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ON PRODUCTS IN A COMPLEX[1]

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1. Introduction

In classical homology theory, founded by Poincaré, the fundamental operation is that of forming the boundary ∂A^p of a chain A^p. This is found by multiplying the coefficients of A^p into a matrix of incidence. Algebraically, an equally obvious operation, using the same matrix of incidence, forms the "coboundary" δA^{p-1} from a given A^{p-1}. It has recently been discovered that the algebraic part of the theory of intersections of chains in a manifold, when interpreted with the other operation, could be generalized to arbitrary complexes. It is the object of this paper to give a complete treatment of the fundamentals of this theory. We use a general type of complex and general coefficient groups, and prove the required invariance theorems. Parts of the paper are new only in form. Various notions used here appear first in Tucker's thesis, [12].[2]

In Part I we define, after Tucker, the complexes to be used. The cohomology groups are defined, and then elementary properties of "dual homomorphisms" are given. The latter are used throughout the paper.

Suppose we ask for a product of p-chains A^p and q-chains B^q, giving $(p + q)$-chains $A^p \smile B^q$, which shall have topological significance. A p-cell times a q-cell far away from the p-cell should certainly give nothing; hence (P_1) of §5 is a natural assumption. Considering δ as the fundamental operation, if we wish the multiplication to give a result in the cohomology groups, we must have cocycle $\smile$ cocycle $=$ cocycle. Hence $\delta(A \smile B)$ must be expressible in terms such as $\delta A \smile B$ and $A \smile \delta B$. (P_2) is the natural form.[3] Suppose we ask that a vertex times itself equal itself. (Hence the γ of §5 is 1.) Then (using Theorem 1), at least in an ordinary connected complex, the products exist, and when carried out in the cohomology groups, are uniquely determined (Theorem 5). This may be considered the fundamental theorem of the present paper.

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[1] Presented to the American Mathematical Society, under a different title, March 27, 1937. An outline of the paper appeared in Proc. Nat. Ac. Sci., vol. 23 (1937), pp. 285-291.

[2] The numbers in square brackets refer to the bibliography at the end of the paper.

[3] Note that the dimension in each term is correct. It is easily seen that the signs must alternate in one of the terms. We might call $\smile$ and $\frown$ "cup" and "cap". Equivalent formulas (see also (Q_2) and (19.2)) occur in a fundamental manner in the classical theory; see for instance Lefschetz, [10], pp. 111, 169 and 226. The same formula appears in the theory of differential forms.

Another product, $A^p \frown B^q = C^{q-p}$, is considered; it is algebraically equivalent to the $\smile$ product. We also consider briefly a particular definition of the products which may be used in simplicial complexes. The rest of Part II is devoted to proving other elementary properties and to showing how general coefficient groups may be used.

In Part III we answer two questions. First, in a polyhedron, are homology and cohomology groups and products independent of the particular simplicial subdivision chosen? The proof §16 that they are (which may be considered well-known) is relatively simple; the considerations show that the groups and products may be associated with an abstract space. Secondly, to find the groups and products, it is often very inconvenient to have to use simplicial subdivisions; but we must then show that a general complex gives the same theory as a simplicial subdivision. This combinatorial theorem (Theorem 14) occupies the rest of Part III. It turns out that complexes which may be used for determining the groups often may not be used for determining the products.[4] However, most of the invariance proof may be carried out in the general type of complex. Similar (but slightly weaker) theorems for homology groups have been proved by Tucker, [12], and Alexandroff-Hopf, [4], Ch. VI.

The relation of the two products to intersection theory in a manifold is considered briefly in Part IV. In contrast with Čech, [5], we use the classical method of dual complexes. In Part V, the products are considered in product complexes and in Euclidean space. As an application of preceding results, some mapping theorems are proved, due in part to H. Hopf.

Some special topics are considered in the Appendix.

Historical note. The coboundary of a chain, when a passage to the limit is applied, becomes the derived of a covariant alternating tensor (compare Alexander, [1]). In this form, of course, it has long been known. From the algebraic standpoint, cocycles appear in a different form in the "pseudocycles" of S. Lefschetz, [10]. Cocycles may be interpreted as cycles in the "dual complex," considered in papers by W. Mayer[5] and A. W. Tucker, [12]. An application of cocycles in their direct form was given in our note on Sphere-spaces.[6]

The work of L. Pontrjagin on character groups led to the realization that not only the homology but also the cohomology groups might be important. At the International Topological Conference, Moscow, 1935, J. W. Alexander and A. Kolmogoroff presented papers giving not only the theory of cohomology groups (with different notations), but also defining a product (for simplicial

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[4] For instance, in a torus, for determining the groups, we may subdivide into one vertex, two 1-cells, and one 2-cell; but for determining the products, we must have say four vertices, eight 1-cells, and four 2-cells. (Or we may use Part V.)

[5] Monatsh. f. Math. u. Phys., vol. 36 (1929), pp. 1–42 and 219–258.

[6] Proc. Nat. Ac. of Sci., vol. 21 (1935), pp. 464–468, §6.

complexes) in the groups.[7] It appears that D. van Danzig[8] and E. Čech also had a portion of these results. However, the Kolmogoroff-Alexander product was not wholly satisfactory, it being too large by a numerical factor. In studying their product at the end of 1935, the author discovered the $\smile$ product of §6. In an effort to generalize a theorem of H. Hopf (see footnote 17), the remaining results of §6 and §7 were found.[9] At the same time, Čech discovered the same products; see [5]. Alexander studied the revised $\smile$ product; see [3]. Finally, the present Part II may be considered an outgrowth of Čech's paper [5]. Our assumptions (P_1) and (P_2) are closely allied to those of Čech, §2. Our proof of Theorem 5 was obtained after a study of a corresponding proof in Čech's paper. In the mean time, H. Freudenthal, [6], also found the results of §7, and studied the relation of these products to other known products.

Recently S. Lefschetz[10] has shown that the products of §6, when properly translated into the residual space of a sphere containing the complex, become the classical intersections. As in Tucker, [12], he gives postulates which a "chain-product" should satisfy. Neither author proves a uniqueness theorem (such as our Theorem 5).

