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A new holomorphic invariant and uniqueness of Kähler–Ricci solitons

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Abstract. In this paper, a new holomorphic invariant is defined on a compact Kähler manifold with positive first Chern class and nontrivial holomorphic vector fields. This invariant generalizes the Futaki invariant. We prove that this invariant is an obstruction to the existence of Kähler–Ricci solitons. In particular, using this invariant together with the main result in [TZ1], we solve completely the uniqueness problem of Kähler–Ricci solitons. Two functionals associated to the new holomorphic invariant are also discussed. The main result here was announced in [TZ2].

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0. Introduction

The purpose of this paper is to introduce a new holomorphic invariant and apply it to studying the uniqueness of Kähler–Ricci solitons on compact Kähler manifolds.

Let M be an n-dimensional compact complex manifold with positive first Chern class $c_1(M) > 0$. Let

$$g = \sum g_{i\overline{j}} dz^i \otimes d\overline{z}^j$$

be a Kähler metric on M with its Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum g_{i\overline{j}} dz^i \wedge d\overline{z}^j$$

representing $c_1(M)$. Since the Ricci-form $\mathrm{Ric}(\omega_g)$ of ω_g also represents $c_1(M)$, there is a smooth function h_g such that

$$\mathrm{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} h_g.$$

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Let $\eta(M)$ be the Lie algebra which consists of all holomorphic vector fields on M. Then, for any holomorphic vector field X on M, by the Hodge Theorem, there is a unique smooth complex-valued function $\theta_X(g)$ of M such that

$$\begin{cases} i_X\omega_g = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\theta_X(g) \\ \int_M e^{\theta_X(g)}\omega_g^n = \int_M \omega_g^n, \end{cases}$$

where $\frac{\omega_g^n}{n!} = \frac{\omega_g\wedge\cdots\wedge\omega_g}{n!}$ is the volume form of g. We define a linear functional from $\eta(M)$ into $\mathbb{C}$ by

$$F_X(v) = \int_M v(h_g - \theta_X(g))e^{\theta_X(g)}\omega_g^n, \ v \in \eta(M).$$

We will first show that this functional defines a *holomorphic invariant* on M (cf. Proposition 1.1).

The invariant $F_X(\cdot)$ can be defined for any holomophic vector field X on M. In particular, if $X \equiv 0$, the invariant is just the *Futaki invariant* in [F1] (The excellent reference for extensive discussions of the Futaki invariant can be found in Futaki's book [F2]). It is well-known that there are compact Kähler manifolds M with $c_1(M) > 0$ and nonvanishing Futaki invariant, for example, $\mathbb{C}P^n\#\overline{\mathbb{C}P^n}$ does have nonvanishing Futaki invariant ([KS]). The new holomorphic invariant $F_X(\cdot)$ can compensate this defect somehow. For example, on each $\mathbb{C}P^n\#k\overline{\mathbb{C}P^n}(1 \le k \le n)$, there exists a unique holomorphic vector field X such that the invariant $F_X(\cdot)$ vanishes (cf. Proposition 2.2).

The invariant $F_X(\cdot)$ is an obstruction to the existence of Kähler–Ricci solitons (cf. Proposition 3.1), just as the Futaki invariant is an obstruction to the existence of Kähler–Einstein metrics. With help of this observation, we can solve completely the uniqueness problem of Kähler–Ricci solitons. It was proved in [TZ1] that the Kähler–Ricci soliton is unique modulo a reductive subgroup of the holomorphic automorphism group for a fixed holomorphic vector field on any compact Kähler manifold.

A Kähler metric g on a compact complex manifold M is called a *Kähler–Ricci soliton* if there is a holomorphic vector field X on M such that the Kähler form ω_g of g satisfies

$$\mathrm{Ric}(\omega_g) - \omega_g = L_X\omega_g,$$

where L_X denotes the Lie derivative along X. In particular, if $X = 0$, g is a Kähler–Einstein metric. Ricci solitons have been studied extensively in recent years ([H1], [C2], [T2], [TZ1], [Zh], etc.). One motivation is that they are very closely related to the limiting behavior of solutions of PDE which arise from the geometric analysis, such as the Hamilton's Ricci flow equation ([H2]) and certain complex Monge–Ampère equations associated to Kähler–Einstein metrics ([T2]). Ricci solitons extend naturally Einstein metrics. Examples of nontrivial Kähler–Ricci solitons (not Kähler–Einstein metrics) were found on certain Kähler manifolds by N. Koiso for compact case ([Ko]), and H. Cao ([C1]), and H. Pedersen, C. Tonnesen-Friedman and G. Valent ([PTV]) for noncompact case.

Our main theorem can be stated as follows (cf. Theorem 3.2).

Uniqueness Theorem. *There is at most one Kähler–Ricci soliton on a compact complex manifold M modulo the identity component* $\mathrm{Aut}^\circ(M)$ *of holomorphic automorphisms group* $\mathrm{Aut}(M)$ *of M, more precisely, if g and g' are two Kähler–Ricci solitons with respect to two holomorphic vector fields X and X' on M, respectively, then there exists an element $\sigma \in \mathrm{Aut}^\circ(M)$ such that*

$$\omega_g = \sigma^* \omega_{g'} \text{ and } X = (\sigma^{-1})_*(X').$$

The above theorem extends Bando and Mabuchi's theorem on the uniqueness of Kähler–Einstein metrics with positive first Chern class ([BM]). Note that the uniqueness of Kähler–Einstein metrics was proved by E. Calabi in 50's on Kähler manifolds with non-positive first Chern class.

The organization of this paper is as follows. In Section 1, we introduce the new holomorphic invariant (cf. Proposition 1.1). In Section 2, another version of new holomorphic invariant is discussed. In Section 3, we first show that the new holomorphic invariant is an obstruction to the existence of Kähler–Ricci solitons (cf. Proposition 3.1), then we complete the proof of the uniqueness theorem of Kähler–Ricci solitons (cf. Theorem 3.2). In Section 4, we revisit a class of the compactifications of C^*-bundles over compact Kähler–Einstein manifolds, and prove that the vanishing of the new holomorphic invariant is a sufficient and necessary condition for the existence of Kähler–Ricci solitons on these manifolds. In Section 5, we introduce two functionals associated to the new holomorphic invariant and prove that these two functionals are bounded from below on any compact complex manifold which admits a Kähler–Ricci soliton (cf. Theorem 5.1). As a corollary, we derive an inequality of the Moser–Trudinger type on such compact complex manifold. In the appendix, another proof of the uniqueness theorem is given.

The main result was announced in [TZ2].

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1. A new holomorphic invariant

In this section, we introduce a new holomorphic invariant. This contains the Futaki invariant as a special case ([F1]).

Let M be an n-dimensional compact complex manifold with positive first Chern class $c_1(M) > 0$. Let g be a Kähler metric on M with the Kähler form $\omega_g \in c_1(M)$. In local coordinates, g is given by $\{g_{i\bar{j}}\}$ and

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum g_{i\bar{j}} dz^i \wedge d\overline{z}^j.$$

Since the Ricci-form

$$\mathrm{Ric}(\omega_g) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\det(g_{i\bar{j}}))$$

also represents $c_1(M)$, there is a smooth function h_g such that

$$\mathrm{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}h_g. \tag{1.1}$$

Let X be a holomorphic vector field on M. Define a $(0,1)$-form $i_X\omega_g$ by

$$i_X\omega_g(u) = \omega(X, u),$$

where u is any smooth complex-valued vector field on M. Note that $i_X\omega_g$ is $\overline{\partial}$-closed. Since $c_1(M) > 0$, there are no nontrivial harmonic (0,1)-forms. By the Hodge Theorem, there is a unique smooth complex-valued function $\theta_X(g)$ of M such that

$$\begin{cases} i_X\omega_g = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\theta_X(g) \\ \int_M e^{\theta_X(g)}\omega_g^n = \int_M \omega_g^n. \end{cases} \tag{1.2}$$

Let $\eta(M)$ be the complex Lie algebra which consists of all holomorphic vector fields on M. For a given Kähler form $\omega_g \in c_1(M)$, we define a linear functional from $\eta(M)$ into $\mathbb{C}$ as follows,

$$F_X(v) = \int_M v(h_g - \theta_X(g))e^{\theta_X(g)}\omega_g^n, \ v \in \eta(M), \tag{1.3}$$

where h_g is the smooth real-valued function defined by (1.1) and $\theta_X(g)$ is the smooth complex-valued function defined by (1.2), respectively. This functional $F_X(\cdot)$ can be defined for any holomophic vector field X on M. In particular, if $X \equiv 0$, the functional is just the Futaki invariant ([F1]). The following proposition shows that this functional defines a holomorphic invariant on M.

Proposition 1.1. *The functional $F_X(\cdot)$ defines a holomorphic invariant on M, i.e., it is independent of the choice of g with the Kähler form $\omega_g \in c_1(M)$.*

Proof. Let g' be another Kähler metric with its Kähler form $\omega_{g'} \in c_1(M)$. Then there is a smooth real-valued function ϕ on M such that

$$\omega_{g'} = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi.$$

Let $\theta_X(g')$ be a smooth complex-valued function on M defined by (1.2) associated to the metric g'. Then

$$\theta_X(g') = \theta_X(g) + X(\phi) + c,$$

for some constant c. We claim

$$\theta_X(g') = \theta_X(g) + X(\phi).$$

Let

$$\omega_{g_s} = \omega_g + (s-1)\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi \ (1 \le s \le 2) \tag{1.4}$$

be a family of Kähler forms on M. Then a direct computation shows

$$\begin{aligned}
&\frac{d}{ds}\int_M e^{\theta_X(g)+(s-1)X(\phi)}\omega_{g_s}^n \\
&= \int_M (\triangle_s\phi + X(\phi))e^{\theta_X(g)+(s-1)X(\phi)}\omega_{g_s}^n \\
&= -\int_M \operatorname{div}(e^{\theta_X(g)+(s-1)X(\phi)}\partial\phi)\omega_{g_s}^n = 0,
\end{aligned}$$

where $\triangle_s$ denote the Laplacian operators associated to Kähler forms ω_{g_s}. It follows

$$\int_M e^{\theta_X(g)+(s-1)X(\phi)}\omega_{g_s}^n = \int_M e^{\theta_X(g)}\omega_g^n,$$

and consequently,

$$\theta_X(g_s) = \theta_X(g) + (s-1)X(\phi), \tag{1.5}$$

where $\theta_X(g_s)$ are smooth complex-valued functions defined by (1.2) associated to metrics g_s. In particular,

$$\theta_X(g') = \theta_X(g_2) = \theta_X(g) + X(\phi).$$

The claim is proved.

