphi_{\max} \\ &\leq C_1 - \varphi_{\max}.\end{aligned}$$

By solving the above ordinary differential inequality, there is a constant C_2 such that $\varphi_{\max} \leq C_2$ and this gives a uniform upper bound for φ. □

Lemma 5.2 *There exists a constant $C > 0$ such that*

$$\frac{\partial \varphi}{\partial t} \leq C. \tag{5.1}$$

Proof Differentiating on both sides of (2.3) we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t}\right) = \Delta \frac{\partial \varphi}{\partial t} + 1 - e^{-t}\mathrm{tr}_\omega(\omega_0 - \chi) - \frac{\partial \varphi}{\partial t}, \tag{5.2}$$

where Δ is the Laplacian operator of the metric g. It can be rewritten as

$$\frac{\partial}{\partial t}\left(e^t \frac{\partial \varphi}{\partial t}\right) = \Delta\left(e^t \frac{\partial \varphi}{\partial t}\right) + e^t - \mathrm{tr}_\omega(\omega_0 - \chi),$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t} + \varphi\right) = \Delta\left(\frac{\partial \varphi}{\partial t} + \varphi\right) + \mathrm{tr}_\omega(\chi) - 1. \tag{5.3}$$

So

$$\frac{\partial}{\partial t}\left(e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t\right) = \Delta\left(e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t\right) - \mathrm{tr}_\omega(\omega_0).$$

Applying the maximum principle, we have

$$e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t \leq C_1$$

for some uniform constant C_1 only depending on the initial data. By the long time existence of the Kähler–Ricci flow, we can always assume $t \geq 1$.

Hence

$$\frac{\partial\varphi}{\partial t} \le \frac{e^{-t}}{1-e^{-t}}\varphi + \frac{C'e^{-t}+te^{-t}+1}{1-e^{-t}} \le C_2$$

for some uniform constant C_2. □

Lemma 5.3 *There exists a constant $C > 0$ such that*

$$|\varphi| \le C. \tag{5.4}$$

Proof It suffices to derive the lower bound for φ. Consider $v(t,z) = \max_X \varphi(t,\cdot) - \varphi(t,z) \ge 0$. Fix $\delta > 0$. For any $p \ge 1$, since both φ and $\frac{\partial\varphi}{\partial t}$ are bounded from above, using (2.3), we have

$$\int_X e^{p\delta v}\left(\omega^2 - \omega_t^2\right) \le \int_X e^{p\delta v}\omega^2 \le C_1 e^{-t}\int_X e^{p\delta v}\omega_0^2. \tag{5.5}$$

Calculate

$$\begin{aligned}
&\int_X e^{p\delta v}\left(\omega^2 - \omega_t^2\right)\\
&= \sqrt{-1}\int_X e^{p\delta v}\partial\overline{\partial}(-v)\wedge(\omega+\omega_t)\\
&= \frac{4\sqrt{-1}}{p\delta}\int_X \partial e^{\frac{p}{2}\delta v}\wedge\overline{\partial}e^{\frac{p}{2}\delta v}\wedge(\omega+\omega_t)\\
&\ge \frac{4\sqrt{-1}}{p\delta}\int_X \partial e^{\frac{p}{2}\delta v}\wedge\overline{\partial}e^{\frac{p}{2}\delta v}\wedge\omega_t\\
&\ge \frac{C_2\sqrt{-1}}{p\delta}e^{-t}\int_X \partial e^{\frac{p}{2}\delta v}\wedge\overline{\partial}e^{\frac{p}{2}\delta v}\wedge\omega_0.
\end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) we obtain

$$\int_X |\nabla e^{\frac{p}{2}\delta v}|^2\omega_0^2 \le C_3 p\delta\int_X e^{p\delta v}\omega_0^2.$$

The Sobolev inequality $\|f\|_{L^4}^2 \le C_4\|f\|_{H^1}^2$ implies that for all $p \ge 1$

$$\|e^{\delta v}\|_{L^{2p}}^p \le C_5\delta p\|e^{\delta v}\|_{L^p}^p.$$

Now we can apply Moser's iteration by successively replacing p by 2^k and letting $k\to\infty$. Then the standard argument shows that

$$\|e^{\delta v}\|_{L^\infty} \le C_6\|e^{\delta v}\|_{L^1}.$$

Then we only need to bound the quantity $\|e^{\delta v}\|_{L^1}$. Note that $A\omega_0 - \sqrt{-1}\partial\overline{\partial}v \ge \chi + e^{-t}(\omega_0-\chi) + \sqrt{-1}\partial\overline{\partial}\varphi > 0$ if we choose $A > 0$ sufficiently large. The lemma is proved if we apply the following proposition. It is proved by the second named author in [Ti2] based on a result in [Hö].

Proposition 5.1 *There exist constants δ, $C > 0$ depending only on (X, ω_0) such that*

$$\int_X e^{-\delta\psi}\omega_0^n \leq C, \tag{5.7}$$

for all $\psi \in C^2(X)$ satisfying $\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi > 0$ and $\sup_X \psi = 0$.

This completes the proof. □

Since $e^t\omega^2 = e^{\frac{\partial\varphi}{\partial t}+\varphi}\Omega$ and $\|\varphi\|_{C^0}$ is uniformly bounded, from the uniform upper bound for $\frac{\partial\varphi}{\partial t}$ we conclude that the normalized volume form $e^t\omega^2$ is uniformly bounded above and a lower bound for it will also give a lower bound for $\frac{\partial\varphi}{\partial t}$.

Lemma 5.4 *There exist constants λ_1, $C > 0$ such that*

$$\frac{1}{C}|S|_h^{2\lambda_1} \leq \frac{e^t\omega^2}{\Omega} \leq C.$$

Proof It suffices to prove the lower bound of the volume form $e^t\omega^2$. Notice that $\log\frac{e^t\omega^2}{\Omega} = \frac{\partial\varphi}{\partial t} + \varphi$ and hence the evolutions for $\log\frac{e^t\omega^2}{\Omega}$ andφ are prescribed by

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log\frac{e^t\omega^2}{\Omega} = \mathrm{tr}_\omega(\chi) - 1 \ \text{ and} \tag{5.8}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\varphi = \mathrm{tr}_\omega(\omega_t) + \log\frac{e^t\omega^2}{\Omega} - \varphi - 2. \tag{5.9}$$

Combining the above equations, at any point $(t, z) \in [0, \infty) \times X_{reg}$ there exists $\lambda > 0$ such that

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\left(\log\frac{e^t\omega^2}{\Omega} + 2A\varphi - \lambda_1\log|S|_h^2\right)\\
&= 2A\mathrm{tr}_\omega(\omega_t) + \mathrm{tr}_\omega(\chi) - \lambda_1\mathrm{tr}_\omega(Ric(h)) + 2A\log\frac{e^t\omega^2}{\Omega} - 2A\varphi - (4A+1)\\
&\geq A\mathrm{tr}_\omega(\omega_t) + 2A\log\frac{e^t\omega^2}{\Omega} + \mathrm{tr}_\omega(A\omega_t - \lambda_1 Ric(h)) - C_1(A+1)\\
&\geq A\mathrm{tr}_\omega(\omega_t) + 2A\log\frac{e^t\omega^2}{\Omega} - C_1(A+1)
\end{aligned}$$

if we choose A sufficiently large. Suppose on each time interval $[0, T]$, the minimum of $\log\frac{e^t\omega^2}{\Omega} + 2A\varphi - \lambda_1\log|S|_h^2$ is achieved at $(t_0, z_0) \in [0, T] \times X_{reg}$, then by the maximum principle at (t_0, z_0) we have

$$\mathrm{tr}_\omega(\omega_t)(t_0, z_0) \leq 2\log\frac{\Omega}{e^t\omega^2}(t_0, z_0) + C_2. \tag{5.10}$$

But for some $\lambda > 0$ we have at (t_0, z_0)

$$C_2 + 2\log\frac{\Omega}{e^t\omega^2} \geq \mathrm{tr}_\omega(\omega_t) \geq 2\left(\frac{\omega_t^2}{\omega^2}\right)^{\frac{1}{2}}$$
$$\geq 2\left(\frac{\Omega}{e^t\omega^2}\right)^{\frac{1}{2}}\left(\frac{\chi\wedge\omega_0}{\Omega}\right)^{\frac{1}{2}} \geq C_3\left(|S|_h^{2\lambda}\frac{\Omega}{e^t\omega^2}\right)^{\frac{1}{2}},$$

where the second inequality follows from the elementary inequality $a^2+b^2 \geq 2ab$ by diagonalizing both ω_t and ω at (t_0, z_0). For each $\delta > 0$, we have the following elementary inequality

$$\log x \leq x^\delta + C_\delta \quad \text{for all } x > 0.$$

It follows that at (t_0, z_0), we have for some small $\delta < \frac{1}{2}$

$$\left(|S|_h^{2\lambda}\frac{\Omega}{e^t\omega^2}\right)^{\frac{1}{2}} \leq C_4\left(\left(\frac{\Omega}{e^t\omega^2}\right)^{\delta} + 1\right)$$

and by multiplying $|S|_h^{2\delta\lambda_1}$,

$$\left(|S|_h^{2\lambda+4\delta\lambda_1}\frac{\Omega}{e^t\omega^2}\right)^{\frac{1}{2}} \leq C_4\left(\left(|S|_h^{2\lambda_1}\frac{\Omega}{e^t\omega^2}\right)^{\delta} + 1\right).$$

We have $2\lambda + 4\delta\lambda_1 = 2\lambda_1$ if λ_1 is chosen by $\lambda_1 = \frac{\lambda}{1-2\delta}$. Therefore $|S|_h^{2\lambda_1}\frac{\Omega}{e^{t_0}\omega^2}(t_0, z_0) \leq C_5$ and

$$\frac{e^t\omega^2}{|S|_h^{2\lambda_1}\Omega}e^{\varphi}(t, z) \geq \frac{e^{t_0}\omega^2}{|S|_h^{2\lambda_1}\Omega}e^{\varphi}(t_0, z_0).$$

Both φ and $\frac{e^{t_0}\omega^2}{|S|_h^{2\lambda_1}\Omega}(t_0, z_0)$ are uniformly bounded from below, hence the lemma is proved. □

This also shows that there is a uniform lower bound for $\frac{\partial\varphi}{\partial t}$ with at worst log poles near the singular fibres.