I. Preliminaries

2. The complexes used

Complexes, in topology, are certain algebraic structures which may be given a geometric significance. For a given algebraic structure to be geometrically realizable, certain conditions must be satisfied. For instance, we may demand that there be a simplicial subdivision (algebraically defined); any simplicial complex determines a geometric complex in Euclidean space. Complexes of this nature we shall say "admit a simplicial subdivision" (see Part III). To define a product theory in the complex, stronger conditions are necessary; a complex satisfying these conditions "admits a product theory." Most of the paper will be concerned with these complexes.

A *complex K admitting a product theory* is a system as follows. It has cells[11]

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7 See [2], [8] and [9]. The Kolmogoroff-Alexander product is not that given in [2].

8 See Recueil Math., Moscow, vol. 1 (43), (1936), pp. 672–674. Cohomology groups with general coefficient groups are studied by Steenrod, Am. Journal of Math., vol. 58 (1936), pp. 661–701.

9 These results were applied in classifying the maps of a 3-sphere into a 2-sphere; see Bull. Am. Math. Soc., vol. 42 (1936), p. 338. They were communicated in letters to L. Zippin early in 1936.

10 Bull. Am. Math. Soc., vol. 43 (1937), pp. 345–359, §5, (d). It should be noted that without a postulate such as our (P_1), the product may give pratically any product in the cohomology groups.

11 We assume the cells are finite in number; however, most of the results extend to the infinite case, at least with the proper definitions.

σ_i^p of dimension p. There are two types of relations. A cell may be a *face* of a cell of higher dimension, and is a face of itself. Two cells σ_i^{p-1} and σ_j^p of neighboring dimension have an *incidence number* ${}^p\partial_j^i$, which is an integer. In terms of these numbers we define boundaries ∂A^p etc. as usual (see §3). The *closure* $\bar{\sigma}_i^p$ of a cell σ_i^p is the set of all its faces. The *star* St (σ) of σ is the set of cells with σ as a face. The closure and star of any subcomplex are defined similarly. We say a cycle $A^p = \sum \alpha_i \sigma_i^p$ is *boundary-like* if $p > 0$ or if $p = 0$ and $\sum \alpha_i = 0$ (or $I \cdot A^1 = 0$; see (2.1) and (3.2)). The definition as here given is useful only in the case that δI contains no 1-cells which have two vertices as faces; see Lemma 5.

We make the following assumptions.

(K_1) If σ_1 is a face of σ_2 and σ_2 is a face of σ_3, then σ_1 is a face of σ_3.

(K_2) If ${}^p\partial_j^i \neq 0$, then σ_i^{p-1} is a face of σ_j^p.

(K_3) $\partial\partial A^p = 0$ always (or equivalently, $\delta\delta A^p = 0$).[12]

(K_4) Each boundary-like cycle (with integer coefficients) in any $\bar{\sigma}_i^p$ bounds a chain in $\bar{\sigma}_i^p$.

Note that, using (4.7) below, (K_3) is equivalent to the vanishing (for all A^p, B^{p-2}) of any of $\partial\partial A^p \cdot B^{p-2}$, $\partial A^p \cdot \delta B^{p-2}$, $A^p \cdot \delta\delta B^{p-2}$, or $\delta\delta B^{p-2}$.

Certain elementary properties of these complexes are the following:

LEMMA 1. *If $p > 0$, then $\partial\sigma_i^p \neq 0$.*

For if $\partial\sigma_i^p = 0$, then σ_i^p is a cycle in $\bar{\sigma}_i^p$. By (K_4), it must bound a $(p + 1)$-chain in $\bar{\sigma}_i^p$. But $\bar{\sigma}_i^p$ contains no $(p + 1)$-cells.

LEMMA 2. *Every cell of dimension > 0 has a lower dimensional face, and hence a vertex as a face.*

This follows from the last lemma and (K_2) and (K_1).

LEMMA 3. *Each 1-cell has just one or two vertices as faces.*

By the last lemma, we need merely prove that any σ^1 has not three vertices a, b, c as faces. If it had, then by (K_4), $b - a = k\partial\sigma^1$ and $c - b = l\partial\sigma^1$. Hence $l(b - a) = k(c - b)$, which is impossible (as $k \neq 0$).

The definitions of open and closed subcomplexes are as usual, in terms of "being a face of"; for σ_1 and σ_2 in the subcomplex K' of K, we say σ_2 is a face of σ_1 in K' if it is in K.

LEMMA 4. *If K admits a product theory, so does every closed subcomplex.*

The proof is simple.

Note that, if

$$I = \sum \sigma_i^0 = \text{sum of all the vertices of } K, \tag{2.1}$$

we may have $\delta I \neq 0$. But see Theorem 2. If K is connected, and $\delta I = 0$, the only 0-cocycles are the multiples of I.

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[12] Complexes satisfying (K_1), (K_2) and (K_3) are exactly those considered by Tucker, [12]. If (K_4) is satisfied also, and $\delta I = 0$ (that is, K is "augmentable"), then Tucker shows that K admits a simplicial subdivision and hence is geometrically realizable.

LEMMA 5. *If each* 1-*cell is on two vertices, then* $\delta I = 0$ *and* K *admits a simplicial subdivision* (see §13).

For take any 1-cell σ; say $\partial\sigma = \alpha a + \beta b$. As $a - b$ is boundary-like, there is a k such that $a - b = k\partial\sigma$; hence $\beta = -\alpha$. Therefore $\delta I \cdot \sigma = I \cdot \partial\sigma = \alpha + (-\alpha) = 0$, and $\delta I = 0$. The last statement has been proved by Tucker (see footnote 12).

3. Homology and cohomology groups[13]

The *boundary* and *coboundary* of chains are defined by

$$(3.1)\qquad \partial\left(\sum\nolimits_i \alpha_i\sigma_i^p\right) = \sum\nolimits_{i,j} \alpha_i{}^p\partial_i^j\sigma_j^{p-1}, \qquad \delta\left(\sum\nolimits_i \alpha_i\sigma_i^p\right) = \sum\nolimits_{i,j} \alpha_i{}^{p+1}\partial_j^i\sigma_j^{p+1}.$$

The α_i are integers or elements of an abelian group G.[14] The chain A^p is a *cycle* [*cocycle*] if $\partial A^p = 0$ [$\delta A^p = 0$]. We say

A is *homologous to* B, $A \sim B$, if $A - B$ is a boundary,
A is *cohomologous to* B, $A \backsim B$, if $A - B$ is a coboundary.