Let $h_{g'}$ be a smooth real-valued function defined by (1.1) associated to the metric g'. Then one can check

$$h_{g'} = h_g - \log\frac{\omega_{g'}^n}{\omega_g^n} - \phi + \text{const}.$$

Now we shall prove

$$\begin{aligned}
&\int_M v(h_g - \theta_X(g))e^{\theta_X(g)}\omega_g^n \\
&= \int_M v(h_{g'} - \theta_X(g'))e^{\theta_X(g')}\omega_{g'}^n, \ \forall\ X, v \in \eta(M).
\end{aligned} \tag{1.6}$$

Let

$$h_s = h_g - \log\frac{\omega_{g_s}^n}{\omega_g^n} - (s-1)\phi, \ 1 \le s \le 2. \tag{1.7}$$

Then h_s satisfies

$$\mathrm{Ric}(\omega_{g_s}) - \omega_{g_s} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}h_s, \tag{1.8}$$

and

$$\frac{dh_s}{ds} = -(\triangle_s\phi + \phi). \tag{1.9}$$

Define

$$f(s) = \int_M v(h_s - \theta_X(g_s))e^{\theta_X(g_s)}\omega_{g_s}^n.$$

Observe that

$$i_v(\omega_{g_s}) = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\psi$$

for some smooth complex-valued function ψ. Then by using (1.5) and (1.9), we have

$$\begin{aligned}\frac{df(s)}{ds} &= \int_M v(-\triangle_s\phi - \phi - X(\phi))e^{\theta_X(g_s)}\omega_{g_s}^n\\ &\quad + \int_M (\triangle_s\phi + X(\phi)) \cdot v(h_s - \theta_X(g_s))e^{\theta_X(g_s)}\omega_{g_s}^n.\end{aligned}$$

Taking integration by parts, we get

$$\begin{aligned}\frac{df(s)}{ds} &= -\int_M < \overline{\partial}\psi, \overline{\partial}(\overline{\triangle_s\phi + X(\phi)}) >_{\omega_{g_s}} e^{\theta_X(g_s)}\omega_{g_s}^n\\ &\quad - \int_M < \overline{\partial}\psi, \overline{\partial}\phi >_{\omega_{g_s}} e^{\theta_X(g_s)}\omega_{g_s}^n\\ &\quad + \int_M (\triangle_s\phi + X(\phi))(v(h_s - \theta_X(g_s)))e^{\theta_X(g_s)}\omega_{g_s}^n\\ &= \int_M (\triangle_s\phi + X(\phi))(\triangle_s\psi + v(\theta_X(g_s)))e^{\theta_X(g_s)}\omega_{g_s}^n\\ &\quad + \int_M \psi(\triangle_s\phi + X(\phi))e^{\theta_X(g_s)}\omega_{g_s}^n\\ &\quad + \int_M (\triangle_s\phi + X(\phi))(v(h_s - \theta_X(g_s)))e^{\theta_X(g_s)}\omega_{g_s}^n\\ &= \int_M (\triangle_s\phi + X(\phi))(\triangle_s\psi + \psi + v(h_s))e^{\theta_X(g_s)}\omega_{g_s}^n.\end{aligned} \tag{1.10}$$

On the other hand, for the fixed metric $g = g_s$ and any point $x \in M$, one can choose a local coordinate near x such that $g_{i\overline{j}} = \delta_{\overline{i}j}$ at x. Let

$$p = \triangle_s\psi + \psi + v(h_s).$$

Then by using the Ricci identity and (1.8), we get

$$\begin{aligned} & p_{\bar{j}}(x) \\ &= (\psi_{i\bar{i}} + \psi + \psi_{\bar{i}}(h_s)_i)_{\bar{j}} \\ &= \psi_{i\bar{i}\bar{j}} + \psi_{\bar{j}} + \psi_{\bar{i}}(h_s)_{i\bar{j}} \\ &= \psi_{\bar{i}\bar{j}i} - \psi_{\bar{k}}R_{k\bar{j}} + \psi_{\bar{j}} + \psi_{\bar{i}}(R_{i\bar{j}} - g_{i\bar{j}}) \\ &= 0, \end{aligned}$$

and consequently,

$$\overline{\partial} p(x) = 0. \tag{1.11}$$

By using the integration by parts together with (1.11), we get from (1.10),

$$\frac{df(s)}{ds} = -\int_M < \overline{\partial} p, \overline{\partial}\phi >_{\omega_{g_s}} e^{\theta_X(g_s)}\omega_{g_s}^n \equiv 0.$$

This shows $f(1) = f(2)$, so (1.6) is true. Proposition 1.1 is proved. □

2. Another formation of the holomorphic invariant

In this section, we give another formulation of the holomorphic invariant defined in last section, by which we will prove that there exists a unique holomorphic vector field such that the corresponding holomorphic invariant vanishes on the reductive Lie algebra generated by holomorphic vector fields. We will keep the notations in last section.

First we notice that $\theta_X = \theta_X(g)$ defined by (1.2) satisfies (compared with (1.11)),

$$\overline{\partial}(\Delta\theta_X + X(h_g) + \theta_X) = 0,$$

where $\triangle$ denotes the Laplacian operator associated to the Kähler form ω_g. Then we can renormalize θ_X to be $\tilde{\theta}_X$ by adding a constant such that

$$\tilde{\theta}_X = -\Delta\tilde{\theta}_X - X(h_g). \tag{2.1}$$

Clearly, this new normalization is equal to the condition

$$\int_M \tilde{\theta}_X e^{h_g}\omega_g^n = 0.$$

Lemma 2.1. *Let* $\omega_{g'} = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi \in c_1(M) > 0$ *be a Kähler form on* M *and* $h_{g'}$ *be defined by* (1.1) *in Section 1 associated to* $\omega_{g'}$. *Let* $\tilde{\theta}_X(g')$ *be a smooth complex-valued function defined by*

$$\begin{cases} i_X\omega_{g'} = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\tilde{\theta}_X(g') \\ \int_M \tilde{\theta}_X(g')e^{h_{g'}}\omega_{g'}^n = 0. \end{cases} \tag{2.2}$$

Then $\tilde{\theta}_X(g') = \tilde{\theta}_X + X(\phi)$.

Proof. Let $\omega_{g_s}(1 \leq s \leq 2)$ and $h_s(1 \leq s \leq 2)$ be a family of Kähler forms and functions defined by (1.4) and (1.7) in Section 1, respectively. Let $\tilde{\theta}_X(g_s)$ be a family of smooth complex-valued functions defined by (2.2) associated to ω_{g_s}. Then

$$\tilde{\theta}_X(g_s) = \tilde{\theta}_X + (s-1)X(\phi) + c_s$$

for some constants $c_s(1 \leq s \leq 2)$ and satisfy (2.1) associated to Kähler forms ω_{g_s}.

Let

$$G(s) = \int_M (\tilde{\theta}_X + (s-1)X(\phi) + c_s)e^{h_s}\omega_{g_s}^n.$$

Then by (1.7), we have

$$G(s) = \int_M (\tilde{\theta}_X + (s-1)X(\phi) + c_s)e^{-(s-1)\phi + h_g}\omega_g^n.$$

Differentiating the above on s and integrating by parts, we get

$$\begin{aligned}\frac{dG(s)}{ds} &= \int_M \Big(X(\phi) + \frac{d}{ds}c_s - (\tilde{\theta}_X + (s-1)X(\phi) + c_s)\phi\Big)e^{-(s-1)\phi + h_g}\omega_g^n \\ &= \int_M \Big(X(\phi) + \frac{d}{ds}c_s - \tilde{\theta}_X(g_s)\phi\Big)e^{h_s}\omega_{g_s}^n \\ &= \Big(\frac{d}{ds}c_s\Big)\int_M e^{h_s}\omega_{g_s}^n.\end{aligned}$$

Since $G(s) \equiv 0$, we conclude $c_s \equiv$ const., and consequently $c_s \equiv 0$. Hence

$$\tilde{\theta}_X(g') = \tilde{\theta}_X(g_2) = \tilde{\theta}_X + X(\phi). \qquad \square$$

Let $Z \in \eta(M)$ and $\tilde{\theta}_Z$ be a smooth complex-valued function defined by (2.2) with respect to Z. We introduce a functional on $\eta(M)$ by

$$f(Z) = \int_M e^{\tilde{\theta}_Z}\omega_g^n. \tag{2.3}$$

Since

$$\begin{aligned}&\int_M e^{\tilde{\theta}_Z + Z(\phi)}\omega_\phi^n \\ &= \int_M e^{\tilde{\theta}_Z}\omega_g^n + \int_0^1 \int_M (\triangle'\phi + Z(\phi))e^{\tilde{\theta}_Z + tZ(\phi)}\omega_{t\phi}^n \wedge dt,\end{aligned}$$

then by using integration by parts, we have

$$\int_M e^{\tilde{\theta}_Z + Z(\phi)}\omega_\phi^n = \int_M e^{\tilde{\theta}_Z}\omega_g^n,$$

where $\triangle'$ are the Laplacian operators associated to Kähler forms $\omega_{t\phi} = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(t\phi)$. It follows from the above and Lemma 2.1 that $f(Z)$ is independent of choices of Kähler metrics with the Kähler class $c_1(M)$.

