A Construction of Seifert Surfaces by Differential Geometry

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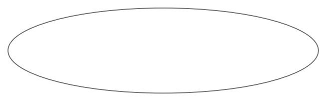
Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Supreedee Dangskul*)

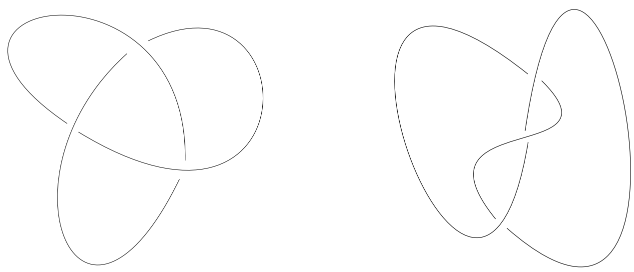
Lay Summary

A *knot* is a mathematical object that can be thought of a piece of string in space with the two ends fused together. The simplest example of a knot is the *unknot*, which is an untangled circle.



Unknot

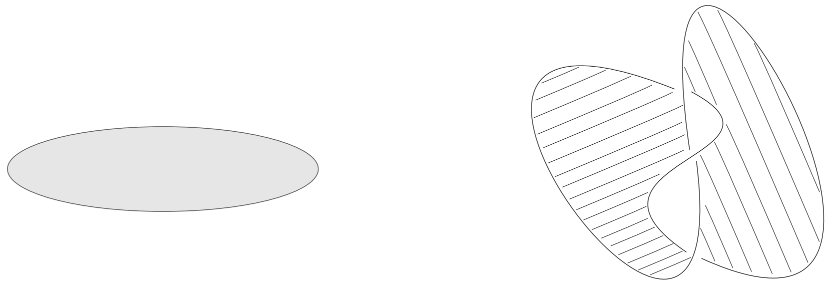
A *trefoil knot* is a more interesting example. The following pictures are both drawings of the trefoil knot; they are mathematically equivalent even though they appear different. Two knots are considered to be the same if one can be picked up and twisted in space (without cutting or gluing) to look like the second.



Trefoil knots

Knots have been studied in Edinburgh since the days of Peter Guthrie Tait 140 years ago. Following Kelvin, he thought that atoms could be modelled by knots. In pursuit of this history, Tait initiated the classification of knots.

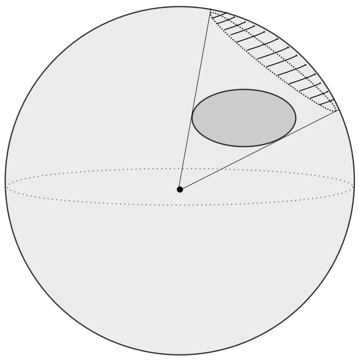
Topologists have proved that every knot is the boundary of a surface in space. Such surfaces are called *Seifert surfaces* for the knot, after the German mathematician Herbert Seifert, who first proved this 80 years ago. It is obvious that the unknot has a Seifert surface, but not at all obvious for the trefoil knot and even less obvious for more complicated knots.



Seifert surfaces of the unknot and a trefoil knot

Seifert surfaces are used in the classification of knots. One may create a Seifert surface for a knot by dipping the knot into soap water; the soap bubble is a Seifert surface for the knot.

In this thesis, we shall be concerned with mathematical constructions of Seifert surfaces. We introduce a new construction using the notion of *solid angle* of a bounded object in space, measured from a reference point: this is the proportion of the area of the shadow cast by the object from the point on the surface of a large sphere containing the object.



Solid angle

We use solid angles to define a canonical differentiable function from the complement of the knot to the circle. For almost all the points in the circle the union of the inverse image of the point and the knot is a Seifert surface, all points of which have the same solid angle. In other words, a Seifert surface in our construction is an iso-surface, where the quantity measured is the solid angle. Our work also makes use of linking numbers, as introduced by Gauss and Maxwell.

In general, a knot in $(n+2)$-space can be defined as an n-sphere in $(n+2)$-space. When $n = 1$, this is a knot in 3-space discussed earlier. It is possible that our construction can be generalised for knots in higher dimensions. Our construction of Seifert surfaces by differential geometry might eventually be used to study the mathematical properties of Seifert surfaces with minimal properties, such as soap bubbles.

Abstract

A Seifert surface for a knot in $\mathbb{R}^3$ is a compact orientable surface whose boundary is the knot. Seifert surfaces are not unique. In 1934 Herbert Seifert provided a construction of such a surface known as the Seifert Algorithm, using the combinatorics of a projection of the knot onto a plane. This thesis presents another construction of a Seifert surface, using differential geometry and a projection of the knot onto a sphere.

Given a knot $K : S^1 \subset R^3$, we construct canonical maps $F : \Lambda_{diff} S^2 \to \mathbb{R}/4\pi\mathbb{Z}$ and $G : \mathbb{R}^3 - K(S^1) \to \Lambda_{diff} S^2$ where $\Lambda_{diff} S^2$ is the space of smooth loops in S^2. The composite

$$FG : \mathbb{R}^3 - K(S^1) \to \mathbb{R}/4\pi\mathbb{Z}$$

is a smooth map defined for each $u \in \mathbb{R}^3 - K(S^1)$ by integration of a 2- form over an extension $D^2 \to S^2$ of $G(u) : S^1 \to S^2$. The composite FG is a surjection which is a canonical representative of the generator $1 \in H^1(\mathbb{R}^3 - K(S^1)) = \mathbb{Z}$. FG can be defined geometrically using the solid angle. Given $u \in \mathbb{R}^3 - K(S^1)$, choose a Seifert surface Σ_u for K with $u \notin \Sigma_u$. It is shown that $FG(u)$ is equal to the signed area of the shadow of Σ_u on the unit sphere centred at u. With this, $FG(u)$ can be written as a line integral over the knot.

By Sard's Theorem, FG has a regular value $t \in \mathbb{R}/4\pi\mathbb{Z}$. The behaviour of FG near the knot is investigated in order to show that FG is a locally trivial fibration near the knot, using detailed differential analysis. Our main result is that $(FG)^{-1}(t) \subset \mathbb{R}^3$ can be closed to a Seifert surface by adding the knot.

Contents

Chapter 1

Introduction

A *closed Seifert surface* for a knot $K : S^1 \subset \mathbb{R}^3$ is a compact orientable surface $\Sigma^2 \subset \mathbb{R}^3$ with boundary $\partial\Sigma = K(S^1)$. Closed Seifert surfaces for a given knot K can be constructed using Seifert's algorithm [16], starting with a choice of knot projection.

Closed Seifert surfaces for a smooth knot $K : S^1 \subset \mathbb{R}^3$ can also be constructed by transversality properties of smooth maps. More explicitly, extend K to an embedding of a tubular neighbourhood $K(S^1) \times D^2 \subset \mathbb{R}^3$ and let

$$X = \mathrm{Cl}_{\mathbb{R}^3}(\mathbb{R}^3 - (K(S^1) \times D^2)) \subset \mathbb{R}^3$$

be the exterior of the knot. There exists a canonical rel ∂ homotopy class of smooth maps

$$(p, \partial p) : (X, \partial X) = (X, K(S^1) \times S^1) \to S^1$$

with ∂p = projection : $\partial X = K(S^1) \times S^1 \to S^1$. The preimage of a regular value $* \in S^1$ of such a smooth map p is a closed Seifert surface $\Sigma = p^{-1}(*) \subset \mathbb{R}^3$ for K. This construction depends on the choice of a tubular neighbourhood, the choice of a map in the rel ∂ homotopy class, and the choice of a regular value.

Let Σ be a closed Seifert surface for a knot K. Then, there exists a smooth embedding

$$K(S^1) \times [0, 1] \hookrightarrow \Sigma,$$

called a collar. This implies that the interior $\Sigma - \partial\Sigma$ is an open surface with the following properties:

- $\mathrm{Cl}_{\mathbb{R}^3}(\Sigma - \partial\Sigma) = \Sigma$ and
- there exists a (topological) embedding $K(S^1) \times [0, 1] \hookrightarrow \Sigma$ such that the restriction

$$K(S^1) \times (0, 1] \hookrightarrow \Sigma - \partial\Sigma$$

is smooth.

This leads to the following definitions. An *open Seifert surface* Σ_0 for a knot $K : S^1 \subset \mathbb{R}^3$ is an open surface in $\mathbb{R}^3$ whose topological boundary is the knot, i.e.,

$\mathrm{Cl}_{\mathbb{R}^3}(\Sigma_0) = \Sigma_0 \cup K(S^1)$. An open Seifert surface Σ_0 is said to be *regular* if there exists a (topological) embedding

$$K(S^1) \times [0,1] \hookrightarrow \Sigma_0 \cup K(S^1)$$

such that the restriction

$$K(S^1) \times (0,1] \hookrightarrow \Sigma_0$$

is smooth.

It is clear that if Σ is a closed Seifert surface for $K : S^1 \subset \mathbb{R}^3$, then $\Sigma_0 = \Sigma - \partial\Sigma$ is a bounded regular open Seifert surface in $\mathbb{R}^3$. Conversely, a bounded regular open Seifert surface Σ_0 for K gives rise to a closed Seifert surface as follows. Since Σ_0 is regular, we consider an embedding $\Theta : K(S^1) \times [0,1] \hookrightarrow \Sigma_0 \cup K(S^1)$ such that the restriction

$$\Theta| : K(S^1) \times [\varepsilon, 1] \hookrightarrow \Sigma_0,$$

for some small $\varepsilon > 0$, is smooth. Hence,

$$\Sigma = \Sigma_0 - \Theta(K(S^1) \times (0, \varepsilon))$$

has boundary $K(S^1) \times \{\varepsilon\} \cong K(S^1)$, and therefore is a closed Seifert surface for K (technically, it is a closed Seifert surface for an ε-copy of K).

The main purpose of this thesis is to construct a closed Seifert surface for a smooth knot K requiring fewer choices, using the knot complement $\mathbb{R}^3 - K(S^1)$, of which the knot exterior X is a deformation retract. We shall define a smooth map

$$\mathbb{R}^3 - K(S^1) \to S^1$$

and then show that the preimage of some regular value is a bounded regular open Seifert surface for K.

Main Construction [Chapter 4, Chapter 5] *For any smooth knot $K : S^1 \subset \mathbb{R}^3$ we construct a smooth map*

$$FG : \mathbb{R}^3 - K(S^1) \to S^1$$

such that $\Sigma_0 = (FG)^{-1}(*)$ is a bounded regular open Seifert surface for K, where $*$ is a regular value of FG. Therefore, a closed Seifert surface Σ for K will be obtained from Σ_0 as discussed above.

The map FG above is composed of two maps F and G defined as follows:

$$F : \Lambda_{diff} S^2 \to \mathbb{R}/4\pi\mathbb{Z} \quad ; \quad \lambda \mapsto \int_{D^2} (\delta\lambda)^* \omega$$

where $\Lambda_{diff} S^2$ is the space of smooth loops $\lambda : S^1 \to S^2$, $\delta\lambda : D^2 \to S^2$ is a smooth extension of λ, ω is a volume 2-form on S^2 with $\int_{S^2} \omega = 4\pi$; and

$$G : \mathbb{R}^3 - K(S^1) \to \Lambda_{diff} S^2 \quad ; \quad u \mapsto \lambda_u$$

where $\lambda_u : S^1 \to S^2$ is the loop given by

$$\lambda_u(x) = \frac{K(x) - u}{\|K(x) - u\|} \ (x \in S^1)$$

For each $u \in \mathbb{R} - K(S^1)$, choose a Seifert surface Σ_u with $u \notin \Sigma$. Define $\Pi_u : \Sigma_u \to S^2$ by

$$\Pi_u(y) = \frac{y - u}{\|y - u\|}.$$

It turns out that FG can be computed by

$$FG(u) = \int_{\Sigma_u} \Pi_u^* \omega.$$

For computational purposes, by Stokes' Theorem, the formula for FG can be expressed as a line integral

$$FG(u) = \int_{K(S^1)} (\Pi_u|_{\text{im } K})^* \, \eta \tag{1.1}$$

where η is a 1-form on $S^2 - \{z\}$, for some $z \in S^2$, with $d\eta = \omega$. Moreover, given a parametrisation $\gamma : [a, b] \to \mathbb{R}^3$ of the knot K, it can be shown that

$$FG(u) = \int_a^b \frac{\left(\frac{\gamma(t) - u}{\|\gamma(t) - u\|} \right) \times z}{\|\gamma(t) - u\| \left(1 - \frac{\gamma(t) - u}{\|\gamma(t) - u\|} \cdot z \right)} \dot{\gamma}(t) dt \tag{1.2}$$

where z is a point in S^2 with $z \neq \dfrac{\gamma(t) - u}{\|\gamma(t) - u\|}$ for all $t \in [a, b]$, and the formula is independent of z.

In practice, it is quite hard to actually compute FG for particular knots. The formula (1.2) allows us to compute FG for the simplest knot, an unknot, using elliptic integrals.

We shall be particularly concerned with the behaviour of FG near the knot K.

Let us introduce the following terminology. A map $q : T \to S^1$ is a *locally trivial fibration* if for each $s \in S^1$ there exists an open neighbourhood $V \subset S^1$ of s such that the following diagram

$$\begin{tikzcd} q^{-1}(V) \arrow[r, "\cong"] \arrow[d, "q"'] & q^{-1}(s) \times V \arrow[dl, "proj"] \\ V & \end{tikzcd}$$

commutes.

Main Theorem [Theorem 7.1.1, Corollary 7.1.2] *For any knot $K \subset \mathbb{R}^3$ and sufficiently small tubular neighbourhood $T = K \times (D^2 - \{0\}) \subset \mathbb{R}^3 - K$ of K with the core removed, the restriction*

$$FG|_T : T \to S^1$$

is a locally trivial fibration. A regular value $* \in S^1$ of FG is in particular a regular value of $FG|_T$, and the open Seifert surface $\Sigma = (FG)^{-1}(*) \subset \mathbb{R}^3$ is regular, i.e., there is a diffeomorphism

$$(FG)^{-1}(*) \cap T = (FG|_T)^{-1}(*) \cong K \times (0, 1].$$

The proof of the Main Theorem uses nontrivial analysis. The result is not obvious even for an unknot.

Here is the outline of the thesis:

- Chapter 2 contains basic definitions and some background knowledge and facts used throughout the thesis – for instance, elementary knot theory, transversality, loop spaces, solid angles, etc.

- Chapter 3 describes the two constructions of closed and open Seifert surfaces, Seifert's algorithm and the transversality construction.

- Chapter 4 introduces our Main Construction. It begins with the definition of the map $F : \Lambda_{diff}S^2 \to S^1$ and its properties followed by that of $G : \mathbb{R}^3 - K(S^1) \to \Lambda_{diff}S^2$. We also investigate some properties of the composite map $FG : \mathbb{R}^3 - K(S^1) \to S^1$.

- Chapter 5 introduces another approach to our Main Construction. More precisely, we show that the map FG gives the area of a shadow cast by a chosen closed Seifert surface. This approach is more computable and we are able to derive a line-integral formula of FG.

- Chapter 6 carries out some computations for the unknot U. The formula of FG for U can be expressed in terms of elliptic integrals. We refer to the result of Paxton, see [12], in order to compute the solid angle of a standard circular disc. We study the behaviour of FG near the unknot and finally show that the open surface $(FG)^{-1}(*)$ is regular near U.

- Chapter 7 extends the regularity results of Chapter 6 from the unknot K to an arbitrary smooth knot K, proving the Main Theorem. For this purpose we divide K into parts to see that the partial derivatives of K and U are close in a small tubular neighbourhood.

- Chapter 8 discusses some possible future work.

Chapter 2

Preliminaries

2.1 Knots

Definition 2.1.1. *Let X be any space. A* path *in X is a continuous map from $[0,1]$ to X. A* loop *in X is a path that sends 0 and 1 to the same point.*

We remark that any loop in X induces a map $S^1 \to X$ by composing with

$$[0,1] \to S^1 \quad ; \quad t \mapsto e^{2\pi i t}.$$

Thus a loop may be defined as a map with domain S^1.

Definition 2.1.2. *A* knot *in $\mathbb{R}^3$ (or S^3) is an injective loop $S^1 \hookrightarrow \mathbb{R}^3$ (or $S^1 \hookrightarrow S^3$). A knot is said to be* smooth *if its embedding is smooth.*

Example 2.1.3. *The path*

$$p : [0,1] \to \mathbb{R}^3 \quad ; \quad t \mapsto (\sin 2\pi t + 2\sin 4\pi t, \cos 2\pi t - 2\cos 4\pi t, -\sin 6\pi t)$$

defines a trefoil knot. Clearly, p is smooth.

Definition 2.1.4. *A* tubular neighbourhood *of a knot K in $\mathbb{R}^3$ or S^3 is an embedding $T : S^1 \times D^2 \hookrightarrow \mathbb{R}^3$ (or $S^1 \times D^2 \hookrightarrow S^3$) such that $T\big(S^1 \times \{0\}\big) = K(S^1)$. A tubular neighbourhood T may be regarded as its image $T(K) := T(S^1 \times D^2) = K(S^1) \times D^2$, and we may simply call it T.*

For simplicity, we assume that our knot is *tame* if there exists a tubular neighbourhood for our knot. By the Tubular neighbourhood theorem, Theorem 10.19 in [7], every smooth knot is tame.

Definition 2.1.5. *A* meridian *of a knot K is a loop in $\mathbb{R}^3 - K(S^1)$ homotopic to a loop of the form*

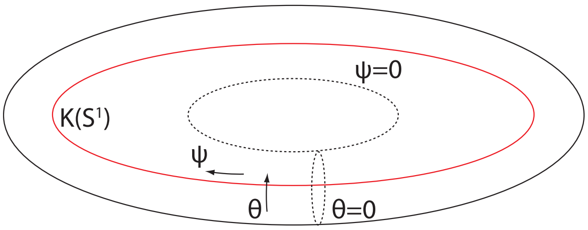
$$[0,1] \to \{K(z)\} \times S^1 \quad ; \quad t \mapsto \big(K(z), e^{2\pi i t}\big)$$

for some $z \in S^1$. A canonical longitude *of a knot K is a loop in $\mathbb{R}^3 - K(S^1)$ homotopic to a loop of the form*

$$[0,1] \to K(S^1) \times \{z'\} \quad ; \quad t \mapsto \left(K(e^{2\pi i t}), z'\right)$$

for some $z' \in S^1$.

Example 2.1.6. *Consider an unknot $K : S^1 \hookrightarrow \mathbb{R}^3$, coloured in red, and a tubular neighbourhood $T : S^1 \times D^2 \hookrightarrow \mathbb{R}^3$ with $T\left(S^1 \times \{0\}\right) = K(S^1)$*



with parametrisation

$$S^1 \times D^2 \hookrightarrow \mathbb{R}^3 \quad ; \quad (\psi, (r, \theta)) \mapsto ((2+\cos\psi)r\cos\theta, (2+\cos\psi)r\sin\theta, \sin\psi)\,.$$

This solid torus is obtained by rotating the disc $(x-2)^2 + z^2 \leqslant 1$ about the z- axis. Setting $\psi = 0, r = 1$ and $\theta = 0, r = 1$, we have the loop $\theta \mapsto (3\cos\theta, 3\sin\theta, 0)$ as a meridian and the loop $\psi \mapsto (2+\cos\psi, 0, \sin\psi)$ as a canonical longitude of the unknot, respectively.

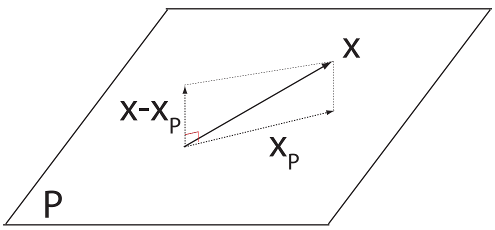
2.2 Knot projections

We may project a knot in $\mathbb{R}^3$ onto a surface − a plane or a sphere, for example. This makes it easy to visualise the knot. In this section, we present two kinds of projection, linear projections and radial projections.

For every plane $P \subset \mathbb{R}^3$, every $x \in \mathbb{R}^3$ has a unique decomposition

$$x = x_P + x_P^{\perp}$$

where $x_P \in P$ and $x_P^{\perp}$ is perpendicular to P.



Definition 2.2.1. *Given a plane P, the* linear projection L_P *of $S \subset \mathbb{R}^3$ onto P is given by*

$$L_P : S \to P \quad ; \quad x \mapsto x_P.$$

We may omit mentioning the plane P and the subset S if they are clearly understood.

Note that $x_P = x - x_P^{\perp}$. We can give an explicit formula for the linear projection onto a plane as follows.

Proposition 2.2.2. *Let $P \subset \mathbb{R}^3$ be a plane with equation $ax + by + cz = d$. The linear projection of $\mathbb{R}^3$ onto P is given by the formula*

$$L_P(x_0, y_0, z_0) = (x_0, y_0, z_0) - \frac{ax_0 + by_0 + cz_0 - d}{a^2 + b^2 + c^2}(a, b, c)$$

for all $x_0, y_0, z_0 \in \mathbb{R}$.

Proof. First notice that the distance between the origin and the plane P is $\frac{|d|}{a^2 + b^2 + c^2}$. Then translating P by $-\frac{d}{a^2 + b^2 + c^2}(a, b, c)$ gives the plane $P' = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\}$. Since the vector (a, b, c) is normal to P', the orthogonal projection of $(x_0, y_0, z_0) \in \mathbb{R}^3$ to this plane is

$$\frac{(x_0, y_0, z_0) \cdot (a, b, c)}{a^2 + b^2 + c^2}(a, b, c).$$

This implies that

$$(x_0, y_0, z_0)_{P'}^{\perp} = (x_0, y_0, z_0) - \frac{(x_0, y_0, z_0) \cdot (a, b, c)}{a^2 + b^2 + c^2}(a, b, c).$$

Translating $(x_0, y_0, z_0)_{P'}^{\perp}$ back with $\frac{d}{a^2 + b^2 + c^2}(a, b, c)$, we have

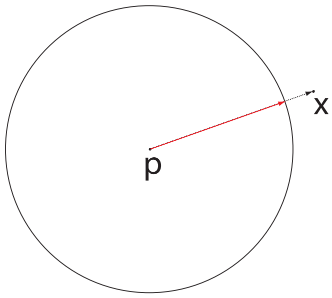
$$\begin{aligned}(x_0, y_0, z_0)_P^{\perp} &= (x_0, y_0, z_0)_{P'}^{\perp} + \frac{d}{a^2 + b^2 + c^2}(a, b, c) \\ &= (x_0, y_0, z_0) - \frac{(x_0, y_0, z_0) \cdot (a, b, c)}{a^2 + b^2 + c^2}(a, b, c) + \frac{d}{a^2 + b^2 + c^2}(a, b, c).\end{aligned}$$

□

Definition 2.2.3. *Given a subset $S \subset \mathbb{R}^3$ and a point $p \notin S$, the* radial projection R_p *of S from p is a map*

$$R_p : S \to S^2 \quad ; \quad x \mapsto \frac{x - p}{\|x - p\|}$$

We may omit mentioning the point p and the subset S if they are clearly understood.



We remark that for the radial projection from the point p, we view p as the origin and draw a unit sphere about p to obtain the projection. Intuitively, this projection gives the image of S on S^2 when we look from p.

Linear or radial projections are not always injective. The linear projection of $\mathbb{R}^3$ onto a plane collapses all the points on a line perpendicular to the plane to a point on the plane. The radial projection of $\mathbb{R}^3$ from a point p collapses all the points on a line passing through p to a pair of antipodal points on S^2.