As $\partial\partial A^p = 0$, $\delta\delta A^p = 0$ (see (K_3)), we may define as usual the homology and cohomology groups. Using the coefficient group G, we denote these by ${}^p\mathbf{H}^G$, ${}^p\mathbf{H}_G$. The group of p-chains of a complex with integer coefficients will be denoted by L^p.

We define the *scalar product* of two chains of the same dimension by

$$(3.2)\qquad \left(\sum \alpha_i\sigma_i^p\right)\cdot\left(\sum \beta_i\sigma_i^p\right) = \sum \alpha_i\beta_i.$$

Note that $A^p\cdot\sigma^p$ is the coefficient of σ^p in the chain A^p.

4. Dual homomorphisms[15]

Let G and G' be free groups with *fixed* sets of generators $a_1, \cdots, a_p$ and $a_1', \cdots, a_q'$. Then to any matrix of integers $\|\phi_{ij}\|$ $(i = 1, \cdots, p, j = 1, \cdots, q)$ correspond homomorphisms of G into G' and of G' into G, defined by

$$(4.1)\qquad \phi a_i = \sum\nolimits_j \phi_{ij}a_j', \qquad \phi' a_i' = \sum\nolimits_j \phi_{ji}a_j.$$

If ϕ is any homomorphism of G into G', then $\phi a_i = \sum \phi_{ij}a_j'$ for some integers ϕ_{ij}; thus the matrix $\|\phi_{ij}\|$ is defined, and hence the homomorphism ϕ'. Each of ϕ, ϕ' determines the other uniquely; the matrix of one is the transposed matrix of the other. We call these homomorphisms *dual*, and write

$$(4.2)\qquad \phi' = D(\phi), \qquad \phi = D(\phi') = D(D(\phi)).$$

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[13] For further details, see for instance Whitney, [13].

[14] Except where otherwise stated, we use integers: $G = I_0$.

[15] Compare the Appendix, §27. Dual homomorphisms correspond to adjoint linear transformations in algebra. They have been used in topology by Tucker, [12], §25.

If ϕ maps G into G' and ψ maps G' into G'', then $\theta = \psi\phi$ maps G into G''. Its matrix is clearly

$$\|\theta\| = \|\psi\| \, \|\phi\|: \qquad \theta_{ij} = \sum_k \psi_{ik}\phi_{kj}\,.$$

On transposing, we find

$$\theta' = D(\theta) = D(\psi\phi) = D(\phi)D(\psi) = \phi'\psi'. \tag{4.3}$$

Linear combinations of homomorphisms are defined as usual. We find

$$D(c\phi + d\psi) = cD(\phi) + dD(\psi). \tag{4.4}$$

We shall apply these homomorphisms to the groups L^p, the individual cells forming the generators. Clearly *∂ and δ are dual.* If ϕ maps $L^p(K)$ into $L^{p+k}(K')$ for each p (k fixed), and $\phi' = D(\phi)$, then

$$\delta\phi'A' = \phi'\delta A' \text{ (all } A'\text{) if and only if } \partial\phi A = \phi\partial A \text{ (all } A\text{)}. \tag{4.5}$$

For one relation follows from the other on taking duals. These relations hold in particular for simplicial maps. Using (3.2), we have

$$A\cdot\phi'B' = \phi A\cdot B' \text{ if and only if } \phi' = D(\phi). \tag{4.6}$$

For

$$\phi\sigma_i\cdot\sigma_j' = \left(\sum_k \phi_{ik}\sigma_k'\right)\cdot\sigma_j' = \phi_{ij}\,, \qquad \sigma_i\cdot\phi'\sigma_j' = \phi_{ji}'\,.$$

Hence

$$A^p\cdot\delta B^{p-1} = \partial A^p\cdot B^{p-1}. \tag{4.7}$$

REMARK. The above definitions and results hold equally well if G and G' are vector spaces with fixed bases.

Finally, note that

$$A\cdot B = 0 \text{ if } \delta A = 0 \text{ and } B \sim 0 \text{ or if } A \backsim 0 \text{ and } \partial B = 0. \tag{4.8}$$

For if $\delta A = 0$ and $B = \partial C$, then $A\cdot B = A\cdot\partial C = \delta A\cdot C = 0$, etc. Hence, if $\delta A = 0$ and $B \sim B'$, then $A\cdot B = A\cdot B'$, etc.

II. THE PRODUCTS

5. Definition and properties of products

We shall use only integer coefficients until §11. Corresponding to each p-cell σ_i^p, q-cell σ_j^q, and $(p+q)$-cell σ_k^{p+q}, we wish to find an integer ${}^{pq}\Gamma_k^{ij}$, such that the following properties hold.

(Γ_1). If σ_i^p and σ_j^q are not both faces of σ_k^{p+q}, then ${}^{pq}\Gamma_k^{ij} = 0$.

(Γ_2). For all p, q, i, j, k,

$$\sum_l {}^{p+q+1}\partial_k^l \, {}^{pq}\Gamma_l^{ij} = \sum_l {}^{p+1}\partial_l^i \, {}^{p+1,q}\Gamma_k^{lj} + (-1)^p \sum_l {}^{q+1}\partial_l^j \, {}^{p,q+1}\Gamma_k^{il}.$$

(Γ_3). For some integer γ, and all q and j,

$$\textstyle\sum_i {}^{0q}\Gamma_j^{ij} = \gamma.$$

The important case is the case $\gamma = 1$; after Part II, we always take $\gamma = 1$.