Let $F'_X(v)$ be the differential of $f(\cdot)$ at X with respect to $v \in \eta(M)$. Then

$$F'_X(v) = \int_M \tilde{\theta}_v e^{\tilde{\theta}_X} \omega_g^n. \tag{2.4}$$

This is clearly independent of choices of Kähler metrics with the Kähler class $c_1(M)$, and so a holomorphic invariant. Moreover, using (2.1) for function $\tilde{\theta}_v$ and integration by parts, we deduce

$$F'_X(v) = -\int_M v(h_g - \tilde{\theta}_X) e^{\tilde{\theta}_X} \omega_g^n. \tag{2.5}$$

Since $\tilde{\theta}_X$ is the same as θ_X modulo const., we see that $F'_X(\cdot)$ is just a multiple of the holomorphic invariant $F_X(\cdot)$ defined in Section 1. In particular, $F_X(\cdot)$ vanishes on $\eta(M)$ if and only if $F'_X(\cdot) \equiv 0$ on $\eta(M)$.

The new version $F'_X(\cdot)$ of $F_X(\cdot)$ will give us more information. We recall some notation. Let K be a maximal compact subgroup of the identity component $\mathrm{Aut}^\circ(M)$ of holomorphic automorphisms group $\mathrm{Aut}(M)$. Then the Chevalley decomposition allows us to write $\mathrm{Aut}^\circ(M)$ as a semidirect product ([FM]),

$$\mathrm{Aut}^\circ(M) = \mathrm{Aut}_r(M) \propto R_u, \tag{2.6}$$

where $\mathrm{Aut}_r(M)$ is a reductive algebraic subgroup of $\mathrm{Aut}^\circ(M)$ and the complexification of K, and R_u is the unipotent radial of $\mathrm{Aut}^\circ(M)$. Let $\eta(M), \eta_r(M), \eta_u(M)$ and $\kappa(M)$ be the Lie algebras of $\mathrm{Aut}(M), \mathrm{Aut}_r(M), R_u$ and K, respectively. From the decomposition (2.6), we obtain

$$\eta(M) = \eta_r(M) + \eta_u(M). \tag{2.7}$$

Lemma 2.2. *There exists a unique holomorphic vector field $X \in \eta_r(M)$ with $\mathrm{Im}(X) \in \kappa(M)$ such that*

$$F'_X(v) = 0, \ \forall\ v \in \eta_r(M),$$

where $\mathrm{Im}(X)$ denotes the imaginary part of X.

Proof. Since $F'_X(\cdot)$ is a linear functional on $\eta_r(M)$, we may choose a K-invariant Kähler metric g and $v \in \eta_r(M)$ with $\mathrm{Im}(v) \in \kappa(M)$ to compute $F'_X(v)$. Let $Z \in \eta_r(M)$ with $\mathrm{Im}(Z) \in \kappa(M)$. Then

$$L_Z\omega_g = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\tilde{\theta}_Z$$

and

$$L_{\overline{Z}}\omega_g = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\overline{\tilde{\theta}_Z}).$$

It follows

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\tilde{\theta}_Z - \overline{\tilde{\theta}_Z}) = L_{Z-\overline{Z}}\omega_g = 0.$$

This shows that $\tilde{\theta}_Z$ is a real-valued function, and consequently $f(Z)$ is a convex functional on $(\eta_r(M), \mathbb{R})$. Since $F'_X(\cdot)$ is the differential of f at X, it suffices to prove that $f(Z)$ is proper, i.e., $f(Z)$ diverges to infinity as Z tends to ∞.

Let $Z_i \in \eta_r(M)$ with $\mathrm{Im}(Z_i) \in \kappa(M)$, $i = 1, \dots, m$, be a base of $(\eta_r(M), \mathbb{R})$ and $\{Z_l\}$ a sequence of holomorphic vector fields in $(\eta_r(M), \mathbb{R})$ so that $\int_M |Z_l|^2_{\omega_g}\omega_g^n \to +\infty$ as $l \to \infty$. Then there are m sequences of numbers $\{t_l^i\}$ such that $Z_l = \sum_{i=1}^m t_l^i Z_i$. Without loss of generality, we may assume that there is a subsequence $\{l_k\}$ such that

$$|t_{l_k}^1| \geq |t_{l_k}^i|,\ i = 2, \dots m, \text{ and } |t_{l_k}^1| \to \infty,$$

and $\left\{\frac{|t_{l_k}^i|}{|t_{l_k}^1|}\right\}$ are all convergent for any $i = 2, \dots, m$ as $l_k \to \infty$. Furthermore, we may also assume $t_{l_k}^i > 0$, $i = 1, \dots, m$, since we can use $-Z_i$ to replace Z_i if necessary. Then it follows that

$$Z_1 + \sum_{i=2}^m \frac{t_{l_k}^i}{t_{l_k}^1} Z_i \to Z_0, \quad \text{as } l_k \to \infty$$

for some holomorphic vector field $Z_0 \in (\eta_r(M), \mathbb{R})$.

Let $\tilde{\theta}_{Z_0}$ be a smooth function defined by (2.2) with respect to Z_0. Then we see that $\tilde{\theta}_{Z_0}$ is real-valued and there is an open set $U \subset M$ such that $\tilde{\theta}_{Z_0} > 0$ on U. It follows

$$\tilde{\theta}_{Z_1} + \sum_{i=2}^m \frac{t_{l_k}^i}{t_{l_k}^1}\tilde{\theta}_{Z_i} > \varepsilon > 0, \text{ on } U, \tag{2.8}$$

as l_k are sufficiently large, where $\tilde{\theta}_{Z_i}$, $i = 1, \dots, m$, are all real-valued functions defined by (2.2) with respect to Z_i. Hence we get

$$\begin{aligned} f(Z_{l_k}) &= \int_M \exp\Big(\sum_{i=1}^m t_{l_k}^i \tilde{\theta}_{Z_i}\Big)\omega_g^n \\ &= \int_M \exp\Big(t_{l_k}^1\Big(\tilde{\theta}_{Z_1} + \sum_{i=2}^m \frac{t_{l_k}^i}{t_{l_k}^1}\tilde{\theta}_{Z_i}\Big)\Big)\omega_g^n \\ &\geq \int_U e^{\varepsilon t_{l_k}^1}\omega_g^n \to \infty, \text{ as } l_k \to \infty. \end{aligned}$$

This shows that $f(Z)$ is proper since the sequence $\{Z_l\}$ is arbitrary, and consequently, it has a unique critical point $X \in (\eta_r(M), \mathbb{R})$ such that $F'_X(\cdot) \equiv 0$ on

$(\eta_r(M), \mathbb{R})$. Therefore, there is a unique holomorphic vector field $X \in \eta_r(M)$ with $\mathrm{Im}(X) \in \kappa(M)$ such that $F'_X(\cdot) \equiv 0$ on $\eta_r(M)$. □

Proposition 2.1. *There exists a unique holomorphic vector field $X \in \eta_r(M)$ with $\mathrm{Im}(X) \in \kappa(M)$ such that the holomorphic invariant $F_X(\cdot)$ defined in Section 1 vanishes on $\eta_r(M)$. Moreover, X is either zero or an element of the center of $\eta_r(M)$, and*

$$F_X([u, v]) = 0, \ \forall \ u \in \eta_r(M) \ \text{and} \ v \in \eta(M). \tag{2.9}$$

In particular, $F_X(\cdot)$ is a Lie character on $\eta_r(M)$.

Proof. The proof in the first part of proposition comes from Lemma 2.2 and (2.5) immediately. For the remaining part of the proposition, we consider the following two cases separately.

1). Suppose that the center of $\eta_r(M)$ is zero. Then $\eta_r(M) = [\eta_r(M), \eta_r(M)]$. Since the Futaki invariant $F(\cdot)$ is a character of $\eta(M)$, we get $F(v) = 0$ for any $v \in \eta_r(M)$ ([F1]). By the uniqueness result in the first part, we see that X must be zero, and consequently the holomorphic invariant $F_X(\cdot)$ is just the Futaki invariant. In particular, (2.9) is true. The proposition is completed.

2). Suppose that the center $\eta_c(M)$ of $\eta_r(M)$ is not zero. We consider the functional $f(Z)$ restricted on $\eta_c(M)$. Then as in the proof of Lemma 2.2, one can prove that there exists a unique holomorphic vector field $X' \in \eta_c(M)$ with $\mathrm{Im}(X') \in \kappa(M)$ such that the holomorphic invariant $F_{X'}(\cdot)$ vanishes on $\eta_c(M)$. Now we claim that the invariant $F_{X'}(\cdot)$ satisfies (2.9).