Definition 2.2.4. *Let L_P be the linear projection of $S \subset \mathbb{R}^3$ onto P and R_p be the radial projection of $S \subset \mathbb{R}^3$ from p. Assume that q, r belong to* im (L_P) *or* im (R_p)*. A point q is said to be a* double point *of the projection if at least two points in S are projected to q. Similarly, a point r is said to be a* triple point *of the projection if at least three points in S are projected to r.*

Example 2.2.5. *(i) Consider the linear projection of $S = \{(a, b, 0) \in \mathbb{R}^3 \mid a, b = 0, 1, 2\}$ onto the plane $x + y = 4$. The point $(2, 2, 0)$ is a triple point. The points $(3/2, 5/2, 0)$ and $(5/2, 3/2, 0)$ are double points.*

(ii) Every point in S^2 is a double point of the radial projection of

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = k \quad \text{for} \quad k = 1, 2\}$$

from the origin. In this case, there is no triple point. □

Let us now consider the linear or radial projections of a knot. By definition, given a knot $K : S^1 \hookrightarrow \mathbb{R}^3$, the linear projection of K onto a plane P is the linear projection $L_P : K(S^1) \to P$. This induces the composite

$$S^1 \overset{K}{\hookrightarrow} \mathbb{R}^3 \overset{\tilde{L}_P}{\to} P$$

where $\tilde{L}_P$ is the linear projection of $\mathbb{R}^3$ onto P. Similarly, if $p \notin K(S^1)$, the radial projection of K from p is the radial projection $R_p : K(S^1) \to S^2$; this induces the composite

$$S^1 \overset{K}{\hookrightarrow} \mathbb{R}^3 \overset{\tilde{R}_p}{\to} S^2$$

where $\tilde{R}_p$ is the radial projection of $\mathbb{R}^3$ from p.

Definition 2.2.6. *The linear projection of a knot $K : S^1 \to \mathbb{R}^3$ onto a plane P is the composite $\tilde{L}_P K : S^1 \to P$. The radial projection of a knot $K : S^1 \to \mathbb{R}^3$ from a point $p \notin K(S^1)$ is the composite $\tilde{R}_p K : S^1 \to S^2$.*

We remark that the linear projection of K onto P gives a loop in P, and the radial projection of K from $p \notin K(S^1)$ is a loop in S^2. We next introduce some "nice" projections of a knot.

Definition 2.2.7. *A linear (or radial) projection of a knot is said to be* regular *if there are only a finite number of double points and no triple points. A regular linear (or radial) projection of a knot is called a* linear knot projection *(or* radial knot projection*).*

A linear knot projection is usually called a *knot diagram*. If the type of projection is clear, we may omit the word "linear" or "radial" for convenience.

Example 2.2.8. *In general, a projection of a knot is not a knot projection. For instance, a knot diagram of a standard unit circle in the xy-plane projected onto xz-plane is not regular since it contains infinitely many double points.* □

2.3 Linking number

We give two definitions of linking number defined via homology and knot diagrams.

Proposition 2.3.1. *Let K be a knot in $\mathbb{R}^3$. Then, $H_1(\mathbb{R}^3 - K(S^1)) \cong \mathbb{Z}$ is generated by the class of meridians. The result also holds for knots in S^3.*

Proof. Let T be a tubular neighbourhood of K and $X = \mathrm{Cl}_{\mathbb{R}^3}\left(\mathbb{R}^3 - T\right)$. Note that $H_1(\mathbb{R}^3 - K(S^1)) \cong H_1(X)$ and $\partial X = \partial T = X \cap T \cong K(S^1) \times S^1$. Now consider

$$\begin{tikzcd} K(S^1) \times S^1 \arrow[r, "i"] \arrow[d, "j"'] & T \arrow[d] \\ X \arrow[r] & \mathbb{R}^3 \end{tikzcd}$$

where i and j are the inclusion maps. The Mayer-Vietoris sequence is

$$\cdots \longrightarrow H_2(\mathbb{R}^3) \longrightarrow H_1(K(S^1) \times S^1) \xrightarrow{(i_*, j_*)} H_1(T) \oplus H_1(X) \longrightarrow H_1(\mathbb{R}^3) \longrightarrow \cdots.$$

By Kunneth's formula and the fact that $H_2(\mathbb{R}^3) \cong 0 \cong H_1(\mathbb{R}^3)$, we have the exact sequence

$$0 \longrightarrow H_1(K(S^1)) \oplus H_0(K(S^1)) \longrightarrow H_1(K(S^1)) \oplus H_1(X) \longrightarrow 0.$$

By exactness,

$$H_1(K(S^1)) \oplus H_0(K(S^1)) \cong H_1(K(S^1)) \oplus H_1(X),$$

and thus

$$H_1(\mathbb{R}^3 - K(S^1)) \cong H_1(X) \cong H_0(K(S^1)) \cong \mathbb{Z}.$$

Note that if μ is a simple closed curve in ∂T which bounds a disc in T (meridian of ∂T), then we have $i_*[\mu] = 0$, implying that $j_*[\mu]$ is a generator of $H_1(\mathbb{R}^3 - K(S^1)) \cong H_1(X)$. □

Definition 2.3.2. *Let K and L be two disjoint knots in $\mathbb{R}^3$. The embedding $L : S^1 \hookrightarrow R^3 - K(S^1)$ induces*

$$L_* : H_1(L) = \mathbb{Z} \to H_1\left(\mathbb{R}^3 - K(S^1)\right) = \mathbb{Z},$$

and the homological linking number of K and L is defined by $L_*(1)$, denoted by Linking(K, L). The linking number of two disjoint knots in S^3 is defined in the same fashion, using $H_1\left(S^3 - K(S^1)\right) = \mathbb{Z}$.

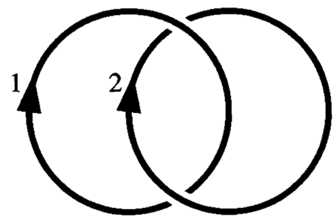
For any two disjoint smooth knots, we can define the linking number geometrically. Two knot projections are said to be *transverse* if they have a finite number of intersection points, and at each intersection their tangent vectors span a plane. Each transverse intersection point is a double point that gives a *crossing* for the projection as follows. If x, y get mapped to a transverse intersection point q under a projection, we say that x is *over* y if $||x - q|| > ||y - q||$, and say that x is *under* y if $||x - q|| < ||y - q||$. We can also add the notion of *under crossing* or *over crossing*, and assign to each crossing a sign $\pm$ depending on the orientation of those two knots with the following rules:

Definition 2.3.3. *Let K and L be oriented knots in $\mathbb{R}^3$ or S^3. If they have transverse knot projections, the* transverse linking number *of K and L is the sum of the signs of all crossings where K crosses under L.*

We remark that this definition of linking number does not depend on the knot projection.

Example 2.3.4. *(i) If two unknots are not linked, we can project them onto the same plane such that there are no crossings; so the transverse linking number of those unknots is zero.*

(ii) A Hopf link consists of two unknots linked according to the following diagram

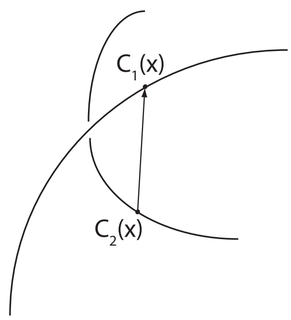


There are only two crossings in the diagram of these two oriented knots, at the top for Knot 2 going under Knot 1 and at the bottom for the other way around. By Definition 2.3.3, the transverse linking number is equal to 1. If we change the orientation of Knot 1, then the transverse linking number is equal to -1. Hence, this linking number depends on the orientation.

□

A proof showing that the two definitions of linking number are equivalent can be found on Page 132 in [14].

2.4 Gauss linking integral



Definition 2.4.1. *Let C_1 and C_2 be disjoint loops in $\mathbb{R}^3$. The Gauss map of C_1 and C_2 is defined by*

$$\Psi_{C_1,C_2} : S^1 \times S^1 \to S^2; (x,y) \mapsto R_{C_2(y)}\left(C_1(x)\right) = \frac{C_1(x) - C_2(y)}{\|C_1(x) - C_2(y)\|}.$$

For each $y \in S^1$, the Gauss map Ψ_{C_1,C_2} defines a loop

$$\Psi_{C_1,C_2}(-,y) : S^1 \to S^2 \quad ; \quad x \mapsto \Psi_{C_1,C_2}(x,y)$$

which is obtained by seeing $C_1(S^1)$ from $C_2(y)$. Hence the Gauss map Ψ_{C_1,C_2} gives a collection of the radial projections of $C_1(S^1)$ onto a sphere seen from each point along C_2.

Let us recall the definition of the degree of a continuous map. The degree of a map $f : M \to N$ between closed connected oriented n-dimensional manifolds is defined via

$$f_* : H_n(M) \to H_n(N) \quad ; \quad f_*([M]) = (\deg f)\,[N],$$

where $[M]$ and $[N]$ are the fundamental classes of M and N, respectively. If $f : M \to N$ is a smooth map between closed connected oriented smooth n-manifolds, then

$$\deg f = \Sigma_{q \in f^{-1}(p)} \left(\mathrm{sgn}\ d_q f\right)$$

where p is a regular value of f, and $d_q f$ is the differential of f at q. It is also shown that

$$\int_M f^*(\omega) = \deg f \int_N \omega$$

where ω is any n-form on N with pullback n-form $f^*(\omega)$. See [3] and Chapter 11 in [8] for detailed descriptions.

The following proposition provides a relationship between Gauss maps and linking number, see [13] and Chapter 11 in [8].

Proposition 2.4.2. (Gauss linking integral) *Let K and L be two disjoint knots. Then, we have*

$$\deg \Psi_{K,L} = \mathrm{Linking}(K, L) = \frac{1}{4\pi} \int_{S^1 \times S^1} \Psi^*_{K,L} \left(\mathbf{Vol}_{S^2}\right)$$

where $\mathbf{Vol}_{S^2}$ is the volume 2-form on S^2 with $\int_{S^2} \mathbf{Vol}_{S^2} = 4\pi$.

If α and β are parametrisations of K and L respectively, then

$$\int_{S^1 \times S^1} \Psi^*_{K,L} \left(\mathbf{Vol}_{S^2}\right) = \int_K \int_L \frac{\det\left(\alpha(s) - \beta(t), \alpha'(s), \beta'(t)\right)}{||\alpha(s) - \beta(t)||^3} dtds,$$

see Theorem 11.14 in [8].

Example 2.4.3. *Consider two knots K and L in $\mathbb{R}^3 \cup \{\infty\} = S^3$ with parametrisations*

$$\alpha : [0, 2\pi] \to \mathbb{R}^3 \quad ; \quad s \mapsto (\cos s, \sin s, 0)$$

and

$$\beta : [-\infty, +\infty] \to \mathbb{R}^3 \cup \{\infty\} \quad ; \quad t \mapsto (0, 0, -t)$$

for K and L, respectively. Note that the knot L is the z-axis whose two ends are identified at infinity and K, L are disjoint. Then K and L form the Hopf link. By

Proposition (2.4.2), *we have*

$$\begin{aligned}
\text{Linking}(K,L) &= \frac{1}{4\pi}\int_0^{2\pi}\int_{-\infty}^{+\infty}\frac{\det\begin{pmatrix}\cos s & \sin s & t\\ -\sin s & \cos s & 0\\ 0 & 0 & 1\end{pmatrix}}{(1+t^2)^{3/2}}dtds\\
&= \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{2\pi}{(1+t^2)^{3/2}}dt\\
&= \frac{1}{4\pi}(2\pi)\left[\frac{t}{\sqrt{1+t^2}}\right]_{-\infty}^{+\infty}\\
&= 1.
\end{aligned}$$

□

2.5 Loop spaces

Let X be a topological space. The loop space ΛX is the set of maps $S^1 \to X$ with compact-open topology. If X is pointed with base point x_0, the pointed loop space ΩX is the subspace of ΛX consisting of pointed maps $\left(S^1, 1\right) \to (X, x_0)$. The space ΩX has a natural base point the constant map $S^1 \to x_0$. The reduced suspension of X is defined as the quotient space

$$\Sigma X = X \wedge S^1 = \frac{X \times S^1}{X \vee S^1}$$

with base point the equivalence class containing $(x_0, 1)$. Let (Y, y_0) be another pointed space. There is a one-to-one correspondence between the spaces of pointed maps

$$\Phi : \text{Map}\,(X, \Omega Y) \cong \text{Map}\,(\Sigma X, Y)\,,$$

sending

$$f : (X, x_0) \to \Omega Y \quad \text{with} \quad f(x) : \left(S^1, 1\right) \to (Y, y_0)$$

to

$$\Phi_f : \Sigma X \to Y \quad ; \quad (x, z) \mapsto f(x)(z) \in Y.$$

It is not hard to see that two equivalent pairs in ΣX get mapped to the same point in Y. Note that if $f \simeq f'$ via $h_t : X \to \Omega Y$, then $\Phi_f \simeq \Phi_{f'}$ via $\Phi_{h_t} : \Sigma X \to Y$. This implies that there is an isomorphism between the sets of equivalence classes of pointed maps

$$[X, \Omega Y] \cong [\Sigma X, Y]\,. \tag{2.1}$$

Proposition 2.5.1. *The fundamental group of ΩS^2 is $\mathbb{Z}$.*

Proof. By (2.1), taking $X = S^1$ and $Y = S^2$ yields

$$\left[S^1, \Omega S^2\right] \cong \left[\Sigma S^1, S^2\right] \cong \left[S^2, S^2\right].$$

Since each equivalence class of maps $S^2 \to S^2$ is determined by its degree, it follows that $\left[S^2, S^2\right] \cong \mathbb{Z}$; and hence

$$\pi_1\left(\Omega S^2\right) \cong \left[S^1, \Omega S^2\right] \cong \mathbb{Z}.$$

□

If X is a path-connected space, then the fundamental group of ΛX is also computable, using the homotopy exact sequence of the fibre bundle

$$\Omega X \to \Lambda X \xrightarrow{p} X$$

with $p(\lambda) = \lambda(1)$. This gives rise to a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_{n+1}(X, x_0) \longrightarrow \pi_n(\Omega X, [(x_0, 1)]) \longrightarrow \pi_n(\Lambda X, [(x_0, 1)]) \longrightarrow \pi_n(X, x_0) \longrightarrow \cdots,$$

see Theorem 4.41 in [3]. Consider the section

$$X \to \Lambda X \quad ; \quad x \mapsto (z \mapsto x) \quad \left(z \in S^1\right).$$

Its composite with p is the identity map on X

$$X \to \Lambda X \xrightarrow{p} X \quad ; \quad x \mapsto (z \mapsto x) \mapsto x.$$

The section induces the inverse homomorphism of $p_* : \pi_n(\Lambda X, [(x_0, 1)]) \to X$, making the exact sequence split. Thus

$$\pi_n(\Lambda X, [(x_0, 1)]) \cong \pi_n(\Omega X, [(x_0, 1)]) \oplus \pi_n(X, x_0) \tag{2.2}$$

for all n.

Proposition 2.5.2. *The fundamental group of ΛS^2 is $\mathbb{Z}$.*

Proof. Taking $X = S^2$, by (2.2) and Proposition 2.5.1, we have

$$\pi_1\left(\Lambda S^2\right) \cong \pi_1\left(\Omega S^2\right) \oplus \pi_1\left(S^2\right) \cong \mathbb{Z}.$$

□

We next investigate the smooth case. Let $\Lambda_{\text{diff}} X$ denote the subspace of ΛX of smooth loops, and $\Omega_{\text{diff}} X$ the subspace of ΩX of smooth pointed loops in X.

Proposition 2.5.3. *If X is a compact metric space, then the inclusion map*

$$\iota : \Lambda_{diff} X \hookrightarrow \Lambda X$$

is a homotopy equivalence. In particular, $\pi_1\left(\Lambda_{diff} X\right) \cong \pi_1(\Lambda X)$. This also holds for the pointed case.

Proof. Given a map $f : S^1 \to X$, by the Smooth Approximation Theorem, Theorem 11.8 in [1], (also Chapter 2 in [4]) f is homotopic to a smooth map $g : S^1 \to X$. The map g can be chosen very close to f, so that there is a continuous choice of smooth approximations. If $j : \Lambda X \to \Lambda_{\mathrm{diff}} X$ is such a continuous choice of smooth maps, then

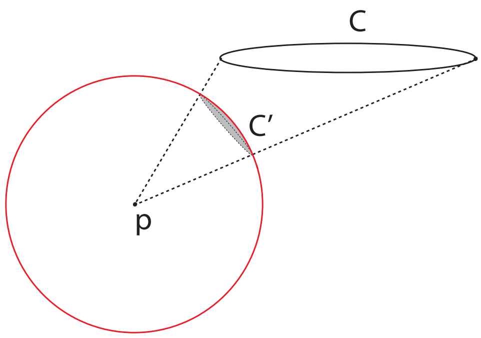
$$\iota j \simeq \mathrm{id}_{\Lambda X} : \Lambda X \to \Lambda X \quad \text{and} \quad j\iota \simeq \mathrm{id}_{\Lambda_{\mathrm{diff}} X} : \Lambda_{\mathrm{diff}} X \to \Lambda_{\mathrm{diff}} X.$$

In the pointed case, given a pointed map $f' : S^1 \to X$, by the Smooth Approximation Theorem, f' is homotopic to a smooth map $g' : S^1 \to X$ relative to the base point. The rest follows as in the previous case. □

Corollary 2.5.4. *The fundamental groups of $\Lambda_{diff} S^2$ and $\Omega_{diff} S^2$ are $\mathbb{Z}$.*

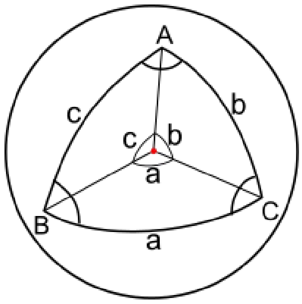
2.6 Solid angle

Definition 2.6.1. *Given an oriented loop C in $\mathbb{R}^3$ and a point $p \in \mathbb{R}^3$ disjoint from C, the normalised vector from p to each point of C traces another oriented loop C' on the unit 2-sphere with centre at p. The* solid angle *of C subtended at p is measured by the spherical surface area enclosed by C'. The sign of the solid angle depends on the choice of the spherical area, on the left or right of the curve.*



In general, given an oriented loop and a point, it is nontrivial to compute the solid angle. Chapter 6 illustrates some computation for an unknot involving elliptic integrals. If the loop consists of a finite number of line segments, it is possible to compute it.

Example 2.6.2. *Given a planar triangle in $\mathbb{R}^3$ and x a point disjoint from the triangle, we can perform the radial projection of the triangle ABC from x. The three angles in this triangle are also denoted by A, B and C. The side lengths of the spherical arcs* *are denoted by* a, b *and* c $-$ *they are also equal to the three angles at the centre of the sphere* $-$ *as in the figure below.*



The solid angle of the given planar triangle is, by definition, equal to the spherical area of ABC*, i.e,*

$$\textit{Solid angle} = A + B + C - \pi.$$

This quantity is known as the spherical excess. The values A, B *and* C *are related to* a, b *and* c *by the cosine rules*

$$\begin{aligned}\cos A &= \frac{\cos a - \cos b \cos c}{\sin b \sin c};\\ \cos B &= \frac{\cos b - \cos a \cos c}{\sin a \sin c};\\ \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b},\end{aligned}$$

where a, b *and* c *can be computed directly from the plane triangle. See more detailed information in [10] and [19].*

Another description regarding solid angles appears in *A Treatise On Electricity and Magnetism* $-$ *Volume II*, [9], by James Clerk Maxwell. He gave several methods to compute the solid angle, one of which comes from physics. It turns out that the solid angle of an oriented loop subtended at a point can be regarded as the magnetic potential of a shell of unit strength whose boundary is the loop. Thus, the solid angle is equal to the work done by bringing a unit magnetic pole from infinity to the given point against the magnetic force from the shell. Let $C : [0, 1] \to \mathbb{R}^3$ be a loop and

$$P : (0, 1] \to \mathbb{R}^3 \quad ; \quad t \mapsto (\xi(t), \eta(t), \zeta(t))$$

be a curve from infinity to the given point $P(1) = (\xi(1), \eta(1), \zeta(1))$ that does not pass through the shell. The solid angle is given by the formula

$$\int\!\!\int -\frac{1}{r^3} \det \begin{pmatrix} \xi - x & \eta - y & \zeta - z \\ \frac{d\xi}{ds} & \frac{d\eta}{ds} & \frac{d\zeta}{ds} \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{pmatrix} ds dt, \tag{2.3}$$

where $r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}$, and the integral with respect to s and t

means integrating along P and C, respectively. Moreover, this integral is independent from the choice of the curve P as long as P does not pass through the shell.

Chapter 3

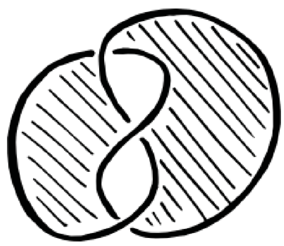
Seifert surfaces and their constructions

In this chapter, we first introduce the notion of closed and open Seifert surfaces. We next discuss a classical construction of such a closed surface invented by Seifert. We end the chapter with a construction of a (closed or open) Seifert surface using transversality.

3.1 Closed and open Seifert surfaces

Definition 3.1.1. *A* closed Seifert surface Σ *of a knot* K *in* $\mathbb{R}^3$ *(or* S^3*) is a compact orientable 2-manifold embedded in* $\mathbb{R}^3$ *(or* S^3*) such that* $\partial\Sigma = K(S^1)$.

Example 3.1.2.



It is not hard to see that the shaded surface has 2 sides; so it is orientable. The boundary of the surface is a trefoil knot. Hence, this is a closed Seifert surface of a trefoil knot. □

Let us introduce the notion of *open* Seifert surfaces. Recall that x is a *topological boundary point* of a subspace A of a topological space X if for each open neighbourhood U of x in X,

$$U \cap A \neq \varnothing \quad \text{and} \quad U \cap (X - A) \neq \varnothing.$$

The set of topological boundary points of A is called the *topological boundary* of A in X. The topological boundary is not canonical – it depends on the ambient space. Note that two concepts of topological boundary and boundary of manifolds are different.

For example, $S^1 \subset \mathbb{R}^2$ is a 1-manifold without boundary with topological boundary S^1. The topological space D^1 has empty topological boundary, but it is a surface with boundary S^1.

Proposition 3.1.3. *If X is an n-manifold with nonempty boundary embedded into $\mathbb{R}^{n+k}$, then ∂X is the topological boundary of X in $\mathbb{R}^{n+k}$. Moreover, the topological boundary of X is independent from k.*

Proof. Since ∂X is the boundary of X, there exists an embedding $\partial X \times [0, \infty) \hookrightarrow X$. This implies that for each $x \in \partial X$ and for each open neighbourhood U of x in $\mathbb{R}^{n+k}$, $U \cap X \neq \varnothing$.

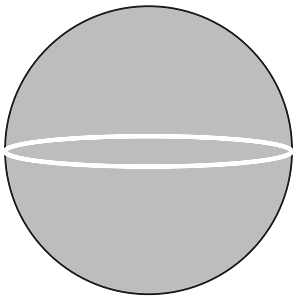
Supposing there were some open neighbourhood V of x in $\mathbb{R}^{n+k}$ such that $V \cap \big(\mathbb{R}^{n+k} - X\big) = \varnothing$, the manifold X would contain an $(n+k)$-dimensional subspace. This is a contradiction. □

We are now ready to define an open Seifert surface.

Definition 3.1.4. *An* open Seifert surface *of a knot is an orientable embedded 2-manifold having the knot as a topological boundary.*

Example 3.1.5. *(i) If Σ is a closed Seifert surface of a knot in $\mathbb{R}^3$ or S^3, then $\Sigma - \partial\Sigma$ is an open Seifert surface.*

(ii) The closure in $\mathbb{R}^3$ or S^3 of an open Seifert surface is not always a closed Seifert surface. Let K be an unknot defined as the standard unit circle on the xy-plane. Clearly, $K(S^1)$ is a subspace of S^2. Then $S^2 - K(S^1)$ is an open Seifert surface of K because the topology boundary of $S^2 - S^1$ in $\mathbb{R}^3$ is $K(S^1)$. Since the closure of $S^2 - S^1$ in $\mathbb{R}^3$ is S^2, it is not a closed Seifert surface of K.



$S^2 - S^1$

□

In this chapter we present two methods for constructing a closed Seifert surface.

3.2 A combinatorial construction of a closed Seifert surface

In 1934 Seifert, [16], showed the existence of a closed Seifert surface of a knot:

Theorem 3.2.1. *Every knot has a closed Seifert surface.*

The proof proceeds by constructing a closed Seifert surface for a knot. This construction is called *Seifert's algorithm* and the steps are as follows.

(1) Choose a knot projection and orient the knot;

(2) Remove the crossings by joining each incoming strand to the adjacent outgoing strand, creating a finite number of circles, called *Seifert circles*;

(3) Fill in the interior of each circle to obtain a disc;

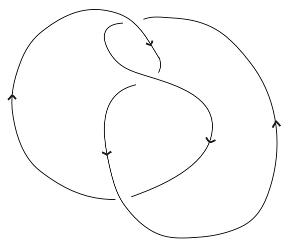
(4) Attach twisted bands to those discs according to the removed crossings.