In terms of the quantities ${}^{pq}\Gamma_k^{ij}$, we define two bilinear operations on chains, as follows.

$$\sigma_i^p \smile \sigma_j^q = \textstyle\sum_k {}^{pq}\Gamma_k^{ij} \sigma_k^{p+q}, \tag{5.1}$$

$$\sigma_j^q \frown \sigma_k^{p+q} = \textstyle\sum_l {}^{pq}\Gamma_k^{lj} \sigma_l^p. \tag{5.2}$$

Clearly

$$(\sigma_i^p \smile \sigma_j^q)\cdot \sigma_k^{p+q} = {}^{pq}\Gamma_k^{ij} = \sigma_i^p \cdot (\sigma_j^q \frown \sigma_k^{p+q}), \tag{5.3}$$

and hence

$$(A^p \smile B^q)\cdot C^{p+q} = A^p \cdot (B^q \frown C^{p+q}). \tag{5.4}$$

Two products $\smile$ and $\frown$ correspond if and only if (5.4) holds. Given one of $\smile$, $\frown$, we can find the corresponding Γ by (5.3), and hence the other.

The above properties translate into the following, for the $\smile$ product.

(P_1) $\sigma_i^p \smile \sigma_j^q$ is a $(p + q)$-chain in St $(\sigma_i^p)\cdot$St (σ_j^q).[16]

(P_2) $\delta(A^p \smile B^q) = \delta A^q \smile B^q + (-1)^p A^p \smile \delta B^q$.

(P_3) For some integer γ, $I \smile \sigma_j^q = \gamma\sigma_j^q$ (all σ_j^q).

Also, for the $\frown$ product,

(Q_1) $\sigma_j^q \frown \sigma_k^{p+q}$ is a p-chain in $\overline{\text{St}\ (\sigma_j^q)\cdot\sigma_k^{p+q}}$.

(Q_2) $\partial(A^q \frown B^{p+q}) = (-1)^p \delta A^q \frown B^{p+q} + A^q \frown \partial B^{p+q}$.

(Q_3) For some integer γ, $I\cdot(\sigma_j^q \frown \sigma_j^q) = \gamma$ (all σ_j^q).

We shall prove the equivalence of the three sets of properties. (P_1) is clearly equivalent to (Γ_1). If we write (P_2) with σ_i^p and σ_j^q, use (5.1), and consider the coefficient of σ_k^{p+q+1} on each side, we obtain (Γ_2); conversely, (Γ_2) implies (P_2). (Γ_3) and (P_3) are clearly equivalent. Suppose (Γ_1) holds; we shall prove (Q_1). If σ_j^q is not a face of σ_k^{p+q}, then ${}^{pq}\Gamma_k^{ij} = 0$ for every i, hence $\sigma_j^q \frown \sigma_k^{p+q} = 0$, and (Q_1) holds. If σ_j^q is a face of σ_k^{p+q}, then St $(\sigma_j^q)\cdot\sigma_k^{p+q} = \sigma_k^{p+q}$; by (Γ_1), $\sigma_j^q \frown \sigma_k^{p+q}$ is a chain in $\bar\sigma_k^{p+q} = \overline{\text{St}\ (\sigma_j^q)\cdot\sigma_k^{p+q}}$. Suppose (Q_1) holds. If σ_j^q is not a face of σ_k^{p+q}, then St $(\sigma_j^q)\cdot\sigma_k^{p+q} = 0$, $\sigma_j^q \frown \sigma_k^{p+q} = 0$, and ${}^{pq}\Gamma_k^{ij} = 0$ (all i). If σ_i^p is not a face of σ_k^{p+q}, then (Q_1) shows that it does not occur in $\sigma_j^q \frown \sigma_k^{p+q}$; hence ${}^{pq}\Gamma_k^{ij} = 0$. (Q_2) is seen to be equivalent to (Γ_2), if we replace p by $p + 1$. Finally, (Q_3) is equivalent to (Γ_3).

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[16] $S\cdot T$ is the subcomplex of K containing those cells in both S and T. Thus $\text{St}(\sigma_1)\cdot\sigma_2 = \sigma_2$ if σ_1 is a face of σ_2, and $= 0$ otherwise. Čech, [5], assumes merely that the chain is in $\text{St}(\sigma_i^p)$.

THEOREM 1. *If K is connected and $\delta I = 0$, then* (Γ_3) *is a consequence of* (Γ_1) *and* (Γ_2).

By Lemmas 1 and 2, it is sufficient to show (using the (P_i)) that if

$$\partial\sigma^q = \alpha\sigma^{q-1}\ (\alpha \neq 0), \qquad I \smile \sigma^{q-1} = \gamma\sigma^{q-1}, \qquad I \smile \sigma^q = \gamma'\sigma^q,$$

then $\gamma = \gamma'$. (That $I \smile \sigma^h = \theta\sigma^h$ for some θ follows from (P_1).) By (P_2),

$$\delta(I \smile \sigma^{q-1}) = I \smile \delta\sigma^{q-1} = I \smile \alpha\sigma^q + \cdots = \alpha\gamma'\sigma^q + \cdots ;$$

also

$$\delta(I \smile \sigma^{q-1}) = \delta\gamma\sigma^{q-1} = \gamma\alpha\sigma^q + \cdots ,$$

as $\alpha \neq 0$, $\gamma = \gamma'$.

THEOREM 2. For all σ_j^q, $\delta I \smile \sigma_j^q = 0$.

For, by (P_2) and (P_3),

$$\delta I \smile \sigma_j^q = \delta(I \smile \sigma_j^q) - I \smile \delta\sigma_j^q = \delta\gamma\sigma_j^q - \gamma\delta\sigma_j^q = 0.$$

THEOREM 3. *If $\delta I = 0$, then*

$$I \smile \sigma^p = \sigma^p \smile I = I \frown \sigma^p = \gamma\sigma^p \qquad (\text{all } \sigma^p). \tag{5.5}$$

First, $\sigma^0 \smile I = \sigma^0 \smile \sigma^0 = I \smile \sigma^0 = \gamma\sigma^0$. Suppose $\sigma^{p-1} \smile I = \gamma\sigma^{p-1}$ and $\delta\sigma^{p-1} = \alpha\sigma^p + \cdots\ (\alpha \neq 0)$. Clearly $\sigma^p \smile I = \theta\sigma^p$ for some θ. Then as $\delta I = 0$, (P_2) and (P_3) give

$$\delta(\sigma^{p-1} \smile I) = \delta\sigma^{p-1} \smile I = \alpha\theta\sigma^p + \cdots ,$$
$$\delta(\sigma^{p-1} \smile I) = \delta(\gamma\sigma^{p-1}) = \gamma\alpha\sigma^p + \cdots ,$$

and $\theta = \gamma$. That $\sigma^p \smile I = \gamma\sigma^p$ now follows, by Lemma 1. The last relation is proved similarly, considering $\partial(I \frown \sigma^p)$.