Lemma 5.5 *There exists a constant* $C > 0$ *such that*

$$\frac{\partial\varphi}{\partial t} \geq \lambda_1 \log|S|_h^2 - C. \tag{5.11}$$

Proof By Lemma 5.4, we have

$$\frac{\partial\varphi}{\partial t} = \log\frac{e^t\omega^2}{\Omega} - \varphi \geq \log|S|_h^{2\lambda_1} - \varphi - C.$$

Then the lemma is proved by the fact that φ is uniformly bounded. □

5.2 Partial second order estimates In this section, we slightly modify the proof of the parabolic Schwarz lemma to derive a partial second order estimate. This will imply that along the Kähler–Ricci flow (1.1) the metric collapses along the fibre direction exponentially fast outside the singular fibres.

Lemma 5.6 (The partial second order estimate) *For any $\delta > 0$ there exists a constant $C > 0$ depending on δ such that*

$$\mathrm{tr}_{\omega}(\chi) \leq \frac{C}{|S|_h^{2\delta}}. \tag{5.12}$$

Proof By Lemma 5.5, for any $\delta > 0$ there exists a constant $C_1 > 0$ such that

$$|S|_h^{2\delta}\frac{\partial\varphi}{\partial t} \geq -C_1.$$

Let $u = g^{i\bar{j}}\chi_{i\bar{j}}$. Following the similar calculation in Sect. 4, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right)\big(\log|S|_h^{2\delta}u - 3A\varphi\big)\\ &\leq -2Au - 3A\frac{\partial\varphi}{\partial t} + \delta\mathrm{tr}_{\omega}(Ric(h)) + C_2A\\ &\leq -Au - 3A\frac{\partial\varphi}{\partial t} + C_2A\end{aligned}$$

for A sufficiently large. On each time interval $[0, T]$, the maximum of $\log|S|_h^{2\delta}u - A\varphi$ must be achieved at some point $(t_0, z_0) \in [0, T] \times X_{reg}$ because $\log|S|_h^{2\delta}u - A\varphi$ tends to $-\infty$ near the singular fibres. By the maximum principle we have

$$\big(|S|_h^{2\delta}u\big)(t_0, z_0) \leq -3\left(|S|_h^{2\delta}\frac{\partial\varphi}{\partial t}\right)(t_0, z_0) + C_3 \leq C_4$$

and for any $(t, z) \in [0, T] \times X_{reg}$

$$\big(|S|_h^{2\delta}u\big)(t, z) \leq \big(|S|_h^{2\delta}u\big)(t_0, z_0)e^{3A(\varphi(t,z)-\varphi(t_0,z_0))} \leq C_4e^{3A(\varphi(t,z)-\varphi(t_0,z_0))}.$$

Since $|\varphi|$ is uniformly bounded, we can conclude that $|S|_h^{2\delta}u$ is uniformly bounded and the theorem is proved. □

Corollary 5.1 *Let X_s be a non-singular fibre for any $s \in \Sigma_{reg}$. Then along the Kähler–Ricci flow* (1.1), *ω decays exponentially fast on X_s. Furthermore if Δ_s is the Laplacian on X_s with respect to $\omega_0|_{X_s}$, then there exist constants λ_2, $C > 0$ such that*

$$-e^{-t} \leq \Delta_s\varphi \leq \frac{Ce^{-t}}{|S|_h^{2\lambda_2}}. \tag{5.13}$$

Proof Applying the partial second order estimate, we have

$$0 < e^{-t} + \Delta_s\varphi = \frac{\omega|_{X_s}}{\omega_0|_{X_s}} = \frac{\omega\wedge\chi}{\omega_0\wedge\chi}$$
$$= \left(\frac{\omega\wedge\chi}{\omega^2}\right)\left(\frac{\omega^2}{\omega_0\wedge\chi}\right) \le \frac{1}{2}\mathrm{tr}_\omega(\chi)\left(\frac{\omega^2}{\omega_0\wedge\chi}\right) \le \frac{Ce^{-t}}{|S|_h^{2\lambda_2}}$$

for some uniform constants $C, \lambda_2 > 0$. This proves the corollary. □

The partial second-order estimate enables us to derive the following strong partial C^0-estimate.

Corollary 5.2 *There exist constants* $\lambda_3, C > 0$ *such that for all* $s \in \Sigma_{reg}$

$$|\sup_{X_s}\varphi - \inf_{X_s}\varphi| \le \frac{Ce^{-t}}{|S|_h^{2\lambda_3}}.$$

Proof Let $\theta(s)$ be the smooth family of standard flat metrics on the elliptic fibres over Σ_{reg} such that $\int_{X_s}\theta(s) = \int_{X_s}\omega_0$ for all $s \in \Sigma_{reg}$. Therefore $\theta(s) = \omega_{\mathrm{SF}}|_{X_s}$. Let $\Delta_{\theta(s)}$ be the Laplacian of $\theta(s)$ on each nonsingular fibre X_s. By Green's formula, we have

$$\varphi - \frac{1}{\int_{X_s}\theta(s)}\int_{X_s}\varphi\theta(s) = \int_{X_s}\Delta_{\theta(s)}\varphi(y)(G_s(x,y) + A_s)\theta(s),$$

where $G_s(\cdot,\cdot)$ is Green's function with respect to $\theta(s)$ and $A_s = \inf_{X_s\times X_s} G_s(\cdot,\cdot)$. Since $(X_s, \theta(s))$ is a flat torus, one can easily show that Green's function $G_s(\cdot,\cdot)$ is uniformly bounded below by a multiple of $\mathrm{Diam}^2(X_s,\theta(s))$ (cf. [Si], p. 137). However the diameter $\mathrm{diam}(X_s,\theta(s))$ might blow up near the singular fibres and actually there exist constants $\lambda > 0$ and C_1 such that

$$\mathrm{diam}(X_s,\theta(s)) \le \frac{C_1}{|S|_h^{\lambda}}.$$

Therefore $A_s \ge -\frac{C_2}{|S|_h^{2\lambda}}$ for some constant $C_2 > 0$ and we have on each nonsingular fibre X_s,

$$|\sup_{X_s}\varphi - \inf_{X_s}\varphi| \le C_3\sup_{X_s}|\Delta_{\theta(s)}\varphi||S|_h^{-2\lambda}.$$

But for some $\mu > 0$, C_4 and $C_5 > 0$ we have

$$|\Delta_{\theta(s)}\varphi| = |\Delta_s\varphi|\left|\frac{\omega_0|_{X_s}}{\theta(s)}\right|$$
$$= |\Delta_s\varphi|\frac{\omega_0\wedge\chi}{\theta(s)\wedge\chi}\bigg|_{X_s} \le C_4|\Delta_s\varphi|\frac{\Omega}{\theta(s)\wedge\chi}\bigg|_{X_s} \le \frac{C_5e^{-t}}{|S|_h^{\mu}},$$

where the last inequality follows from Corollary 5.1 and Lemma 3.4. This completes the proof of the corollary. □

5.3 Gradient estimates In this section we will adapt the gradient estimate in [ChYa] and the argument in [Pe2,SeTi] to obtain a uniformbound for $|\nabla \frac{\partial\varphi}{\partial t}|_g$and the scalar curvature R. Let $u = \frac{\partial\varphi}{\partial t} + \varphi = \log \frac{e^t\omega^2}{\Omega}$. The evolution equation for u is given by

$$\frac{\partial u}{\partial t} = \Delta u + \mathrm{tr}_\omega(\chi) - 1. \tag{5.14}$$

We will obtain a gradient estimate for u, which will help us bound the scalar curvature from below. Note that u is uniformly bounded from above, so we can find a constant $A > 0$ such that $A - u \geq 1$.

Theorem 5.1 *There exist constants* λ_4, λ_5, $C > 0$ *such that*

1. $|S|_h^{2\lambda_4}|\nabla u|^2 \leq C(A-u)$,
2. $-|S|_h^{2\lambda_5}\Delta u \leq C(A-u)$,

where ∇ *is the gradient operator with respect to the metric* g *and* $|\cdot| = |\cdot|_g$.

Proof Standard computation gives the following evolution equations for $|\nabla u|^2$ and Δu.

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2 = |\nabla u|^2 + (\nabla \mathrm{tr}_\omega(\chi)\cdot\overline{\nabla}u + \overline{\nabla}\mathrm{tr}_\omega(\chi)\cdot\nabla u) \tag{5.15}$$
$$-|\nabla\nabla u|^2 - |\overline{\nabla}\nabla u|^2,$$
$$\left(\frac{\partial}{\partial t} - \Delta\right)\Delta u = \Delta u + g^{i\bar{l}}g^{k\bar{j}}R_{k\bar{l}}u_{i\bar{j}} + \Delta\mathrm{tr}_\omega(\chi), \tag{5.16}$$

where $V\cdot W = \langle V, W\rangle_g$ for $V, X \in TX$.

On the other hand, $\nabla_i\nabla_{\bar{j}}u = -R_{i\bar{j}} - \chi_{i\bar{j}}$, so

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Delta u = \Delta u - |\nabla\overline{\nabla}u|^2 - g^{i\bar{l}}g^{k\bar{j}}\chi_{i\bar{j}}u_{k\bar{l}} + \Delta\mathrm{tr}_\omega(\chi).$$

We shall now prove the first inequality. Let

$$H = |S|_h^{2\lambda_4}\left(\frac{|\nabla u|^2}{A-u} + \mathrm{tr}_\omega(\chi)\right) = H_1 + H_2,$$

where $H_1 = |S|_h^{2\lambda_4}(\frac{|\nabla u|^2}{A-u})$ and $H_2 = |S|_h^{2\lambda_4}\mathrm{tr}_\omega(\chi)$.