Let $v \in \eta(M)$ and σ_t be one parameter subgroup generated by $\mathrm{Re}(v)$. Then by using the fact $X' \in \eta_c(M)$, for any $\tau \in \mathrm{Aut}_r(M)$, we have

$$\begin{aligned}
&\int_M (\tau \cdot \sigma_t \cdot \tau^{-1})^* (h_g - \theta_{X'}) e^{\theta_{X'}} \omega_g^n \\
&= \int_M (h_g - \theta_{X'})((\tau \cdot \sigma_t \cdot \tau^{-1})^{-1})^* (e^{\theta_{X'}} \omega_g^n) \\
&= \int_M (h_g - \theta_{X'})((\tau \cdot \sigma_t)^{-1})^* (\tau^* (e^{\theta_{X'}} \omega_g^n)) \\
&= \int_M (\tau \cdot \sigma_t)^* (h_g - \theta_{X'}) \tau^* (e^{\theta_{X'}} \omega_g^n) \\
&= \int_M (\sigma_t)^* (\tau^* (h_g - \theta_{X'})) \tau^* (e^{\theta_{X'}} \omega_g^n) \\
&= \int_M (\sigma_t)^* (\tau^* h_g - \theta_{X'}(\tau^* \omega_g)) e^{\theta_{X'}(\tau^* \omega_g)} (\tau^* \omega_g)^n,
\end{aligned} \tag{2.10}$$

where $\theta_{X'}$ is a smooth function defined by (2.2) with respect to X'. Differentiating (2.10) at $t = 0$, and using Proposition 1.1, we get

$$F_{X'}(\mathrm{Ad}_\tau v) = F_{X'}(v). \tag{2.11}$$

Let $u \in \eta_r(M)$ with $\mathrm{Im}(u) \in \kappa(M)$ and $\tau = \tau_s$ be one parameter subgroup generated by $\mathrm{Re}(u)$. Then differentiating (2.11) at $s = 0$, we have

$$F_{X'}([u,v]) = 0, \ \forall\ u \in \eta_r(M) \text{ and } v \in \eta(M). \tag{2.12}$$

The claim is proved.

From (2.12), we see that $F_{X'}(\cdot)$ vanishes on $\eta_r(M)$. Then by the uniqueness, we conclude $X = X'$. Hence, $F_X(\cdot)$ also satisfies (2.9), and in particular, $F_X(\cdot)$ is a Lie character on $\eta_r(M)$. □

In general, the Futaki invariant may not vanish on a compact Kähler manifold with $c_1(M) > 0$, for example, $\mathbb{C}P^n \# \overline{\mathbb{C}P^n}$ is such a manifold ([KS]). By using Proposition 2.1, we can prove

Proposition 2.2. *Let $M_k = \mathbb{C}P^n \# k\overline{\mathbb{C}P^n} (1 \leq k \leq n)$ be the blowing-up of $\mathbb{C}P^n$ at generic k points.* (*Here k points are called generic if such points could not be belonged to a $(k-2)$-dimensional subplane of $\mathbb{C}P^n$.*) *Then there exists a unique holomorphic vector field $X \in \eta_r(M_k)$ with $\mathrm{Im}(X) \in \kappa(M_k)$ such that the corresponding holomorphic invariant $F_X(\cdot)$ vanishes on $\eta(M_k)$.*

Proof. By Proposition 2.1, we see that there exists a unique holomorphic vector field $X \in \eta_r(M_k)$ with $\mathrm{Im}(X) \in \kappa(M_k)$ such that the holomorphic invariant $F_X(\cdot)$ vanishes on $\eta_r(M_k)$. Thus it suffices to prove $F_X(v) = 0$ for any $v \in \eta_u(M_k)$ by the decomposition (2.7).

Let

$$g(M_k) = \{(a_{ij}) \in gl(n+1, \mathbb{C}) | a_{ij} = 0,\ j \neq i,\ j = 1, \ldots, k\},$$

$$g_r(M_k) = \{(a_{ij}) \in g(M_k) |\ a_{ij} = 0, i = 1, \ldots, k, j = k+1, \ldots, n+1\},$$

and

$$g_u(M_k) = \{(a_{ij}) \in g(M_k) |\ a_{ii} = 0, i = 1, \ldots, k \quad a_{ij} = 0, i, j = k+1, \ldots, n+1\},$$

be three Lie subalgebras of $gl(n+1, \mathbb{C})$. Then it is easy to see $\eta(M_k) \cong g(M_k)/\mathbb{C}^*$, $\eta_r(M_k) \cong g_r(M_k)/\mathbb{C}^*$, and $\eta_u(M_k) \cong g_u(M_k)/\mathbb{C}^*$.

Let $A_{ij} = (a_{kl}) \in g_u(M_k)$ such that $a_{kl} = 1$, if $k = i$ and $l = j$, and $a_{kl} = 0$, otherwise. Then $\{A_{ij}\}$ is basis of $g_u(M_k)$. Moreover,

$$[B, A_{ij}] = (\lambda_i - \lambda_j) A_{ij},$$

where $B = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1}) \in g_r(M_k)$ with $\lambda_i \neq \lambda_j$ for any $i \neq j$. Hence we can choose a basis $\{v_i\}_{i=1,\ldots,\Lambda}$ of $\eta_u(M_k)$ and an element $u \in \eta_r(M_k)$ such that

$$[u, v_i] = a_i v_i,\ \forall\ i = 1, \ldots, \Lambda, \tag{2.13}$$

where $a_i \neq 0$ are some complex-valued numbers.

By (2.13) and (2.9) in Proposition 2.1, we have

$$F_X(v_i) = \frac{1}{a_i} F_X([u, v_i]) = 0, \ \forall\ i = 1, \ldots, \Lambda,$$

and consequently,

$$F_X(v) = 0, \ \forall\ v \in \eta_u(M_k).$$

The proposition is proved. □

Problem 2.1. *Let M be a compact Kähler manifold with $c_1(M) > 0$ and $\eta(M) \neq 0$. Does there always exist a unique holomorphic vector field X contained in a reductive Lie subalgebra of $\eta(M)$ such that the corresponding holomorphic invariant $F_X(\cdot)$ vanishes on $\eta(M)$?*

3. Uniqueness of Kähler–Ricci solitons

In this section, we solve completely the uniqueness problem of Kähler–Ricci solitons by using the new holomorphic invariant introduced in Section 1. In our previous paper [TZ1], we prove the uniqueness of Kähler–Ricci soliton for a fixed holomorphic vector field.

First, we shows that the new holomorphic invariant provides an obstruction to the existence of Kähler–Ricci solitons.

Let g be a Kähler–Ricci soliton with respect to a holomorphic vector field X on M. Then by definition, the Kähler form ω_g satisfies the following

$$\mathrm{Ric}(\omega_g) - \omega_g = L_X \omega_g, \tag{3.1}$$

where L_X denotes the Lie derivative along X.

Proposition 3.1. *If M admits a Kähler–Ricci soliton ω_g with respect to a holomorphic vector field X. Then the corresponding holomorphic invariant $F_X(\cdot)$ defined by* (1.3) *in Section 1 vanishes, i.e.,*

$$F_X(v) = 0, \ \forall\ v \in \eta(M). \tag{3.2}$$

Proof. By Proposition 1.1, it suffices to prove that $F_X(\cdot)$ vanishes under the choice of the Kähler–Ricci soliton g. Let h_g be a smooth real-valued function and $\theta_X(g)$ a smooth complex-valued function defined by (1.1) and (1.2) in Section 1, respectively. Since

$$L_X \omega_g = \partial i_X(\omega_g),$$

then

$$L_X\omega_g = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\theta_X(g). \tag{3.3}$$

By the maximal principle together with (3.1), we get

$$h_g - \theta_X(g) = \text{const.}. \tag{3.4}$$

Now (3.2) follows from the definition of the integral in (1.3) immediately. □

From (3.1), we see that if g is a Kähler–Ricci soliton with respect to a holomorphic vector field X, then the (1,1)-form $L_X(\omega_g)$ is real-valued , i.e., $L_{\mathrm{Im}(X)}\omega_g = 0$, where $\mathrm{Im}(X)$ denotes the imaginary part of X. Therefore, $\mathrm{Im}(X)$ generates a one-parameter family of isometries of (M, ω_g). Let K be a maximal compact subgroup of the identity component $\mathrm{Aut}^\circ(M)$ of $\mathrm{Aut}(M)$ containing such a one-parameter family of isometries and $\mathrm{Aut}_r(M) \subset \mathrm{Aut}^\circ(M)$ be the complexification of K. Then $\mathrm{Aut}_r(M)$ is a reductive algebraic subgroup of $\mathrm{Aut}(M)$ with a reductive Lie subalgebra $\eta_r(M)$ of $\eta(M)$. Clearly, $X \in \eta_r(M)$ and $\mathrm{Im}(X) \in \kappa(M)$, where $\kappa(M)$ is the Lie algebra of K.

In [TZ1], we proved the following uniqueness theorem of Kähler–Ricci solitons for a fixed holomorphic vector field by solving certain complex Monge–Ampère equations.

Theorem 3.1 ([TZ1]). *Let $X \in \eta_r(M)$. Then the Kähler–Ricci soliton on M with respect to X is unique modulo $\mathrm{Aut}_r(M)$. Precisely, if g and g' are two Kähler–Ricci solitons with respect to the holomorphic vector field X, then there exists an element $\sigma \in \mathrm{Aut}_r(M)$ such that*

$$\omega_g = \sigma^*\omega_{g'}.$$

Theorem 3.2 (Uniqueness Theorem). *There is at most one Kähler–Ricci soliton on M modulo $\mathrm{Aut}^\circ(M)$, more precisely, if g and g' are two Kähler–Ricci solitons on M with respect to two holomorphic vector fields X and X', respectively, then there exists a holomorphic automorphism $\sigma \in \mathrm{Aut}^\circ(M)$ such that*

$$\omega_g = \sigma^*\omega_{g'} \text{ and } X = (\sigma^{-1})_*(X').$$

Proof. Let g and g' be two Kähler–Ricci solitons with respect to two holomorphic vector fields X and X' on M, respectively. Then both $\mathrm{Im}(X)$ and $\mathrm{Im}(X')$ generate a one-parameter family of isometries of (M, ω_g) and $(M, \omega_{g'})$. Let K and K' be two maximal compact subgroup of the identity component $\mathrm{Aut}^\circ(M)$ of $\mathrm{Aut}(M)$ containing each one-parameter family of isometries, respectively. Since K' is conjugate to K ([Iw]), there exists a holomorphic automorphism $\tau_1 \in \mathrm{Aut}^\circ(M)$ such that $(\tau_1^{-1})_*(X') = \mathrm{Ad}_{\tau_1^{-1}}(X') \in \eta_r(M)$, where $\mathrm{Ad}_{\tau_1^{-1}}$ is the adjoint action on $\eta(M)$ induced by τ^{-1}. Clearly, $\tau_1^*\omega_{g'}$ is still a Kähler–Ricci soliton with respect to $Y = (\tau_1^{-1})_*(X')$ and $\mathrm{Im}(Y)$ is contained in $\kappa(M)$. Hence by Proposition 3.1, we see that both $F_X(\cdot)$ and $F_Y(\cdot)$ vanish on $\eta(M)$. By using the uniqueness result about the holomorphic vector field in Proposition 2.1, we prove

$$X = Y = \mathrm{Ad}_{\tau_1^{-1}}(X'). \tag{3.5}$$

On the other hand, by Theorem 3.1, we see that there exists a holomorphic automorphism $\tau_2 \in \mathrm{Aut}_r(M)$ such that

$$\omega_g = (\tau_1 \cdot \tau_2)^* \omega_{g'}.$$

Since $\mathrm{Ad}_{\tau_1^{-1}}(Y)$ is contained in the center of $\eta_r(M)$ by Proposition 2.1 (see also Lemma 2.2 in [TZ1]), then by (3.5), we also have

$$X = \mathrm{Ad}_{\tau_2^{-1}}(\mathrm{Ad}_{\tau_1^{-1}}(X')) = \mathrm{Ad}_{(\tau_1\tau_2)^{-1}}(X') = (\tau_1\tau_2)_*^{-1}(X').$$