THEOREM 4. *The $\smile$ and $\frown$ products define products among the cohomology and homology groups, thus*:

cohomology class $\smile$ cohomology class = cohomology class,

cohomology class $\frown$ homology class = homology class.

Explicit formulas are given in (11.6) and (11.7). That cocycle $\smile$ cocycle = cocycle, cocycle $\frown$ cycle = cycle, follows from (P_2) and (Q_2). Also

$$A_1 \smile B \backsim A_2 \smile B \quad \text{if} \quad A_1 \backsim A_2 \quad \text{and} \quad \delta B = 0, \tag{5.6}$$

$$A \smile B_1 \backsim A \smile B_2 \quad \text{if} \quad B_1 \backsim B_2 \quad \text{and} \quad \delta A = 0, \tag{5.7}$$

$$A_1 \frown B \sim A_2 \frown B \quad \text{if} \quad A_1 \backsim A_2 \quad \text{and} \quad \partial B = 0, \tag{5.8}$$

$$A \frown B_1 \sim A \frown B_2 \quad \text{if} \quad B_1 \sim B_2 \quad \text{and} \quad \delta A = 0. \tag{5.9}$$

For instance, if $A_2 - A_1 = \delta C$ and $\partial B = 0$, then by (Q_2),

$$A_2 \frown B - A_1 \frown B = \delta C \frown B = \pm\partial(C \frown B).$$

Hence the definitions in the theorem are unique.

THEOREM 5. *For each integer γ, the products of Theorem 4 exist in any complex and are uniquely determined.*

This will be proved in §8 and §9. The relation between products with different γ's is given in Theorems 9 and 10.

In §10 we shall prove the two *associative* laws and the *commutative* law: *If K admits a simplicial subdivision* (see §13), *then*

$$A \smile (B \smile C) \backsim (A \smile B) \smile C \quad \text{if} \quad \delta A = \delta B = \delta C = 0, \tag{5.10}$$

$$A \frown (B \frown C) \sim (A \smile B) \frown C \quad \text{if} \quad \delta A = \delta B = \partial C = 0, \tag{5.11}$$

it being understood in (5.11) that $\frown$ and $\smile$ have the same γ. *If $\delta I = 0$, then*

$$A^p \smile B^q \backsim (-1)^{pq} B^q \smile A^p \quad \text{if} \quad \delta A^p = \delta B^q = 0. \tag{5.12}$$

THEOREM 6. *Let K and K′ be simplicial, let f be a simplicial map of K into K′, and let f′ be the dual of f. Then*

$$f'(A' \smile B') \backsim f'A' \smile f'B' \quad \textit{if} \quad \delta A' = \delta B' = 0, \tag{5.13}$$

$$f(f'A' \frown B) \sim A' \frown fB \quad \textit{if} \quad \delta A' = \partial B = 0. \tag{5.14}$$

These generalize a theorem of Hopf to arbitrary complexes.[17] For the proof, see §10. For the use of different coefficient groups, see §11.

6. The products in simplicial complexes

If K is simplicial, a very simple definition of Γ, $\smile$, $\frown$ is possible. Order the vertices of K in a fixed manner. Each simplex σ^p may now be written in the normal form $\sigma^p = x_{i_0} \cdots x_{i_p}$, $i_0 < \cdots < i_p$. We define, if $x_{i_0} \cdots x_{i_p} \cdots x_{i_{p+q}}$ is a simplex,

$$^{pq}\Gamma^{i_0 \cdots i_p, i_p \cdots i_{p+q}}_{i_0 \cdots i_p \cdots i_{p+q}} = 1 \qquad (i_0 < \cdots < i_p < \cdots < i_{p+q}), \tag{6.1}$$

and $\Gamma = 0$ for any other triple of simplexes. (The meaning of the Γ should be clear.) In terms of $\smile$ and $\frown$, this gives for instance (if $x_1x_2x_3x_4$ is a simplex)

$$x_1x_2 \smile x_2x_3x_4 = x_1x_2x_3x_4, \qquad x_2x_3x_4 \frown x_1x_2x_3x_4 = x_1x_2 .$$

(Γ_1) and (Γ_3) obviously hold. A simple calculation gives (Γ_2); see Alexander, [4].

For these products we clearly have

$$A \smile (B \smile C) = (A \smile B) \smile C, \qquad A \frown (B \frown C) = (A \smile B) \frown C. \tag{6.2}$$

7. Products and simplicial maps

Having ordered the vertices of K', order those of K so that

$$\text{if } f(x_i) = x'_{\theta(i)}, \text{ then } \theta(i) < \theta(j) \text{ implies } i < j. \tag{7.1}$$

---

[17] H. Hopf, [7], Satz I.