Calculate

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)H_1\\
&=|S|_h^{2\lambda_4}\left(\frac{|\nabla u|^2-|\nabla\nabla u|^2-|\overline{\nabla}\nabla u|^2+(\nabla \mathrm{tr}_\omega(\chi)\cdot\overline{\nabla}u+\overline{\nabla}\mathrm{tr}_\omega(\chi)\cdot\nabla u)}{A-u}\right)\\
&\quad-2|S|_h^{2\lambda_4}\frac{|\nabla u|^4}{(A-u)^3}-\left(\Delta|S|_h^{2\lambda_4}\right)\frac{|\nabla u|^2}{A-u}\\
&\quad-\left(\nabla|S|_h^{2\lambda_4}\cdot\overline{\nabla}\left(\frac{|\nabla u|^2}{A-u}\right)+\overline{\nabla}|S|_h^{2\lambda_4}\cdot\nabla\left(\frac{|\nabla u|^2}{A-u}\right)\right)\\
&\quad-|S|_h^{2\lambda_4}\left(\frac{\nabla|\nabla u|^2\cdot\overline{\nabla}u}{(A-u)^2}+\frac{\overline{\nabla}|\nabla u|^2\cdot\nabla u}{(A-u)^2}\right)+|S|_h^{2\lambda_4}(\mathrm{tr}_\omega(\chi)-1)\frac{|\nabla u|^2}{(A-u)^2}.
\end{aligned}$$

Rewrite

$$\nabla|S|_h^{2\lambda_4}\cdot\overline{\nabla}\left(\frac{|\nabla u|^2}{A-u}\right)=\nabla|S|_h^{2\lambda_4}\cdot\overline{\nabla}\left(\frac{H_1}{|S|_h^{2\lambda_4}}\right)$$

and

$$\frac{\nabla|\nabla u|^2\cdot\overline{\nabla}u}{(A-u)^2}=\epsilon\frac{\nabla|\nabla u|^2\cdot\overline{\nabla}u}{(A-u)^2}+\frac{1-\epsilon}{(A-u)^2}\left(\nabla\left(\frac{(A-u)H_1}{|S|_h^{2\lambda_4}}\right)\cdot\overline{\nabla}u\right)$$

for small $\epsilon>0$. Then the evolution equation for H_1 is given by

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)H_1\\
&=|S|_h^{2\lambda_4}\left(\frac{|\nabla u|^2-|\nabla\nabla u|^2-|\overline{\nabla}\nabla u|^2+(\nabla \mathrm{tr}_\omega(\chi)\cdot\overline{\nabla}u+\overline{\nabla}\mathrm{tr}_\omega(\chi)\cdot\nabla u)}{A-u}\right)\\
&\quad-\epsilon|S|_h^{2\lambda_4}\left(\frac{\nabla|\nabla u|^2\cdot\overline{\nabla}u}{(A-u)^2}+\frac{\overline{\nabla}|\nabla u|^2\cdot\nabla u}{(A-u)^2}\right)-2\epsilon|S|_h^{2\lambda_4}\frac{|\nabla u|^4}{(A-u)^3}\\
&\quad-\frac{1-\epsilon}{(A-u)}(\nabla H_1\cdot\overline{\nabla}u+\overline{\nabla}H_1\cdot\nabla u)\\
&\quad-\frac{1}{|S|_h^{2\lambda_4}}\left(\nabla H_1\cdot\overline{\nabla}|S|_h^{2\lambda_4}+\overline{\nabla}H_1\cdot\nabla|S|_h^{2\lambda_4}\right)+2\left(\frac{\left|\nabla|S|_h^{2\lambda_4}\right|^2}{|S|_h^{2\lambda_4}}\right)\left(\frac{|\nabla u|^2}{A-u}\right)\\
&\quad+(1-\epsilon)\left(\nabla|S|_h^{2\lambda_4}\cdot\overline{\nabla}u+\overline{\nabla}|S|_h^{2\lambda_4}\cdot\nabla u\right)\frac{|\nabla u|^2}{(A-u)^2}\\
&\quad-\left(\Delta|S|_h^{2\lambda_4}\right)\frac{|\nabla u|^2}{A-u}+|S|_h^{2\lambda_4}(\mathrm{tr}_\omega(\chi)-1)\frac{|\nabla u|^2}{(A-u)^2}.
\end{aligned}$$

Also

$$\left(\frac{\partial}{\partial t}-\Delta\right)H_2=|S|_h^{2\lambda_4}\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_\omega(\chi)-\big(\Delta|S|_h^{2\lambda_4}\big)\mathrm{tr}_\omega(\chi)$$
$$-\big(\nabla|S|_h^{2\lambda_4}\cdot\overline{\nabla}\mathrm{tr}_\omega(\chi)+\overline{\nabla}|S|_h^{2\lambda_4}\cdot\nabla\mathrm{tr}_\omega(\chi)\big).$$

Therefore the evolution equation for H is given by

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)H\\
&=|S|_h^{2\lambda_4}\left(\frac{|\nabla u|^2-|\nabla\nabla u|^2-|\overline{\nabla}\nabla u|^2+(\nabla\mathrm{tr}_\omega(\chi)\cdot\overline{\nabla}u+\overline{\nabla}\mathrm{tr}_\omega(\chi)\cdot\nabla u)}{A-u}\right)\\
&\quad-\epsilon|S|_h^{2\lambda_4}\left(\frac{\nabla|\nabla u|^2\cdot\overline{\nabla}u}{(A-u)^2}+\frac{\overline{\nabla}|\nabla u|^2\cdot\nabla u}{(A-u)^2}\right)-2\epsilon|S|_h^{2\lambda_4}\frac{|\nabla u|^4}{(A-u)^3}\\
&\quad-\frac{2(1-\epsilon)}{A-u}Re(\nabla H\cdot\overline{\nabla}u)-\frac{2}{|S|_h^{2\lambda_4}}Re\big(\nabla H\cdot\overline{\nabla}|S|_h^{2\lambda_4}\big)\\
&\quad+2(1-\epsilon)Re\big(\nabla u\cdot\overline{\nabla}|S|_h^{2\lambda}\big)\frac{|\nabla u|^2}{(A-u)^2}\\
&\quad+\frac{2(1-\epsilon)}{A-u}Re\big(\nabla u\cdot\overline{\nabla}\big(|S|_h^{2\lambda_4}\mathrm{tr}_\omega(\chi)\big)\big)+\frac{2}{|S|_h^{2\lambda_4}}\big|\nabla|S|_h^{2\lambda_4}\big|^2\mathrm{tr}_\omega(\chi)\\
&\quad-\big(\Delta|S|_h^{2\lambda_4}\big)\frac{|\nabla u|^2}{A-u}-\big(\Delta|S|_h^{2\lambda_4}\big)\mathrm{tr}_\omega(\chi)+|S|_h^{2\lambda_4}(\mathrm{tr}_\omega(\chi)-1)\frac{|\nabla u|^2}{(A-u)^2}\\
&\quad+2\big|\nabla|S|_h^{2\lambda_4}\big|^2\frac{|\nabla u|^2}{|S|_h^{2\lambda_4}(A-u)}+|S|_h^{2\lambda_4}\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_\omega(\chi).
\end{aligned}$$

For the last term, by the calculation in the proof of Theorem 4.3, there exists a constant $C_1>0$ such that

$$|S|_h^{2\lambda_4}\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_\omega(\chi)\le|S|_h^{2\lambda_4}\left(\mathrm{tr}_\omega(\chi)+C_1(\mathrm{tr}_\omega(\chi))^2-\frac{|\nabla\mathrm{tr}_\omega(\chi)|^2}{\mathrm{tr}_\omega(\chi)}\right).$$

Also for any $\epsilon>0$ there exists a constant $C_2>0$ independent of ϵ such that

$$\begin{aligned}
&-\epsilon|S|_h^{2\lambda_4}\left(\frac{\nabla|\nabla u|^2\cdot\overline{\nabla}u}{(A-u)^2}+\frac{\overline{\nabla}|\nabla u|^2\cdot\nabla u}{(A-u)^2}\right)\\
&\le\epsilon C_2|S|_h^{2\lambda_4}\frac{|\nabla u|^2(|\nabla\nabla u|^2+|\overline{\nabla}\nabla u|^2)^{\frac12}}{(A-u)^2}\\
&\le\frac{\epsilon}{2}|S|_h^{2\lambda_4}\frac{|\nabla u|^4}{(A-u)^2}+\frac{\epsilon}{2}(C_2)^2|S|_h^{2\lambda_4}\frac{|\nabla\nabla u|^2+|\overline{\nabla}\nabla u|^2}{(A-u)^2}.
\end{aligned}$$

Since $|S|_h^2$ can be considered as functions pulled back from the base, there exist constants $C_3, C_4 > 0$ such that

$$\left|\nabla |S|_h^{2\lambda_4}\right|^2 \leq C_3 |S|_h^{4\lambda_4 - 2} \mathrm{tr}_\omega(\chi)$$

and

$$\left|\Delta |S|_h^{2\lambda_4}\right| \leq C_4 |S|_h^{2\lambda_4 - 2} \mathrm{tr}_\omega(\chi).$$

Also note that $|S|_h^{2\delta} \mathrm{tr}_\omega(\chi)$ and$|S|_h^{2\delta} u$ are uniformly bounded on X for any $\delta > 0$.