These 4 steps give a surface bounded by the knot. We explain why this surface is orientable as follows. In Step (3), we can assign $\pm$ to those discs depending on the orientation of the Seifert circles; if it is counterclockwise, assign $+$. Hence, according to Steps (2) and (3), two adjacent discs must have opposite signs, and if two adjacent discs are nested then they must have the same sign. In Step (4), we can see that each attaching results a two-sided surface. Since the number of crossings is finite, the resulting surface must be orientable.

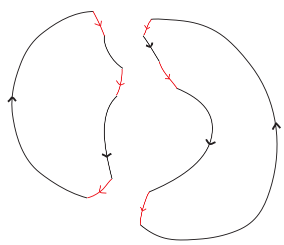
Notice also that this construction of a closed Seifert surface depends on the knot diagram.

Example 3.2.2. *We will perform Seifert's algorithm to produce a closed Seifert surface of a trefoil knot.*

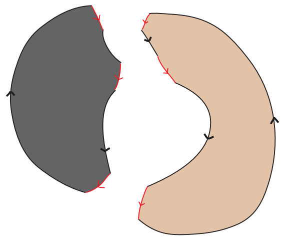
(1) Choose a knot diagram of the trefoil knot and orient the knot.



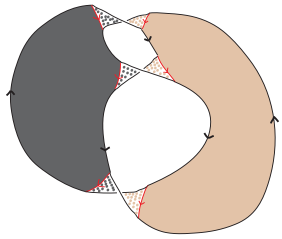
(2) Now we remove all the crossings and join the red strands according to the orientation.



(3) Each circle spans a disc.



(4) Attach three twisted bands corresponding to the three crossings removed in Step (2).



□

3.3 A transversality construction of a closed Seifert surface

This construction is a direct consequence of the regular value theorem, see Lemma 1 in [11]. Recall that $c \in N$ is a regular value of a smooth function $f : M \to N$ if the differential $d_x f$ is surjective for all $x \in f^{-1}(c)$.

Theorem 3.3.1. (Regular value theorem) *Let M^m and N^n be differential manifolds and c be a regular value of a smooth map $f : M \to N$. Then $f^{-1}(c)$ is a submanifold of M of dimension $m - n$. If $g : (M^m, \partial M) \to (N^n, \partial N)$ is a smooth map between manifolds with boundary and c is a regular value of both g and $g| : \partial M to \partial N$, then $g^{-1}(c)$ is a manifold with boundary $(g|)^{-1}(c)$.*

Any knot in $\mathbb{R}^3$ can be viewed as a knot in $\mathbb{R}^3 \cup \{\infty\} = S^3$, and vice versa. Hence, for simplicity, let us work with knots in S^3.

Now let $K : S^1 \hookrightarrow S^3$ be a smooth knot and let X denote the knot exterior $\mathrm{Cl}_{S^3}(S^3 - (K(S^1) \times D^2))$ with boundary $\partial X = K(S^1) \times S^1$. By the regular value theorem, if a smooth map $f : X \to S^1$ has the restriction

$$f|_{\partial X} : K(S^1) \times S^1 \to S^1 \quad ; \quad (x, y) \mapsto y$$

and $z \in S^1$ is a regular value of both f and $f|_{\partial X}$, then

$$\big(\Sigma, K(S^1) \times \{z\}\big) = \big(f^{-1}(z), f|_{\partial X}^{-1}(z)\big)$$

is a closed Seifert surface for the knot K. Observe that $f|_{\partial X} = p_0 : K(S^1) \times S^1 \to S^1$ is the projection map onto S^1, and f is then an extension of p_0. This implies that if we can extend p_0 over X, then we obtain a Seifert surface for the knot K.

Let us now prove a fact if an extension $f : X \to S^1$ of p_0 exists.

Proposition 3.3.2. *If $f : X \to S^1$ is an extension of the projection map p_0, then the induced homomorphism $f_* : H_1(X) \to H_1(S^1)$ is given by linking number, i.e., for any knot L in S^3 disjoint from K, we have*

$$f_*([L]) = \mathrm{Linking}(K, L).$$

Proof. Since $H_1(X) = \mathbb{Z}$ is generated by the class of meridians, it is enough to show that f_* maps any meridian of K to 1. Fixing a point $x \in K(S^1)$, let m be the inclusion $S^1 \hookrightarrow \{x\} \times S^1 \subset X$ that defines a meridian of the knot K. Since $fm = p_0 m$, it follows that $f_*([m]) = \deg fm = 1$. □

Let f and m be defined as in the previous proposition. If $g : X \to S^1$ is another smooth map such that $g_* : H_1(X) \to H_1(S^1)$ is given by linking number, then f and g are homotopic. To show this, we use the following fact.

Proposition 3.3.3. *Let Y be any space and $\mathrm{Map}(Y, S^1)$ denote the set of all maps $Y \to S^1$. The following statements hold:*

(i) $\mathrm{Map}(Y, S^1)$ is an abelian group with $(f + g)(y) = f(y) \cdot g(y)$, where $\cdot$ is the multiplication on S^1. So is the set of homotopy classes $[Y, S^1]$ of maps $Y \to S^1$.

(ii) The group $[Y, S^1]$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(H_1(Y), \mathbb{Z})$ via $[f] \mapsto f_* : H_1(Y) \to \mathbb{Z}$.

Proof. (i) Obvious.

(ii) It is clear that $f \mapsto f_*$ is a homomorphism. Now, given a homomorphism $\varphi : H_1(Y) \to \mathbb{Z}$, we can construct a map $g : X \to S^1$ such that $g_* = \varphi$. See Theorem 7.1, Section 7, Chapter 2 in [5] for the proof. □

By Proposition 3.3.3, f and g are homotopic since they have the same induced homomorphism.

We are now ready to state the existence theorem of a closed Seifert surface.

Theorem 3.3.4. *Let $K : S^1 \hookrightarrow S^3$ be a knot and X be the knot exterior of K. Then there exists a unique homotopy class of maps $X \to S^1$ which induces*

$$H_1(X) \to H_1(S^1) = \mathbb{Z} \quad ; \quad [L] \mapsto \mathrm{Linking}(K, L)$$

for every knot $L : S^1 \hookrightarrow S^3 - K(S^1)$. In particular, a smooth map in this homotopy class determines a closed Seifert surface for K as a preimage of a regular value.

We have already shown the uniqueness of the homotopy class. It remains to explain how one can extend the projection map $p_0 : \partial X = K(S^1) \times S^1 \to S^1$ over X; this will be Proposition 3.3.6. The following lemma plays an important role in the proof of the proposition.

Lemma 3.3.5. *The Poincaré dual* $[l]^* \in H^1(\partial X)$ *of a canonical longitude of* K *corresponds to the induced homomorphism*

$$(p_0)_* : H_1(\partial X) \to H_1(S^1) = \mathbb{Z}.$$

Proof. The homology group $H^1(\partial X) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by the class of meridians $[m]$ and the class of canonical longitudes $[l]$. Also, we know that

$$[m] \cap [l]^* = [m \cap l] = 1 \quad \text{and} \quad [l] \cap [l]^* = 0.$$

Since

$$(p_0)_* \in \mathrm{Hom}_{\mathbb{Z}}(H_1(\partial X), \mathbb{Z}) = H^1(\partial X)$$

and

$$(p_0)_*([x]) = [x] \cap (p_0)_* \in H_1(S^1) = \mathbb{Z}$$

for all $[x] \in H_1(\partial X)$, it follows that

$$[m] \cap (p_0)_* = \deg\ p_0 m = 1 \quad \text{and} \quad [l] \cap (p_0)_* = \deg\ p_0 l = 0.$$

Thus, $(p_0)_* \in H^1(\partial X)$ is the Poincaré dual $[l]^*$ of the canonical longitude l. $\square$

Proposition 3.3.6. *The projection map* $p_0 : \partial X = K(S^1) \times S^1 \to S^1$ *extends to a map* $X \to S^1$.

Proof. We know that the homotopy class of p_0 corresponds to

$$(p_0)_* \in \mathrm{Hom}_{\mathbb{Z}}(H_1(\partial X, \mathbb{Z})) = H^1(\partial X).$$

Consider the *Poincaré-Lefschetz duality* diagram, see 6.25 in [20],

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(X) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^1(\partial X) \arrow[r] \arrow[d, "PD"] & H^2(X, \partial X) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_2(X, \partial X) \arrow[r] & H_1(\partial X) \arrow[r, "i_*"'] & H_1(X) \arrow[r] & \cdots ,
\end{tikzcd}$$

where both rows are exact sequences of cohomology and homology groups for the pair $(X, \partial X)$. We shall show that $(p_0)_*$ belongs to the image of $i^* : H^1(X) \to H^1(\partial X)$. By Lemma 3.3.5, $(p_0)_*$ is the Poincaré dual $[l]^* \in H^1(\partial X)$ of the class of canonical longitudes $[l]$. Hence, $PD((p_0)_*) = [l]$. Since $i_*([l])$ becomes trivial in $H_1(X)$, it follows that

$$(p_0)_* \in \ker(H^1(\partial X) \to H^2(X, \partial X)) = \mathrm{im}\ i^*.$$

Since $[\partial X, S^1] \cong H^1(\partial X)$, there exists an extension $X \to S^1$ of p_0. $\square$

We remark that the extension of p_0 from Proposition 3.3.6 may not be smooth. However, such an extension is homotopic rel ∂X to a smooth map $f : X \to S^1$.

We have already constructed a Seifert surface, embedded in S^3, for K which appears as the preimage of a regular value of a smooth map

$$f : \mathrm{Cl}_{S^3}\left(S^3 - (K(S^1) \times D^2)\right) \to S^1$$

extending $p_0 : K(S^1) \times S^1 \to S^1 \quad ; \quad (x, y) \mapsto y$. Now consider the restriction

$$f| : \mathrm{Cl}_{\mathbb{R}^3}\left(\mathbb{R}^3 - (K(S^1) \times D^2)\right) \to S^1$$

of f. We can see that $f|$ is a smooth extension of p_0 over the knot exterior in $\mathbb{R}^3$ because we ignore only the point $\infty \in \mathrm{Cl}_{S^3}\left(S^3 - (K(S^1)\right)$. Hence, if $* \neq f(\infty)$ is a regular value of f, then $(f|)^{-1}(*)$ is a closed Seifert surface embedded in $\mathbb{R}^3$ for K.

3.4 A transversality construction of an open Seifert surface

In Section 3.3, a closed Seifert surface in $\mathbb{R}^3$ is obtained as the preimage of a regular value of a smooth map

$$f| : \mathrm{Cl}_{\mathbb{R}^3}\left(\mathbb{R}^3 - (K(S^1) \times D^2)\right) \to S^1$$

extending the projection

$$p_0 : K(S^1) \times S^1 \to S^1 \quad ; \quad (x, y) \mapsto y \ .$$

Here we study a similar situation for open Seifert surfaces.

Consider the knot complement $\mathbb{R}^3 - K(S^1)$ of the knot K. We shall show that an open Seifert surface of K can also be obtained as the preimage of a regular value of a smooth map

$$g : \mathbb{R}^3 - K(S^1) \to S^1$$

with some certain property. Notice that we drop the condition that g is an extension of p_0.

Proposition 3.4.1. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be a knot and $m : S^1 \hookrightarrow \mathbb{R}^3 - K(S^1)$ be a meridian of K. If $g : \mathbb{R}^3 - K(S^1) \to S^1$ is a smooth map such that*

$$g_* : H_1\left(\mathbb{R}^3 - K(S^1)\right) = \mathbb{Z} \to H_1(S^1) = \mathbb{Z} \quad ; \quad [\mu] \mapsto \mathrm{Linking}(K, m)$$

where $[\mu]$ is the homology class representing m, then the preimage $g^{-1}(c)$ of a regular value $c \in S^1$ is an open Seifert surface for K.

Proof. By the regular value theorem, the preimage $\Sigma_0 = g^{-1}(c)$ is an open surface embedded into $\mathbb{R}^3 - K(S^1)$. It remains to show that $K(S^1)$ is a topological boundary of Σ_0 in $\mathbb{R}^3$.

Let $z \in K(S^1)$ and U be an open neighbourhood in $\mathbb{R}^3$ containing z. Then, U must contain a small meridian $m : S^1 \hookrightarrow \mathbb{R}^3 - K(S^1)$ of the knot K. Recall that the naturality of the Hurewicz map h between the fundamental group and the first homology group, i.e., the diagram

$$\begin{tikzcd}
\pi_1\left(\mathbb{R}^3 - K(S^1)\right) \arrow[r] \arrow[d, "h"'] & \pi_1(S^1) = \mathbb{Z} \arrow[d, "\cong"] \\
H_1\left(\mathbb{R}^3 - K(S^1)\right) \arrow[r, "g_*"] & H_1(S^1) = \mathbb{Z}
\end{tikzcd}$$

is commutative. Note that if $[m] \in \pi_1\left(\mathbb{R}^3 - K(S^1)\right)$, then $[gm] \in \pi_1\left(S^1\right)$ is equal to

$$g_*\left(h([m])\right) = g_*(\mu) = \mathrm{Linking}(K, m) = 1;$$

hence $gm(S^1) = S^1$ and $m(S^1)$ must intersect Σ_0. This implies $U \cap \Sigma_0 \neq \varnothing$.

Suppose that U were contained in Σ_0. Then, $gm(S^1)$ would be $\{c\}$ and $[gm] \in \pi_1(S^1) = \mathbb{Z}$ would be equal to 0, a contradiction. Thus, $U \cap \left(\mathbb{R}^3 - \Sigma_0\right) \neq \varnothing$. $\square$

We remark that Proposition 3.3.3 implies that if $g' : \mathbb{R}^3 - K(S^1) \to S^1$ is another smooth map such that

$$g'_* : H_1\left(\mathbb{R}^3 - K(S^1)\right) = \mathbb{Z} \to H_1(S^1) = \mathbb{Z} \quad ; \quad [\mu] \mapsto \mathrm{Linking}(K, m),$$

then g and g' are homotopic.

One may ask: when is a closed Seifert surface obtained from an open counterpart? We first notice that the condition "g does not have to be an extension of p_0" we have dropped weakens the geometry of the open Seifert surface $\Sigma_0 = g^{-1}(c)$ in the sense that Π may be wild near the knot, and in that case it cannot be compactified to be a closed Seifert surface. Thus, if Σ_0 behaves "nicely" near the knot, then a closed Seifert surface can be obtained from Σ_0.

We say that an open Seifert surface Σ_0 for the knot K is *regular* near K if there exists a (topological) embedding

$$K(S^1) \times [0, 1] \to \Sigma_0 \cup K(S^1)$$

such that the restriction

$$K(S^1) \times (0, 1] \to \Sigma_0$$

is smooth.

Therefore, the answer to the question above is that if an open Seifert surface Σ_0 for the knot K is bounded and regular near K, then a closed Seifert surface for K can be obtained form Σ_0.

Chapter 4

Definition of the map FG

The aim of this chapter is to define a map

$$\mathbb{R}^3 - K(S^1) \to S^1$$

that induces an isomorphism between the homology groups, $\mathbb{Z}$. Such a map is constructed as the composition of two maps

$$G : \mathbb{R}^3 - K(S^1) \to \Lambda_{diff} S^2$$

and

$$F : \Lambda_{diff} S^2 \to \mathbb{R}/4\pi\mathbb{Z} \cong S^1.$$

The map G depends on the given knot K whereas the map F is independent of K.

4.1 Definition of F

We wish to associate to each smooth loop in S^2 a real number modulo 4π. For each smooth loop $\lambda : S^1 \to S^2$, we associate

$$F(\lambda) = \int_{D^2} \delta\lambda^* \omega \in \mathbb{R}$$

where $\delta\lambda : D^2 \to S^2$ is a smooth extension of λ and ω is a volume 2-form on S^2 with $\int_{S^2} \omega = 4\pi$.

The extension $\delta\lambda$ exists since λ is nullhomotopic, but it is not unique. Hence, different extensions may be associated with different real numbers. It turns out, however, that the difference between those numbers is a multiple of 4π.

Proposition 4.1.1. *The real number $F(\lambda)$ is uniquely defined in $\mathbb{R}/4\pi\mathbb{Z} \cong S^1$, independent of the extension $\delta\lambda$.*

Proof. Let $\delta\lambda' : D^2 \to S^2$ be another extension of λ. Define

$$g = \delta\lambda \cup -\delta\lambda' : D^2 \cup_{S^1} -D^2 \to S^2.$$

Notice that $S^2 \cong D^2 \cup_{S^1} -D^2$ and we can select the orientation on S^2 so that g is orientation-preserving. We then have

$$\begin{aligned}
\int_{S^2} g^*(\omega) &= \int_{D^2} \delta\lambda^*\omega + \int_{-D^2} \delta\lambda'^*\omega \\
&= \int_{D^2} \delta\lambda^*\omega - \int_{D^2} \delta\lambda'^*\omega.
\end{aligned}$$

Since

$$\int_{S^2} g^*\omega = \deg g \int_{S^2} \omega \quad \text{and} \quad \int_{S^2} \omega = 4\pi,$$

we have

$$\int_{D^2} \delta\lambda^*\omega - \int_{D^2} \delta\lambda'^*\omega = 4\pi \deg g \in 4\pi\mathbb{Z}.$$

□

In order to emphasise that $F(\lambda)$ represents an equivalence class in $\mathbb{R}/4\pi\mathbb{Z}$, we may write

$$F(\lambda) = \int_{D^2} \delta\lambda^*\omega \mod 4\pi.$$

Example 4.1.2. *In simple cases, we can construct an extension of a loop in S^2 easily. For instance, the unit circle*

$$\lambda : S^1 \to S^2 \quad ; \quad \theta \mapsto (\cos\theta, \sin\theta, 0)$$

can be extended as

$$\delta\lambda : D^2 \to S^2 \quad ; \quad (\theta, r) \mapsto (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$$

with image the upper hemisphere. The standard volume 2-form on S^2 is

$$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

The pullback form $\delta\lambda^*(\omega)$ is

$$\begin{aligned}
\delta\lambda^*(\omega) &= r\cos\theta d(r\sin\theta) \wedge d(\sqrt{1-r^2}) + r\sin\theta d(\sqrt{1-r^2}) \wedge d(r\cos\theta) \\
&\quad + \sqrt{1-r^2} d(r\cos\theta) \wedge d(r\sin\theta) \\
&= \frac{r}{\sqrt{1-r^2}} dr \wedge d\theta.
\end{aligned}$$

Thus,

$$F(\lambda) = \int_{D^2} \frac{r}{\sqrt{1-r^2}} dr \wedge d\theta = \int_0^{2\pi} \int_0^1 \frac{r}{\sqrt{1-r^2}} dr d\theta = 2\pi.$$

□

Remark 4.1. Any injective loop (simple closed curve) in S^2 divides S^2 into two connected components. If $\lambda : S^1 \to S^2$ is such a loop, then we are able to choose an extension $\delta\lambda : D^2 \to S^2$ such that $\delta\lambda(D^2 - S^1)$ is one of the two connected components of $S^2 - \operatorname{im} \lambda$. This implies that $F(\lambda)$ is equal to the (signed) area of that connected component.

4.2 Properties of F

This section shows that the map F induces isomorphisms between both fundamental groups and homology groups.

Proposition 4.2.1. *The induced homomorphism*

$$F_* : \pi_1 \left(\Lambda_{diff} S^2\right) \to \pi_1 \left(S^1\right)$$

is an isomorphism.

Proof. Let $i : \Omega_{diff} S^2 \hookrightarrow \Lambda_{diff} S^2$ be the inclusion. It is sufficient to show that the restriction $F|_{\Omega_{diff} S^2} = Fi$ induces an isomorphism

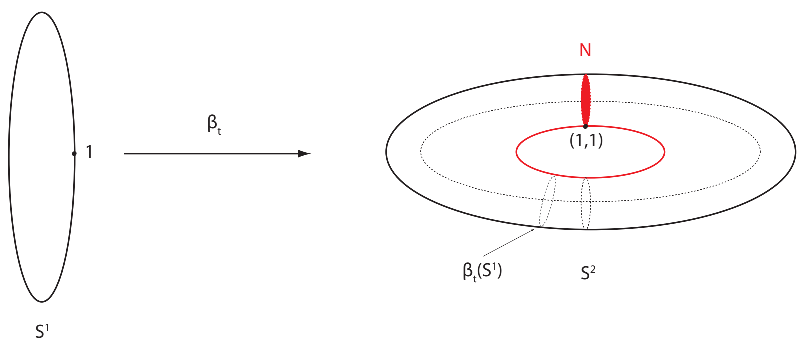
$$\left(F|_{\Omega_{diff} S^2}\right)_* = F_* i_* : \pi_1 \left(\Omega_{diff} S^2\right) \to \pi_1 \left(S^1\right)$$

since $i_* : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism.

Let N be the north pole of S^2. A generator of $\pi_1 \left(\Omega_{diff} S^2\right) \cong \mathbb{Z}$ is given by the loop

$$\beta : S^1 \to \Omega_{diff} S^2 \quad ; \quad t \mapsto (\beta_t : s \mapsto s \wedge t)$$

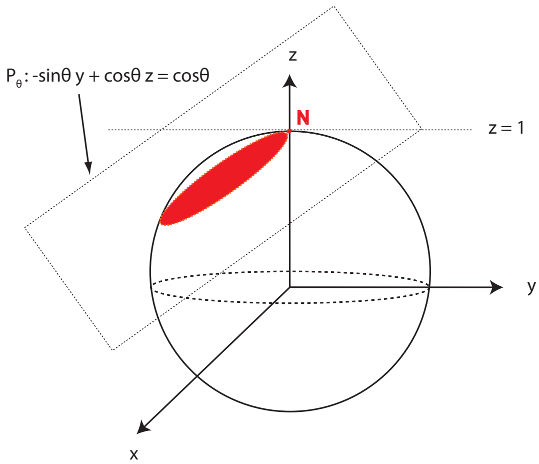
where $s \wedge t \in S^1 \wedge S^1 = \dfrac{S^1 \times S^1}{S^1 \vee S^1} = S^2$.



From the picture, the red part is identified as the base point N of S^2. For each $t \in S^1$, the loop β_t is based at N. If $t = 1 \in S^1$, the loop

$$\beta_1 : s \mapsto s \wedge 1 = N$$

is a constant loop at the base point N. Let $P_\theta : -\sin\theta y + \cos\theta z = \cos\theta$ be the plane obtained by rotating the plane $z = 1$ anticlockwise about the line $\{y = 0, z = 1\}$ with angle $\theta \in [0, \pi)$ and let C_θ be the intersection of P_θ and S^2.



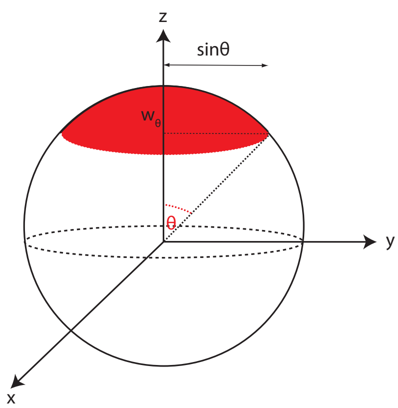
Note that if (a, b, c) is a point in the intersection, then

$$a^2 + (b + \sin\theta\cos\theta)^2 + \left(c - \cos^2\theta\right)^2 = \sin^2\theta.$$

So, the intersection is the circle of radius $\sin\theta$ lying on S^2 with centre $\left(0, -\sin\theta\cos\theta, \cos^2\theta\right)$. There is a one-to-one correspondence between β_t and C_θ for all $t \in S^1$ and $\theta \in [0, \pi)$. By Remark 4.1, we know that $F(\beta_t)$ is equal to the area of one of the two connected components of $S^2 - \operatorname{im}\beta_t$. For each $t \in S^1$, consider $\theta \in [0, \pi)$ corresponding to t and

$$W_\theta := \left\{(x, y, z) \in S^2 | \quad (x, y, z) \in P_{\theta'} \quad \text{for some} \quad 0 \leqslant \theta' < \theta\right\}.$$

We would like to find the area of W_θ. To do so, we rotate W_θ to the standard position.