Using the products of §6, we shall prove (5.13) and (5.14) in the form:

THEOREM 7. *In simplicial complexes, using the products of* §6,

$$f'(A' \smile B') = f'A' \smile f'B', \qquad f(f'A' \frown B) = A' \frown fB. \tag{7.2}$$

To prove the first equation, take any $\sigma_i'^p$ and $\sigma_j'^q$; say

$$\sigma_i'^p = x_0' \cdots x_p', \qquad \sigma_j'^q = x_p' \cdots x_{p+q}';$$

if the vertices of the simplexes are not related as shown, then clearly both sides of the equation (with these simplexes) vanish. By definition of f',

$$f'\sigma_i'^p = \sum x_{\alpha_0} \cdots x_{\alpha_p}, \qquad \theta(\alpha_h) = h,$$

$$f'\sigma_j'^q = \sum x_{\beta_p} \cdots x_{\beta_{p+q}}, \qquad \theta(\beta_h) = h.$$

Hence

$$f'\sigma_i'^p \smile f'\sigma_j'^q = \sum x_{\alpha_0} \cdots x_{\alpha_p} \cdots x_{\alpha_{p+q}}, \qquad \theta(\alpha_h) = h.$$

This is clearly also $f'(\sigma_i'^p \smile \sigma_j'^q)$.[18] To prove the second equation, we use the first, (4.6) and (5.4): For any cell σ' (of the proper dimension),

$$\begin{aligned}\sigma' \cdot f(f'A' \frown B) = f'\sigma' \cdot (f'A' \frown B) &= (f'\sigma' \smile f'A') \cdot B = f'(\sigma' \smile A') \cdot B \\ &= (\sigma' \smile A') \cdot fB = \sigma' \cdot (A' \frown fB).\end{aligned}$$

8. Construction of the products

In a general complex, it is most convenient to construct the $\frown$ product. Take any integer γ. We shall construct all $\sigma_i^q \frown \sigma_j^{p+q}$ in succession for $p = 0, 1, 2, \cdots$. First consider $p = 0$. Set $\sigma_i^q \frown \sigma_j^q = 0$ for $j \neq i$. Let $\sigma_j^q \frown \sigma_j^q$ be any 0-chain in $\bar{\sigma}_j^q$ the sum of whose coefficients is γ; this is possible, by Lemma 2. The required properties hold so far. Suppose all $\sigma_i^q \frown \sigma_j^{r+q}$ are properly constructed for all q and all $r < p$; we must construct each $\sigma_i^q \frown \sigma_j^{p+q}$. If σ_i^q is not a face of σ_j^{p+q}, set $\sigma_i^q \frown \sigma_j^{p+q} = 0$. As $\mathrm{St}(\sigma_i^q) \cdot \bar{\sigma}_j^{p+q}$ has no cells, both sides of (Q_2) with σ_i^q and σ_j^{p+q} vanish. Now suppose σ_i^q is a face of σ_j^{p+q}. Set

$$C^{p-1} = (-1)^p \delta\sigma_i^q \frown \sigma_j^{p+q} + \sigma_i^q \frown \partial\sigma_j^{p+q}; \tag{8.1}$$

this is a chain in $\bar{\sigma}_j^{p+q}$.

Suppose first that $p = 1$. Then for some α, $\delta\sigma_i^q = \alpha\sigma_j^{p+q} + \cdots$, and

$$\begin{aligned} I \cdot C^{p-1} &= -I \cdot (\alpha\sigma_j^{p+q} \frown \sigma_j^{p+q}) + I \cdot (\sigma_i^q \frown \alpha\sigma_i^q) \\ &= -\alpha\gamma + \alpha\gamma = 0. \end{aligned} \tag{8.2}$$

Hence, by (K_4), we may choose the 1-chain $\sigma_i^q \frown \sigma_j^{p+q}$ in $\bar{\sigma}_j^{p+q}$ so that its boundary is C^{p-1}. Then (Q_1) through (Q_3) hold.

---

[18] By using joins, and the 0-chains $X_h = f'x_h'$, one could give a direct formal proof.

Suppose next that $p > 1$. Then (Q_2) gives

$$\partial C^{p-1} = (-1)^p[(-1)^{p-1}\delta\delta\sigma_i^q \frown \sigma_j^{p+q} + \delta\sigma_i^q \frown \partial\sigma_j^{p+q}] \tag{8.3}$$
$$+ (-1)^{p-1}\delta\sigma_i^q \frown \partial\sigma_j^{p+q} + \sigma_i^q \frown \partial\partial\sigma_j^{p+q} = 0.$$

Hence, by (K_4), we may choose $\sigma_i^q \frown \sigma_j^{p+q}$ in $\bar\sigma_j^{p+q}$ with the boundary C^{p-1}. The properties again hold.

We remark that any $\frown$ product may be constructed in this manner.

9. Uniqueness of the products

We first prove

THEOREM 8. *Let $\frown$ be any product with $\gamma = 0$. Then there is a bilinear operation $\wedge$ such that*

(R_1) $\sigma_i^q \wedge \sigma_j^{p+q}$ is a $(p+1)$-chain in $\overline{\mathrm{St}(\sigma_i^q)\cdot\sigma_j^{p+q}}$.

(R_2) $\sigma_i^q \frown \sigma_j^q = \partial(\sigma_i^q \wedge \sigma_j^q)$.

(R_3) $\sigma_i^q \frown \sigma_j^{p+q} = \partial(\sigma_i^q \wedge \sigma_j^{p+q}) + (-1)^p\delta\sigma_i^q \wedge \sigma_j^{p+q} + \sigma_i^q \wedge \partial\sigma_j^{p+q}$ for $p > 0$.

We shall construct $\wedge$ for $p = 0, 1, \cdots$. As $\gamma = 0$, we can construct $\sigma_i^q \wedge \sigma_j^q$ so (R_2) holds. Suppose all $\sigma_i^q \wedge \sigma_j^{r+q}$ are constructed for all q and $r < p$; we shall construct $\sigma_i^q \wedge \sigma_j^{p+q}$. We make it 0 if σ_i^q is not a face of σ_j^{p+q}. Suppose it is. Set

$$C^p = \sigma_i^q \frown \sigma_j^{p+q} - (-1)^p\delta\sigma_i^q \wedge \sigma_j^{p+q} - \sigma_i^q \wedge \partial\sigma_j^{p+q}.$$

By (Q_2), and (R_3) solved for $\partial(\sigma_i^q \wedge \sigma_j^{p+q})$,

$$\partial C^p = (-1)^p\delta\sigma_i^q \frown \sigma_j^{p+q} + \sigma_i^q \frown \partial\sigma_j^{p+q} - (-1)^p[\delta\sigma_i^q \frown \sigma_j^{p+q} - \delta\sigma_i^q \wedge \partial\sigma_j^{p+q}]$$
$$- [\sigma_i^q \frown \partial\sigma_j^{p+q} - (-1)^{p-1}\delta\sigma_i^q \wedge \partial\sigma_j^{p+q}] = 0.$$

Hence C^p is a cycle in $\bar\sigma_j^{p+q}$, and (as $p > 0$) we may choose $\sigma_i^q \wedge \sigma_j^{p+q}$ with C^p as boundary.