Applying the Cauchy–Schwarz inequality repeatedly and choosing ϵ sufficiently small, there exists a constant $C_5 > 0$ such that

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right) H \\ &\leq -\epsilon |S|_h^{2\lambda_4} \frac{|\nabla u|^4}{2(A-u)^3} - \frac{2(1-\epsilon)}{A-u} Re(\nabla H \cdot \overline{\nabla} u) \\ &\quad - \frac{2}{|S|_h^{2\lambda_4}} Re\left(\nabla H \cdot \overline{\nabla} |S|_h^{2\lambda_4}\right) + C_5.\end{aligned}$$

Suppose that H achieves its maximum at t (t_0, z_0) on $[0, T] \times X_{reg}$. By the maximum principle,

$$\nabla H(t_0, z_0) = 0$$

and

$$|S|_h^{2\lambda_4} \frac{|\nabla u|^4}{(A-u)^3}(t_0, z_0) \leq C_4$$

for some uniform constant $C_4 > 0$.

Since H_2 is uniformly bounded, there exists a constant $C_6 > 0$ such that

$$H(t_0, z_0) \leq C_6$$

so that H is uniformly bounded on $[0, \infty) \times X$.

This completes the proof of (1).

Now we can prove the second inequality by making use of the first one. Let $\lambda_5 \geq 2\lambda_4$ be sufficiently large and

$$K = -|S|_h^{2\lambda_5}\left(\frac{\Delta u}{A-u}\right) + 4|S|_h^{2\lambda_5}\left(\frac{|\nabla u|^2}{A-u}\right) = K_1 + 4K_2,$$

where $K_1 = -|S|_h^{2\lambda_5}\left(\frac{\Delta u}{A-u}\right)$ and $K_2 = 4|S|_h^{2\lambda_5}\left(\frac{|\nabla u|^2}{A-u}\right)$.

K is uniformly bounded from below since K_2 is uniformly bounded from the previous gradient estimate, the Ricci curvature $Ric(\omega)$ is uniformly bounded from below and

$$-\sqrt{-1}\partial\overline{\partial}u = Ric(\omega) + \chi, \qquad -\Delta u = R + \mathrm{tr}_\omega(\chi) \geq R.$$

The evolution equation for $K_1 = -|S|_h^{2\lambda_5}\left(\frac{\Delta u}{A-u}\right)$ is given by

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)K_1\\
&= |S|_h^{2\lambda_5}\left(\frac{-\Delta u + |\nabla\overline{\nabla}u|^2 + g^{i\overline{j}}g^{k\overline{l}}\chi_{i\overline{j}}u_{k\overline{l}} - \Delta\mathrm{tr}_\omega(\chi)}{A-u}\right) + \Delta|S|_h^{2\lambda_5}\left(\frac{\Delta u}{A-u}\right)\\
&\quad -\frac{2}{A-u}Re(\nabla K_1\cdot\overline{u}) - \frac{2}{|S|_h^{\lambda_5}}Re\left(\nabla K_1\cdot\overline{\nabla}|S|_h^{2\lambda_5}\right)\\
&\quad -|S|_h^{2\lambda_5}(\mathrm{tr}_\omega(\chi)-1)\frac{\Delta u}{(A-u)^2}\\
&\quad -2\left|\nabla|S|_h^{2\lambda_5}\right|^2\frac{\Delta u}{|S|_h^{2\lambda_5}(A-u)} - \frac{2\Delta u}{(A-u)^2}Re\left(\nabla u\cdot\overline{\nabla}|S|_h^{2\lambda_5}\right).
\end{aligned}$$

The evolution equation for K_2 is given in the earlier calculation.

Then the evolution of K is given by

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)K\\
&= |S|_h^{2\lambda_5}\left(\frac{4|\nabla u|^2 - \Delta u - 4|\nabla\nabla u|^2 - 3|\nabla\overline{\nabla}u|^2 + g^{i\overline{j}}g^{k\overline{l}}\chi_{i\overline{j}}u_{k\overline{l}} - \Delta\mathrm{tr}_\omega(\chi) + 8Re(\nabla\mathrm{tr}_\omega(\chi)\cdot\overline{\nabla}u)}{A-u}\right)\\
&\quad + \Delta|S|_h^{2\lambda_5}\left(\frac{\Delta u}{A-u}\right) - 4\left(\Delta|S|_h^{2\lambda_5}\right)\frac{|\nabla u|^2}{A-u}\\
&\quad + 4|S|_h^{2\lambda_4}(\mathrm{tr}_\omega(\chi)-1)\frac{|\nabla u|^2}{(A-u)^2} - \frac{2}{A-u}Re(\nabla K\cdot\overline{u})\\
&\quad - \frac{2}{|S|_h^{\lambda_5}}Re\left(\nabla K\cdot\overline{\nabla}|S|_h^{2\lambda_5}\right) - |S|_h^{2\lambda_5}(\mathrm{tr}_\omega(\chi)-1)\frac{\Delta u}{(A-u)^2}\\
&\quad - 2\left|\nabla|S|_h^{2\lambda_5}\right|^2\frac{\Delta u}{|S|_h^{2\lambda_5}(A-u)} - \frac{2\Delta u}{(A-u)^2}Re\left(\nabla u\cdot\overline{\nabla}|S|_h^{2\lambda_5}\right)\\
&\quad + 8\left(\frac{\left|\nabla|S|_h^{2\lambda_4}\right|^2}{|S|_h^{2\lambda_4}}\right)\left(\frac{|\nabla u|^2}{A-u}\right) + 8Re\left(\nabla|S|_h^{2\lambda_4}\cdot\overline{\nabla}u\right)\frac{|\nabla u|^2}{(A-u)^2}.
\end{aligned}$$

By the calculation in the proof of Theorem 4.3 we have

$$\begin{aligned}
-\Delta\mathrm{tr}_\omega(\chi) = &- g^{i\overline{l}}g^{k\overline{j}}R_{k\overline{l}}\chi_{i\overline{j}} - g^{i\overline{l}}g^{k\overline{j}}g^{p\overline{q}}\chi_{i\overline{j},p}\chi_{k\overline{l},\overline{q}}\\
&+ g^{i\overline{j}}g^{k\overline{l}}S_{\alpha\overline{\beta}\gamma\overline{\delta}}f_i^\alpha f_{\overline{j}}^{\overline{\beta}}f_k^\gamma f_{\overline{l}}^{\overline{\delta}}
\end{aligned}$$

$$
\begin{aligned}
&= g^{i\bar{l}} g^{k\bar{j}} (u_{k\bar{l}} + \chi_{k\bar{l}}) \chi_{i\bar{j}} - g^{i\bar{l}} g^{k\bar{j}} g^{p\bar{q}} \chi_{i\bar{j},p} \chi_{k\bar{l},\bar{q}} \\
&\quad + g^{i\bar{j}} g^{k\bar{l}} S_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^{\alpha} f_{\bar{j}}^{\bar{\beta}} f_k^{\gamma} f_{\bar{l}}^{\bar{\delta}} \\
&\le \epsilon |\overline{\nabla}\nabla u|^2 + C_{\epsilon} (\mathrm{tr}_{\omega}(\chi))^2 - \frac{|\nabla \mathrm{tr}_{\omega}(\chi)|^2}{\mathrm{tr}_{\omega}(\chi)}
\end{aligned}
$$

for some sufficiently small $\epsilon > 0$ and a constant $C_{\epsilon} > 0$ depending on ϵ.

Also there exists a constant $C_7 > 0$ such that

$$
|\Delta u|^2 \le C_7 |\nabla \overline{\nabla} u|^2.
$$

We can assume that $|S|_h^{\lambda_5} |\nabla u|^2$ is uniformly bounded if we choose $\lambda_5 \ge 2\lambda_4$ sufficiently large.

By applying the Cauchy–Schwarz inequality repeatedly, there exists a constant $C_8 > 0$ such that

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right) K \\
&\le -|S|_h^{2\lambda_5} \frac{|\nabla\overline{\nabla} u|^2}{A - u} - \frac{2}{A - u} Re(\nabla K \cdot \overline{u}) \\
&\quad - \frac{2}{|S|_h^{\lambda_5}} Re\big(\nabla K \cdot \overline{\nabla} |S|_h^{2\lambda_5}\big) + C_8.
\end{aligned}
$$

Suppose that H achieves its maximum at (t_0, z_0) on $[0, T] \times X_{reg}$, by the maximum principle,

$$
\nabla K(t_0, z_0) = 0
$$

and so

$$
|S|_h^{2\lambda_5} \frac{|\nabla\overline{\nabla} u|^2}{A - u} (t_0, z_0) \le C_8.
$$

Since K_2 is uniformly bounded, there exists a constant C_9 such that on $[0, \infty] \times X$

$$
K(t_0, z_0) \le C_9.
$$

This completes the proof of (2). □

By the volume estimate we have the following immediate corollary.

Corollary 5.3 *For any $\delta > 0$, there exists $C_{\delta} > 0$ such that*

1. $|S|_h^{2\lambda_4 + \delta} |\nabla u|^2 \le C_{\delta}$,
2. $-|S|_h^{2\lambda_5 + \delta} \Delta u \le C_{\delta}$.

Now we are in the position to prove a uniform bound for the scalar curvature. The following corollary tells that the Kähler–Ricci flow will collapse with bounded scalar curvature away from the singular fibres.

Corollary 5.4 *Along the Käher-Ricci flow* (1.1) *the scalar curvature R is uniformly bounded on any compact subset of X_{reg}. More precisely, there exist constants λ_6, $C > 0$ such that*

$$-C \leq R \leq \frac{C}{|S|_h^{2\lambda_6}}. \tag{5.17}$$

Proof It suffices to give an upper bound for R by Proposition 2.1. Notice that $R_{i\bar{j}} = -u_{i\bar{j}} - \chi_{i\bar{j}}$ and then

$$R = -\Delta u - \mathrm{tr}_\omega(\chi).$$

By Corollary 5.3 and the partial second order estimate, there exist constants λ_6, $C > 0$ such that

$$R \leq \frac{C}{|S|_h^{2\lambda_6}}.$$

□

It will be interesting to know if the Ricci curvature is uniformly bounded on any compact subset of X_{reg}. It is not expected to be true for the bisectional curvature. For example, we can choose $X = X_1 \times X_2$ where X_1 is a Calabi–Yau manifold and X_2 is a compact Kähler manifold of $c_1(X_2) < 0$. We can also choose the initial metric $\omega_0(x_1, x_2) = \omega_1(x_1) + \omega_2(x_2)$ where $Ric(\omega_1) = 0$ and $Ric(\omega_2) = -\omega_2$. Then along the Kähler–Ricci flow (1.1), the solution $\omega(t, \cdot)$ is given by

$$\omega(t, x_1, x_2) = e^{-t}\omega_1(x_1) + \omega_2(x_2).$$

The bisectional curvature of ω_t will blow up along time if the bisectional curvature of ω_1 on X_1 does not vanish.