Let $\varphi \in [0, 2\pi]$ and $\gamma \in [0, \pi]$ represent the angles in spherical coordinates. Then the area of W_θ can be computed by the integral

$$\begin{aligned}\int_0^\theta \int_0^{2\pi} \sin\gamma d\varphi d\gamma &= 2\pi \int_0^\theta \sin\gamma d\gamma \\ &= 2\pi\left(1 - \cos\theta\right).\end{aligned}$$

Hence

$$F\left(\beta_t\right) = 2\pi\left(1 - \cos\theta\right) \in [0, 4\pi).$$

Note that there is a one-to-one correspondence between $\theta \in [0, \pi)$ and $2\pi(1-\cos\theta) \in [0, 4\pi)$. Thus, there is a one-to-one correspondence between $t \in S^1$ and $2\pi(1-\cos\theta) \in [0, 4\pi)$. This implies that

$$F_*\left(\beta\right) : S^1 \to \mathbb{R}/4\pi\mathbb{Z} \quad ; \quad t \mapsto 2\pi(1-\cos\theta)$$

is a generator of $\pi_1\left(S^1\right)$. Therefore $\left(F|_{\Omega_{diff}S^2}\right)_*$ maps a generator of $\pi_1\left(\Omega_{diff}S^2\right)$ to a generator of $\pi_1\left(S^1\right)$, and hence an isomorphism. $\square$

Corollary 4.2.2. *The induced map* $F_*^H : H_1\left(\Lambda_{diff}S^2\right) \cong \mathbb{Z} \to H_1(S^1) \cong \mathbb{Z}$ *is an isomorphism.*

Proof. It follows from Hurewicz's theorem and the naturality of Hurewicz maps. $\square$

4.3 Definition of G

Given a smooth knot $K : S^1 \subset \mathbb{R}^3$, we wish to associate to each point in $\mathbb{R}^3 - K(S^1)$ a loop in S^2. Define

$$G : \mathbb{R}^3 - K(S^1) \to \Lambda_{diff}S^2 \quad ; \quad u \mapsto \left(G(u) : S^1 \to S^2\right)$$

with

$$G\left(u\right)\left(y\right) = \frac{K\left(y\right) - u}{\left\|K\left(y\right) - u\right\|}.$$

Since K is smooth, so is $G(u)$ for all $u \in \mathbb{R}^3 - K(S^1)$. Hence, the definition of G is well-defined. We remark that the definition of the map G depends on the knot.

Geometrically, G is the collection of the projections of the knot K onto S^2 from all the points in $\mathbb{R}^3 - K(S^1)$.

Example 4.3.1. *Let* $K : S^1 \to \mathbb{R}^3$ *be the unknot in* $\mathbb{R}^3$ *given by*

$$K(\theta) = (\cos\theta, \sin\theta, 0)$$

for $\theta \in [0, 2\pi)$, and let $u = (u_1, u_2, u_3) \in \mathbb{R}^3 - K(S^1)$. By the definition of G, we have

$$G(u)(\theta) = \frac{(\cos\theta - u_1, \sin\theta - u_2, -u_3)}{\sqrt{(\cos\theta - u_1)^2 + (\sin\theta - u_2)^2 + u_3^2}}.$$

If $u_1 = u_2 = 0$, we have

$$G(0, 0, u_3)(\theta) = \frac{(\cos\theta, \sin\theta, -u_3)}{\sqrt{1 + u_3^2}}.$$

This matches our geometric intuition that we see a circle when we look at the unknot from a point on z-axis.

It is more interesting when $u_3 = 0$. Notice that

$$G(u_1, u_2, 0)(\theta) = \frac{(\cos\theta - u_1, \sin\theta - u_2, 0)}{\sqrt{u_1^2 + u_2^2 + 1 - 2u_1\cos\theta - 2u_2\sin\theta}}.$$

If $u_1^2 + u_2^2 < 1$, then $G(u_1, u_2, 0)(\theta)$ is injective. If $u_1^2 + u_2^2 > 1$, then there will be two values of θ projected to the same point in S^2. Imagine that we look at the unknot from a point on the xy- plane. We see a circle (the unknot) if we are inside the open unit disc, but we see only an arc if we are outside.

It is slightly more complicated when $u_1, u_2, u_3 \neq 0$. In this case, we see an ellipse. To see this, we draw a cone having the unknot as the base and having u as the vertex. The image $G(u)(S^1)$ is the intersection of this cone and the unit sphere centred at u. Equivalently, $G(u)(S^1)$ is obtained by intersecting the cone with some plane perpendicular to the radius vector of this unit sphere. Since the plane is not parallel to the base of the cone, the intersection is an ellipse. $\square$

The example above shows that G gives a collection of the projections of the unknot onto S^2 from all the points outside the unknot. At most points, the projections are injective loops.

4.4 Properties of G

We investigate some properties of the induced homomorphisms of G on the level of fundamental groups and homology groups.

Proposition 4.4.1. *The induced homomorphism*

$$G_* : \pi_1\left(\mathbb{R}^3 - K(S^1)\right) \to \pi_1\left(\Lambda_{diff} S^2\right) \cong \mathbb{Z}$$

sends any meridian m of the knot K to 1. Moreover, if $L : S^1 \hookrightarrow \mathbb{R}^3 - K(S^1)$ is another knot, then $G_*([L]) = \mathrm{Linking}(K, L)$.

Proof. Let $m : S^1 \hookrightarrow \mathbb{R}^3 - K(S^1)$ be a meridian of K. Then, $\mathrm{Linking}(K, m) = 1$. Note that for each $x, y \in S^1$,

$$G\left(m(y)\right)(x) = \frac{K(x) - m(y)}{\|K(x) - m(y)\|} = \Psi_{K,m}(x, y)$$

where $\Psi_{K,m}$ is the Gauss map of K and m. Since $G_*\left([m]\right) = [Gm] \in \pi_1\left(\Lambda_{diff} S^2\right)\right) \cong \mathbb{Z}$ and Gm defines the Gauss map $\Psi_{K,m} : S^1 \times S^1 \to S^2$, it follows that $[Gm]$ represents the degree of $\Psi_{K,m}$, i.e.,

$$[Gm] = \deg \Psi_{K,m} = \text{Linking}(K, m) = 1.$$

The second statement follows by replacing the meridian m by the knot $L : S^1 \hookrightarrow \mathbb{R}^3 - K(S^1)$. $\square$

Corollary 4.4.2. *The induced homomorphism $G_*^H : H_1\left(\mathbb{R}^3 - K(S^1)\right) \cong \mathbb{Z} \to H_1\left(\Lambda_{diff} S^2\right) \cong \mathbb{Z}$ of G is an isomorphism.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_1\left(\mathbb{R}^3 - K(S^1)\right) & \xrightarrow{G_*} & \pi_1\left(\Lambda_{diff} S^2\right)\right) \cong \mathbb{Z} \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \cong} \\
H_1\left(\mathbb{R}^3 - K(S^1)\right) & \xrightarrow{G_*^H} & H_1\left(\Lambda_{diff} S^2\right)\right) \cong \mathbb{Z}
\end{array}$$

where h is the Hurewicz map. We know that h sends all the meridians to the homology class of meridians $[\mu] \in H_1\left(\mathbb{R}^3 - K(S^1)\right) \cong \mathbb{Z}$ and the class $[\mu]$ is a generator of $H_1\left(\mathbb{R}^3 - K(S^1)\right)$. Since

$$G_*^H\left([\mu]\right) = G_*[m] = [Gm] = 1,$$

G_*^H sends a generator of $H_1\left(\mathbb{R}^3 - K(S^1)\right) \cong \mathbb{Z}$ to a generator of $H_1\left(\Lambda_{diff} S^2\right) \cong \mathbb{Z}$. Thus, G_*^H is an isomorphism. $\square$

4.5 Properties of FG

We have already seen some properties of the maps F and G regarding their induced homomorphisms. We next study the induced homomorphism

$$(FG)_*^H : H_1\left(\mathbb{R}^3 - K(S^1))\right) \cong \mathbb{Z} \to H_1(S^1) \cong \mathbb{Z}$$

of the composite map FG.

Proposition 4.5.1. *The induced homomorphism*

$$(FG)_*^H : H_1\left(\mathbb{R}^3 - K(S^1))\right) \cong \mathbb{Z} \to H_1(S^1) \cong \mathbb{Z}$$

is the isomorphism given by

$$(FG)^H_*([L]) = \text{Linking}(K, L)$$

for every knot $L : S^1 \hookrightarrow \mathbb{R}^3 - K(S^1)$.

Proof. By Corollary 4.2.2,

$$F^H_* : H_1\left(\Lambda_{diff} S^2\right) \to H_1\left(S^1\right))$$

is an isomorphism, since F_* is an isomorphism between the fundamental groups. By Corollary 4.4.2,

$$G^H_* : H_1\left(\mathbb{R}^3 - K(S^1)\right) \to H_1\left(\Lambda_{diff} S^2\right)$$

is an isomorphism. Thus, $(FG)^H_* = F^H_* G^H_*$ is also an isomorphism.

Let $m : S^1 \hookrightarrow \mathbb{R}^3 - K(S^1)$ be a meridian of K with $[m] \in \pi_1\left(\mathbb{R}^3 - K(S^1)\right)$ and let $[\mu] \in H_1\left(\mathbb{R}^3 - K(S^1)\right)$ be the class corresponding to $[m]$ via Hurewicz map h. We wish to show that $(FG)^H_*([\mu]) = 1 = \text{Linking}(K, m)$. By Proposition 4.4.1, we know that $G_*[m] = [Gm] \in \pi_1\left(\Lambda_{diff} S^2\right) \cong \mathbb{Z}$ represents 1. Hence,

$$(FG)^H_*([\mu]) = F^H_* G^H_*([\mu]) = F^H_* G^H_* h([m]) = F^H_*(hG_*([m])) = F^H_*(h[Gm]).$$

Since $h[Gm] \in H_1(\Lambda_{diff} S^2)$ is a generator and F^H_* is an isomorphism, we obtain $(FG)^H_*([\mu]) = 1 = \text{Linking}(K, m)$.

Let L be a knot in $\mathbb{R}^3 - K(S^1)$ with $[L] \in \pi_1(\mathbb{R}^3 - K(S^1))$. Then, $h[L] \in H_1(\mathbb{R}^3 - K(S^1))$. Since

$$h[L] = \text{Linking}(K, L)[\mu],$$

we have

$$(FG)^H_*(h[L]) = \text{Linking}(K, L)(FG)^H_*([\mu]) = \text{Linking}(K, L)$$

as desired.

□

A geometric interpretation of the composite function FG can be described when K is an unknot.

Example 4.5.2. *By Example 4.3.1, $G(u)$ is an injective loop for most points u. However, if u is on the xy-plane with $\|u\| > 1$, then $G(u)$ is not injective.*

Now we consider those points u at which $G(u)$ is injective. By Remark 4.1, $G(u)$ divides S^2 into two connected components and $F(G(u))$ is equal to the (signed) area of one of those. Hence, we may say that $FG(u)$ is equal to the (signed) area of a region on S^2 enclosed by $G(u)$.

Let u be on the xy-plane with $\|u\| > 1$ and let $u' \in \mathbb{R}^3 - K(S^1)$ be a point close to u. Observe that $G(u')$ encloses a small region or almost all of S^2. By continuity, we may guess that $FG(u)$ would be 0 or 4π ($0 = 4\pi \in 4\pi\mathbb{Z}$). It will be computed explicitly using a certain formula.

Corollary 4.5.3. *The map $FG : \mathbb{R}^3 - K(S^1) \to \mathbb{R}/4\pi\mathbb{Z}$ is surjective.*

We shall show later in Corollary 5.3.8 that FG is a smooth map, but now we would like to use that fact to state the following proposition.

Proposition 4.5.4. *Let $K : S^1 \subset \mathbb{R}^3$ be a smooth knot. If $t \in \mathbb{R}/4\pi\mathbb{Z}$ is a regular value of FG, then $(FG)^{-1}(t)$ is an open Seifert surface for K.*

Proof. It follows directly from Propositions 3.4.1 and 4.5.1. $\square$

Chapter 5

Definition of FG via solid angle

The definition of F, in Chapter 4, involves an extension of a loop in S^2. When the loop is not injective, there is not a direct way to relate FG and the area of some region on S^2. In this chapter, we use the notion of solid angle, see Section 2.6, to define the map

$$\Phi : \mathbb{R}^3 - K(S^1) \to \mathbb{R}/4\pi\mathbb{Z} \cong S^1$$

and we shall show that $\Phi = FG$. The map Φ is more geometric and computable. We are also able to derive a formula for Φ in terms of a line integral over the knot.

5.1 Definition of Φ

Let K be a smooth knot in $\mathbb{R}^3$. For each $u \in \mathbb{R}^3 - K(S^1)$, we choose a closed Seifert surface Σ_u with $u \notin \Sigma_u$ and define

$$\Pi_u : \Sigma_u \to S^2 \quad ; \quad y \mapsto \frac{y-u}{\|y-u\|}.$$

Then, im Π_u is the projection of Σ_u onto S^2 from u. Notice also that

$$\Pi_u(\partial\Sigma_u) = \Pi_u(K(S^1)) = G(u)(S^1).$$

Now we define $\Phi : \mathbb{R}^3 - K(S^1) \to \mathbb{R}/4\pi\mathbb{Z} \cong S^1$ by

$$\Phi(u) = \int_{\Sigma_u} \Pi_u^*(\omega)$$

where

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$$

is a volume 2-form on S^2 with $\int_{S^2} \omega = 4\pi$.

We need to show that the definition of Φ is well-defined. First notice that we can always select a closed Seifert surface Σ_u such that $u \notin \Sigma_u$. If u belongs to a closed Seifert surface, then we slightly push a small neighbourhood of u so that it avoids u.

Hence, Σ_u exists. We next show that $\Phi(u)$ does not depend on the choice of Σ_u.

Proposition 5.1.1. *If Σ_u and Σ'_u are two distinct closed Seifert surfaces for K that both avoid u, then the difference*

$$\int_{\Sigma_u} \Pi_u^*(\omega) - \int_{\Sigma'_u} (\Pi'_u)^*(\omega)$$

is a multiple of 4π.

Proof. Since $\partial\Sigma_u = \partial\Sigma'_u = \text{im } K$, we can form a closed surface

$$C = \Sigma_u \sqcup -\Sigma'_u$$

by taking a disjoint union and identifying $\partial\Sigma_u$ with $-\partial\Sigma'_u$, and also form a map

$$f = \Pi_u \sqcup -\Pi'_u : C \to S^2$$

Since

$$4\pi \deg f = \deg f \int_{S^2} \omega = \int_C f^*(\omega),$$

we have

$$\int_{\Sigma_u} \Pi_u^*(\omega) - \int_{\Sigma'_u} (\Pi'_u)^*(\omega) = \int_C f^*(\omega) = 4\pi \deg f \in 4\pi\mathbb{Z}.$$

□

We may write

$$\Phi(u) = \int_{\Sigma_u} \Pi_u^*(\omega) \mod 4\pi$$

to emphasise that $\Phi(u)$ represents an equivalence class in $\mathbb{R}/4\pi\mathbb{Z}$.

The quantity $\Phi(u)$ is, by definition, the signed area of $\Pi_u(\Sigma_u) \subset S^2$. Equivalently, $\Phi(u)$ is equal to the signed area of the shadow of Σ_u on the unit sphere with centre u. Thus, $\Phi(u)$ is the solid angle of Σ_u subtended at u.

5.2 Φ is equal to FG

Recall from Chapter 4 that

$$FG(u) = \int_{D^2} \delta G(u)^*(\omega) \in \mathbb{R}/4\pi\mathbb{Z}$$

where $\delta G(u) : D^2 \to S^2$ is a smooth extension of the loop $G(u) : S^1 \to S^2$.

If K is an unknot and $G(u)$ is an injective loop in S^2, then $\delta G(u)$ and Σ_u can be chosen such that $\delta G(u)(D^2) = \Pi_u(\Sigma_u)$. In this case,

$$FG(u) = \int_{D^2} \delta G(u)^*(\omega) = \int_{\Sigma_u} \Pi_u^*(\omega) = \Phi(u).$$

We next prove that $\Phi = FG$ for any smooth knot $K : S^1 \subset \mathbb{R}^3$.

Theorem 5.2.1. *Let $u \in \mathbb{R}^3 - K(S^1)$ and Σ_u be a closed Seifert surface for K that avoids u. Then*

$$FG(u) = \int_{\Sigma_u} \Pi_u^*(\omega) \mod 4\pi.$$

Proof. We show that for each $u \in \mathbb{R}^3 - K(S^1)$

$$\int_{D^2} \delta G(u)^*(\omega) = \int_{\Sigma_u} \Pi_u^*(\omega)$$

by reducing the domain of integration to $\partial D^2 = S^1$ for the left integral and $\partial \Sigma_u = \text{im } K$ for the right integral. We need the two following lemmas:

Lemma 5.2.2. *Let $\lambda : S^1 \to S^2$ be a smooth loop. Then there exists an extension $\delta\lambda : D^2 \to S^2$ of λ which is not surjective.*

Proof. Notice that λ is not surjective because the preimage of a regular value of λ cannot be of codimension 2.

Let $z \in S^2$ be a point outside im λ. Since $\pi_1(S^2 - \{z\})$ is trivial, there exists a smooth map

$$\delta\lambda : D^2 \to S^2 - \{z\} \subset S^2$$

extending λ. The extension $\delta\lambda$ is clearly not surjective, considered as a map to S^2. $\square$

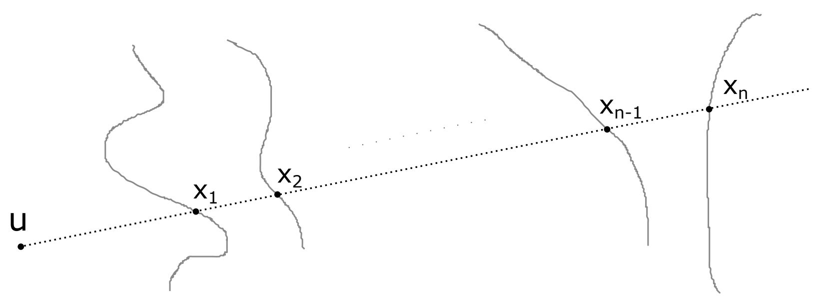
We remark that, from the proof of Lemma 5.2.2, we can choose an extension of $\delta\lambda : D^2 \to S^2$ which misses out z for any $z \in S^2 - \text{im } \lambda$.

Lemma 5.2.3. *For each $u \in \mathbb{R}^3 - K(S^1)$, we can choose a closed Seifert surface Σ_u such that $u \notin \Sigma_u$ and $\Pi_u : \Sigma_u \to S^2$ is not surjective.*

Proof. Let Σ_u be a closed Seifert surface that avoids u. We shall modify Σ_u so that a certain straight line from u to infinity does not meet the modified surface.

Consider a ray $r_u : [0, \infty) \to \mathbb{R}^3$ with $r(0) = u$. With small perturbation, let us assume that im $r_u \cap \Sigma_u$ is a finite set, consisting of $x_1, x_2, \ldots, x_n$ (if the intersection is empty, we are done). Notice that we can always choose r_u such that $x_i \notin K(S^1)$ since $G(u) : S^1 \to S^2$ is not surjective. In addition, we assume that $\|u - x_i\|$ is increasing with respect to i, i.e.

$$\|u - x_i\| < \|u - x_{i+1}\|.$$



For each $i \in \{1, 2, \ldots, n\}$, choose a small 2-disc $D_i \subset \Sigma_u$ containing x_i and define a small tube

$$T_i : D_i \times [0, 1] \hookrightarrow \mathbb{R}^3 \quad ; \quad (a, t) \mapsto a + t\varepsilon_i(u - x_i)$$

for some $\varepsilon_i > 1$ (just over 1). Notice that $T_i(D_i \times \{0\}) = D_i \subset \Sigma_u$ and im T_i contains the line segment between u and x_i. In addition, we assume that

$$D_i \subsetneqq D_{i+1} \quad \text{and} \quad \varepsilon_i < \varepsilon_{i+1}$$

for all i, and D_n is chosen so small that im $T_n - D_n$ does not intersect Σ_u. With all this, we obtain

$$\text{im } T_i \subsetneqq \text{im } T_{i+1}$$

for all i.

Now we remove each $D_i - \partial D_i \subset \Sigma_u$ and glue $\partial(\text{im } T_i) - D_i$ back along ∂D_i. By our construction, all $\partial(\text{im } T_i) - D_i$ are disjoint and they intersect Σ_u only at ∂D_i. Note that the resulting space is a surface with corners. We may have to smooth all the corners to obtain a new closed Seifert surface with the required property. $\square$

For each $u \in \mathbb{R}^3 - K(S^1)$ and for each closed Seifert surface for K that avoids u, the image of Π_u is the shadow of the closed Seifert surface on S^2. By Lemma 5.2.3, we can see that if $z \in S^2$ is not in the shadow of $K(S^1)$, there exists a closed Seifert surface Σ_u such that $z \notin \Pi_u(\Sigma_u)$.

We are now ready to prove the theorem. By Lemmas 5.2.2 and 5.2.3, we can choose an extension $\delta G(u) : D^2 \to S^2$ of the loop $G(u)$ and a closed Seifert surface Σ_u avoiding u such that the images $G(u)(D^2)$ and $\Pi_u(\Sigma_u)$ miss out the same point z for some $z \in S^2$. Consider the restriction

$$\omega' = (\omega)|_{S^2 - \{z\}}$$

of ω. The form ω' is an exact 2-form on $S^2 - \{z\}$ since $H^1_{dR}(S^2 - \{z\})$ is trivial. Then, there is a 1-form η on $S^2 - \{z\}$ such that $d\eta = \omega'$. By Stokes' Theorem, we have

$$\int_{D^2} \delta G(u)^*(\omega) = \int_{D^2} \delta G(u)^*(\omega') = \int_{D^2} \delta G(u)^*(d\eta) = \int_{D^2} d(\delta G(u)^*(\eta)) = \int_{S^1} G(u)^*(\eta)$$

and

$$\int_{\Sigma_u} \Pi_u^*(\omega) = \int_{\Sigma_u} \Pi_u^*(\omega') = \int_{\Sigma_u} \Pi_u^*(d\eta) = \int_{\Sigma_u} d\Pi_u^*(\eta) = \int_{K(S^1)} (\Pi_u|_{K(S^1)})^*\eta.$$

The two integrals above are equal because $K : S^1 \to K(S^1) \subset \mathbb{R}^3$ is of degree 1 and

$G(u) = \Pi_u K$, i.e.,

$$\int_{S^1} G(u)^*(\eta) = \int_{S^1} K^* \Pi_u^*(\eta) = \int_{K(S^1)} (\Pi_u|_{K(S^1)})^* \eta.$$

□

We remark that FG and Φ may be used interchangeably, but FG will be preferable.

5.3 A line-integral formula

The proof of Theorem 5.2.1 paves the way for a line-integral formula. It has been shown that

$$FG(u) = \int_{K(S^1)} (\Pi_u|_{K(S^1)})^* \eta \mod 4\pi \tag{5.1}$$

where η is a 1-form on $S^2 - \{z\}$ for some $z \in S^2$, having the property that

$$d\eta = \omega' = \omega|_{S^2 - \{z\}}.$$

In this chapter, we compute an explicit formula of the line integral (5.1). In order to do so, we find an explicit expression of the 1-form η and compute the pullback form $(\Pi_u|_{K(S^1)})^* \eta$ explicitly.