Let $\sigma = \tau_1\tau_2$. Then the theorem is proved. □

4. Remark on the Koiso's examples

In this section, we discuss the existence and uniqueness of Kähler–Ricci solitons on a class of compactifications of C^*-bundles over compact Kähler–Einstein manifolds in terms of our new holomorphic invariant. These manifolds were first studied by E. Calabi ([Ca]) for extremal metrics in 1982 and by Koiso and Sanake for Kähler–Einstein metrics in 1986 ([KS]), and lately by Koiso for Kähler–Ricci solitons in 1990 ([Ko]). We first recall some notations, which can be found in either [KS] or [Ko].

Let $p : L \to M$ be a holomorphic line bundle over a compact Kähler–Einstein manifold M with positive first Chern class $c_1(M)$ and a Hermitian metric h on L. Denote by $\mathring{L}$ the open subset $L \setminus \{0 - \text{section}\}$. Let $r \in C^\infty(\mathring{L})$ be defined by $r(l) = \log \|l\|_h (l \in \mathring{L})$, where $\|.\|_h$ is the norm induced by h.

Let $t(r)$ be a smooth monotone increasing function with respect to r so that $\min t < 0 < \max t$. For any one-parameter family of Riemannian metrics g_t on M, we consider a Riemannian metric on $\mathring{L}$ of the form

$$\tilde{g} = dt^2 + (dt \cdot \tilde{J})^2 + p^* g_t, \tag{4.1}$$

where $\tilde{J}$ is the standard almost complex structure of L.

Let H be the real vector field on $\overset{\circ}{L}$ corresponding to R^*-action on $\overset{\circ}{L}$. Define

$$u(t)^2 = \tilde{g}(H,H) \quad \text{and} \quad U(t) = \int_0^t u(s)ds.$$

Then by a result in [KS], one sees that $\tilde{g}$ is a Kähler metric if only if g_0 is Kähler and $g_t = g_0 - U(t)B$, where B is the curvature of L with respect to h.

Throughout this section, we assume that

(1) g_0 is a Kähler–Einstein metric of M so that its Kähler form $\omega_{g_0} \in c_1(M)$ and the eigenvalues of B with respect to g_0 are constant on M;

(2) $\tilde{L}$ is a compactification of $\overset{\circ}{L}$ and $\tilde{g}$ denotes the restriction of a Kähler metric $\tilde{g}$ (still denoted by the same symbol) of $\tilde{L}$ to $\overset{\circ}{L}$;

(3) the Kähler form of $\tilde{g}$ of $\tilde{L}$ represents the first Chern class of $\tilde{L}$.

Lemma 4.1 ([KS]). *Let $X = H - \sqrt{-1}\tilde{J}H$. Then there exists a Kähler–Ricci soliton of the form* (4.1) *with respect to the holomorphic vector field aX on $\tilde{L}$ if and only if*

$$f(a) = \int_{\min\, U}^{\max U} e^{2aU} Q(U) U dU = 0, \tag{4.2}$$

where

$$Q(U) = q(t) = det(I - U(t) g_0^{-1} B) > 0. \tag{4.3}$$

Lemma 4.2. *Let $X = H - \sqrt{-1}\tilde{J}H$. Then there exists a Kähler–Ricci soliton of the form* (4.1) *with respect to the holomorphic vector field aX on $\tilde{L}$ if and only if the corresponding holomorphic invariant $F_{aX}(\cdot)$ defined by* (1.3) *vanishes, i.e.,*

$$F_{aX}(v) = 0,\ \forall\ v \in \eta(M). \tag{4.4}$$

Proof. By Proposition 3.1 and Lemma 4.1, it suffices to prove that (4.2) is equivalent to

$$F_{aX}(X) = 0.$$

Let $\tilde{g}$ be a Kähler metric of the form (4.1) and $\theta_X = \theta_X(\tilde{g})$ a complex-valued function defined by (1.2) in Section 1 associated to the metric $\tilde{g}$. Since

$$i_X \omega_{\tilde{g}}(\overline{X}) = \frac{\sqrt{-1}}{2\pi} \tilde{g}_{0\overline{0}} dz^0 \wedge d\overline{z}^0 (X, \overline{X}) = \frac{\sqrt{-1}}{2\pi}(2u^2),$$

we have

$$X = u(t)\frac{d}{dt} - \sqrt{-1}u(t)\tilde{J}\frac{d}{dt},$$

and

$$H(\theta_X) = X(\theta_X) = \overline{X(\theta_X)} = \overline{\partial}\theta_X(\overline{X}) = 2u^2. \tag{4.5}$$

It follows

$$\theta_X = 2U + c \tag{4.6}$$

for some constant c. In particular, θ_X is a real-valued function.

Let $h = h_{\tilde{g}}$ be the smooth function defined by (1.1) in Section 1 associated to $\tilde{g}$. Then, by a result in [KS], we have

$$\frac{d}{dU}\phi + \frac{\phi}{Q}\frac{d}{dU}Q + 2U + H(h) = 0, \tag{4.7}$$

where $\phi = \phi(U) = u^2$, $Q = Q(U)$ is defined in (4.3).

Since $\theta_{aX} = a\theta_X + c_a$ for some constants c_a, by using (4.5), (4.6) and (4.7), one can compute

$$\begin{aligned}
&F_{aX}(X) = e^{c_a}\int_{\hat{L}} X(h - a\theta_X)e^{a\theta_X}\omega_{\tilde{g}}^n \\
&= \mathrm{Vol}(M, g_0)e^{c_a}\int_{\min t}^{\max t} H(h - a\theta_X)e^{a\theta_X}uq dt \\
&= \mathrm{Vol}(M, g_0)e^{c_a}\int_{\min U}^{\max U} H(h - a\theta_X)e^{a\theta_X}QdU \\
&= -\mathrm{Vol}(M, g_0)e^{c_a}\int_{\min U}^{\max U} \Big(\frac{d}{dU}\phi + \frac{\phi}{Q}\frac{d}{dU}Q + 2U + aH(\theta_X)\Big)e^{a\theta_X}QdU \qquad (4.8)\\
&= -\mathrm{Vol}(M, g_0)e^{c_a}\int_{\min U}^{\max U} \Big(\frac{d}{dU}(\phi Q) + 2a\phi Q + 2UQ\Big)e^{a\theta_X}dU \\
&= -\mathrm{Vol}(M, g_0)e^{c_a}\int_{\min U}^{\max U} \Big(\frac{d}{dU}(e^{2aU}\phi Q) + 2e^{2aU}UQ\Big)e^{a\theta_X - 2aU}dU \\
&= -2\mathrm{Vol}(M, g_0)e^{ac + c_a}\int_{\min U}^{\max U} e^{2aU}UQdU.
\end{aligned}$$

This shows that $F_{aX}(X) = 0$ if and only if (4.2) is true. Lemma 4.2 is proved. □

Remark 4.1. From (4.2), we see

$$\frac{df(a)}{da} = 2\int_{\min\ U}^{\max\ U} e^{2aU}Q(U)U^2dU > 0.$$

This shows that there exists only one a_0 such that $f(a_0) = 0$. By Proposition 3.1 and Lemma 4.2, $F_{aX}(.) \equiv 0$ if and only if $a = a_0$. Furthermore, there exists a Kähler–Einstein metric on $\tilde{L}$ if and only if $a_0 = 0$, i.e., the Futaki invariant vanishes.

Combining Lemma 4.2, Proposition 3.1 and Proposition 2.1, we prove

Proposition 4.1. *Let $\tilde{L}$ be a compactification of C^*-bundle satisfying the assumptions* (1), (2) *and* (3). *Then there exists a Kähler–Ricci soliton with respect to a holomorphic vector field X on $\tilde{L}$ if and only if X is contained in a reductive Lie subalgebra of $\eta(\tilde{L})$ with $\mathrm{Im}(X)$ generating a compact one-parameter subgroup of $\mathrm{Aut}(\tilde{L})$ and the corresponding holomorphic invariant $F_X(\cdot)$ vanishes.*

Example 4.1. $\mathbb{C}P^{n+1}\#\overline{\mathbb{C}P^{n+1}}$.