5.4 Second order estimates In this section, we prove a second order estimate for the potential φ along the Kähler–Ricci flow. First we will prove a formula which allows us to commute the $\partial\overline{\partial}$ operator and the push-forward operator for smooth functions on X. Integrating along each fibre with respect to the initial metric ω_0, we get a function on Σ

$$\overline{\varphi} = \frac{1}{\mathrm{Vol}(X_s)} \int_{X_s} \varphi\omega_0.$$

This can be considered as a push-forward of φ.

Lemma 5.7 *Let φ be a smooth function defined on X, then we have*

$$\partial\overline{\partial}\Big(\int_{X_s}\varphi\omega_0\Big)=\int_{X_s}\partial\overline{\partial}\varphi\wedge\omega_0. \tag{5.18}$$

Proof It suffices to prove that the push forward and $\partial\overline{\partial}$ commute. Let $\pi:\mathcal{M}\to B$ be an analytic deformation of a complex manifold $M_0=\pi^{-1}(0)$ and let $M_t=\pi^{-1}(t)$. Choose a sufficiently small neighborhood $\Delta\subset B$ such that $M_\Delta=\pi^{-1}(\Delta)=\cup(\Delta\times U_i)$ with local coordinates $(z_1^i,\ldots,z_n^i,t)$, where z^i is the coordinate on U_i and t on Δ. Now choose any test function ζ on B with $\operatorname{supp}\zeta\subset\Delta$ and a partition of unity ρ_i with $\operatorname{supp}\rho_i\subset\Delta\times U_i$. Let $\varphi_i=\rho_i\varphi$. We calculate

$$\begin{aligned}\int_{M_\Delta}f^*\zeta\partial\overline{\partial}\varphi\wedge\omega&=\sum_i\int_{\Delta\times U_i}\partial\overline{\partial}f^*\zeta\wedge\varphi_i\omega=\int_\Delta\partial\overline{\partial}\zeta\Big(\sum_i\int_{U_i}\varphi_i\omega\Big)\\&=\int_\Delta\partial\overline{\partial}\zeta\Big(\int_{M_t}\varphi\omega\Big)=\int_\Delta\zeta\partial\overline{\partial}\Big(\int_{M_t}\varphi\omega\Big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{M_\Delta}f^*\zeta\partial\overline{\partial}\varphi\wedge\omega&=\int_{M_\Delta}f^*\zeta\sum_i(\partial\overline{\partial}\varphi_i\wedge\omega)=\sum_i\int_{\Delta\times U_i}f^*\zeta\partial\overline{\partial}\varphi_i\wedge\omega\\&=\int_\Delta\zeta\Big(\sum_i\int_{U_i}\partial\overline{\partial}\varphi_i\wedge\omega\Big)=\int_\Delta\zeta\Big(\int_{M_t}\partial\overline{\partial}\varphi\wedge\omega\Big).\end{aligned}$$

Therefore

$$\int_\Delta\zeta\partial\overline{\partial}\int_{M_t}\varphi\omega=\int_\Delta\zeta\Big(\int_{M_t}\partial\overline{\partial}\varphi\wedge\omega\Big)$$

for any test function f and hence

$$\partial\overline{\partial}\Big(\int_{M_t}\varphi\omega\Big)=\int_{M_t}\partial\overline{\partial}\varphi\wedge\omega.$$

□

Lemma 5.8 *There exists a constant $C>0$ such that*

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\log\operatorname{tr}_{\omega_0}(\omega)\le C(\operatorname{tr}_\omega(\omega_0)+1). \tag{5.19}$$

Proof Choose a normal coordinate system for g_0 such that g is diagonalized. By straightforward calculation we have

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\operatorname{tr}_{\omega_0}(\omega)\le-\operatorname{tr}_{\omega_0}(\omega)-\sum_{i,j,k}g^{i\bar{i}}g^{j\bar{j}}g_0^{k\bar{k}}g_{i\bar{j},k}g_{j\bar{i},\bar{k}}+C\operatorname{tr}_{\omega_0}(\omega)\operatorname{tr}_\omega(\omega_0). \tag{5.20}$$

It can also be shown that

$$
\begin{aligned}
|\nabla \mathrm{tr}_{\omega_0}(\omega)|^2 &= \sum_{i,j,k} g^{k\bar{k}} g_{i\bar{i},k} g_{j\bar{j},k} \\
&\le \sum_{i,j} \Big(\sum_k g^{k\bar{k}} |g_{i\bar{i},k}|^2 \Big)^{\frac{1}{2}} \Big(\sum_k g^{k\bar{k}} |g_{j\bar{j},k}|^2 \Big)^{\frac{1}{2}} \\
&\le \Big(\sum_i \Big(\sum_k g^{k\bar{k}} |g_{i\bar{i},k}|^2 \Big)^{\frac{1}{2}} \Big)^2 \\
&= \Big(\sum_i (g_{i\bar{i}})^{\frac{1}{2}} \Big(\sum_k g^{i\bar{i}} g^{k\bar{k}} |g_{i\bar{i},k}|^2 \Big)^{\frac{1}{2}} \Big)^2 \\
&\le \mathrm{tr}_{\omega_0}(\omega) \sum_{k,i} g^{i\bar{i}} g^{k\bar{k}} |g_{k\bar{i},i}|^2 \\
&\le \mathrm{tr}_{\omega_0}(\omega) \sum_{i,j,k} g^{i\bar{i}} g^{k\bar{k}} g_{i\bar{k},j} g_{k\bar{i},\bar{j}}. \qquad (5.21)
\end{aligned}
$$

Combined with the above inequalities, the lemma follows by calculating $(\frac{\partial}{\partial t} - \Delta) \log \mathrm{tr}_{\omega_0}(\omega)$. □

Lemma 5.9

$$
\Delta(e^t(\varphi - \overline{\varphi})) \le -\mathrm{tr}_{\omega}(\omega_0) + \frac{1}{\mathrm{Vol}(X_s)} \mathrm{tr}_{\omega}\Big(\int_{X_s} \omega_0^2 \Big) + 2e^t. \qquad (5.22)
$$

Proof Applying (5.18), we have

$$
\begin{aligned}
\Delta(\varphi - \overline{\varphi}) &= \mathrm{tr}_{\omega}(\omega - \omega_t) - \mathrm{tr}_{\omega}\Big(\frac{1}{\mathrm{Vol}(X_s)} \int_{X_s} \partial\overline{\partial}\varphi \wedge \omega_0 \Big) \\
&= 2 - \mathrm{tr}_{\omega}(\omega_t) - \frac{1}{\mathrm{Vol}(X_s)} \mathrm{tr}_{\omega}\Big(\int_{X_s} \omega \wedge \omega_0 - \int_{X_s} \omega_t \wedge \omega_0 \Big) \\
&\le 2 - e^{-t} \mathrm{tr}_{\omega}(\omega_0) + \frac{e^{-t}}{\mathrm{Vol}(X_s)} \mathrm{tr}_{\omega}\Big(\int_{X_s} \omega_0^2 \Big) - (1 - e^{-t}) \mathrm{tr}_{\omega}(\chi) \\
&\quad + (1 - e^{-t}) \frac{\int_{X_s} \omega_0}{\mathrm{Vol}(X_s)} \mathrm{tr}_{\omega}(\chi) \\
&\le 2 - e^{-t} \mathrm{tr}_{\omega}(\omega_0) + \frac{e^{-t}}{\mathrm{Vol}(X_s)} \mathrm{tr}_{\omega}\Big(\int_{X_s} \omega_0^2 \Big).
\end{aligned}
$$

□

Theorem 5.2 (Second order estimates) *There exist constants λ_7, A, $C > 0$ such that*

$$
\mathrm{tr}_{\omega_0}(\omega)(t, z) \le C e^{A e^t (\varphi - \overline{\varphi})(t,z) - \frac{A}{|S|_h^{2\lambda_7}(t,z)} \inf_{X \times [0,T]} \left(|S|_h^{2\lambda_7} e^s (\varphi - \overline{\varphi}) \right)} + C. \qquad (5.23)
$$

Proof Put $H = |S|_h^{2\lambda_7}(\log \mathrm{tr}_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi}))$. We will apply the maximum principle on the evolution of H. There exists a constant $C_1 > 0$ such that

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)H \\
&= |S|_h^{2\lambda_7}\left(\frac{\partial}{\partial t} - \Delta\right)\big(\log \mathrm{tr}_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})\big) \\
&\quad -\left(\nabla H \cdot \frac{\overline{\nabla}|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}} + \overline{\nabla} H \cdot \frac{\nabla|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}}\right) \\
&\quad + \frac{\big|\nabla|S|_h^{2\lambda_7}\big|^2}{|S|_h^{2\lambda_7}}\big(\log \mathrm{tr}_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})\big) \\
&\quad -\big(\Delta|S|_h^{2\lambda_7}\big)\big(\log \mathrm{tr}_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})\big) \\
&\le C_1|S|_h^{2\lambda_7}\mathrm{tr}_\omega(\omega_0) - A|S|_h^{2\lambda_7}\mathrm{tr}_\omega(\omega_0) - \left(\nabla H \cdot \frac{\overline{\nabla}|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}} + \overline{\nabla} H \cdot \frac{\nabla|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}}\right) \\
&\quad + |S|_h^{2\lambda_7}\left(-Ae^t(\varphi - \overline{\varphi}) - Ae^t\frac{\partial(\varphi - \overline{\varphi})}{\partial t} + \frac{1}{\mathrm{Vol}(X_s)}\mathrm{tr}_\omega\left(\int_{X_s}\omega_0^2\right)\right) \\
&\quad + C_1e^t + \frac{\big|\nabla|S|_h^{2\lambda_7}\big|^2}{|S|_h^{2\lambda_7}}\big(\log \mathrm{tr}_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})\big) \\
&\quad -\big(\Delta|S|_h^{2\lambda_7}\big)\big(\log \mathrm{tr}_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})\big).
\end{aligned}
$$