Let $z = (a, b, c) \in S^2$ be a point outside $\Pi_u(\Sigma_u)$. As before, let

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$$

be a volume 2-form on S^2, and let

$$\omega'_{(a,b,c)} = \omega|_{\{(a,b,c)\}} \quad \text{and} \quad \omega'_{(0,0,1)} = \omega|_{\{(0,0,1)\}}$$

be the 2-forms restricted on $S^2 - \{(a, b, c)\}$ and $S^2 - \{(0, 0, 1)\}$, respectively. In fact, ω may be viewed as a 2-form on any subset of $\mathbb{R}^3$. Hence ω, $\omega'_{(a,b,c)}$ and $\omega'_{(0,0,1)}$ have the same expression

$$x dy \wedge dz + y dz \wedge dx + z dx \wedge dy,$$

but $\omega'_{(a,b,c)}$ and $\omega'_{(0,0,1)}$ may be rearranged into other forms. We shall find a 1-form η on $S^2 - \{(a, b, c)\}$ such that $d\eta = \omega'_{(a,b,c)}$ as follows.

$$\begin{tikzcd} S^2 - \{(a,b,c)\} \arrow[d, "R"'] \arrow[dr, "T_{(a,b,c)}"] & \\ S^2 - \{(0,0,1)\} \arrow[r, "T_{(0,0,1)}"'] & \mathbb{R}^2 \end{tikzcd}$$

Define the stereographic projection

$$T_{(a,b,c)} : S^2 - \{(a,b,c)\} \to \mathbb{R}^2$$

as the composite of $T_{(0,0,1)}$ and R where

$$T_{(0,0,1)} : S^2 - \{(0,0,1)\} \to \mathbb{R}^2 \quad ; \quad (x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

is the stereographic projection of $S^2 - \{(0,0,1)\}$ and R is the rotation in $\mathbb{R}^3$ with rotation matrix ($a \neq \pm 1$)

$$[R] = \begin{pmatrix} \sqrt{1-a^2} & \frac{-ab}{\sqrt{1-a^2}} & \frac{-ac}{\sqrt{1-a^2}} \\ 0 & \frac{c}{\sqrt{1-a^2}} & \frac{-b}{\sqrt{1-a^2}} \\ a & b & c \end{pmatrix}.$$

If $a = \pm 1$, the rotation matrix is

$$\begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}.$$

Since $T_{(0,0,1)}$ and R are diffeomorphisms, so is $T_{(a,b,c)}$. We start with computing the 2-form on $\mathbb{R}^2$ corresponding to $\omega'_{(0,0,1)}$.

Proposition 5.3.1.

$$\left(T^{-1}_{(0,0,1)}\right)^* \omega'_{(0,0,1)} = \frac{-4}{(\mathbf{x}^2+\mathbf{y}^2+1)^2} d\mathbf{x} \wedge d\mathbf{y}.$$

Proof. The inverse of $T_{(0,0,1)}$ is given by

$$T^{-1}_{(0,0,1)}(\mathbf{x},\mathbf{y}) = \left(\frac{2\mathbf{x}}{\mathbf{x}^2+\mathbf{y}^2+1}, \frac{2\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}, 1 - \frac{2}{\mathbf{x}^2+\mathbf{y}^2+1}\right).$$

Here,

$$x = \frac{2\mathbf{x}}{\mathbf{x}^2+\mathbf{y}^2+1}, \quad y = \frac{2\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1} \quad \text{and} \quad z = 1 - \frac{2}{\mathbf{x}^2+\mathbf{y}^2+1}.$$

Setting $t = \dfrac{2}{\mathbf{x}^2+\mathbf{y}^2+1}$, we have $dt = -t^2(\mathbf{x}d\mathbf{x} + \mathbf{y}d\mathbf{y})$. Note that

$$dx = d(\mathbf{x}t) = \mathbf{x}dt + td\mathbf{x} \quad \text{and} \quad dy = d(\mathbf{y}t) = \mathbf{y}dt + td\mathbf{y}.$$

Hence,

$$\begin{aligned}
\left(T_{(0,0,1)}^{-1}\right)^{*}\omega'_{(0,0,1)} &= \mathbf{x}td(\mathbf{y}t)\wedge(-dt)+\mathbf{y}t(-dt)\wedge d(\mathbf{x}t)+(1-t)d(\mathbf{x}t)\wedge d(\mathbf{y}t)\\
&= \mathbf{x}tdt\wedge d\mathbf{y}+\mathbf{y}td\mathbf{x}\wedge dt+(t^2-t^3)d\mathbf{x}\wedge d\mathbf{y}\\
&= \mathbf{x}t(-\mathbf{x}t^2d\mathbf{x}\wedge d\mathbf{y})+\mathbf{y}t(-\mathbf{y}t^2d\mathbf{x}\wedge d\mathbf{y})+(t^2-t^3)d\mathbf{x}\wedge d\mathbf{y}\\
&= (-t^3(\mathbf{x}^2+\mathbf{y}^2+1)+t^2)d\mathbf{x}\wedge d\mathbf{y}\\
&= -t^2d\mathbf{x}\wedge d\mathbf{y}=\frac{-4}{(\mathbf{x}^2+\mathbf{y}^2+1)^2}d\mathbf{x}\wedge d\mathbf{y}.
\end{aligned}$$

□

The 2-form $\left(T_{(0,0,1)}^{-1}\right)^{*}\omega'_{(0,0,1)}$ is exact. So it is the differential of some 1-form on $\mathbb{R}^2$. As in the previous proposition, set

$$t=\frac{2}{\mathbf{x}^2+\mathbf{y}^2+1}.$$

It is not hard to see that

$$d\left(\frac{2\mathbf{y}d\mathbf{x}-2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}\right)=d(\mathbf{y}td\mathbf{x}-\mathbf{x}td\mathbf{y})=-t^2d\mathbf{x}\wedge d\mathbf{y}=\frac{-4}{(\mathbf{x}^2+\mathbf{y}^2+1)^2}d\mathbf{x}\wedge d\mathbf{y}.$$

We now claim that

$$T^{*}_{(a,b,c)}\left(\frac{-4d\mathbf{x}\wedge d\mathbf{y}}{(\mathbf{x}^2+\mathbf{y}^2+1)^2}\right)=\omega'_{(a,b,c)}. \tag{5.2}$$

If this is true, we will have

$$\begin{aligned}
FG(u) &= \int_{\Sigma_u}\Pi_u^{*}\omega'_{(a,b,c)}\\
&= \int_{\Sigma_u}\Pi_u^{*}T^{*}_{(a,b,c)}\left(\frac{-4d\mathbf{x}\wedge d\mathbf{y}}{(\mathbf{x}^2+\mathbf{y}^2+1)^2}\right)\\
&= \int_{\Sigma_u}\Pi_u^{*}T^{*}_{(a,b,c)}d\left(\frac{2\mathbf{y}d\mathbf{x}-2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}\right)\\
&= \int_{K(S^1)}(\Pi_u|_{K(S^1)})^{*}T^{*}_{(a,b,c)}\left(\frac{2\mathbf{y}d\mathbf{x}-2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}\right)
\end{aligned}$$

and

$$\eta=T^{*}_{(a,b,c)}\left(\frac{2\mathbf{y}d\mathbf{x}-2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}\right).$$

Notice that

$$T^{*}_{(a,b,c)}\left(\frac{-4d\mathbf{x}\wedge d\mathbf{y}}{(\mathbf{x}^2+\mathbf{y}^2+1)^2}\right)=R^{*}T^{*}_{(0,0,1)}\left(\frac{-4d\mathbf{x}\wedge d\mathbf{y}}{(\mathbf{x}^2+\mathbf{y}^2+1)^2}\right)=R^{*}\omega'_{(0,0,1)}. \tag{5.3}$$

Hence, in order to show (5.2), we only need to show that the rotation R preserves the form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$.

Proposition 5.3.2. *Let R be any rotation in $\mathbb{R}^3$. Then,*

$$R^*\omega = \omega.$$

Here, ω may be viewed as a 2-form on any subset of $\mathbb{R}^3$.

Proof. Let R_i be the i-th row of the matrix $[R]$ and C_j the j-th column of the matrix $[R]$ for $i, j \in \{1, 2, 3\}$. We may think of R_i and C_j as vectors in $\mathbb{R}^3$, i.e.,

$$R_i = \begin{pmatrix} r_{i1} \\ r_{i2} \\ r_{i3} \end{pmatrix} \quad \text{and} \quad C_i = \begin{pmatrix} r_{1j} \\ r_{2j} \\ r_{3j} \end{pmatrix},$$

where r_{ij} is the entry $[R]$ from the i-th row and the j-th column. Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

be the position vector of (x, y, z). Hence,

$$\begin{aligned} R^*\omega &= (R_1 \cdot \mathbf{x})d(R_2 \cdot \mathbf{x}) \wedge d(R_3 \cdot \mathbf{x}) + (R_2 \cdot \mathbf{x})d(R_3 \cdot \mathbf{x}) \wedge d(R_1 \cdot \mathbf{x}) \\ &+ (R_3 \cdot \mathbf{x})d(R_1 \cdot \mathbf{x}) \wedge d(R_2 \cdot \mathbf{x}) \\ &= (R_1 \cdot \mathbf{x})(R_2 \cdot d\mathbf{x}) \wedge (R_3 \cdot d\mathbf{x}) + (R_2 \cdot \mathbf{x})(R_3 \cdot d\mathbf{x}) \wedge (R_1 \cdot d\mathbf{x}) \\ &+ (R_3 \cdot \mathbf{x})(R_1 \cdot d\mathbf{x}) \wedge (R_2 \cdot d\mathbf{x}) \\ &= (R_1 \cdot \mathbf{x})(R_2 \times R_3) \cdot \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} + (R_2 \cdot \mathbf{x})(R_3 \times R_1) \cdot \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} \\ &+ (R_3 \cdot \mathbf{x})(R_1 \times R_2) \cdot \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix}. \end{aligned}$$

Since $[R]$ is an orthogonal matrix with determinant 1, we have $[R]^{-1} = [R]^T$ and $[R] = adj[R]$, which implies that

$$R_1 \times R_2 = R_3, \quad R_2 \times R_3 = R_1 \quad \text{and} \quad R_3 \times R_1 = R_2.$$

Thus,

$$\begin{aligned}
R^*\omega &= ((R_1\cdot\mathbf{x})r_{11} + (R_2\cdot\mathbf{x})r_{21} + (R_3\cdot\mathbf{x})r_{31})\,dy\wedge dz\\
&\quad + ((R_1\cdot\mathbf{x})r_{12} + (R_2\cdot\mathbf{x})r_{22} + (R_3\cdot\mathbf{x})r_{32})\,dz\wedge dx\\
&\quad + ((R_1\cdot\mathbf{x})r_{13} + (R_2\cdot\mathbf{x})r_{23} + (R_3\cdot\mathbf{x})r_{33})\,dx\wedge dy\\
&= ((C_1\cdot C_1)x + (C_2\cdot C_1)y + (C_3\cdot C_1)z)\,dy\wedge dz\\
&\quad + ((C_1\cdot C_2)x + (C_2\cdot C_2)y + (C_3\cdot C_2)z)\,dz\wedge dx\\
&\quad + ((C_1\cdot C_3)x + (C_2\cdot C_3)y + (C_3\cdot C_3)z)\,dx\wedge dy\\
&= xdy\wedge dz + ydz\wedge dx + zdx\wedge dy\\
&= \omega.
\end{aligned}$$

□

Corollary 5.3.3. *The map $FG : \mathbb{R}^3 - \mathrm{im}\ K \to \mathbb{R}/4\pi\mathbb{Z} \cong S^1$ can be expressed as the line integral*

$$FG(u) = \int_{K(S^1)} (\Pi_u|_{K(S^1)})^*\eta$$

where $\eta = T^*_{(a,b,c)}\left(\dfrac{2\mathbf{y}d\mathbf{x} - 2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}\right)$.

Proof. We just verify that $d\eta = \omega'_{(a,b,c)}$. By Proposition 5.3.1 and Equation (5.3), we have

$$d\eta = T^*_{(a,b,c)}d\left(\frac{2\mathbf{y}d\mathbf{x} - 2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}\right) = R^*T^*_{(0,0,1)}\left(\frac{-4d\mathbf{x}\wedge d\mathbf{y}}{(\mathbf{x}^2+\mathbf{y}^2+1)^2}\right) = R^*\omega'_{(0,0,1)} = \omega'_{(a,b,c)}.$$

The rightmost equality follows from Proposition 5.3.2. □

To find an explicit expression of

$$(\Pi_u|_{K(S^1)})^*\eta = (\Pi_u|_{K(S^1)})^*R^*T^*_{(0,0,1)}\left(\frac{2\mathbf{y}d\mathbf{x} - 2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}\right),$$

we calculate one pullback form at a time.

Proposition 5.3.4.

$$T^*_{(0,0,1)}\left(\frac{2\mathbf{y}d\mathbf{x} - 2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2+1}\right) = \frac{ydx - xdy}{1-z}.$$

Proof. Recall that

$$T_{(0,0,1)} : S^2 - \{0,0,1\} \to \mathbb{R}^2 \quad ; \quad (x,y,z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Here,

$$\mathbf{x} = \frac{x}{1-z} \quad \text{and} \quad \mathbf{y} = \frac{y}{1-z}.$$

Using $x^2 + y^2 + z^2 = 1$, we have

$$\frac{2}{\mathbf{x}^2 + \mathbf{y}^2 + 1} = \frac{2}{\left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2 + 1} = 1 - z.$$

Hence,

$$\begin{aligned} T^*_{(0,0,1)}\left(\frac{2\mathbf{y}d\mathbf{x} - 2\mathbf{x}d\mathbf{y}}{\mathbf{x}^2 + \mathbf{y}^2 + 1}\right) &= yd\left(\frac{x}{1-z}\right) - xd\left(\frac{y}{1-z}\right) \\ &= y\left(\frac{dx}{1-z} - \frac{xd(1-z)}{(1-z)^2}\right) - x\left(\frac{dy}{1-z} - \frac{yd(1-z)}{(1-z)^2}\right) \\ &= \frac{ydx - xdy}{1-z}. \end{aligned}$$

□

Proposition 5.3.5.

$$\begin{aligned} R^*\left(\frac{ydx - xdy}{1-z}\right) &= \frac{(cy - bz)dx + (az - cx)dy + (bx - ay)dz}{1 - (ax + by + cz)} \\ &= \frac{\det\begin{pmatrix} dx & dy & dz \\ x & y & z \\ a & b & c \end{pmatrix}}{1 - (ax + by + cz)} \end{aligned}$$

Proof. Recall that if $a \neq \pm 1$, then

$$R(x, y, z) = \left(\frac{(1 - a^2)x - aby - acz}{\sqrt{1 - a^2}}, \frac{cy - bz}{\sqrt{1 - a^2}}, ax + by + cz\right).$$

Computing $R^*(ydx - xdy)$, we have

$$\begin{aligned} &\frac{cy - bz}{1 - a^2}((1 - a^2)dx - abdy - acdz) - \frac{(1 - a^2)x - aby - acz}{1 - a^2}(cdy - bdz) \\ &\qquad = (cy - bz)dx + (ab^2 z - (1 - a^2)cx + ac^2 z)\frac{dy}{1 - a^2} \\ &\qquad + (-ac^2 y + (1 - a^2)bx - ab^2 y)\frac{dz}{1 - a^2}. \end{aligned}$$

Using $\frac{b^2 + c^2}{1 - a^2} = 1$, we have

$$R^*\left(\frac{ydx - xdy}{1-z}\right) = \frac{(cy - bz)dx + (az - cx)dy + (bx - ay)dz}{1 - (ax + by + cz)}.$$

If $a = \pm 1$, the rotation is given by

$$(x, y, z) \mapsto (\mp z, y, \pm x)$$

and hence the pullback form is

$$\frac{\pm(zdy - ydz)}{1 \mp x}.$$

□

Proposition 5.3.6. *Let* $\mathbf{y} = (y_1, y_2, y_3) \in K(S^1)$*. Then, we have*

$$
\begin{aligned}
(\Pi_u|_{K(S^1)})^*\eta &= (\Pi_u|_{K(S^1)})^* \left(\frac{(cy - bz)dx + (az - cx)dy + (bx - ay)dz}{1 - (ax + by + cz)} \right) \\
&= \frac{\det \begin{pmatrix} dy_1 & dy_2 & dy_3 \\ y_1 - u_1 & y_2 - u_2 & y_3 - u_3 \\ a & b & c \end{pmatrix}}{\|\mathbf{y} - u\| \left(\|\mathbf{y} - u\| - (a(y_1 - u_1) + b(y_2 - u_2) + c(y_3 - u_3)) \right)}.
\end{aligned}
$$

Proof. For each $u = (u_1, u_2, u_3) \in \mathbb{R}^3 - K(S^1)$, recall that

$$\Pi_u(y_1, y_2, y_3) = \frac{(y_1 - u_1, y_2 - u_2, y_3 - u_3)}{\|\mathbf{y} - u\|}$$

for all $\mathbf{y} = (y_1, y_2, y_3) \in \Sigma_u$. Here,

$$x = \frac{y_1 - u_1}{\|\mathbf{y} - u\|}, \quad y = \frac{y_2 - u_2}{\|\mathbf{y} - u\|} \quad \text{and} \quad z = \frac{y_3 - u_3}{\|\mathbf{y} - u\|}.$$

Then,

$$(\Pi_u|_{K(S^1)})^*\eta = \frac{\det \begin{pmatrix} d\left(\frac{y_1 - u_1}{\|\mathbf{y} - u\|}\right) & d\left(\frac{y_2 - u_2}{\|\mathbf{y} - u\|}\right) & d\left(\frac{y_3 - u_3}{\|\mathbf{y} - u\|}\right) \\ \frac{y_1 - u_1}{\|\mathbf{y} - u\|} & \frac{y_2 - u_2}{\|\mathbf{y} - u\|} & \frac{y_3 - u_3}{\|\mathbf{y} - u\|} \\ a & b & c \end{pmatrix}}{1 - \left(a\frac{y_1 - u_1}{\|\mathbf{y} - u\|} + b\frac{y_2 - u_2}{\|\mathbf{y} - u\|} + c\frac{y_3 - u_3}{\|\mathbf{y} - u\|} \right)}.$$

Note that

$$d\left(\frac{y_i - u_i}{\|\mathbf{y} - u\|}\right) = \frac{dy_i}{\|\mathbf{y} - u\|} - \frac{(y_i - u_i)d\|\mathbf{y} - u\|}{\|\mathbf{y} - u\|^2}$$

for all i. Multiplying the second row by $\dfrac{d\|\mathbf{y} - u\|}{\|\mathbf{y} - u\|}$ and adding to the first row, we have

$$(\Pi_u|_{K(S^1)})^*\eta = \frac{\det \begin{pmatrix} \frac{dy_1}{\|\mathbf{y} - u\|} & \frac{dy_2}{\|\mathbf{y} - u\|} & \frac{dy_3}{\|\mathbf{y} - u\|} \\ \frac{y_1 - u_1}{\|\mathbf{y} - u\|} & \frac{y_2 - u_2}{\|\mathbf{y} - u\|} & \frac{y_3 - u_3}{\|\mathbf{y} - u\|} \\ a & b & c \end{pmatrix}}{1 - \left(a\frac{y_1 - u_1}{\|\mathbf{y} - u\|} + b\frac{y_2 - u_2}{\|\mathbf{y} - u\|} + c\frac{y_3 - u_3}{\|\mathbf{y} - u\|} \right)},$$

and hence

$$(\Pi_u|_{K(S^1)})^*\eta = \frac{\dfrac{1}{\|\mathbf{y}-u\|^2}\det\begin{pmatrix} dy_1 & dy_2 & dy_3 \\ y_1-u_1 & y_2-u_2 & y_3-u_3 \\ a & b & c \end{pmatrix}}{1-\left(a\dfrac{y_1-u_1}{\|\mathbf{y}-u\|}+b\dfrac{y_2-u_2}{\|\mathbf{y}-u\|}+c\dfrac{y_3-u_3}{\|\mathbf{y}-u\|}\right)}.$$

□

Theorem 5.3.7. *Let $u=(u_1,u_2,u_3)\in\mathbb{R}^3-K(S^1)$ and $z=(a,b,c)\in S^2-G(u)(S^1)$. Then, $FG:\mathbb{R}^3-K(S^1)\to\mathbb{R}/4\pi\mathbb{Z}\cong S^1$ can be expressed as the line integral*

$$\begin{aligned} FG(u_1,u_2,u_3) &= \int_{K(S^1)} \frac{\det\begin{pmatrix} dy_1 & dy_2 & dy_3 \\ y_1-u_1 & y_2-u_2 & y_3-u_3 \\ a & b & c \end{pmatrix}}{\|\mathbf{y}-u\|\left(\|\mathbf{y}-u\|-(a(y_1-u_1)+b(y_2-u_2)+c(y_3-u_3))\right)} \\ &= \int_{K(S^1)} \frac{\left(\dfrac{\mathbf{y}-u}{\|\mathbf{y}-u\|}\times z\right)\cdot D\mathbf{y}}{\|\mathbf{y}-u\|\left(1-\dfrac{\mathbf{y}-u}{\|\mathbf{y}-u\|}\cdot z\right)}. \end{aligned}$$

Moreover, this formula is independent of the choice of z.

Proof. It follows from previous propositions. □

It is not obvious from the original definition of FG that it is a smooth map, but from this formula we can see that this is indeed the case. The formula above proves it.

Corollary 5.3.8. *The map $FG:\mathbb{R}^3-K(S^1)\to\mathbb{R}/4\pi\mathbb{Z}$ is a smooth map.*

Proof. As we integrate along the knot, it is sufficient to verify that the integrand

$$\frac{\det\begin{pmatrix} dy_1 & dy_2 & dy_3 \\ y_1-u_1 & y_2-u_2 & y_3-u_3 \\ a & b & c \end{pmatrix}}{\|\mathbf{y}-u\|\left(\|\mathbf{y}-u\|-(a(y_1-u_1)+b(y_2-u_2)+c(y_3-u_3))\right)}$$

is smooth at each point $u\in\mathbb{R}^3-K(S^1)$. Since the factor $1/\|\mathbf{y}-u\|$ is smooth on $\mathbb{R}^3-K(S^1)$, it remains to show that there exists a small neighbourhood V of u in $\mathbb{R}^3-K(S^1)$ and $z=(a,b,c)\in S^2$ such that

$$\frac{\mathbf{y}-u'}{\|\mathbf{y}-u'\|}\neq z$$

for all $\mathbf{y}\in K(S^1)$ and $u'\in V$.

Choose $z \notin K(S^1)$ such that $\|z\| = 1$ and

$$-1 \leqslant \frac{\mathbf{y} - u}{\|\mathbf{y} - u\|} \cdot z \leqslant M < 1$$

for all $\mathbf{y} \in K(S^1)$. Hence, $\dfrac{\mathbf{y} - u'}{\|\mathbf{y} - u'\|} \cdot z$ cannot jump to 1 when u' is very close to u. In other words, we can choose a neighbourhood V which is so small that the dot product $\dfrac{\mathbf{y} - u'}{\|\mathbf{y} - u'\|} \cdot z$ is away from 1 for all $\mathbf{y} \in K(S^1)$ and $u' \in V$. □

5.4 Bounded pre-images

Sard's theorem says that the set of critical values of any smooth map has Lebesgue measure zero. This implies that the smooth map FG must have a regular value, say $t \in \mathbb{R}/4\pi\mathbb{Z}$. By Thom-Sard transversality theorem, the pre-image $(FG)^{-1}(t)$ is an orientable open surface in $\mathbb{R}^3 - K(S^1)$, which consists of all the points $u \in \mathbb{R}^3 - K(S^1)$ with the property that a closed Seifert surface Σ_u casts the same (signed) shadow area t on the unit sphere. Geometry suggests that if u is far from the origin, then the shadow area cast by the closed Seifert surface will be small.