Let H be the hyperplane line bundle over $\mathbb{C}P^n$ and H^{-1} be its dual line bundle. Then $L = H^{-1} \oplus I$ is a two-dimensional holomorphic vector bundle and $\tilde{L} = P(L)$ is a $\mathbb{C}P^1$-projective bundle over $\mathbb{C}P^n$. Let S_0 and S_∞ be $\{0-\text{section}\}$ and $\{\infty{-}\text{section}\}$ of H^{-1} respectively. Then $\overset{\circ}{H}{}^{-1} = H^{-1} \setminus S_0 \cong \mathbb{C}^{n+1} \setminus 0$ and $\tilde{L} = \overset{\circ}{H}{}^{-1} \cup S_0 \cup S_\infty$, and consequently $\tilde{L} \cong \mathbb{C}P^{n+1}\#\overline{\mathbb{C}P^{n+1}}$. Hence by Lemma 4.2, Remark 4.1 and Theorem 3.2, there exists a unique Kähler–Ricci soliton metric (modulo the holomorphic transformations group) on $\mathbb{C}P^{n+1}\#\overline{\mathbb{C}P^{n+1}}$ with respect to some holomorphic vector field induced by the fiber of $P(L)$.

Example 4.2. $\tilde{L} = P(L \oplus I)$, where $L = p_1^* H^{k_1} \otimes p_2^* H^{k_2}$.

Let n_1 and n_2 be two positive integers. Let H_i be the hyperplane line bundle over $\mathbb{C}P^{n_i}$, $i = 1, 2$. Denote by $p_i : \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \to \mathbb{C}P^{n_i}$ the projection to i-th factor. Let L be the holomorphic line bundle over $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$ given by

$$L = p_1^* H^{k_1} \otimes p_2^* H^{k_2},$$

where $|k_1| \leq n_1$ and $|k_2| \leq n_2$ are integers. Put $\tilde{L}$ the total space of projective bundle $P(L \oplus I)$. Then $c_1(\tilde{L}) > 0$ and $\eta(\tilde{L}) \cong gl(n_1+1, C) + gl(n_2+1, C) + c$ (cf. [F2]), where c is generated by the holomorphic vector field $X = H - \sqrt{-1}\tilde{J}H$ as before. Hence by Lemma 4.2, Remark 4.1 and Theorem 3.2, there exists a unique Kähler–Ricci soliton metric with respect to aX for some a on $\tilde{L}$ modulo $\mathrm{Aut}^\circ(\tilde{L})$.

5. Two functionals associated to the holomorphic invariant

In this section, we introduce two functionals which integrate the new holomorphic invariant defined in Section 1, then by using the arguments in the proof of uniqueness theorem in [TZ1], we prove both functionals are bounded from below if the underlying manifold admits a Kähler–Ricci soliton.

Let g be a K-invariant Kähler metric and X be a holomorphic vector field of M. Let h_g and $\theta_X = \theta_X(\omega_g)$ are two smooth real-valued functions defined by (1.1) and (1.2) in Section 1, respectively. Set

$$\mathcal{M}_X(\omega_g) = \{\phi \in C^\infty(M) |\ \omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi > 0,\ \mathrm{Im}(X)(\phi) = 0\}.$$

We recall the following functional on $\mathcal{M}_X(\omega_g)$ from [Zh] and [TZ1],

$$\begin{aligned} F_{\omega_g}(\phi) &= J(\phi) - \frac{1}{V}\int_M \phi e^{\theta_X}\omega_g^n - \log\left(\frac{1}{V}\int_M e^{h_g-\phi}\omega_g^n\right) \\ &= -\frac{1}{V}\int_0^1\int_M \dot{\phi}_t e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n \wedge dt - \log\left(\frac{1}{V}\int_M e^{h_g-\phi}\omega_g^n\right), \end{aligned} \tag{5.1}$$

where $J_{\omega_g}(\phi) = \frac{1}{V}\int_0^1\int_M \dot{\phi}_t(e^{\theta_X}\omega_g^n - e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n) \wedge dt$ and ϕ_t $(0 \le t \le 1)$ is a path connecting 0 to ϕ in $\mathcal{M}_X(\omega_g)$. Note that $F_{\omega_g}(\phi)$ is independent of the choice of path ϕ_t. Moreover, one can check that for any two ϕ and ψ in $\mathcal{M}_X(\omega_g)$, the following cocycle condition is satisfied,

$$F_{\omega_g}(\psi) = F_{\omega_g}(\phi) + F_{\omega_\phi}(\psi - \phi), \tag{5.2}$$

where

$$\begin{aligned} &F_{\omega_\phi}(\psi - \phi) \\ &= -\frac{1}{V}\int_0^1\int_M \dot{\phi}_t e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n \wedge dt - \log\left(\frac{1}{V}\int_M e^{h_{\omega_\phi}-(\psi-\phi)}\omega_\phi^n\right). \end{aligned}$$

Here ϕ_t $(0 \le t \le 1)$ is a path connecting ϕ to ψ in $\mathcal{M}_X(\omega_g)$ and h_{ω_ϕ} is a smooth real-valued function defined by (1.1) in Section 1 associated to the Kähler form ω_ϕ.

The next functional can be regarded as a generalization of Mabuchi's K-energy, which integrates the holomorphic invariant $F_X(\cdot)$ ([Ma]),

$$\begin{aligned} &\mu_{\omega_g}(\phi) \\ &= -\frac{1}{V}\int_0^1\int_M \dot{\phi}_t[\mathrm{R}(\phi_t) - n - \mathrm{tr}_{\omega_{\phi_t}}(\nabla_{\omega_{\phi_t}} X) \\ &+ X(h_{\omega_{\phi_t}} - \theta_X(\omega_{\phi_t}))]e^{\theta_X(\omega_{\phi_t})}\omega_{\phi_t}^n \wedge dt \\ &= \frac{\sqrt{-1}}{2\pi V}\int_0^1\int_M \partial(h_{\omega_{\phi_t}} - \theta_X(\omega_{\phi_t})) \wedge \overline{\partial}\phi_t e^{\theta_X(\omega_{\phi_t})}\omega_{\phi_t}^{n-1} \wedge dt, \end{aligned} \tag{5.3}$$

where $\theta_X(\omega_{\phi_t}) = \theta_X + X(\phi_t)$ and $\mathrm{R}(\phi_t)$ is the scalar curvature of ω_{ϕ_t} and ϕ_t $(0 \le t \le 1)$ is a path connecting 0 to ϕ in $\mathcal{M}_X(\omega_g)$. One can show that $\mu_{\omega_g}(\phi)$ is well-defined, in fact, it can be represented by $F_{\omega_g}(\phi)$.

Lemma 5.1. *We have the following identity*

$$\begin{aligned}&\mu_{\omega_g}(\phi)\\&= F_{\omega_g}(\phi) - \frac{1}{V}\int_M (h_{\omega_\phi} - \theta_X - X(\phi))e^{\theta_X + X(\phi)}\omega_\phi^n + \frac{1}{V}\int_M (h_g - \theta_X)e^{\theta_X}\omega_g^n,\end{aligned}$$

where h_{ω_ϕ} is normalized by

$$\int_M e^{h_{\omega_\phi}}\omega_\phi^n = V.$$

It follows that

$$\mu_{\omega_g}(\phi) \geq F_{\omega_g}(\phi) - C.$$

Proof. The argument is originally due to [DT] (see also [T2]). We can rewrite (5.3) as follows,

$$\begin{aligned}&\mu_{\omega_g}(\phi)\\&= -\frac{n}{V}\int_0^1\int_M \dot{\phi}_t\Big[\mathrm{Ric}(\omega_{\phi_t}) - \mathrm{Ric}(\omega_g) - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}X(\phi_t)\\&+ \frac{\sqrt{-1}}{2\pi}\partial(h_{\omega_{\phi_t}} - X(\phi_t) - h_g + \phi_t)\wedge\overline{\partial}\theta_X(\omega_{\phi_t})\Big]e^{\theta_X(\omega_{\phi_t})}\omega_{\phi_t}^{n-1}\wedge dt\\&- \frac{n}{V}\int_0^1\int_M \dot{\phi}_t\Big[\mathrm{Ric}(\omega_g) - \omega_g - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\theta_X\\&+ \frac{\sqrt{-1}}{2\pi}\partial(h_g - \theta_X)\wedge\overline{\partial}\theta_X(\omega_{\phi_t})\Big]e^{\theta_X(\omega_{\phi_t})}\omega_{\phi_t}^{n-1}\wedge dt\\&- \frac{n}{V}\int_0^1\int_M \dot{\phi}_t\Big[\omega_g - \omega_{\phi_t} - \frac{\sqrt{-1}}{2\pi}\partial\phi_t\wedge\overline{\partial}\theta_X(\omega_{\phi_t})\Big]e^{\theta_X(\omega_{\phi_t})}\omega_{\phi_t}^{n-1}\wedge dt.\end{aligned}$$

Note that

$$h_{\omega_{\phi_t}} - h_g = -\log\Big(\frac{\omega_{\phi_t}^n}{\omega_g^n}\Big) - \phi_t + \text{ const.}. \tag{5.4}$$

Then integrating by parts, we can get

$$
\begin{aligned}
&\mu_{\omega_g}(\phi)\\
&= -\frac{1}{V}\int_0^1\int_M \log\Big(\frac{e^{\theta_X}\omega_g^n}{e^{\theta_X+X(\phi)}\omega_{\phi_t}^n}\Big)\frac{d}{dt}(e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n)\\
&- \frac{1}{V}\int_0^1\int_M (h_g-\theta_X)\frac{d}{dt}(e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n)\\
&- \frac{1}{V}\int_0^1\int_M \phi_t\frac{d}{dt}(e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n)\\
&= \frac{1}{V}\int_M \log\Big(\frac{e^{\theta_X+X(\phi)}\omega_\phi^n}{e^{\theta_X}\omega_g^n}\Big)e^{\theta_X+X(\phi)}\omega_\phi^n - (I(\phi)-J(\phi))\\
&+ \frac{1}{V}\int_M (h_g-\theta_X)(e^{\theta_X}\omega_g^n - e^{\theta_X+X(\phi)}\omega_\phi^n).
\end{aligned}
$$