We may form linear combinations of $\frown$ products (and also of $\smile$ products) by defining

$$A(\alpha_1 \frown_1 + \alpha_2 \frown_2)B = \alpha_1(A \frown_1 B) + \alpha_2(A \frown_2 B). \tag{9.1}$$

By applying this to any $\sigma_j^q \frown \sigma_j^q$, we find

$$\gamma(\alpha_1 \frown_1 + \alpha_2 \frown_2) = \alpha_1\gamma(\frown_1) + \alpha_2\gamma(\frown_2). \tag{9.2}$$

THEOREM 9. *For any two products $\frown$ and $\frown'$,*

$$\gamma(\frown)(A^p \frown' B^q) \sim \gamma(\frown')(A^p \frown B^q) \quad \text{if} \quad \delta A^p = \partial B^q = 0. \tag{9.3}$$

As a consequence, if v^p and u^q denote cohomology and homology classes, then

$$\gamma(\frown)(v^p \frown' u^q) = \gamma(\frown')(v^p \frown u^q). \tag{9.4}$$

To prove this, set $\frown'' = \gamma(\frown)\frown' - \gamma(\frown')\frown$. Then $\gamma(\frown'') = 0$. By Theorem 8, (R_2) or (R_3), $A^p \frown'' B^q \sim 0$, from which (9.3) follows.

THEOREM 10. *Theorem 9 holds for the* $\smile$ *product. Thus*

$$\gamma(\smile)(A^p \smile' B^q) \backsim \gamma(\smile')(A^p \smile B^q) \quad \textit{if} \quad \delta A^p = \delta B^q = 0. \tag{9.5}$$

As above, it is sufficient to show that if $\gamma(\smile) = 0$, then $A^p \smile B^q \backsim 0$ for all cocycles A^p, B^q. Let $\frown$ correspond to $\smile$; then $\gamma(\frown) = 0$, and we may construct $\wedge$. Construct a corresponding $\vee$ (using relations of the form (5.1), (5.2)); then, as in (5.4),

$$(D^{p+1} \vee E^q)\cdot F^{p+q} = D^{p+1}\cdot(E^q \wedge F^{p+q}). \tag{9.6}$$

Now for $p > 0$ and any cell $\sigma = \sigma_k^{p+q}$, as $A\cdot\partial C = \delta A\cdot C$,

$$\begin{aligned}(A \smile B)\cdot\sigma = A\cdot(B \frown \sigma) &= A\cdot[\partial(B \wedge \sigma) + (-1)^p\delta B \wedge \sigma + B \wedge \partial\sigma]\\ &= (\delta A \vee B)\cdot\sigma + (-1)^p(A \vee \delta B)\cdot\sigma + \delta(A \vee B)\cdot\sigma,\end{aligned}$$

and hence

$$A^p \smile B^q = \delta(A^p \vee B^q) + \delta A^p \vee B^q + (-1)^p A^p \vee \delta B^q; \tag{9.7}$$

if $p = 0$, then $A^0 \smile B^q = \delta A^0 \vee B^q$. Hence if A and B are cocycles, then $A \smile B \backsim 0$.

10. Proof of properties in §5

To prove (5.12), we define a new product, $\smile'$, by

$$A^p \smile' B^q = (-1)^{pq}B^q \smile A^p.$$

Clearly (P_1) holds for $\smile'$. To prove (P_2), (P_2) for $\smile$ gives

$$\begin{aligned}\delta(A^p \smile' B^q) &= (-1)^{pq}[\delta B^q \smile A^p + (-1)^q B^q \smile \delta A^p]\\ &= (-1)^{pq+(q+1)p}A^p \smile' \delta B^q + (-1)^{pq+q+q(p+1)}\delta A^p \smile' B^q,\end{aligned}$$

which reduces to the desired formula. (P_3) is a consequence of Theorem 3. (5.12) now follows from (9.5).

To prove (5.10) and (5.11), take a simplicial subdivision K' of K, and define new products in K, using those of §6 in K', by (15.3) and (15.6). The properties now follow at once from (6.2). (Compare the proof of Theorem 14.)

To prove Theorem 6, we apply (9.5) and (7.2): Using $\smile_0$ for the product of §6,[19]

$$f'(A' \smile B') \backsim \gamma(\smile)f'(A' \smile_0 B') = \gamma(\smile)(f'A' \smile_0 f'B') \backsim f'A' \smile f'B'.$$

The other relation is proved similarly.

---

[19] If $C' \backsim D'$, then $D' - C' = \delta E'$, and

$$f'D' - f'C' = f'\delta E' = \delta f'E', \qquad f'C' \backsim f'D'.$$

11. The products, general coefficient groups

Let G be an abelian group. Let $\mathrm{Ch}_Y(X)$ be the group of homomorphisms (characters) of the group X into the group Y. Then $H = \mathrm{Ch}_G(G)$ is a ring with the definitions (writing $h\cdot g$ for $h(g)$)

$$(11.1) \qquad (h + h')\cdot g = h\cdot g + h'\cdot g, \qquad\qquad (hh')\cdot g = h\cdot(h'\cdot g).$$

If we use the coefficient group G in chains with which we form cycles, then the most useful coefficients for chains from which cocycles are formed are those of H.[20]

EXAMPLES. Let I_0, I_m, R, $\mathfrak{R}$, R_1, $\mathfrak{R}_1$ denote the groups of integers, integers mod m, real numbers, rational numbers, reals mod 1, and rationals mod 1. All but the last two are rings. Then, using $\approx$ for isomorphic, examples of G and corresponding H are

(α) $G \approx I_0$, $H \approx I_0$.

(β) $G \approx I_m$, $H \approx I_m$ $(m \geqq 1)$.

(γ) $G \approx R$ or $\mathfrak{R}$, $H \approx R$ or $\mathfrak{R}$.