Proposition 5.4.1. *If t is a regular value of FG and $t \neq 0$, then the pre-image $(FG)^{-1}(t)$ is a bounded surface.*

Proof. Suppose that $(FG)^{-1}(t)$ is not bounded when $t \neq 0$. Then, for each $R > 0$, there is a point $u \in \mathbb{R}^3 - K(S^1)$ with $\|u\| = R$ such that $FG(u) = t$. We show that this contradicts the fact that

$$\lim_{\|u\| \to \infty} FG(u) = 0.$$

Let $w : [-l, l] \to \mathbb{R}^3$ be a smooth arc-length parametrisation of the knot K. Then, by Theorem 5.3.7, we have

$$FG(u) = \int_{-l}^{l} \frac{\left(\dfrac{w(s) - u}{\|w(s) - u\|} \times z\right) \cdot \dot{w}(s)}{\|w(s) - u\| \left(1 - \dfrac{w(s) - u}{\|w(s) - u\|} \cdot z\right)} ds.$$

We next consider all the points $u \in \mathbb{R}^3 - K(S^1)$ such that $\|u\|$ is sufficiently large – assume $\|u\| \geqslant R_0$. Since R_0 is large, we can choose z_1 and z_2 with $\|z_1\| = 1 = \|z_2\|$ and $m > 0$ such that for each $u \in \mathbb{R}^3 - K(S^1)$ with $\|u\| \geqslant R_0$, we have

$$0 < m \leqslant \left|1 - \frac{w(s) - u}{\|w(s) - u\|} \cdot z_1\right| \quad \text{or} \quad 0 < m \leqslant \left|1 - \frac{w(s) - u}{\|w(s) - u\|} \cdot z_2\right|$$

for all $s \in [-l, l]$. Note that for each $u \in \mathbb{R}^3 - K(S^1)$ with $\|u\| \geqslant R_0$,

$$\begin{aligned}\left| \frac{\left(\frac{w(s)-u}{\|w(s)-u\|} \times z \right) \cdot w'(s)}{\|w(s)-u\| \left(1 - \frac{w(s)-u}{\|w(s)-u\|} \cdot z \right)} \right| &\leqslant \frac{1}{\|w(s)-u\| \left| 1 - \frac{w(s)-u}{\|w(s)-u\|} \cdot z \right|} \\ &\leqslant \frac{1}{m|\|w(s)\| - \|u\||}\end{aligned}$$

for all $s \in [-l, l]$, where $z = z_1$ or z_2. Since $\dfrac{1}{|\|w(s)\| - \|u\||}$ is also bounded, the dominated convergence theorem yields

$$\lim_{\|u\| \to \infty} |FG(u)| \leqslant \int_{-l}^{l} \lim_{\|u\| \to \infty} \frac{ds}{m|\|w(s)\| - \|u\||} = 0.$$

□

We will see later that in the case when K is the unknot, $(FG)^{-1}(0)$ is not bounded. One may ask if the converse of the proposition is true in general. We do not know it yet.

Chapter 6

Analysis of FG for an unknot

This chapter focuses on computation and behaviour of the map FG for the standard unit circle on the xy-plane, which plays the role of the unknot. It turns out that explicit formulae can be written in terms of elliptic integrals, see [12]. Our main goal is to use FG to construct a closed Seifert surface for the unknot, Proposition 6.4.5. We first introduce the definition of elliptic integrals and state some facts that will be used in the computation, and then derive formulae of FG in terms of complete elliptic integrals. In the final section, we investigate the behaviour of FG near the unknot.

Throughout this chapter, let U denote the unknot in $\mathbb{R}^3$ parametrised by

$$\gamma(t) = (\cos t, \sin t, 0)$$

for $t \in [-\pi, \pi]$.

6.1 Elliptic Integrals

This section is based on *Handbook of Elliptic Integrals for Engineers and Scientists*, see [2].

Definition 6.1.1. *Let $\varphi \in [0, \pi/2]$. For any $k \in [0, 1]$, the* complementary modulus k' *of k is defined by $k' = \sqrt{1 - k^2}$.*

1. *The integral*
$$F(\varphi, k) = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$
is called an elliptic integral of the first kind. *If $\varphi = \pi/2$, it is called a* complete elliptic integral of the first kind, *denoted by $\mathbf{K}(k) := F(\pi/2, k)$.*

2. *The integral*
$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 t}\; dt$$
is called an elliptic integral of the second kind. *If $\varphi = \pi/2$, it is called a* complete elliptic integral of the second kind, *denoted by $E(k) := E(\pi/2, k)$.*

3. *The integral*
$$\Pi(\varphi, \alpha^2, k) = \int_0^{\varphi} \frac{dt}{(1-\alpha^2 \sin^2 t)\sqrt{1-k^2\sin^2 t}}$$
is called an elliptic integral of the third kind*. If* $\varphi = \pi/2$*, it is called a* complete elliptic integral of the third kind*, denoted by* $\Pi(\alpha^2, k) := \Pi(\pi/2, \alpha^2, k)$*.*

4. *The Heuman's Lambda function* $\Lambda_0(\beta, k)$ *can be defined by the formula*
$$\Lambda_0(\beta, k) = \frac{2}{\pi}\left(E(k)F(\beta, k') + \mathbf{K}(k)E(\beta, k') - \mathbf{K}(k)F(\beta, k')\right).$$

Remark 6.1.

- The integrals $F(\varphi, k)$ and $\Pi(\varphi, \alpha^2, k)$ may not be integrable for some values. For example, if $\varphi = \pi/2$ and $k = 1$, then $F(\pi/2, 1) = \mathbf{K}(1)$ is not integrable.
- Some special values of elliptic integrals and the Heuman's Lambda function are
$$\begin{aligned} E(0,k) = F(0,k) = \Pi(0,\alpha^2,k) &= 0 \\ E(\varphi,0) = F(\varphi,0) = \Pi(\varphi,0,0) &= \varphi \\ \mathbf{K}(0) = E(0) = \pi/2, \quad E(1) &= 1 \\ \Lambda_0(\beta,0) = \sin\beta, \quad \Lambda_0(0,k) &= 0 \\ \Lambda_0(\beta,1) = 2\beta/\pi, \quad \Lambda_0(\pi/2,k) &= 1 \\ \Lambda_0(-\beta,k) = -\Lambda_0(\beta,k). \end{aligned}$$

Although $\mathbf{K}(k)$ blows up at $k = 1$, we know how fast it does so when k approaches 1 from below; see (10) in [18] on Page 318.

Proposition 6.1.2.
$$\mathbf{K}(k) = \ln\frac{4}{\sqrt{1-k^2}} + O\left((1-k^2)\ln\sqrt{1-k^2}\right) \quad \text{as} \quad k \to 1^-.$$

Corollary 6.1.3.
$$\lim_{k\to 1^-}\left(\mathbf{K}(k) - \ln\frac{4}{\sqrt{1-k^2}}\right) = 0.$$

Since the complete elliptic integrals $\mathbf{K}(k)$ and $E(k)$ vary smoothly in the variable k, we can differentiate them using the formulae on Page 282 in [2]
$$\frac{d}{dk}\mathbf{K}(k) = \frac{E(k) - (k')^2\mathbf{K}(k)}{k(k')^2} \tag{6.2}$$
and
$$\frac{d}{dk}E(k) = \frac{E(k) - \mathbf{K}(k)}{k} \tag{6.3}$$

where $k' = \sqrt{1-k^2}$. Heuman's Lambda function $\Lambda_0(\beta, k)$ depends smoothly on both β and k. Hence, the partial derivatives of $\Lambda_0(\beta, k)$ can be computed by the formulae on Page 284 in [2]:

$$\frac{\partial}{\partial k}\Lambda_0(\beta, k) = \frac{2(E(k) - \mathbf{K}(k))\sin\beta\cos\beta}{\pi k\sqrt{1-k'^2\sin^2\beta}} \tag{6.4}$$

and

$$\frac{\partial}{\partial \beta}\Lambda_0(\beta, k) = \frac{2(E(k) - k'^2\sin^2\beta\mathbf{K}(k))}{\pi\sqrt{1-k'^2\sin^2\beta}}. \tag{6.5}$$

6.2 Computation for the unknot U

Recall that the unknot U has the parametrisation $\gamma : [-\pi, \pi] \to \mathbb{R}^3$ given by

$$\gamma(t) = (\cos t, \sin t, 0).$$

For convenience, let us simply set $U := U([-\pi, \pi]) \subset \mathbb{R}^3$.

For each $u \in \mathbb{R}^3 - U$, we choose a point $z \in S^2$ with $z \notin \operatorname{im}\Pi_u$ to obtain the formula in Theorem 5.3.7. Observe that, for most points u, we are able to find a closed Seifert surface Σ_u for U such that $\Pi_u(\Sigma_u)$ misses out the north pole $(0,0,1) \in S^2$. However, if

$$u \in \{(u_1, u_2, u_3)| \quad u_1^2 + u_2^2 = 1 \quad \text{and} \quad u_3 < 0\},$$

then $(0,0,1) \in \Pi_u(\Sigma_u)$ $-$ in this case, we can choose a closed Seifert surface whose image under Π_u misses out the south pole $(0,0,-1)$.

Let us fix $z = (0,0,1)$ and consider all the points

$$u \in \mathbb{R}^3 - \{(u_1, u_2, u_3)| \quad u_1^2 + u_2^2 = 1 \quad \text{and} \quad u_3 < 0\}.$$

By Theorem 5.3.7, we have

$$\begin{aligned}
&FG(u_1, u_2, u_3) \\
&= \int_{-\pi}^{\pi} \frac{\det\begin{pmatrix} -\sin t & \cos t & 0 \\ \cos t - u_1 & \sin t - u_2 & -u_3 \\ 0 & 0 & 1 \end{pmatrix} dt}{\|\gamma(t) - u\|\left(\|\gamma(t) - u\| + u_3\right)} \\
&= \int_{-\pi}^{\pi} \frac{(u_1\cos t + u_2\sin t - 1)dt}{1 + \|u\|^2 - 2u_1\cos t - 2u_2\sin t + u_3(\sqrt{1 + \|u\|^2 - 2u_1\cos t - 2u_2\sin t})}.
\end{aligned} \tag{6.6}$$

Writing

$$u_1 = \|u\|\cos\theta\sin\varphi, \quad u_2 = \|u\|\sin\theta\sin\varphi \quad \text{and} \quad u_3 = \|u\|\cos\varphi$$

for some $\theta \in [0, 2\pi)$ and $\varphi \in [0, \pi]$, we have

$$
\begin{aligned}
&FG(u_1, u_2, u_3)\\
&= \int_{-\pi}^{\pi} \frac{(\|u\| \sin\varphi \cos(t-\theta) - 1)dt}{1 + \|u\|^2 - 2\|u\| \sin\varphi \cos(t-\theta) + \|u\| \cos\varphi \sqrt{1 + \|u\|^2 - 2\|u\| \sin\varphi \cos(t-\theta)}}\\
&= \int_{-\pi}^{\pi} \frac{(\|u\| \sin\varphi \cos t - 1)dt}{1 + \|u\|^2 - 2\|u\| \sin\varphi \cos t + \|u\| \cos\varphi \sqrt{1 + \|u\|^2 - 2\|u\| \sin\varphi \cos t}}.
\end{aligned}
$$

This shows that FG does not depend on θ. Thus, for each circle parallel to the unknot, FG is constant on that circle. Hence,

$$FG(u_1, u_2, u_3) = FG(\sqrt{u_1^2 + u_2^2}, 0, u_3).$$

With this, we assume in addition that $u_2 = 0$; so the formula (6.6) becomes

$$FG(u_1, 0, u_3) = \int_{-\pi}^{\pi} \frac{(u_1 \cos t - 1)dt}{1 + u_1^2 + u_3^2 - 2u_1 \cos t + u_3 \sqrt{1 + u_1^2 + u_3^2 - 2u_1 \cos t}}. \tag{6.7}$$

We have some special cases where we can compute the integral explicitly.

1. If $u_3 = 0$, we use the identity

   $$\cos t = \frac{1 - \tan^2(t/2)}{1 + \tan^2(t/2)}$$

   and deal with improper integrals; there are two situations:

   - $|u_1| < 1$: we have

     $$FG(u_1, 0, 0) = \left[-\frac{t}{2} - \arctan\left(\frac{1 + |u_1|}{1 - |u_1|} \tan\frac{t}{2} \right) \right]_{-\pi}^{\pi} = -2\pi;$$

   - $|u_1| > 1$: we have

     $$FG(u_1, 0, 0) = \left[-\frac{t}{2} + \arctan\left(\frac{|u_1| + 1}{|u_1| - 1} \tan\frac{t}{2} \right) \right]_{-\pi}^{\pi} = 0.$$

2. If $u_1 = u_2 = 0$, then we have

   $$FG(0, 0, u_3) = -\int_{-\pi}^{\pi} \frac{dt}{1 + u_3^2 + u_3\sqrt{1 + u_3^2}} = \frac{-2\pi}{1 + u_3^2 + u_3\sqrt{1 + u_3^2}}.$$

Let us consider the general case (u_2 still assumed to be 0). We simplify the integrand of (6.7) as follows:

$$\begin{aligned}
&\frac{u_1\cos t-1}{1+u_1^2+u_3^2-2u_1\cos t+u_3\sqrt{1+u_1^2+u_3^2-2u_1\cos t}}\\
&=\frac{u_1\cos t-1}{u_3}\left(\frac{1}{\sqrt{1+u_1^2+u_3^2-2u_1\cos t}}-\frac{1}{\sqrt{1+u_1^2+u_3^2-2u_1\cos t}+u_3}\right)\\
&=\frac{u_1\cos t-1}{u_3}\left(\frac{1}{\sqrt{1+u_1^2+u_3^2-2u_1\cos t}}-\frac{\sqrt{1+u_1^2+u_3^2-2u_1\cos t}-u_3}{1+u_1^2-2u_1\cos t}\right)\\
&=\frac{u_1\cos t-1}{u_3}\left(\frac{-u_3^2}{(1+u_1^2-2u_1\cos t)\sqrt{1+u_1^2+u_3^2-2u_1\cos t}}+\frac{u_3}{1+u_1^2-2u_1\cos t}\right)\\
&=\frac{-u_3(u_1\cos t-1)}{(1+u_1^2-2u_1\cos t)\sqrt{1+u_1^2+u_3^2-2u_1\cos t}}+\frac{u_1\cos t-1}{1+u_1^2-2u_1\cos t}.
\end{aligned}$$

Set

$$C(u_1):=\int_{-\pi}^{\pi}\frac{u_1\cos t-1}{1+u_1^2-2u_1\cos t}=\begin{cases}0 & \text{if } \ |u_1|>1\\ -\pi & \text{if } \ u_1=\pm1\\ -2\pi & \text{if } \ |u_1|<1\end{cases}.$$

Then, we have ($u_3\neq 0$)

$$FG(u_1,0,u_3)=\int_{-\pi}^{\pi}\frac{-u_3(u_1\cos t-1)dt}{(1+u_1^2-2u_1\cos t)\sqrt{1+u_1^2+u_3^2-2u_1\cos t}}+C(u_1) \qquad (6.8)$$

for $u\in\mathbb{R}^3-U$.

Proposition 6.2.1. *1. Let*

$$u=(u_1,u_2,u_3)\in\mathbb{R}^3-\{(u_1,u_2,u_3)|\quad u_1^2+u_2^2=1 \quad \text{and} \quad u_3<0\}.$$

- *If* $u_3\neq 0$, *then*

$$\begin{aligned}
&FG(u_1,u_2,u_3)=FG(\sqrt{u_1^2+u_2^2},0,u_3)\\
&=\int_{-\pi}^{\pi}\frac{-u_3(\sqrt{u_1^2+u_2^2}\cos t-1)dt}{(1+u_1^2+u_2^2-2\sqrt{u_1^2+u_2^2}\cos t)\sqrt{1+\|u\|^2-2\sqrt{u_1^2+u_2^2}\cos t}}+C(\sqrt{u_1^2+u_2^2})
\end{aligned}$$

where

$$C(\sqrt{u_1^2+u_2^2}):=\begin{cases}0 & \text{if } \ u_1^2+u_2^2>1\\ -\pi & \text{if } \ u_1^2+u_2^2=1\\ -2\pi & \text{if } \ u_1^2+u_2^2<1\end{cases}.$$

- *If* $u_3 = 0$, *then*

$$FG(u_1, u_2, 0) = C(\sqrt{u_1^2 + u_2^2}) = \begin{cases} 0 & \text{if} \quad u_1^2 + u_2^2 > 1 \\ -2\pi & \text{if} \quad u_1^2 + u_2^2 < 1 \end{cases}.$$

2. If $u = (u_1, u_2, u_3) \in \{(u_1, u_2, u_3) | \quad u_1^2 + u_2^2 = 1 \quad \text{and} \quad u_3 < 0\}$, *then*

$$\begin{aligned} FG(u_1, u_2, u_3) &= -FG(u_1, u_2, -u_3) = -FG(1, 0, -u_3) \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{u_3 dt}{\sqrt{2 + u_3^2 - 2\cos t}} + \pi. \end{aligned}$$

6.3 Formulae of FG in terms of elliptic integrals

As in Section 6.2, we assume that $u_1 > 0$ and $u_2 = 0$. To write $FG(u_1, 0, u_3)$ in terms of elliptic integrals, from (6.8), we shift t by π and then obtain

$$FG(u_1, 0, u_3) = \int_0^{2\pi} \frac{u_3(1 + u_1 \cos t)dt}{(1 + u_1^2 + 2u_1 \cos t)\sqrt{1 + u_1^2 + u_3^2 + 2u_1 \cos t}} + C(u_1).$$

Using $\cos 2\theta = 1 - 2\sin^2 \theta$, the formula becomes

$$\begin{aligned} &FG(u_1, 0, u_3) \\ &= 2u_3 \int_0^{\pi} \frac{(1/2 + u_1^2/2 + u_1 \cos t) - (u_1^2/2 - 1/2)}{(1 + u_1^2 + 2u_1 \cos t)\sqrt{1 + u_1^2 + u_3^2 + 2u_1 \cos t}} dt + C(u_1) \\ &= u_3 \int_0^{\pi} \frac{dt}{\sqrt{1 + u_1^2 + u_3^2 + 2u_1 \cos t}} \\ &- u_3 \int_0^{\pi} \frac{u_1^2 - 1}{(1 + u_1^2 + 2u_1 \cos t)\sqrt{1 + u_1^2 + u_3^2 + 2u_1 \cos t}} dt + C(u_1) \\ &= \frac{2u_3}{\sqrt{(1 + u_1)^2 + u_3^2}} \int_0^{\pi/2} \frac{dt}{\sqrt{1 - \frac{4u_1}{(1 + u_1)^2 + u_3^2} \sin^2 t}} \\ &- \frac{2u_3(u_1^2 - 1)}{(1 + u_1)^2\sqrt{(1 + u_1)^2 + u_3^2}} \int_0^{\pi/2} \frac{dt}{\left(1 - \frac{4u_1}{(1 + u_1^2)} \sin^2 t\right)\sqrt{1 - \frac{4u_1}{(1 + u_1)^2 + u_3^2} \sin^2 t}} + C(u_1) \\ &= \frac{2u_3}{\sqrt{(1 + u_1)^2 + u_3^2}} \mathbf{K}\left(\sqrt{\frac{4u_1}{(1 + u_1)^2 + u_3^2}}\right) \\ &+ \frac{2u_3(1 - u_1)}{(1 + u_1)\sqrt{(1 + u_1)^2 + u_3^2}} \Pi\left(\frac{4u_1}{(1 + u_1)^2}, \sqrt{\frac{4u_1}{(1 + u_1)^2 + u_3^2}}\right) + C(u_1). \end{aligned} \tag{6.9}$$

We may write the formula in terms of Heuman's Lambda function Λ_0 using the formula

$$\Pi\left(\alpha^2, k\right) = \frac{\pi}{2} \frac{\alpha \Lambda_0(\xi, k)}{\sqrt{(\alpha^2 - k^2)(1 - \alpha^2)}}$$

where

$$\xi = \arcsin\sqrt{\frac{\alpha^2-k^2}{\alpha^2(1-k^2)}},$$

see [2] on Page 228 and [12]. We then have ($u_3 \neq 0$)

$$\Pi\left(\frac{4u_1}{(1+u_1)^2}, \sqrt{\frac{4u_1}{(1+u_1)^2+u_3^2}}\right)$$
$$= \frac{\pi}{2}\frac{\sqrt{\frac{4u_1}{(1+u_1)^2}}\Lambda_0\left(\arcsin\frac{|u_3|}{\sqrt{(1-u_1)^2+u_3^2}}, \sqrt{\frac{4u_1}{(1+u_1)^2+u_3^2}}\right)}{\sqrt{\frac{4u_1u_3^2(1-u_1)^2}{(1+u_1)^4((1+u_1)^2+u_3^2)}}}$$

and (6.9) becomes

$$FG(u_1,0,u_3) = C(u_1) + \frac{2u_3}{\sqrt{(1+u_1)^2+u_3^2}}\mathbf{K}\left(\sqrt{\frac{4u_1}{(1+u_1)^2+u_3^2}}\right)$$
$$+\pi\Lambda_0\left(\arcsin\frac{|u_3|}{\sqrt{(1-u_1)^2+u_3^2}}, \sqrt{\frac{4u_1}{(1+u_1)^2+u_3^2}}\right)\frac{u_3(1-u_1)|1+u_1|}{|u_3||1-u_1|(1+u_1)}. \quad (6.10)$$

Proposition 6.3.1. *1. Let*

$$u = (u_1,u_2,u_3) \in \mathbb{R}^3 - \{(u_1,u_2,u_3)| \quad u_1^2+u_2^2 = 1 \quad \text{and} \quad u_3 < 0\}.$$

- *If $u_3 \neq 0$, then*

$$FG(u_1,u_2,u_3) = FG(\sqrt{u_1^2+u_2^2}, 0, u_3)$$
$$= C(\sqrt{u_1^2+u_2^2}) + \frac{2u_3}{\sqrt{(1+\sqrt{u_1^2+u_2^2})^2+u_3^2}}\mathbf{K}\left(\sqrt{\frac{4\sqrt{u_1^2+u_2^2}}{(1+\sqrt{u_1^2+u_2^2})^2+u_3^2}}\right)$$
$$+\pi\Lambda_0\left(\arcsin\frac{|u_3|}{\sqrt{(1-\sqrt{u_1^2+u_2^2})^2+u_3^2}}, \sqrt{\frac{4\sqrt{u_1^2+u_2^2}}{(1+\sqrt{u_1^2+u_2^2})^2+u_3^2}}\right)\frac{u_3(1-\sqrt{u_1^2+u_2^2})}{|u_3||1-\sqrt{u_1^2+u_2^2}|}$$

where

$$C(\sqrt{u_1^2+u_2^2}) := \begin{cases} 0 & \text{if } \; u_1^2+u_2^2 > 1 \\ -\pi & \text{if } \; u_1^2+u_2^2 = 1 \\ -2\pi & \text{if } \; u_1^2+u_2^2 < 1 \end{cases}.$$

- *If $u_3 = 0$, then*

$$FG(u_1,u_2,0) = C(\sqrt{u_1^2+u_2^2}) = \begin{cases} 0 & \text{if } \; u_1^2+u_2^2 > 1 \\ -2\pi & \text{if } \; u_1^2+u_2^2 < 1 \end{cases}.$$

2. If $u = (u_1, u_2, u_3) \in \{(u_1, u_2, u_3) | \quad u_1^2 + u_2^2 = 1 \quad \text{and} \quad u_3 < 0\}$, *then*

$$FG(u_1, u_2, u_3) = -FG(u_1, u_2, -u_3) = -FG(1, 0, -u_3) = \pi + \frac{2u_3}{\sqrt{4 + u_3^2}} \mathbf{K} \left(\sqrt{\frac{4}{4 + u_3^2}} \right).$$

Another approach in computing the solid angle for an unknot was given by F. Paxton, see [12]. He showed that the solid angle subtended at a point P with height L from the unknot and with distance r_0 from the axis of the unknot is equal to

$$\begin{cases} 2\pi - \frac{2L}{R_{max}} \mathbf{K}(k) - \pi\Lambda_0(\xi, k) & \text{if} \quad r_0 < 1 \\ \pi - \frac{2L}{R_{max}} \mathbf{K}(k) & \text{if} \quad r_0 = 1 \\ -\frac{2L}{R_{max}} \mathbf{K}(k) + \pi\Lambda_0(\xi, k) & \text{if} \quad r_0 > 1 \end{cases}$$

where $R_{max} = \sqrt{(1 + r_0)^2 + L^2}$ and $\xi = \arctan \frac{L}{|1 - r_0|}$. Writing L, r_0 and ξ in terms of u_1, u_2 and u_3, his and our results agree.

We remark that the computation of the solid angle of the unknot was also studied by Maxwell. He gave the formulae in terms of infinite series, see Page 331-334, Chapter XIV in [9].

6.4 Behaviour of FG near U

Let T be the tubular neighbourhood of the smooth knot K with the core removed. Recall that a map $q : T \to S^1$ is a *locally trivial fibration* if for each $s \in S^1$ there exists an open neighbourhood $V \subset S^1$ of s such that the following diagram

$$\begin{tikzcd} q^{-1}(V) \arrow[r, "\cong"] \arrow[d, "q"'] & q^{-1}(s) \times V \arrow[dl, "proj"] \\ V & \end{tikzcd}$$

commutes.

Our main goal in this chapter is to construct a closed Seifert surface for U by showing that FG is a locally trivial fibration near U. We borrow the result from Theorem 7.2.1, which says that FG is a locally trivial fibration near U if its partial derivative with respect to the meridional coordinate never vanishes.