Using the fact

$$h_{\omega_\phi} - h_g = -\log\Big(\frac{\omega_\phi^n}{\omega_g^n}\Big) - \phi - \log\Big(\frac{1}{V}\int_M e^{h_g-\phi}\omega_g^n\Big),$$

we derive

$$
\begin{aligned}
&\mu_{\omega_g}(\phi)\\
&= -\frac{1}{V}\int_M \phi e^{\theta_X+X(\phi)}\omega_\phi^n - (I(\phi)-J(\phi)) - \log\Big(\frac{1}{V}\int_M e^{h_g-\phi}\omega_g^n\Big)\\
&+ \frac{1}{V}\int_M (h_g-\theta_X)e^{\theta_X}\omega_g^n - \frac{1}{V}\int_M (h_{\omega_\phi}-\theta_X-X(\phi))e^{\theta_X+X(\phi)}\omega_\phi^n\\
&= F_{\omega_g}(\phi) + \frac{1}{V}\int_M (h_g-\theta_X)e^{\theta_X}\omega_g^n - \frac{1}{V}\int_M (h_{\omega_\phi}-\theta_X-X(\phi))e^{\theta_X+X(\phi)}\omega_\phi^n.
\end{aligned}
\tag{5.5}
$$

On the other hand,

$$\int_M e^{h_{\omega_\phi}}\omega_\phi^n = V,$$

and $e^{\theta_X+X(\phi)}$ is uniformly bounded (cf. [TZ1]), we have

$$\frac{1}{V}\int_M e^{h_{\omega_\phi}-\theta_X-X(\phi)}\omega_\phi^n \le C'.$$

Then it follows from the concavity of logarithmic function,

$$\frac{1}{V}\int_M (h_{\omega_\phi}-\theta_X-X(\phi))\omega_\phi^n \le \ln C'. \tag{5.6}$$

Inserting (5.6) into (5.5), we get

$$\mu_{\omega_g}(\phi) \ge F_{\omega_g}(\phi) - C.$$

The lemma is proved. □

In the following, we assume that there exists a Kähler–Ricci soliton ω_{KS} with respect to X on M. Let $\sigma \in \mathrm{Aut}_r(M)$. Then $\sigma^*\omega_{KS}$ is still a Kähler–Ricci soliton with respect to X on M. Since there is a path in $\mathrm{Aut}_r(M)$ from the identity to σ, then by the definition (5.3) (the background metric ω_g is replaced by ω_{KS}), one can show

$$\mu_{\omega_{KS}}(\phi_\sigma) \equiv 0, \ \forall\ \sigma \in \mathrm{Aut}_r(M),$$

where ϕ_σ is defined by $\sigma^*\omega_{KS} = \omega_{KS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi_\sigma$. It follows from Lemma 5.1,

$$F_{\omega_{KS}}(\phi_\sigma) \equiv 0, \ \forall \sigma \in \mathrm{Aut}_r(M). \tag{5.7}$$

The following theorem is our main result in this section.

Theorem 5.1. *Let M be a compact complex manifold which admits a Kähler Ricci soliton ω_{KS} with respect to X. Then both functionals $F_{\omega_g}(\phi)$ and $\mu_{\omega_g}(\phi)$ are bounded from below on $\mathcal{M}_X(\omega_g)$.*

Theorem 5.1 generalizes a result in [DT] in case of a compact Kähler–Einstein manifold with positive scalar curvature. To prove it, we shall introduce certain complex Monge–Ampère equations. In [TZ1], we considered the following complex Monge–Ampère equations with parameter $t \in [0, 1]$:

$$\left\{\begin{array}{l} \det(g_{i\overline{j}} + \phi_{i\overline{j}}) = \det(g_{i\overline{j}})\exp\{h_g - \theta_X - X(\phi) - t\phi\} \\ (g_{i\overline{j}} + \phi_{i\overline{j}}) > 0. \end{array}\right. \tag*{$(5.8)_t$}$$

One can check that $\omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi$ is a Kähler–Ricci soliton, if and only if $\phi + c$ is a solution of $(5.8)_t$ at $t = 1$, where c is some constant.

Lemma 5.2. *Let ϕ_s be solutions of $(5.8)_s$ for $s \le t \le 1$ and*

$$\hat{F}_{\omega_g}(\phi_t) = J_{\omega_g}(\phi_t) - \frac{1}{V}\int_M \phi_t e^{\theta_X}\omega_g^n.$$

Then

$$\hat{F}_{\omega_g}(\phi_t) = -\frac{1}{t}\int_0^t (I_{\omega_g}(\phi_s) - J_{\omega_g}(\phi_s))ds < 0,$$

where

$$I_{\omega_g}(\phi_t) = \frac{1}{V}\int_M \phi_t(e^{\theta_X}\omega_g^n - e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n).$$

Proof. First from the proof of Lemma 3.2 in [TZ1], we can obtain

$$\begin{aligned}\frac{d}{dt}(I_{\omega_g}(\phi_t) - J_{\omega_g}(\phi_t)) &= -\frac{1}{V}\int_M \phi_t \frac{d}{dt}(e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n) \\ &= -\frac{1}{V}\int_M \phi_t(\triangle'\dot{\phi}_t + X(\dot{\phi}_t))e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n.\end{aligned} \tag{5.9}$$

On the other hand, by differentiating $(5.8)_t$ on t, we have

$$\triangle' \dot{\phi}_t + X(\dot{\phi}_t) = -(t\dot{\phi}_t + \phi_t). \tag{5.10}$$

Inserting (5.10) into (5.9) and using $(5.8)_t$, we get

$$\begin{aligned}
&\frac{d}{dt}(I_{\omega_g}(\phi_t) - J_{\omega_g}(\phi_t)) \\
&= \frac{1}{V}\int_M \phi_t(t\dot{\phi}_t + \phi_t)e^{h_g - t\phi_t}\omega_g^n \\
&= \frac{1}{V}\frac{d}{dt}\left(\int_M (-\phi_t)e^{h_g - t\phi_t}\omega_g^n\right) + \frac{1}{V}\int_M \dot{\phi}_t e^{h_g - t\phi_t}\omega_g^n \\
&= \frac{1}{tV}\frac{d}{dt}\left(\int_M t(-\phi_t)e^{h_g - t\phi_t}\omega_g^n\right) \\
&= \frac{1}{tV}\frac{d}{dt}\left(\int_M t(-\phi_t)e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n\right).
\end{aligned}$$

It follows

$$\begin{aligned}
&\frac{d}{dt}(t(I_{\omega_g}(\phi_t) - J_{\omega_g}(\phi_t))) - (I_{\omega_g}(\phi_t) - J_{\omega_g}(\phi_t)) \\
&= \frac{1}{V}\frac{d}{dt}\left(\int_M t(-\phi_t)e^{\theta_X(\phi_t)}\omega_{\phi_t}^n\right).
\end{aligned}$$

Integrating the above inequality from 0 to t, and then dividing t on both sides, we get

$$\begin{aligned}
\hat{F}_{\omega_g}(\phi_t) &= -\frac{1}{V}\int_M \phi e^{\theta_X + X(\phi_t)}\omega_t^n - (I_{\omega_g}(\phi_t) - J_{\omega_g}(\phi_t)) \\
&= -\frac{1}{t}\int_0^t (I_{\omega_g}(\phi_s) - J_{\omega_g}(\phi_s)ds.
\end{aligned}$$

Since

$$I_{\omega_g}(\phi) - J_{\omega_g}(\phi) > 0$$

for any $\phi \in \mathcal{M}_X(\omega_g)$ (cf. [TZ1]), we have $\hat{F}_{\omega_g}(\phi_t) < 0$. □

Proof of Theorem 5.1. By Lemma 5.1, it suffices to prove that $F_{\omega_g}(\phi)$ is bounded from below. Let $\phi_0 \in \mathcal{M}_X(\omega_g)$ such that $\omega_g = \omega_{KS} - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi_0$. In [TZ1], it was proved that there is an element $\sigma \in \mathrm{Aut}_r(M)$ such that

$$\begin{aligned}
\omega_{KS}' = \sigma^*\omega_{KS} &= \omega_\phi + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\widetilde{\psi} \\
&= \omega_{KS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\widetilde{\psi} + \phi - \phi_0)
\end{aligned}$$

and the following complex Monge–Ampère equations

$$\begin{cases} \det(g_{i\bar{j}} + \psi_{i\bar{j}}) = \det(g_{i\bar{j}})\exp\{h_{\omega_\phi} - \theta_X(\omega_\phi) - X(\psi) - t\psi\} \\ (g_{i\bar{j}} + \psi_{i\bar{j}}) > 0 \end{cases} \tag{5.11$_t$}$$

are solvable for any $t \in [0,1]$, while $\psi_1 = \widetilde{\psi} + \text{const.}$ is a solution of $(5.11)_t$ on $t = 1$, where the initial Kähler form ω_g is replaced by ω_ϕ and $\theta_X(\omega_\phi)$ is a smooth real-valued function defined by (1.2) in Section 1 associated to the Kähler form ω_ϕ.

Let ψ_t be solutions of $(5.11)_t$. Since

$$\int_M e^{h_{\omega_\phi} - \psi_1}\omega_\phi^n = \int_M e^{\theta_X(\omega_\phi) + X(\omega_{\psi_1})}\omega_{\psi_1}^n = V,$$

by Lemma 5.2, we have

$$\begin{aligned} F_{\omega'_{KS}}(-\widetilde{\psi}) &= -F_{\omega_\phi}(\psi_1) = -\hat{F}_{\omega_\phi}(\psi_1) \\ &= \int_0^1 (I(\psi_t) - J(\psi_t))dt > 0. \end{aligned}$$

Hence by using the cocycle condition (5.3) and (5.7), we prove

$$\begin{aligned} F_{\omega_g}(\phi) &= F_{\omega_{KS}}(\phi - \phi_0) + F_{\omega_g}(\phi_0) \\ &= F_{\omega_{KS}}(\widetilde{\psi} + \phi - \phi_0) + F_{\omega'_{KS}}(-\widetilde{\psi}) + F_{\omega_g}(\phi_0) \\ &\geq F_{\omega_g}(\phi_0) = C. \end{aligned}$$

□

As a consequence of Theorem 5.1, we obtain the following Moser–Trudinger type inequality on a compact complex manifold with admitting a Kähler–Ricci soliton.