Notice that there exist constants $C_2 > 0$ and C_3 such that

$$
\begin{aligned}
\frac{1}{\mathrm{Vol}(X_s)}\mathrm{tr}_\omega\left(\int_{X_s}\omega_0^2\right) &\le \frac{C_2}{\mathrm{Vol}(X_s)}\mathrm{tr}_\omega\left(\int_{X_s}\Omega\right) \\
&= \frac{C_2}{\mathrm{Vol}(X_s)}\frac{\Omega}{\chi \wedge \omega_{\mathrm{SF}}}\mathrm{tr}_\omega\left(\int_{X_s}\chi \wedge \omega_{\mathrm{SF}}\right) \\
&\le C_3\left(\frac{\Omega}{\omega_{\mathrm{SF}} \wedge \chi}\right)\mathrm{tr}_\omega(\chi).
\end{aligned}
$$

Then $e^t|S|_h^{2\lambda_7}(\varphi - \overline{\varphi})$, $|S|_h^{2\lambda_7}\frac{\partial(\varphi - \overline{\varphi})}{\partial t}$ and $|S|_h^{2\lambda_7}\mathrm{tr}_\omega\left(\int_{X_s}\omega_0^2\right)$ are uniformly bounded if λ_7 is chosen to be sufficiently large. Also we have

$$\Delta|S|_h^{2\lambda_7} \le C_4|S|_h^{2\lambda_7 - 2}\mathrm{tr}_\omega(\chi)$$

and

$$\frac{\big|\nabla|S|_h^{2\lambda_7}\big|^2}{|S|_h^{2\lambda_7}} \le C_4|S|_h^{2\lambda_7 - 2}\mathrm{tr}_\omega(\chi)$$

for a uniform constant $C_4 > 0$. Therefore we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) H \le C_5 |S|_h^{2\lambda_7} \mathrm{tr}_\omega(\omega_0) - A|S|_h^{2\lambda_7} \mathrm{tr}_\omega(\omega_0)$$
$$-\left(\nabla H \cdot \frac{\overline{\nabla}|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}} + \overline{\nabla} H \cdot \frac{\nabla |S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}}\right) + C_5 e^t$$

for some uniform constant C_5. Choose A sufficiently large and assume H achieves its maximum at (t_0, z_0) on $[0, T] \times X_{reg}$. Applying the maximum principle, we have $\nabla H(t_0, z_0) = 0$ and then

$$\left(|S|_h^{2\lambda_7} \mathrm{tr}_\omega(\omega_0)\right)(t_0, z_0) \le C_6 e^{t_0}.$$

This implies

$$\left(|S|_h^{2\lambda_7} \mathrm{tr}_{\omega_0}(\omega)\right)(t_0, z_0) \le C_7.$$

The theorem is then proved by comparing H at any point $(t, z) \in [0, T] \times X_{reg}$ and (t_0, z_0). □

Corollary 5.5 *Let Δ_0 be the Laplace operator associated to ω_0. Then there exist constants λ_8, $C > 0$ such that*

$$-C \le \Delta_0 \varphi \le C e^{\frac{C}{|S|_h^{2\lambda_8}}} + C. \tag{5.24}$$

Proof Notice that $\Delta_0 \varphi = \mathrm{tr}_{\omega_0}\omega - \mathrm{tr}_{\omega_0}(\omega_0) = \mathrm{tr}_{\omega_0}\omega - 2$ and the corollary is an immediate consequence of Theorem 5.2. □

6 Uniform convergence

In this section we will prove a uniform convergence of the Kähler–Ricci flow. Let φ_∞ be the solution in Proposition 3.1 and $\chi_\infty = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$. We also identify $f^*\varphi_\infty$ and $f^*\chi_\infty$ with φ_∞ and χ_∞ for simplicity.

For each $s_i \in \Delta$ and the corresponding singular fibre $X_{s_i} = f^{-1}(s_i)$, we let

$$B_r(s_i) = \{s \in \Sigma \mid dist_\chi(s, s_i) \le r\}$$

be the geodesic ball in Σ centered at s_i for $r > 0$ with respect to the fixed Kähler metric χ. We also let $B_r(X_{s_i}) = f^{-1}(B_r(s_i))$ be a tubular neighborhood of the singular fibre X_{s_i}.

Since φ and φ_∞ are both uniformly bounded on X. Therefore for any $\epsilon > 0$, there exists $r_\epsilon > 0$ with $\lim_{\epsilon\to 0} r_\epsilon = 0$, such that for any $z \in \bigcup_{i=1}^k B_{r_\epsilon}(X_{s_i})$ and $t > 0$ we have

$$\left(\varphi - \varphi_\infty + \epsilon \log |S|_h^2\right)(t, z) < -1$$

and

$$\big(\varphi - \varphi_\infty - \epsilon \log |S|_h^2\big)(t, z) > 1.$$

Let η_ϵ be a cut off function on X_{can} such that $\eta_\epsilon = 1$ on$X_{\text{can}} \setminus \cup_{i=0}^k B_{r_\epsilon}(X_{s_i})$ and $\eta_\epsilon = 0$ on $\cup_{i=0}^k B_{\frac{r_\epsilon}{2}}(X_{s_i})$.

Suppose that the semi-flat closed form is given by $\omega_{\text{SF}} = \omega_0 + \sqrt{-1}\partial\overline{\partial}\rho_{\text{SF}}$ and ρ_{SF} blows up near the singular fibres. We let ρ_ϵ be an approximation for ρ_{SF} given by

$$\rho_\epsilon = (f^*\eta_\epsilon)\rho_{\text{SF}}.$$

We also define $\omega_{SF,\epsilon} = \omega_0 + \sqrt{-1}\partial\overline{\partial}\rho_\epsilon$. Now we define the twisted difference of φ and φ_∞ by

$$\psi_\epsilon^- = \varphi - \varphi_\infty - e^{-t}\rho_\epsilon + \epsilon \log |S|_h^2$$

and

$$\psi_\epsilon^+ = \varphi - \varphi_\infty - e^{-t}\rho_\epsilon - \epsilon \log |S|_h^2.$$

Proposition 6.1 *Let* $\mathcal{A} = \sup_X \left(\frac{\chi\wedge\omega_{\text{SF}}}{\chi_\infty\wedge\omega_{\text{SF}}}\right) = \sup_{X_{\text{can}}}(F^{-1}e^{-\varphi_\infty}) < \infty$. *Then there exists* $\epsilon_0 > 0$ *such that for any* $0 < \epsilon < \epsilon_0$, *there exists* $T_\epsilon > 0$ *such that for any* $z \in X$ *and* $t > T_\epsilon$ *we have*

$$\psi_\epsilon^-(t, z) \le (\mathcal{A} + 3)\epsilon \tag{6.1}$$

and

$$\psi_\epsilon^+(t, z) \ge -(\mathcal{A} + 3)\epsilon. \tag{6.2}$$

Proof The evolution equation for ψ_ϵ^- is given by

$$\frac{\partial\psi_\epsilon^-}{\partial t} = \log \frac{e^t\big(\chi_\infty + \epsilon\chi + e^{-t}\omega_{SF,\epsilon} + \sqrt{-1}\partial\overline{\partial}\psi_\epsilon^-\big)^2}{2\chi_\infty \wedge \omega_{\text{SF}}} - \psi_\epsilon^- + \epsilon \log |S|_h^2. \tag{6.3}$$

Since ρ_ϵ is bounded on X, we can always choose $T_1 > 0$ sufficiently large such that for $t > T_1$

1. $\psi_\epsilon^-(t, z) < -\frac{1}{2}$ on $\cup_{i=1}^k B_{r_\epsilon}(X_{s_i})$,
2. $e^{-t}\frac{\omega_{\text{SF}}^2}{2\chi_\infty\wedge\omega_{\text{SF}}} \le \epsilon$ on $X \setminus \cup_{i=1}^k B_{r_\epsilon}(X_{s_i})$.

We will discuss in two cases for $t > T_1$.

1. If $\psi_{\epsilon,\max}^-(t) = \max_X \psi_\epsilon^-(t, \cdot) = \psi_\epsilon^-(t, z_{\max,t}) > 0$ for all $t > T_1$. Then $z_{\max,t} \in X \setminus \cup_{i=1}^k B_{r_\epsilon}(X_{s_i})$ for all$t > T_1$ and so $\omega_{SF,\epsilon}(z_{\max,t}) = \omega_{\text{SF}}(z_{\max,t})$.

Applying the maximum principle at $z_{\max,t}$, we have

$$
\begin{aligned}
&\frac{\partial \psi_\epsilon^-}{\partial t}(t, z_{\max,t})\\
&\leq \left(\log \frac{e^t\left(\chi_\infty + \epsilon\chi + e^{-t}\omega_{SF,\epsilon}\right)^2}{2\chi_\infty \wedge \omega_{\mathrm{SF}}} - \psi_\epsilon^- + \epsilon \log |S|_h^2 \right)(t, z_{\max,t})\\
&= \left(\log \frac{2(\chi_\infty + \epsilon\chi) \wedge \omega_{SF,\epsilon} + e^{-t}\omega_{SF,\epsilon}^2}{2\chi_\infty \wedge \omega_{\mathrm{SF}}} - \psi_\epsilon^- + \epsilon \log |S|_h^2 \right)(t, z_{\max,t})\\
&= \left(\log \frac{2(\chi_\infty + \epsilon\chi) \wedge \omega_{\mathrm{SF}} + e^{-t}\omega_{\mathrm{SF}}^2}{2\chi_\infty \wedge \omega_{\mathrm{SF}}} - \psi_\epsilon^- + \epsilon \log |S|_h^2 \right)(t, z_{\max,t})\\
&\leq -\psi_\epsilon^-(t, z_{\max,t}) + \log(1 + (\mathcal{A}+1)\epsilon) + \epsilon.
\end{aligned}
$$

Applying the maximum principle again, we have

$$
\psi_\epsilon^- \leq (\mathcal{A}+2)\epsilon + O(e^{-t}) \leq (\mathcal{A}+3)\epsilon, \tag{6.4}
$$

if we choose ϵ sufficiently small in the beginning and then t sufficiently large.