We shall now investigate the behaviour of FG and its partial derivatives near U.

Let us compute FG near the unknot at $(1, 0, 0)$. Write

$$u_1 = 1 + \varepsilon \cos \lambda, \quad u_2 = 0 \quad \text{and} \quad u_3 = \varepsilon \sin \lambda$$

where $\varepsilon > 0$ is sufficiently small and $\lambda \in [0, 2\pi]$. By Proposition 6.3.1, we obtain

$$
\begin{aligned}
&FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda)\\
&= C(1+\varepsilon\cos\lambda) + \frac{2\varepsilon\sin\lambda}{\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}\mathbf{K}\left(\sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right)\\
&- \operatorname{sgn}(\tan\lambda)\pi\Lambda_0\left(\arcsin|\sin\lambda|, \sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right)\\
&= \begin{cases}
\frac{2\varepsilon\sin\lambda}{\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}\mathbf{K}\left(\sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right) - \pi\Lambda_0\left(\lambda, \sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right) \\
\qquad \text{if} \quad \lambda \in [0, \pi/2]; \\
-2\pi + \frac{2\varepsilon\sin\lambda}{\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}\mathbf{K}\left(\sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right) + \pi\Lambda_0\left(\pi-\lambda, \sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right) \\
\qquad \text{if} \quad \lambda \in [\pi/2, \pi]; \\
-2\pi + \frac{2\varepsilon\sin\lambda}{\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}\mathbf{K}\left(\sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right) - \pi\Lambda_0\left(\lambda-\pi, \sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right) \\
\qquad \text{if} \quad \lambda \in [\pi, 3\pi/2]; \\
\frac{2\varepsilon\sin\lambda}{\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}\mathbf{K}\left(\sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right) + \pi\Lambda_0\left(2\pi-\lambda, \sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right) \\
\qquad \text{if} \quad \lambda \in [3\pi/2, 2\pi].
\end{cases}
\end{aligned}
\tag{6.11}
$$

Remark 6.12. By (6.11), when $\lambda = 3\pi/2$, it falls into the third and the fourth cases. Since $\Lambda_0(\pi/2, k) = 1$, we have

$$FG(1,0,\varepsilon) = -3\pi - \frac{2\varepsilon}{\sqrt{4+\varepsilon^2}}\mathbf{K}\left(\sqrt{\frac{4}{4+\varepsilon^2}}\right) = -\frac{2\varepsilon}{\sqrt{4+\varepsilon^2}}\mathbf{K}\left(\sqrt{\frac{4}{4+\varepsilon^2}}\right) + \pi \mod 4\pi.$$

□

With this coordinate system, the point $u = (u_1, u_2, u_3)$ becomes close to U as $\varepsilon \to 0^+$. To understand FG near U, we would like to find

$$\lim_{\varepsilon\to 0^+} FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda).$$

We need the following lemma.

Lemma 6.4.1.

$$\lim_{\varepsilon\to 0^+} \frac{2\varepsilon\sin\lambda}{\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}\mathbf{K}\left(\sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}\right)=0.$$

Proof. Here, set

$$k=\sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}}.$$

Then,

$$\sqrt{1-k^2}=\frac{\varepsilon}{\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}.$$

It is enough to show that

$$\lim_{k\to 1^-}(\sqrt{1-k^2})\mathbf{K}(k)=0.$$

By Proposition 6.1.2, we have

$$(\sqrt{1-k^2})\mathbf{K}(k)=(\sqrt{1-k^2})\ln\frac{4}{\sqrt{1-k^2}}+O\left((1-k^2)^{3/2}\ln\sqrt{1-k^2}\right)\quad \text{as}\quad k\to 1^-.$$

Since

$$\lim_{k\to 1^-}(\sqrt{1-k^2})\ln\sqrt{1-k^2}=0,$$

it follows that

$$\lim_{k\to 1^-}(\sqrt{1-k^2})\mathbf{K}(k)=0.$$

□

Proposition 6.4.2.

$$\lim_{\varepsilon\to 0^+} FG(1+\varepsilon\cos\lambda,0,\varepsilon\sin\lambda)=-2\lambda\quad\in\mathbb{R}/4\pi\mathbb{Z}.$$

Proof. By Equation (6.11) and the previous lemma, we have

$$\lim_{\varepsilon\to 0^+} FG(1+\varepsilon\cos\lambda,0,\varepsilon\sin\lambda)=\begin{cases}-\pi\Lambda_0(\lambda,1) & \text{if}\quad \lambda\in[0,\pi/2]\\ -2\pi+\pi\Lambda_0(\pi-\lambda,1) & \text{if}\quad \lambda\in[\pi/2,\pi]\\ -2\pi-\pi\Lambda_0(\lambda-\pi,1) & \text{if}\quad \lambda\in[2\pi,3\pi/2]\\ \pi\Lambda_0(2\pi-\lambda,1) & \text{if}\quad \lambda\in[3\pi/2,2\pi].\end{cases}$$

Using the identity $\Lambda_0(\beta,1)=\dfrac{2\beta}{\pi}$, we finally obtain

$$\lim_{\varepsilon\to 0^+} FG(1+\varepsilon\cos\lambda,0,\varepsilon\sin\lambda)=-2\lambda\quad\in\mathbb{R}/4\pi\mathbb{Z} \tag{6.13}$$

for all $\lambda\in[0,2\pi]$. □

Next we compute the derivatives of FG with respect to ε and λ near U. As before, let

$$k = \sqrt{\frac{4+4\varepsilon\cos\lambda}{4+4\varepsilon\cos\lambda+\varepsilon^2}} \quad \text{and} \quad k' = \sqrt{1-k^2} = \frac{\varepsilon}{\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}.$$

Proposition 6.4.3.

$$\frac{\partial}{\partial\varepsilon}FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda) = \frac{2\sin\lambda(\mathbf{K}(k)-E(k))}{(1+\varepsilon\cos\lambda)\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}.$$

Proof. Note that

$$\frac{\partial k}{\partial\varepsilon} = \frac{-2\varepsilon(2+\varepsilon\cos\lambda)}{k(4+4\varepsilon\cos\lambda+\varepsilon^2)^2}.$$

Hence, using (6.2)

$$\frac{d}{dk}\mathbf{K}(k) = \frac{E(k)-k'^2\mathbf{K}(k)}{kk'^2},$$

we have

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon}k'\mathbf{K}(k) &= k'\frac{\partial}{\partial\varepsilon}\mathbf{K}(k) + \mathbf{K}(k)\left(\frac{-k}{k'}\right)\frac{\partial k}{\partial\varepsilon} \\
&= \left(\frac{E(k)-k'^2\mathbf{K}(k)}{kk'} - \left(\frac{k^2\mathbf{K}(k)}{kk'}\right)\right)\frac{\partial k}{\partial\varepsilon} \\
&= \left(\frac{E(k)-\mathbf{K}(k)}{kk'}\right)\frac{-2\varepsilon(2+\varepsilon\cos\lambda)}{k(4+4\varepsilon\cos\lambda+\varepsilon^2)^2}.
\end{aligned}$$

Hence,

$$2\sin\lambda\frac{\partial}{\partial\varepsilon}k'\mathbf{K}(k) = \frac{\sin\lambda(2+\varepsilon\cos\lambda)(\mathbf{K}(k)-E(k))}{(1+\varepsilon\cos\lambda)\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}. \tag{6.14}$$

By the formula

$$\frac{d}{dk}\Lambda_0(\arcsin|\sin\lambda|, k) = \frac{2(E(k)-\mathbf{K}(k))|\sin\lambda\cos\lambda|}{\pi k\sqrt{1-k'^2\sin^2\lambda}}$$

in [2], we have

$$\begin{aligned}
\pi\frac{\partial}{\partial\varepsilon}\Lambda_0(\arcsin|\sin\lambda|, k) &= \left(\frac{2(E(k)-\mathbf{K}(k))|\sin\lambda\cos\lambda|}{k\sqrt{1-k'^2\sin^2\lambda}}\right)\left(\frac{-2\varepsilon(2+\varepsilon\cos\lambda)}{k(4+4\varepsilon\cos\lambda+\varepsilon^2)^2}\right) \\
&= \frac{\varepsilon|\sin\lambda\cos\lambda|(\mathbf{K}(k)-E(k))}{(1+\varepsilon\cos\lambda)\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}.
\end{aligned} \tag{6.15}$$

By (6.14) and (6.15), we obtain

$$\frac{\partial}{\partial\varepsilon}FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda) = \frac{2\sin\lambda(\mathbf{K}(k)-E(k))}{(1+\varepsilon\cos\lambda)\sqrt{4+4\varepsilon\cos\lambda+\varepsilon^2}}.$$

□

We observe that as $\varepsilon \to 0^+$, $\mathbf{K}(k) - E(k)$ blows up and is thus unbounded. Also, notice that the sign of $\dfrac{\partial}{\partial\varepsilon}FG$ depends on $sin\lambda$. Hence, $FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda)$ is non-decreasing with respect to ε when $\lambda \in [0,\pi]$ and it is non-increasing when $\lambda \in [\pi, 2\pi]$. Since we know that

$$\lim_{\varepsilon\to 0^+} FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda) = -2\lambda,$$

Dini's theorem, see Theorem 7.13 in [15], yields that $FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda)$ converges uniformly to -2λ on $[0, 2\pi]$. With this, we can extend FG near U over $[0,\varepsilon] \times [0, 2\pi]$ even though FG is not defined at (1,0,0).

We next deal with the derivative of FG with respect to λ.

Proposition 6.4.4.

$$\frac{\partial}{\partial\lambda}FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda) < 0$$

and

$$\lim_{\varepsilon\to 0^+}\frac{\partial}{\partial\lambda}FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda) = -2.$$

Proof. Note that

$$\begin{aligned}\frac{\partial k}{\partial\lambda} &= \frac{1}{2k}\left(\frac{(4+4\varepsilon\cos\lambda+\varepsilon^2)(-4\varepsilon\sin\lambda) - (4+4\varepsilon\cos\lambda)(-4\varepsilon\sin\lambda)}{(4+4\varepsilon\cos\lambda+\varepsilon^2)^2}\right)\\ &= \frac{-k'^3\sin\lambda}{\sqrt{1+\varepsilon\cos\lambda}}.\end{aligned}$$

Again, using the formula in [2], we have

$$\begin{aligned}&\frac{\partial}{\partial\lambda}FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda) = \frac{\partial}{\partial\lambda}(2\sin\lambda k'\mathbf{K}(k) \pm \pi\Lambda_0(\arcsin|\sin\lambda|, k))\\ &= 2\sin\lambda\frac{\partial k}{\partial\lambda}\left(\frac{E(k)-\mathbf{K}(k)}{kk'}\right) + 2k'\mathbf{K}(k)\cos\lambda\\ &- 2\left(\frac{E(k) - k'^2\sin^2\lambda\mathbf{K}(k))}{\sqrt{1-k'^2\sin^2\lambda}}\right) - 2\left(\frac{2(E(k)-\mathbf{K}(k))\sin\lambda\cos\lambda}{k\sqrt{1-k'^2\sin^2\lambda}}\right)\frac{\partial k}{\partial\lambda}.\end{aligned}$$

As $\varepsilon \to 0^+$, we have $k \to 1$, $k' \to 0$. Hence, the only significant term in the above expression is $-\dfrac{2E(k)}{\sqrt{1-k'^2\sin^2\lambda}}$ since $k'\mathbf{K}(k) \to 0$ and $k'E(k) \to 0$ as $\varepsilon \to 0^+$. Thus,

$$\lim_{\varepsilon\to 0^+}\frac{\partial}{\partial\lambda}FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda) = -2E(1) = -2.$$

□

Remark 6.16.

$$\lim_{\varepsilon\to 0^+}\frac{\partial}{\partial\lambda}FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda) = -2 = \frac{\partial}{\partial\lambda}\lim_{\varepsilon\to 0^+} FG(1+\varepsilon\cos\lambda, 0, \varepsilon\sin\lambda).$$

□

In Section 6.2, we have seen that FG has symmetry along any circle that is parallel to U and has centre on the z-axis. Hence, if α is the longitudinal coordinate near U, then $\dfrac{\partial}{\partial \alpha} FG = 0$. The two coordinates we have to deal with are the meridional and radial coordinates λ and ε.

The following proposition is the main result in this chapter.

Proposition 6.4.5. *If $t \in \mathbb{R}/4\pi\mathbb{Z}$ is a regular value of FG with $t \neq 0$, then $FG^{-1}(t)$ is a bounded regular open Seifert surface for U.*

Proof. Let D_0 be the punctured disc of radius ε without the centre $(1,0,0)$. Then, D_0 is a slice of the tubular neighbourhood of U, consisting of the points with distance ε from U. We use the polar coordinates (r, λ) on D_0 where r represents the distance from $(1,0,0)$ and $\lambda \in [0, 2\pi]$ (with 0 and 2π identified, and we may think of λ as the coordinate on S^1) represents the angle.

Let $t \in \mathbb{R}/4\pi\mathbb{Z}$ be a regular value of FG with $t \neq 0$. By Propositions 4.5.4 and 5.4.1, we know that $\Sigma_0 := (FG)^{-1}(t)$ is a bounded open Seifert surface for U. It remains to show that Σ_0 is regular. We can think of $(FG)|_{D_0}$ as

$$(FG)|_{D_0} : (0, \varepsilon] \times [0, 2\pi] \to \mathbb{R}/4\pi\mathbb{Z}$$

with $(FG)|_{D_0}(r, 0) = (FG)|_{D_0}(r, 2\pi)$. Since

$$\lim_{r \to 0^+} FG(1 + r\cos\lambda, 0, r\sin\lambda) = -2\lambda$$

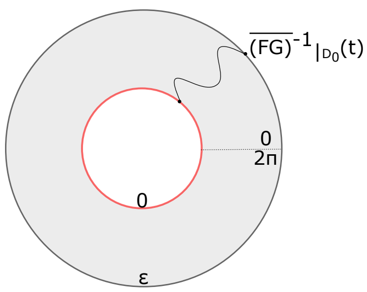
and the convergence is independent of r, we can extend $(FG)|_{D_0}$ over $[0, \varepsilon] \times [0, 2\pi]$ to

$$\overline{(FG)}|_{D_0} : [0, \varepsilon] \times [0, 2\pi] \to \mathbb{R}/4\pi\mathbb{Z}$$

such that $\overline{(FG)}|_{D_0}(0, \lambda) = -2\lambda$. Note that t is also a regular value of both $(FG)|_{D_0}$ and $\overline{(FG)}|_{D_0}$. Hence,

$$[0, 1] \cong \overline{(FG)}|_{D_0}^{-1}(t) \cong (FG)|_{D_0}^{-1}(t) \cup \{x\}$$

for some $x \in U$.



[0,ε]×[0,2π]

This implies that there exists an embedding

$$U \times [0,1] \cong \Big((FG)|_{D_0}^{-1}(t) \times U \Big) \cup U \hookrightarrow (FG)^{-1}(t) \cup U$$

such that

$$U \times (0,1] \cong \Big((FG)|_{D_0}^{-1}(t) \times U \Big) \hookrightarrow (FG)^{-1}(t)$$

is smooth. $\square$

Chapter 7

Main results

This Chapter deals with the general situation, where K is an arbitrary knot in $\mathbb{R}^3$. As in Chapter 6, we shall show that FG is a locally trivial fibration near the knot. This implies that the union of the preimage $(FG)^{-1}(t)$ of a regular value $t \in \mathbb{R}/4\pi\mathbb{Z}$ and K is a closed Seifert surface for the knot.

The work in this chapter is in collaboration with Dr. Maciej Borodzik.

7.1 Statement of results

We shall prove the following.

Theorem 7.1.1. *Let $K \subset \mathbb{R}^3$ be a C^3-smooth knot. Then for a neighbourhood T of K the map $FG\colon T - K \to S^1$ is a locally trivial fibration, whose fibers are diffeomorphic to the product $S^1 \times (0, 1]$.*

Corollary 7.1.2. *If $t \in (0, 4\pi)$ is a regular value of FG, then $FG^{-1}(t)$ is a (possibly disconnected) closed Seifert surface for K.*

The proof takes the remainder of this chapter. Here is a short sketch.

- We introduce local coordinates r, φ, λ in a neighbourhood of the knot K. We may think of the neighbourhood as a small tube around the knot so that r is the distance to the knot, φ is the longitudinal coordinate (increasing as we go around the knot) and λ is meridional coordinate, that is, angle on a plane orthogonal to the knot at a given point.
- Using Proposition 7.2.1 with $M = S^1 \times (0, 1]$, we shall show that $-\dfrac{\partial FG}{\partial \lambda}$ is bounded from below by a positive constant.
- For given point $u \notin K$ in a neighbourhood of K we consider an auxiliary knot K_0, which is a round circle. The corresponding function FG for the knot K_0 will be denoted FG_0. Notice that K_0 depends on the choice of u.

- The main part of the proof is to show that in a neighbourhood of $u \notin K$ we have a bound $\left|\dfrac{\partial FG}{\partial\lambda} - \dfrac{\partial FG_0}{\partial\lambda}\right| < C\varepsilon^{1/5}$, where ε is the distance between u and K, and C is a constant that depends on derivatives of the parametrisation of K, but not on u.

- Since the round circle K_0 is an unknot, we know from Chapter 6 that $\dfrac{\partial FG_0}{\partial\lambda}+2 = O(\varepsilon^{1/5})$ as $\varepsilon \to 0^+$.

- The two above results show that $\dfrac{\partial FG}{\partial\lambda} \sim -2$ if ε is small.

Remark 7.1. A little care should be taken. Our function FG takes values in $\mathbb{R} \mod 4\pi$. However the coordinate λ changes in $\mathbb{R} \mod 2\pi$. Hence the derivative of FG over λ being 2 means that the preimage of FG is locally connected.

7.2 Fibration theorem

We shall prove the following result:

Proposition 7.2.1. *Suppose M is a smooth manifold and $\pi\colon M\times S^1 \to S^1$ is a smooth surjection such that $\dfrac{\partial\pi}{\partial\alpha} > 0$, where α is the second coordinate. Then π is a locally trivial fibration.*

Proof. Choose a Riemannian metric $\langle\cdot,\cdot\rangle$ on $M \times S^1$ preserving the product structure and consider an auxiliary proper function $f\colon M \to \mathbb{R}_{\geq 0}$ (this might be e.g. the square of the distance to a point). Extend f to the whole of $M \times S^1$ so that it depends on the first factor only. The vector field $v = \dfrac{\partial}{\partial\alpha}$ is orthogonal to the gradient of f. Define $w = \left(\dfrac{\partial\pi}{\partial\alpha}\right)^{-1} v$. Then w is orthogonal to f and

$$\langle w, \nabla\pi\rangle = 1. \tag{7.2}$$

As w admits a proper first integral f, the solution of an equation $\dot{x} = w(x)$ exists over the whole of $\mathbb{R}$. Therefore, w defines a flow φ_t on $M \times S^1$. We claim that

$$\pi(\varphi_t(x)) = \pi(x) + t \tag{7.3}$$

for any $x \in M \times S^1$.

To prove (7.3) differentiate both sides over t at $t = 0$. The left hand side becomes $w(\pi)$, that is, the differential of π in the direction of w. This can be written as $\langle w, \nabla\pi\rangle$, by (7.2), it is equal to 1.

Given now (7.3), we notice that φ_t is a diffeomorphism of fibers of π, providing a local trivialisation. □

In general, unless we have some control over f or M, we cannot claim that the fibers of π are a disjoint union of copies of M. However, if f has only finitely many critical points, then an easy exercise shows that π has this property.

7.3 Some facts about curves in $\mathbb{R}^3$

We define a knot K as a C^3-smooth embedding $w : [0, l] \to \mathbb{R}^3$ such that $w(0) = w(l)$, and both first and second derivatives of w at 0 and l agree. In addition, we assume that w is an arc length parametrisation of K, that is, $\|\dot{w}(t)\| \equiv 1$. With this notation, l is the length of the knot. We denote by C_2 the supremum of $\|\ddot{w}\|$ and C_3 the supremum of the third-order derivative of w. We will sometimes consider w as a periodic function on the whole of $\mathbb{R}$ with period l.

Lemma 7.3.1. *There is a constant $\delta_0 > 0$ such that any ball in $\mathbb{R}^3$ of radius δ_0 or smaller intersects K in a connected set: either an arc, or a point, or an empty intersection.*

Proof. It follows from the Lebesgue's Number Lemma. □

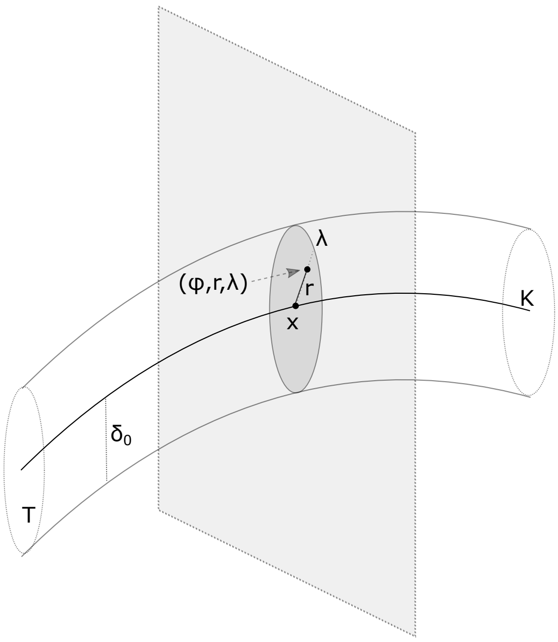
The curvature and the torsion of a C^2-smooth closed curve, by compactness, are bounded. Therefore the following lemma holds.

Lemma 7.3.2. *There exist positive constants D_1 and D_2 such that for any $x \in K$ and for any small $\varepsilon > 0$, the length of K contained in the ball $B(x, \varepsilon)$ is between $D_1\varepsilon$ and $D_2\varepsilon$.*

Proof. One can take $D_1 = 2$. To choose D_2, we use a result regarding distortion. The distortion of a curve in $\mathbb{R}^3$ is the supremum of the quotient between the length between two points on the curve and the distance between two points in $\mathbb{R}^3$. Since the curvature is finite, the distortion is also finite, see Section 7 in [17]. □

7.4 A coordinate system near K

Choose a tubular neighbourhood T of K in $\mathbb{R}^3$. We can think of it as a set of points at distance less or equal to δ_0 from K. In other words, $T - K$ can be viewed as a solid torus without core $S^1 \times (D^2 - \{0\})$. We shall introduce the following coordinate system.



We set $\varphi = \dfrac{l}{2\pi} \bmod 2\pi$ to be the first coordinate going along K in the longitudinal direction. For a point $x \in K$, consider the plane perpendicular to K at x which intersects T along a disk. Then, r is the radial coordinate on the disc representing the distance to the centre of the disc and λ is the angular coordinate. It remains to specify the zero of the λ coordinate. To this end, suppose $\ddot{w} \neq 0$ at each point. Then the direction of the normal vector of w points to the zero value of the λ coordinate.

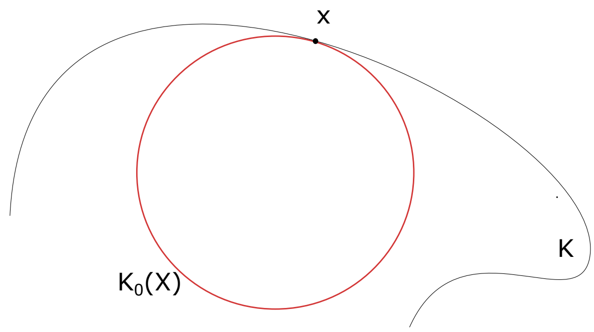
The triple (φ, r, λ) forms a local coordinate system on $T - K$ (we might need to shrink δ_0). This either follows from the Implicit Function Theorem or can be seen geometrically that: for any two points x and x' with $x \neq x'$, the planes through x and x' perpendicular to K do not intersect in T, and each point in T belongs to exactly one such plane.