Corollary 5.1. *Let M be a compact complex manifold which admits a Kähler–Ricci soliton with respect to X. Then there is a uniform constant C such that for any $\phi \in \mathcal{M}_X(\omega_g)$,*

$$\int_M e^{-\phi}\omega_g^n \leq C\exp\Big(J_{\omega_g}(\phi) - \frac{1}{V}\int_M \phi\omega_g^n\Big). \tag{5.12}$$

Lemma 5.3. *For any $\phi \in \mathcal{M}_X(\omega_g)$, there is a uniform constant C such that*

$$\sup_M \phi \leq \frac{1}{V}\int_M \phi e^{\theta_X}\omega_g^n + C. \tag{5.13}$$

Proof. By Yau's Theorem for Calabi's conjecture ([Ya]), we see that there is a Kähler potential function ψ such that ψ solves the complex Monge–Ampère equation,

$$\begin{cases} \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi\right)^n = \omega_\psi^n = e^{\theta_X}\omega_g^n, \\ \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi > 0. \end{cases}$$

Since

$$\triangle_{\omega_\psi}(\phi - \psi) \geq -n,$$

using the Green formula associated to the Kähler form ω_ψ, we get

$$\begin{aligned} \sup_M(\phi - \psi) &\leq \frac{1}{V}\int_M (\phi - \psi)\omega_\psi^n + C \\ &= \frac{1}{V}\int_M (\phi - \psi)e^{\theta_X}\omega_g^n + C \\ &\leq \frac{1}{V}\int_M \phi e^{\theta_X}\omega_g^n + C. \end{aligned}$$

Hence (5.13) follows from the above inequality directly. □

Proof of Corollary 5.1. By Theorem 5.1, we have

$$\int_M e^{-\phi}e^{\theta_X}\omega_g^n \leq C\exp\left(J_{\omega_g}(\phi) - \frac{1}{V}\int_M \phi e^{\theta_X}\omega_g^n\right),$$

for some uniform constant C. Then (5.12) follows from Lemma 5.3 immediately. □

Remark 5.1. In a later paper [CTZ], we will prove a stronger version of Moser–Trudinger type inequality on a compact complex manifold with admitting a Kähler–Ricci soliton. Such a stronger inequality is a sufficient and necessary condition for the existence of Kähler–Ricci solitons.

Appendix. Another proof of Theorem 3.2

In [TZ2], we gave a sketch of proof of Theorem 3.2. The original proof is different to one appeared in Section 3 in this paper. For completeness, we give that proof in details in this appendix. The proof is independent of Proposition 2.1.

The following lemma can be found in [TZ1] (see also Theorem 2.4.3 in [F2], Corollary 2.148 in [Be]).

Lemma A. *Let g be a Kähler–Ricci soliton with respect to a holomorphic vector field $X \in \eta_r(M)$ on M^n. Let L be a linear elliptic operator on $C^\infty(M, \mathbb{C})$ defined by*

$$L(\psi) = \triangle_g\psi + X(\psi) + \psi, \;\; \psi \in C^\infty(M, \mathbb{C}).$$

Then the correspondence

$$\overline{\partial} : \mathrm{Ker}(L) \to \eta(M),$$
$$\psi \mapsto \sum_{i,j=1}^{n} g^{i\bar{j}} \psi_{\bar{j}} \frac{\partial}{\partial z^i},$$

is one-to-one.

Another proof of Theorem 3.2. Let g and g' be two Kähler–Ricci solitons with respect to two holomorphic vector fields X and X' on M, respectively. Then by a result of Iwasawa ([Iw]), we can find a holomorphic automorphism $\sigma \in \mathrm{Aut}^\circ(M)$ such that $\mathrm{Ad}_{\sigma^{-1}}(X') \in \eta_r(M)$. Clearly, $\sigma^* \omega_{g'}$ is a Kähler–Ricci soliton with respect to $(\sigma^{-1})_*(X') = \mathrm{Ad}_{\sigma^{-1}}(X')$. Hence, by Theorem 3.1, we suffice to prove that

$$X = (\sigma^{-1})_*(X').$$

For simplicity, we may assume that $\sigma = \mathrm{Id}$ and $X, X' \in \eta_r(M)$. In particular, $\mathrm{Im}(X') \in \kappa(M)$.

On the contrary, we assume that $X' \neq X$. Let $\theta_X = \theta_X(g)$ and $\theta_{X'} = \theta_{X'}(g)$ are two smooth complex-valued functions defined by (1.2) in Section 1 with respect to X and X', respectively. Clearly, θ_X is a real-valued function since $L_X \omega_g$ is a real-valued (1.1)-form. Since g is K-invariant by a result in Appendix in [TZ1], $L_{X'} \omega_g$ is also a real-valued (1,1)-form. Hence, $\theta_{X'}$ is also a real-valued function on M. Furthermore, by Lemma A, there are $\tilde{\theta}_X = \theta_X + c_1$ and $\tilde{\theta}_{X'} = \theta_{X'} + c_2$ for some constants c_1 and c_2 such that

$$\tilde{\theta}_X \neq \tilde{\theta}_{X'}, \tag{A.1}$$

$$\triangle_g(\tilde{\theta}_X) + X(\tilde{\theta}_X) + \tilde{\theta}_X = 0, \tag{A.2}$$

and

$$\triangle_g(\tilde{\theta}_{X'}) + X(\tilde{\theta}_{X'}) + \tilde{\theta}_{X'} = 0. \tag{A.3}$$

Let h_g be a smooth real-valued function defined by (1.1) in Section 1. We define a function on $[0,1]$ as follows:

$$F_1(a) = \int_M X(h_g - a\tilde{\theta}_X - (1-a)\tilde{\theta}_{X'}) e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n. \tag{A.4}$$

Then by Proposition 3.1, it is clear

$$F_1(1) = e^{c_1} \int_M X(h_g - \theta_X) e^{\theta_X} \omega_g^n = 0. \tag{A.5}$$

Moreover, by Proposition 1.1 and 3.1, we have

$$\begin{aligned} F_1(0) &= e^{c_2} \int_M X(h_g - \theta_{X'}) e^{\theta_{X'}} \omega_g^n \\ &= e^{c_2} \int_M X(h_{g'} - \theta_{X'}(g')) e^{\theta_{X'}(g')} \omega_{g'}^n \\ &= 0, \end{aligned} \tag{A.6}$$

where $h_{g'}$ and $\theta_{X'}(g')$ are two smooth real-valued functions defined by (1.1) and (1.2) respectively in Section 1 associated to the Kähler–Ricci soliton g'.

Since

$$h_g - \theta_X = \text{const.},$$

we have

$$F_1(a) = \int_M (1-a) X(\tilde{\theta}_X - \tilde{\theta}_{X'}) e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n. \tag{A.7}$$

By using integration by parts and (A.2), one can compute

$$\begin{aligned} &\frac{dF_1(a)}{da} \\ &= \int_M [-X(\tilde{\theta}_X - \tilde{\theta}_{X'}) + (1-a)(\tilde{\theta}_X - \tilde{\theta}_{X'}) \cdot X(\tilde{\theta}_X - \tilde{\theta}_{X'})] e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n \\ &= \int_M (\tilde{\theta}_X - \tilde{\theta}_{X'})(\triangle \tilde{\theta}_X + \overline{(aX + (1-a)X')\tilde{\theta}_X}) e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n \\ &+ \int_M (1-a)(\tilde{\theta}_X - \tilde{\theta}_{X'}) \cdot X(\tilde{\theta}_X - \tilde{\theta}_{X'}) e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n \\ &= \int_M (\tilde{\theta}_X - \tilde{\theta}_{X'})(\triangle \tilde{\theta}_X + X(\tilde{\theta}_X)) e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n \\ &= \int_M -(\tilde{\theta}_X - \tilde{\theta}_{X'})\tilde{\theta}_X e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n. \end{aligned} \tag{A.8}$$

Similar to (A.4), we define

$$F_2(a) = \int_M X'(h_g - a\tilde{\theta}_X - (1-a)\tilde{\theta}_{X'}) e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n.$$

Then the above argument shows

$$F_2(0) = F_2(1) = 0 \tag{A.9}$$

and

$$\begin{aligned} \frac{dF_2(a)}{da} &= \int_M (\tilde{\theta}_X - \tilde{\theta}_{X'})(\triangle \tilde{\theta}_{X'} + X(\tilde{\theta}_{X'})) e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n \\ &= -\int_M (\tilde{\theta}_X - \tilde{\theta}_{X'})\tilde{\theta}_{X'} e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n. \end{aligned} \tag{A.10}$$

The last inequality follows from (A.3).

Combining (A.8) and (A.10), we get

$$\frac{d}{da}(F_2(a) - F_1(a)) = \int_M (\tilde{\theta}_X - \tilde{\theta}_{X'})^2 e^{a\tilde{\theta}_X + (1-a)\tilde{\theta}_{X'}} \omega_g^n > 0.$$

Then by (A.6) and (A.9), it follows

$$F_2(a) - F_1(a) > 0, \ \forall\ a > 0,$$

in particular,

$$F_2(1) - F_1(1) > 0,$$

which is impossible, since $F_1(1) = 0$ and $F_2(1) = 0$ by (A.5) and (A.9). The contradiction shows that Theorem 3.2 is true. □

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