2. If there exists $t_0 \geq T_1$ such that $\max_{z\in X} \psi_\epsilon^-(t_0, z) = \psi_\epsilon^-(t_0, z_0) < 0$ for some $z_0 \in X$. Assume t_1 is the first time when $\max_{z\in X, t\leq t_1} \psi_\epsilon^-(t, z) = \psi_\epsilon^-(t_1, z_1) \geq (\mathcal{A}+3)\epsilon$. Then $z_1 \in X \setminus \cup_{i=1}^k B_{r_\epsilon}(X_{s_i})$ and applying the maximum principle we have

$$
\begin{aligned}
\psi_\epsilon^-(t_1, z_1) &\leq \left(\log \frac{2(\chi_\infty + \epsilon\chi) \wedge \omega_{SF,\epsilon} + e^{-t_1}\omega_{SF,\epsilon}^2}{2\chi_\infty \wedge \omega_{\mathrm{SF}}} + \epsilon \log |S|_h^2 \right)(t_1, z_1)\\
&\leq \log(1 + (\mathcal{A}+1)\epsilon) + \epsilon < (\mathcal{A}+2)\epsilon,
\end{aligned}
$$

which contradicts the assumption that $\psi_\epsilon^-(t_1, z_1) \geq (\mathcal{A}+3)\epsilon$. Hence we have

$$
\psi_\epsilon^- \leq (\mathcal{A}+3)\epsilon.
$$

By the same argument we have

$$
\psi_\epsilon^+ \geq -(\mathcal{A}+3)\epsilon.
$$

This completes the proof. □

Proposition 6.2 *We have the point-wise convergence of φ on X_{reg}. That is, for any $z \in X_{reg}$ we have*

$$
\lim_{t\to\infty} \varphi(t, z) = \varphi_\infty(z). \tag{6.5}
$$

Proof By Proposition 6.1, we have for $t > T_\epsilon$

$$\varphi_\infty(t,z) + \epsilon \log |S|_h^2(t,z) - (\mathcal{A}+3)\epsilon \le \varphi(t,z) \le \varphi_\infty(t,z) - \epsilon \log |S|_h^2(t,z) + (\mathcal{A}+3)\epsilon.$$

Then the proposition is proved by letting $\epsilon \to 0$. □

Since we have the uniform zeroth and second order estimates for φ away from the singular fibres, we derive our main theorem.

Theorem 6.1 *Along the Kähler–Ricci flow* (1.1)*, φ converges to the pullback of the unique solution φ_∞ solving equation* (3.11) *on Σ uniformly on any compact subset of X_{reg} in the $C^{1,1}$ topology.*

7 An alternative deformation and large complex structure limit points

Mirror symmetry and the SYZ conjecture make predictions for Calabi–Yau manifolds with "large complex structure limit point" (cf. [StYaZa]). It is believed that in the large complex structure limit, the Ricci-flat metrics should converge in the Gromov–Hausdorff topology to a half-dimensional sphere by collapsing a special Lagrangian torus fibration over this sphere. This holds trivially for elliptic curves and is proved by Gross and Wilson (cf. [GrWi]) in the case of $K3$ surfaces. The method of the proof is to find a good approximation for the Ricci-flat metrics near the large complex structure limit. The approximation metric is obtained by gluing together the Ooguri–Vafa metrics near the singular fibres and a semi-flat metric on the regular part of the fibration. Such a limit metric of $K3$ surfaces is McLean's metric.

In this section, we will apply a deformation for a family of Calabi–Yau metrics and derive Mclean's metric [Mc] without writing down an accurate approximation metric. Such a deformation can be also done in higher dimensions. It will be interesting to have a flow which achieves this limit. The large complex structure limit of a $K3$ surface X can be identified as the mirror to the large Kähler limit of X as shown in [GrWi], so we can fix the complex structure on X and deform the Kähler class to infinity. Let $f : X \to \mathbf{CP}^1$ be an elliptic $K3$ surface. Let $\chi \ge 0$ be the pullback of a Kähler form on $\mathbf{CP}^1$ and ω_0 be a Kähler form on X. We construct a reference Kähler metric $\omega_t = \chi + t\omega_1$ and $[\omega_t]$ tends to $[\chi]$ as $t \to 0$. We can always scale ω_1 so that the volume of each fibre of f with respect to ω_t is t. Suppose that Ω is a Ricci-flat volume form on X with $\partial\overline{\partial}\log\Omega = 0$. Then Yau's proof [Ya2] of the Calabi conjecture yields a unique solution φ_t to the following Monge–Ampère equation for $t \in (0, 1]$

$$\begin{cases} \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t)^2}{\Omega} = C_t \\ \int_X \varphi_t \Omega = 0, \end{cases} \tag{7.1}$$

where $C_t = [\omega_t]^2$. Therefore we obtain a family of Ricci-flat metrics $\omega(t,\cdot) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t$. The following theorem is the main result of this section.

Theorem 7.1 *Let $f : X \to \mathbf{CP}^1$ be an elliptically fibred $K3$ surface with 24 singular fibres of type I_1. Then the Ricci-flat metrics $\omega(t,\cdot)$ converges to the pullback of a Kähler metric $\hat{\omega}$ on $\mathbf{CP}^1$ in any compact subset of X_{reg} in the $C^{1,1}$ topology as $t \to 0$. The Kähler metric $\hat{\omega}$ on $\mathbf{CP}^1$ satisfies the equation*

$$Ric(\hat{\omega}) = \omega_{\mathrm{WP}}. \tag{7.2}$$

Proof All the estimates can be obtained by the same argument in Sect. 5 with little modification. It is relatively easy compared to the Kähler–Ricci flow because there is no $\frac{\partial\varphi}{\partial t}$ term. Let $\mathbf{CP}^1_{reg}$ be the set of all points $s \in \mathbf{CP}^1$ with $f^{-1}(s)$ being a nonsingular fibre. We apply the similar argument in Sect. 6 to prove the uniform convergence of (7.1) away from the singular fibres to the solution $\varphi_0 \in C^0(\mathbf{CP}^1) \cap C^\infty(\mathbf{CP}^1_{reg})$ solving the following equation

$$\frac{\chi + \sqrt{-1}\partial\overline{\partial}\varphi_0}{\chi} = \frac{\Omega}{2\chi \wedge \omega_{\mathrm{SF}}}.$$

Therefore ω_t converges to $\hat{\omega} = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_0$ and $\hat{\omega}$ satisfies (7.2). This completes the proof of the theorem. □

This limit metric $\hat{\omega}$ coincides with McLean's metric as obtained by Gross and Wilson [GrWi]. Their construction is certainly more delicate and gives an accurate approximation near the singular fibres by the Ooguri–Vafa metrics. Also McLean's metric is an example of the generalized Kähler–Einstein metric defined in Definition 3.2 satisfying

$$Ric(\omega) = \lambda\omega + \omega_{\mathrm{WP}}$$

when $\lambda = 0$.

8 Generalizations and problems

8.1 A metric classification for surfaces of non-negative Kodaira dimension In this section we will give a metric classification for surfaces of non-negative Kodaira dimension. Any surface X with nef canonical line bundle K_X must be a minimal surface and $\mathrm{kod}(X) \geq 0$.

Now we assume that X is a minimal surface of non-negative Kodaira dimension.

1. When $\mathrm{kod}(X) = 2$, X is a minimal surface of general type and we have the following theorem.

Theorem 8.1 [TiZha] *If X is a minimal complex surface of general type, then the global solution of the Kähler–Ricci flow converges to a positive current* ω_∞ *which descends to the Kähler–Einstein orbifold metric on its canonical model. In particular,* ω_∞ *is smooth outside finitely many rational curves and has local continuous potential.*

2. When $\mathrm{kod}(X) = 1$, X is a minimal elliptic surface. By Theorem 1.1, the Kähler–Ricci flow deforms any Kähler metric to the unique generalized Kähler–Einstein metric ω_∞ on its canonical model X_{can}.
3. When $\mathrm{kod}(X) = 0$, X is a Calabi–Yau surface. The normalized Kähler–Ricci flow defined in [Ca] deforms any Kähler metric to the unique Ricci-flat Kähler metric in the same Kähler class.

When X is not minimal, the Kähler–Ricci flow (1.1) must develop singularities in finite time. Let ω_0 be the initial Kähler metric and T be the first time when $e^{-t}[\omega_0]-(1-e^{-t})2\pi c_1(X)$ fails to be a Kähler class. The Kähler–Ricci flow has a smooth solution $\omega(t,\cdot)$ on $[0, T)$ (Theorem 2.1) converging to a degenerate metric as t tends to T (cf. [TiZha]). This degenerate metric is actually smooth outsidea subvariety C. Such a C is characterized by the condition that $e^{-T}[\omega_0] - (1 - e^{-T})2\pi c_1(X)$ vanishes along C. This implies that C is a disjoint union of finitely many rational curves with self-intersection -1. Then we can blow down these (-1)-curves and obtain a complex surface X'. Also $e^{-T}[\omega_0] - (1 - e^{-T})2\pi c_1(X)$ descends to a Kähler class on X' and $\omega(T,\cdot)$ descends to a singular Kähler metric ω'_0 on X' with a bounded continuous local potential and a bounded volume form (cf. [TiZha]). We can consider the Kähler–Ricci flow (1.1) on X' with ω'_0 as the initial data. We expect that (1.1) has a unique and smooth solution $\omega'(t,\cdot)$ on $X' \times (0, T')$, where T' is either ∞ or the first time when $[\omega'(t,\cdot)]$ fails to be a Kähler class on X'. If $T' < \infty$, we can repeat the previous procedure and continue the flow (1.1), and we will obtain a minimal complex surface in finite time. Then the flow has a global solution which falls into one of the cases described above. The problem of contracting exceptional divisors by the Kähler–Ricci flow is also addressed in [CaLa].