7.5 A reference unknot at a point x

For each point $x \in K$, we define $K_0(x)$ to be the *reference unknot* for (K, x). This is an unknot bitangent to K, that is, a round circle parametrised by $w_0(t)$ such that $w(t_0) = w_0(t_0) = x$. We assume that the first and second derivative at t_0 of w and w_0 coincide, i.e.,

$$\dot{w}(t_0) = \dot{w}_0(t_0) \quad \text{and} \quad \ddot{w}(t_0) = \ddot{w}_0(t_0).$$

The radius of the circle is the inverse of $\|\ddot{w}(t_0)\|$. In addition, we assume that $\|\ddot{w}(t)\|$ is bounded from below by a non-zero constant $\dfrac{1}{R}$.



Fix a point $x \in K$. The projection of K from x to the unit sphere, the image of the map $\Pi_x \colon t \mapsto \dfrac{w(t)-x}{\|w(t)-x\|}$, cannot fill the whole sphere since Π_x is a smooth map whose codomain has higher dimension. Hence, there is a point z in the sphere such that z misses the image of K. The same argument holds for K replaced by $K_0(x)$.

Lemma 7.5.1. *There exists $\rho' > 0$ such that for any $u \in T$, there exist a point $z \in S^2$ and a neighbourhood U of u in $\mathbb{R}^3$ with $U \cap K \neq \varnothing$ such that*

$$\left\| z - \frac{w(t)-y}{\|w(t)-y\|} \right\| > \frac{1}{\rho'}$$

for all $y \in U$. The lemma also holds for all knots $K_0(x)$ for $x \in U \cap K$.

Proof. Given $u \in T$, choose $x \in K$ that is the closest point to u (if $u \in K$, we choose $x = u$). The projection $\Pi_x : t \mapsto \dfrac{w(t)-x}{\|w(t)-x\|}$ misses some points in S^2; so let z and z' be antipodal points with this property (any differentiable curve in $\mathbb{RP}^2$ is not surjective). In fact, Π_x misses both small neighbourhoods of z and z' in S^2. Let K_x be a small neighbourhood of x in K. Notice that for any $y \in T - K$ near x, $\Pi_y(K - K_x)$ misses both z and z' because $\Pi_x(K - K_x)$ and $\Pi_y(K - K_x)$ do not differ much. Since K_x is almost a straight line, it is clear that $\Pi_y(K_x)$ cannot hit both antipodal points z and z'. Hence, for each $x \in K$ there exist a positive number $\rho'(x)$ and an open neighbourhood U_x of x in $\mathbb{R}^3$ such that for any $y \in U_x$,

$$\left\| z_x - \frac{w(t)-u}{\|w(t)-u\|} \right\| > \frac{1}{\rho'(x)}$$

for some $z_x \in S^2$.

Now we cover K by the union of those U_x's. Since K is compact, we can pass to a finite subcover, $U_{x_1} \cup \cdots \cup U_{x_n} \supset K$. Shrinking further δ_0 if necessary so that T belongs to the union of U_{x_j}'s. Setting $\rho' = \min\{\rho'(x_1), \ldots, \rho'(x_n)\}$, we complete the proof. □

Corollary 7.5.2. *There exists $\rho > 0$ such that for any $u \in T$, there exist a point $z \in S^2$ and a neighbourhood U of u in $\mathbb{R}^3$ with $U \cap K \neq \varnothing$ such that*

$$\left\| 1 - \frac{w(t)-y}{\|w(t)-y\|} \cdot z \right\| > \frac{1}{\rho}$$

for all $y \in U$. The lemma also holds for all knots $K_0(x)$ for $x \in U \cap K$.

7.6 Behaviour of FG for the knot K and for its reference unknots

Recall from Theorem 5.3.7 that the function FG can be expressed as

$$FG(u) = \int_0^l P_z(w(t), u)dt,$$

where

$$P_z(w(t), u) = \frac{\left(\frac{w(t) - u}{\|w(t) - u\|} \times z\right) \cdot \dot{w}(t)}{\|w(t) - u\| \left(1 - \frac{w(t) - u}{\|w(t) - u\|} \cdot z\right)}. \tag{7.4}$$

Here, z is a point in the sphere away from im Π_u. The value of FG modulo 4π does not depend on the choice of z.

Let us describe further about $P_z(w(t), u)$. To be precise, we first fix $u = (u_1, u_2, u_3) \in T - K$ close to $w(t_0) = x \in K$. The reference unknot $K_0(x)$ is then defined as in Section 7.5. As before, z can always be chosen so that both $\Pi_x(K - \{x\})$ and $\Pi_x(K_0 - \{x\})$ miss z. Let $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and

$$P_z(\mathbf{w}, u) = \frac{\left(\frac{\mathbf{w} - u}{\|\mathbf{w} - u\|} \times z\right) \cdot \dot{\mathbf{w}}}{\|\mathbf{w} - u\| \left(1 - \frac{\mathbf{w} - u}{\|\mathbf{w} - u\|} \cdot z\right)}$$

where the map $\mathbf{w} \mapsto \dot{\mathbf{w}}$ is C^2-smooth with property that if $\mathbf{w} = w(t)$ is a curve, then $\dot{\mathbf{w}} = \dot{w}(t)$ is the tangent vector. The function $P_z(\mathbf{w}, u)$ is defined locally; that is, it is defined on a small neighbourhood U of u and x. It should be noted that z may not be fixed for the whole U. However, we can fix z if $\mathbf{w}$ changes by a small amount $-$ in particular, we can fix z if $\mathbf{w}$ varies between $w(t)$ and $w_0(t)$ for all t near t_0. With this, we can differentiate $P_z(\mathbf{w}, u)$ with respect to both $\mathbf{w}_j$ and u_j.

The next lemma follows from the form of $P_z(\mathbf{w}, u)$.

Lemma 7.6.1. *Given $u \in T - K$, the function $P_z(\mathbf{w}, u)$ (respectively its k-th derivative)*[1] *is bounded from above by an expression of the form*

$$E_k \left(\frac{1}{\|\mathbf{w} - u\|^{k+1}}\right) \frac{1}{\left\|1 - \frac{\mathbf{w} - u}{\|\mathbf{w} - u\|} \cdot z\right\|},$$

where E_k is a constant depending on $\|\mathbf{w}\|_{C^{k+1}}$.

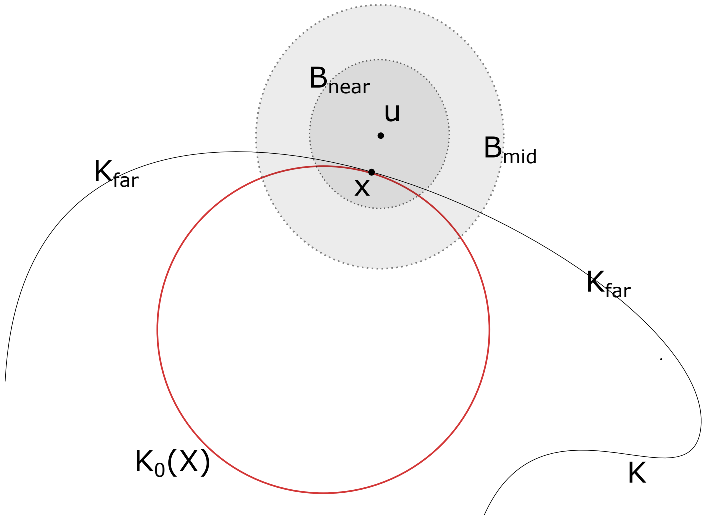
[1] Unless specified explicitly otherwise, we henceforth consider derivatives with respect to u_j or $\mathbf{w}_j$.

Setting $\mathbf{w} = w(t)$ and $E_k^{FG} = \rho E_k$, we obtain.

Corollary 7.6.2. *The k-th derivative of the function $FG(u)$ is bounded by a constant E_k^{FG} times the integral of $\dfrac{1}{\|w(t)-y\|^{k+1}}$ over $[0,l]$.*

Fix a point $u \in T - K$ and let $\varepsilon = r$ be the distance to the knot K. Consider the following balls with centre u: B_{near} has radius $\varepsilon^{3/5}$ and the ball B_{mid} has radius $\varepsilon^{2/5}$. Accordingly, we write $K_{near} = K \cap B_{near}$, $K_{mid} = K \cap (B_{mid} - B_{near})$ and $K_{far} = K \cap (\mathbb{R}^3 - B_{mid})$. We split the interval $[0,l]$ into three parts

$$T_{near/mid/far} = \{t \in [0,l] \colon w(t) \in K_{near/mid/far}\}.$$



By Lemma 7.3.2 the length of T_{near} is bounded from above by $D_2\varepsilon^{3/5}$, while the length of T_{mid} is bounded by $D_2\varepsilon^{2/5}$.

Lemma 7.6.3. *There are constants C_{mid} and C_{far} depending only on δ_0 and the C^2 norm of w such that*

$$\left| \frac{\partial}{\partial u_j} \int_{T_{mid/far}} P_z(w(t), u) dt \right| \leq C_{mid/far} \varepsilon^{-4/5}.$$

Proof. By Lemma 7.6.1 we have

$$\left| \frac{\partial}{\partial u_j} \int_{T_{mid/far}} P_z(w(t), u) dt \right| \leq \int_{T_{mid/far}} E_1 \frac{1}{\|w(t)-u\|^2} \cdot \frac{1}{\left\| 1 - \frac{w(t)-u}{\|w(t)-u\|} \cdot z \right\|} dt.$$

Now for $u \in T - K$ we have $\dfrac{1}{\left\| 1 - \frac{w(t)-u}{\|w(t)-u\|} \cdot z \right\|} < \rho$. Therefore the integrand is bounded by $\dfrac{E_1^{FG}}{\|w(t)-u\|^2}$; compare Corollary 7.6.2.

- For T_{mid}, the measure of T_{mid} is bounded by $D_2\varepsilon^{2/5}$, while $\|w(t)-u\| > \varepsilon^{3/5}$, so the integral is bounded by $D_2 E_1^{FG}\varepsilon^{-4/5}$.
- For T_{far}, the measure of T_{far} is bounded by l and $\|w(t)-u\| > \varepsilon^{2/5}$, so the total contribution is bounded by $lE_1^{FG}\varepsilon^{-4/5}$.

Since E_1^{FG} does not depend on u, we set $C_{mid} = D_2E_1^{FG}$ and $C_{far} = lE_1^{FG}$. □

Next we take care of T_{near}. Following the proof of Lemma 7.6.3, $\dfrac{\partial}{\partial u_j}FG(u)$ is bounded by $D_2E_1^{FG}\varepsilon^{-7/5}$, which is too large. This makes sense – as in Chapter 6 we have already seen that if we go along a very small (of radius ε, for instance) loop around the knot, the total change of the function FG is 4π. Thus, instead of bounding the integral over T_{near} directly, we shall compare the derivative of FG with the derivative of FG_0.

For the point $u \in T - K$ consider the circle $K_0 := K_0(x)$, where $x \in K$ is the nearest point in K to u. The circle K_0 is parametrised by $w_0(t)$ for $t \in [0, l_0]$. For convenience, we assume that $w_0(0) = w(0) = x = w(l) = w_0(l_0)$. Notice also that $\|w(t) - w_0(t)\| \leq C_3t^3$ because w and w_0 agree up to second derivatives.

The FG_0 function for K_0 can be written as the integral

$$FG_0(u) = \int_0^{l_0} P_z(w_0(t), u)dt.$$

Now we assume that $\|u - x\| = \varepsilon$. Similarly to K, we define $K_{0,near} = K_0 \cap B_{near}$, $K_{0,mid} = K \cap (B_{mid} - B_{near})$ and $K_{0,far} = K_0 \cap (\mathbb{R}^3 - B_{mid})$, and the interval $[0, l_0]$ will be split into three parts:

$$T^0_{near/mid/far} = \{t \in [0, l] \colon w(t) \in K_{0,near/0,mid/0,far}\}.$$

The derivative of $FG_0(u)$ is then also split into three integrals over $T^0_{near/mid/far}$. In the following lemma, we bound the integrals over the intervals T^0_{mid} and T^0_{far} as in Lemma 7.6.3.

Lemma 7.6.4. *There are constants $C_{0,mid}$ and $C_{0,far}$ depending only on δ_0 and the C^2 norm of w_0 such that*

$$\left|\frac{\partial}{\partial u_j}\int_{T^0_{mid/far}} P_z(w_0(t), u)dt\right| \leq C_{0,mid/0,far}\varepsilon^{-4/5}.$$

We next compare the contributions of the integrals over T_{near} and T^0_{near} from the knot K and the reference unknot K_0, respectively. First, we notice that $T_{near} = T^0_{near}$.

Lemma 7.6.5. *There is a constant C' depending on ρ and the C^2 norm of w such that*

$$\left|\frac{\partial}{\partial u_j}\int_{T_{near}} P_z(w_0(t), u) - \frac{\partial}{\partial u_j}\int_{T_{near}} P_z(w(t), u)\right| \leq C'\varepsilon^{-3/5}.$$

Proof. Applying the Lagrange mean value theorem to $P_z(\mathbf{w}, u)$ when $\mathbf{w}$ varies between $w(t)$ and $w_0(t)$, we have

$$\left|\frac{\partial}{\partial u_j} P_z(w_0(t), u) - \frac{\partial}{\partial u_j} P_z(w(t), u)\right| \leqslant \left|\frac{\partial^2}{\partial \mathbf{w} \partial u_j} P_z(\xi(t), u)\right| \|w(t) - w_0(t)\|,$$

where $\xi(t)$ belongs to the segment connecting $w(t)$ and $w_0(t)$. Using Lemmas 7.5.1 and 7.6.1, we obtain

$$\left|\frac{\partial^2}{\partial \mathbf{w} \partial u_j} (P_z(w_0(t), u) - P_z(w(t), u))\right| \leq \frac{\rho E_2 C_3 t^3}{\|\xi(t) - u\|^3}.$$

We integrate this over T_{near} with $t \in [-D_2 \varepsilon^{3/5}, D_2 \varepsilon^{3/5}]$ (this is legitimate as w and w_0 are periodic). Notice that $\|\xi(t) - u\| \geq D_3 \varepsilon$ for some $D_3 \in (0, 1)$. With all this, we obtain

$$\left|\int_{T_{near}} \frac{\partial}{\partial u_j} P_z(w_0(t), u) - \int_{T_{near}} \frac{\partial}{\partial u_j} P_z(w(t), u)\right| \leq \frac{\rho D_2^4 E_2 \varepsilon^{12/5}}{D_3^3 \varepsilon^3} \leq \frac{\rho D_2^4 E_2 \varepsilon^{-3/5}}{D_3^3}.$$

We have seen earlier that the constants D_2 and E_2 depend only on w and δ_0. Similarly, the constant D_3 is away from 0 and depends only on the curvature of w. Since the curvature of w is bounded, so is $1/D_3$. We now set $C' = \dfrac{\rho D_2^4 E_2}{D_3^3}$ to complete the proof. □

Corollary 7.6.6. *The difference of the derivatives of FG and FG_0 over u_j is bounded from above by $C_{tot}\varepsilon^{-4/5}$, where C_{tot} does not depend on the choice of the point u.*

Proof. This difference is calculated by integrating $P_z(w(t), u)$ over $T_{near/mid/far}$ and $P_z(w_0(t), u)$ over $T^0_{near/mid/far}$. On $T_{mid/far}$ and $T^0_{mid/far}$ the contribution of each integral is of order $\varepsilon^{-4/5}$, while the difference of the integrals over T_{near} and T^0_{near} is of order $\varepsilon^{-3/5}$. More explicitly,

$$\begin{aligned}
\left|\frac{\partial}{\partial u_j} (FG(u) - FG_0(u))\right| &\leqslant \left|\frac{\partial}{\partial u_j} \int_{T_{mid}} P_z(w(t), u) dt\right| + \left|\frac{\partial}{\partial u_j} \int_{T_{far}} P_z(w(t), u) dt\right| \\
&+ \left|\frac{\partial}{\partial u_j} \int_{T^0_{mid}} P_z(w_0(t), u) dt\right| + \left|\frac{\partial}{\partial u_j} \int_{T^0_{far}} P_z(w_0(t), u) dt\right| \\
&+ \left|\frac{\partial}{\partial u_j} \int_{T_{near}} P_z(w(t), u) - \frac{\partial}{\partial u_j} \int_{T^0_{near}} P_z(w_0(t), u)\right| \\
&\leqslant (C_{mid} + C_{far} + C_{0,mid} + C_{0,far})\, \varepsilon^{-4/5} + C' \varepsilon^{-3/5}.
\end{aligned}$$

Set $C_{tot} = C_{mid} + C_{far} + C_{0,mid} + C_{0,far} + C'$. By previous lemmas and corollaries in Sections 7.5 and 7.6, the constant C_{tot} depends only on δ_0, ρ, w and w_0. Thus, C_{tot} works for all $u \in T - K$. □

Now consider point $x \in K$ and a plane P going through x perpendicular to K.

On this plane there are coordinates r and λ which represent the radius and the angle as mentioned in Section 7.4. Note that these coordinates are the same for K and for $K_0(x)$, because P is also perpendicular to $K_0(x)$ at x by its definition.

Proposition 7.6.7. *Consider the restriction $FG|_{T-K} : T-K \to S^1$ of FG. Then, $\dfrac{\partial}{\partial\lambda}FG < 0$. Therefore, $FG|_{T-K}$ is a locally trivial fibration.*

Proof. Applying the chain rule to $FG - FG_0$ at $u \in T-K$, we have

$$\frac{\partial}{\partial\lambda}\left(FG(u) - FG_0(u)\right) = \sum_{j=1}^{3} \frac{\partial}{\partial u_j}\left(FG(u) - FG_0(u)\right)\frac{\partial u_j}{\partial\lambda}.$$

We know that the polar coordinate (r, λ) is a rotation of the standard polar coordinate in $\mathbb{R}^2$; this implies that $\left|\dfrac{\partial u_j}{\partial\lambda}\right| \leqslant r$. Since $x \in K$ is the nearest point to $u \in T-K$ with $\|u - x\| = \varepsilon$, the radius coordinate of u is ε; that is $\left|\dfrac{\partial u_j}{\partial\lambda}\right| \leqslant \varepsilon$. Hence,

$$\begin{aligned}\left|\frac{\partial}{\partial\lambda}\left(FG(u) - FG_0(u)\right)\right| &\leqslant \sum_{j=1}^{3}\left|\frac{\partial}{\partial u_j}\left(FG(u) - FG_0(u)\right)\right|\left|\frac{\partial u_j}{\partial\lambda}\right| \\ &\leqslant (C\varepsilon^{-4/5})\varepsilon = C\varepsilon^{1/5}\end{aligned}$$

for some $C > 0$ independent of u. Since $\lim_{\varepsilon\to 0^+}\dfrac{\partial}{\partial\lambda}FG_0(u) = -2$, it yields

$$\frac{\partial}{\partial\lambda}FG(u) = -2 + O(\varepsilon^{1/5}) \quad \text{as} \quad \varepsilon \to 0^+.$$

Therefore, $\dfrac{\partial}{\partial\lambda}FG(u) < 0$ for all $u \in T-K$. $\square$

Chapter 8

Prospects

This chapter lists some possible future work regarding our construction.

- **Minimality property**: The genus of a knot is the minimal genus of Seifert surfaces for the knot. Given a knot and a knot projection, we can compute the genus of a Seifert surface for the knot produced from Seifert's Algorithm by the formula

$$\text{genus} = 1 - \frac{s - c + 1}{2}$$

where s is the number of Seifert circles and c is the number of crossings, see Chapter 5 in [14]. However, this Seifert surface may not give the minimal genus. We may ask if a Seifert surface produced from our construction gives the minimal genus of the knot.

- **Construction of Seifert surfaces for knots in higher dimensions**: A smooth knot in $\mathbb{R}^{n+2}$ is a smooth embedding $K : S^n \subset \mathbb{R}^{n+2}$. A (closed) Seifert surface Σ for a knot K in $\mathbb{R}^{n+2}$ is a compact orientable $(n+1)$-manifold embedded in $\mathbb{R}^{n+2}$ with $\partial\Sigma = K(S^n)$. It is possible that Seifert surfaces for knots in $\mathbb{R}^{n+2}$ can be constructed using a similar method as follows.

Let $K : S^n \hookrightarrow \mathbb{R}^{n+2}$ be a smooth n-dimensional knot in $\mathbb{R}^{n+2}$. Consider the composite

$$\mathbb{R}^{n+2} - K(S^n) \xrightarrow{G'} C^\infty(S^n, S^{n+1}) \xrightarrow{F'} \mathbb{R}/4\pi\mathbb{Z} = S^1$$

where

$$F' : C^\infty(S^n, S^{n+1}) \to S^1 \quad ; \quad \lambda \mapsto \int_{D^{n+1}} \delta\lambda^*(\mathrm{Vol}_{S^{n+1}})$$

and

$$G' : \mathbb{R}^{n+2} - K(S^n) \to C^\infty(S^n, S^{n+1}) \quad ; \quad x \mapsto \left(G'(x) : y \mapsto \frac{K(y) - x}{\|K(y) - x\|} \right).$$

Show that if $c \neq 0$ is a regular value of $F'G'$, then $(F'G')^{-1}(c) \cup K(S^n)$ is a Seifert surface for K.

• **Equipotential surfaces**: It has been known since Maxwell's work, [9], that the magnetic potential of a magnetic shell of unit strength bounded by a simple closed curve (knot) can be measured by the solid angle.

The *force surface* mentioned on Page 140 in [6] by Jancewicz is an equipotential surface, the surface of constant potential. He wrote "*a magnetic force around a circuit is the locus of points of a constant solid visual angle of the circuit.*" He discussed a geometric problem regarding the unknot "*What is the locus of points in which the circle is seen at a given constant solid angle?*", and pointed out that this locus cannot be a part of a sphere.

If a knot is regarded as a current inducing a magnetic field, then equipotential surfaces are Seifert surfaces for the knot. We may investigate further the geometric nature of these surfaces.

Bibliography

[1] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

[2] Paul F. Byrd and Morris D. Friedman. *Handbook of elliptic integrals for engineers and scientists*. Die Grundlehren der mathematischen Wissenschaften, Band 67. Springer-Verlag, New York-Heidelberg, 1971. Second edition, revised.

[3] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[4] Morris W. Hirsch. The work of Stephen Smale in differential topology. In *From Topology to Computation: Proceedings of the Smalefest (Berkeley, CA, 1990)*, pages 83–106. Springer, New York, 1993.

[5] Sze-tsen Hu. *Homotopy theory*. Pure and Applied Mathematics, Vol. VIII. Academic Press, New York, 1959.

[6] Bernard Jancewicz. *Multivectors and Clifford algebra in electrodynamics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1988.

[7] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[8] Ib Madsen and Jørgen Tornehave. *From calculus to cohomology*. Cambridge University Press, Cambridge, 1997. de Rham cohomology and characteristic classes.

[9] James Clerk Maxwell. *A treatise on electricity and magnetism. Vol. 2*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, 1998. Reprint of the third (1891) edition.

[10] Oleg Mazonka. Solid angle of conical surfaces, polyhedral cones, and intersecting spherical caps. *http://arxiv.org/abs/1205.1396*, 2012.

[11] John W. Milnor. *Topology from the differentiable viewpoint*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver, Revised reprint of the 1965 original.

[12] F. Paxton. Solid angle calculation for a circular disk. *Rev. Sci. Instrum.*, 30(4):254–258, 1959.

[13] Renzo L. Ricca and Bernardo Nipoti. Gauss' linking number revisited. *J. Knot Theory Ramifications*, 20(10):1325–1343, 2011.

[14] Dale Rolfsen. *Knots and links.* Publish or Perish Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7.

[15] Walter Rudin. *Principles of mathematical analysis.* McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. International Series in Pure and Applied Mathematics.

[16] H. Seifert. Über das Geschlecht von Knoten. *Math. Ann.*, 110(1):571–592, 1935.

[17] John M. Sullivan. Curves of finite total curvature. In *Discrete differential geometry*, volume 38 of *Oberwolfach Semin.*, pages 137–161. Birkhäuser, Basel, 2008.

[18] H. van Haeringen and L. P. Kok. Table errata: *Higher transcendental functions, Vol. II* [McGraw-Hill, New York, 1953; MR **15**, 419] by A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *Math. Comp.*, 41(164):778, 1983.

[19] A. Van Oosterom and J. Strackee. The solid angle of a plane triangle. *IEEE Trans. Biomed. Eng.*, BME-30(2)(8):125–126, 1983.

[20] James W. Vick. *Homology theory*, volume 145 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1994. An introduction to algebraic topology.