8.2 Higher dimensions In this section, we discuss possible generalizations of Theorem 1.1 in higher dimensions. First, as we assumed in Sect. 3, let X be an n-dimensional non-singular algebraic variety such that K_X^m is base point free for m sufficiently large. Then the pluricanonical map defines a holomorphic fibration $f : X \to X_{\mathrm{can}}$ by the linear system $|K_X^m|$, where X_{can} is the canonical model of X.

1. If $\mathrm{kod}(X) = n$, K_X is big and nef. Hence X is a minimal model of general type. The Kähler–Ricci flow will deform any Kähler metric to a singular canonical Kähler–Einstein metric on X (cf. [Ts,TiZha]).
2. If $\mathrm{kod}(X) = 1$, X_{can} is a curve. With little modification of the proof, Theorem 1.1 can be generalized and the Kähler–Ricci flow will converge.

3. If $1 < \text{kod}(X) < n$, the fibration structure of f can be very complicated. A large number of the calculations can be carried out as in this paper and we expect the Kähler–Ricci flow will converge appropriately to the pullback of a canonical metric ω_∞ on the X_{can} such that $Ric(\omega_\infty) = -\omega_\infty + \omega_{\text{WP}}$ on X_{can}°.

In general, when K_X is nef (not necessarily semi-ample), the Kähler–Ricci flow has long time existence. Yet it does not necessarily converge, although the abundance conjecture predicts that K_X^m is globally generated for m sufficiently large. Hence, the problem of convergence of the Kähler–Ricci flow for nef K_X can be considered as the analytic version of the abundance conjecture. If K_X is not nef, the flow will develop finite time singularities. Let ω_0 be the initial Kähler metric and T the first time such that $e^{-t}[\omega_0] - (1-e^{-t})2\pi c_1(X)$ fails to be a Kähler class. The potential $\varphi(T,\cdot)$ is bounded and smooth outside an analytic set of X(cf. [TiZha]). Let X_1 be the metric completion of $\omega(T,\cdot)$. We conjecture that X_1 is an analytic variety, possibly obtained by certain standard algebraic procedures such as a flip. In general, X_1 might have singularities and it is not clear at all how to develop the notion of a weak Ricci flow on a singular variety. Suppose such a procedure can be achieved and the Kähler–Ricci flow can continue on X_1, then after applying the above procedure finitely many times on $X_1, X_2, ..., X_N$, K_{X_N} will be nef and we obtain the minimal model of X. Then the Kähler–Ricci flow has a global solution. We expect that this global solution converges to a generalized Kähler–Einstein metric on the canonical model in a suitable sense.

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References

[Au] Aubin, T.: Equations du type Monge–Ampère sur les variétés Kähleriennes compacts. Bull. Sc. Math. **102**, 119–121 (1976)

[BaMu] Bando, S., Mabuchi, T.: Uniqueness of Einstein Kähler metrics modulo connected group actions. In: Algebraic Geometry, Sendai, 1985. Adv. Stud. Pure Math., vol. 10, pp. 11–40. North-Holland, Amsterdam (1987)

[BaHuPeVa] Barth, W., Hulek, K., Peters, C., Van De Ven, A.: Compact complex surfaces. Springer (2003)

[Ca] Cao, H.: Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds. Invent. Math. **81**(2), 359–372 (1985)

[CaLa] Cascini, P., La Nave, P.: Kähler–Ricci Flow and the Minimal Model Program for Projective Varieties. Preprint

[Ch] Chow, B.: The Ricci flow on the 2-sphere. J. Differ. Geom. **33**(2), 325–334 (1991)

[ChLuTi] Chen, X., Lu, P., Tian, G.: A note on uniformization of Riemann surfaces by Ricci flow. Proc. Am. Math. Soc. **134**(11), 3391–3393 (2006)

[ChTi] Chen, X.X., Tian, G.: Ricci flow on Kähler–Einstein surfaces. Invent. Math. **147**(3), 487–544 (2002)

[ChYa] Cheng, S.Y., Yau, S.T.: Differential equations on Riemannian manifolds and their geometric applications. Commun. Pure Appl. Math. **28**(3), 333–354 (1975)

[ClKoMo] Clemens, H., Kollár, J., Mori, S.: Higher-dimensional complex geometry. Astérisque **166**, 144 pp. (1988)

[FaLu] Fang, H., Lu, Z.: Generalized Hodge metrics and BCOV torsion on Calabi–Yau moduli. J. Reine Angew. Math. **588**, 49–69 (2005)

[GrWi] Gross, M., Wilson, P.M.H.: Large complex structure limits of $K3$ surfaces. J. Differ. Geom. **55**(3), 475–546 (2000)

[Ha1] Hamilton, R.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**(2), 255–306 (1982)

[Ha2] Hamilton, R.: The formation of singularities in the Ricci flow. In: Surveys in Differential Geometry, vol. II (Cambridge, MA, 1993), pp. 7–136. Internat. Press, Cambridge, MA (1995)

[Hö] Hörmander, L.: An introduction to complex analysis in several variables. Van Nostrand, Princeton (1973)

[Ko] Kodaira, K.: On compact complex analytic surfaces I. Ann. Math. (2) **71** (1960), 111–152. II. Ann. Math. (2) **77**, 563–626 (1963). III. Ann. Math. (2) **78**, 1–40 (1963)

[Kol1] Kolodziej, S.: The complex Monge–Ampère equation. Acta Math. **180**(1), 69–117 (1998)

[Kol2] Kolodziej, S.: Stability of solutions to the complex Monge–Ampère equation on compact Kähler manfiolds. Indiana Univ. Math. J. **52**, 667–686 (2003)

[KoSo] Kontsevich, M., Soibelman, Y.: Homological mirror symmetry and torus fibrations. In: Symplectic Geometry and Mirror Symmetry (Seoul, 2000), pp. 203–263. World Sci. Publishing, River Edge, NJ (2001)

[LiYa] Li, P., Yau, S.T.: Estimates of eigenvalues of a compact Riemannian manifold. In: Geometry of the Laplace Operator (Proc. Symp. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979). Proc. Symp. Pure Math., vol. XXXVI, pp. 205–239. Am. Math. Soc., Providence, RI (1980)

[Lu] Lu, Y.: Holomorphic mappings of complex manifolds. J. Differ. Geom. **2**, 299–312 (1968)

[Mi] Miranda, R.: The basic theory of elliptic surfaces. Dottorato di Ricerca in Matematica. Doctorate in Mathematical Research. ETS Editrice, Pisa (1989)

[Mc] McLean, R.: Deformations of calibrated submanifolds. Commun. Anal. Geom. **6**(4), 705–747 (1998)

[OoVa] Ooguri, H., Vafa, C.: Summing up Dirichlet instantons. Phys. Rev. Lett. **77**(16), 3296–3298 (1996)

[Pe1] Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. Preprint, math.DG/0211159

[Pe2] Perelman, G.: Unpublished work on the Kähler–Ricci flow

[PhSt] Phong, D.H., Sturm, J.: On stability and the convergence of the Kähler–Ricci flow. J. Differ. Geom. **72**(1), 149–168 (2006)

[SeTi] Sesum, N., Tian, G.: Bounding scalar curvature and diameter along the Kähler–Ricci flow (after Perelman). Lecture note

[Si] Siu, Y.-T.: Lectures on Hermitian–Einstein metrics for stable bundles and Kähler–Einstein metrics. Birkhäuser, Basel (1987)

[SoTi] Song, J., Tian, G.: Holomorphic Kähler–Einstein metrics. I. Semi-ample Canonical Bundle. Preprint

[SoWe] Song, J., Weinkove, B.: On the convergence and singularities of the J-flow with applications to the Mabuchi energy. To appear in Commun. Pure Appl. Math.

[StYaZa] Strominger, A., Yau, S.T., Zaslow, E.: Mirror symmetry is T-duality. Nucl. Phys. B **479**(1–2), 243–259 (1996)

[Ti1] Tian, G.: Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Peterson-Weil metric. In: Yau, S.T. (ed.) Mathematical Aspects of String Theory. World Scientific (1986)

[Ti2] Tian, G.: On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. Invent. Math. **89**(2), 225–246 (1987)

[Ti3] Tian, G.: On Calabi's conjecture for complex surfaces with positive first Chern class. Invent. Math. **101**(1), 101–172 (1990)

[TiZha] Tian, G., Zhang, Z.: On the Kähler–Ricci flow on projective manifolds of general type. Chin. Ann. Math., Ser. B **27**(2), 179–192 (2006)

[TiZhu] Tian, G., Zhu, X.: Convergence of Kähler–Ricci flow. J. Am. Math. Soc. **20**, 675–699 (2007)

[Ts] Tsuji, H.: Existence and degeneration of Kähler–Einstein metrics on minimal algebraic varieties of general type. Math. Ann. **281**, 123–133 (1988)

[Ya1] Yau, S.T.: A general Schwarz lemma for Kähler manifolds. Am. J. Math. **100**(1), 197–203 (1978)

[Ya2] Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I. Commun. Pure Appl. Math. **31**, 339–411 (1978)

[Ya3] Yau, S.T.: Open problems in geometry. Proc. Sympos. Pure Math. **54**, 1–28 (1993)

[Zh] Zhang, Z.: Thesis at MIT