(δ) $G \approx R_1$ or $\mathfrak{R}_1$, $H \approx I_0$.

(α) is a special case of (β). H is commutative in these cases. If G is a direct sum of such groups, we may find the corresponding H with the following rules:

$$\mathrm{Ch}_Y(X_1 + X_2) \approx \mathrm{Ch}_Y(X_1) + \mathrm{Ch}_Y(X_2).$$

$$\mathrm{Ch}_{Y_1+Y_2}(X) \approx \mathrm{Ch}_{Y_1}(X) + \mathrm{Ch}_{Y_2}(X).$$

Our object here is three-fold. (a) we point out how the products with G and H are defined in terms of the former products. (b) The products are discussed, assuming merely that they satisfy the (P_i) and the (Q_i). (c) In a simplicial complex, for certain groups G and H, when one of $\smile$, $\frown$ is known in the cohomology and homology groups, the other may be determined at once.

(a) If the products with integral coefficients are given, we may set

$$(11.2) \qquad (\textstyle\sum h_i\sigma_i^p)\cdot(\sum g_i\sigma_i^p) = \sum h_i\cdot g_i,$$

$$(11.3) \qquad h\sigma_i^p \smile h'\sigma_j^q = \textstyle\sum_k {}^{pq}\Gamma_k^{ij}\, hh'\sigma_k^{p+q},$$

$$(11.4) \qquad h\sigma_j^q \frown g\sigma_k^{p+q} = \textstyle\sum_l {}^{pq}\Gamma_k^{lj}\, h\cdot g\sigma_l^p.$$

The relations (P_2) and (Q_2), and hence (5.6) through (5.9), continue to hold, so the products are defined among the cohomology and homology groups. If u^p and v^p denote homology and cohomology classes, the explicit definitions are

[20] As $h\cdot(g + g') = h\cdot g + h\cdot g'$, if we consider H as an additive group, then H and G form a "group pair" with respect to G. We could replace H by any subring. Thus, in (γ) below, we could take $H \approx I_0$.

$$v^pA^p \cdot u^pB^p = A^p \cdot B^p, \tag{11.5}$$

$$v^pA^p \smile v^qB^q = v^{p+q}(A^p \smile B^q), \tag{11.6}$$

$$v^qB^q \frown u^{p+q}C^{p+q} = u^p(B^q \frown C^{p+q}). \tag{11.7}$$

(5.4) holds, because of (11.1), and hence

$$(v_i^p \smile v_j^q)\cdot u_k^{p+q} = v_i^p \cdot (v_j^q \frown u_k^{p+q}). \tag{11.8}$$

(P_3) and (Q_3) with $\gamma = 1$ are replaced by the relations

$$hI \smile h'\sigma_j^q = hh'\sigma_j^q, \qquad I\cdot(h\sigma_j^q \frown g\sigma_j^q) = h\cdot g. \tag{11.9}$$

(We could leave out the h in the first.) Thus the integer $\gamma(\smile) = \gamma(\frown)$ is replaced by the operations $\gamma(\smile; h, h') = hh'$, $\gamma(\frown; h, g) = h\cdot g$; if G is the group of integers, then $\gamma = \gamma(\smile; 1, 1) = \gamma(\frown; 1, 1)$. To a general γ correspond general bilinear maps $\gamma(\smile; h, h')$ into H and $\gamma(\frown; h, g)$ into G, and (11.9) becomes

$$hI \smile h'\sigma_j^q = \gamma(\smile; h, h')\sigma_j^q, \qquad I\cdot(h\sigma_j^q \frown g\sigma_j^q) = \gamma(\frown; h, g). \tag{11.10}$$

The relations (5.10) through (5.14) and (5.4) hold, if in (5.12) we assume that H is commutative. For the proofs, we need merely multiply by elements of H and G and sum, using (11.1). The same remark holds for Theorem 2. Suppose ${}^p\partial_i^j = \pm 1$ or 0 in the complex. With the above interpretation for γ, the proofs of Theorems 1 and 3 hold. (Note that only the revised (P_i) and (Q_i) are used here.)

(b) Suppose $\smile$ and $\frown$ products are defined, satisfying the analogues of (P_1) and (Q_1), (P_2) and (Q_2), and (11.10), which we call (P_3') and (Q_3'). That $\smile$ and $\frown$ are bilinear means that there are bilinear maps ${}^{pq}\Phi_k^{ij}$ of (H, G) into G and ${}^{pq}\Psi_k^{ij}$ of (H, H) into H such that

$$h\sigma_i^p \smile h'\sigma_j^q = \textstyle\sum_k {}^{pq}\Psi_k^{ij}(h, h')\sigma_k^{p+q}, \tag{11.11}$$

$$h\sigma_j^q \frown g\sigma_k^{p+q} = \textstyle\sum_l {}^{pq}\Phi_k^{lj}(h, g)\sigma_l^p. \tag{11.12}$$

(P_2) and (Q_2) translate into (Γ_2) with ${}^{pq}\Gamma_k^{ij}$ replaced by ${}^{pq}\Psi_k^{ij}(h, h')$ and ${}^{pq}\Phi_k^{ij}(h, g)$ respectively. We suppose scalar products are defined by (11.2). We say $\smile$ and $\frown$ *correspond* if (5.4) holds. By (11.11) and (11.12), this is so if and only if

$${}^{pq}\Psi_k^{ij}(h, h')\cdot g = h\cdot {}^{pq}\Phi_k^{ij}(h', g). \tag{11.13}$$

(In the simplest case, $\Psi = \gamma hh'$ and $\Phi = \gamma h'\cdot g$ for some integer γ, and (11.13) follows from (11.1).) The products may not be derivable from the products in (a). For instance, if G is the group of real numbers, the ${}^{pq}\Gamma_k^{ij}$ may not be integers.

If $\frown$ *is given, and* $H = \mathrm{Ch}_G(G)$, *then* (11.13) *determines* $\smile$. For, given h and h', the right hand side is a homomorphism of G into itself, and thus determines an element of H; we call this element ${}^{pq}\Psi_k^{ij}(h, h')$. Clearly Ψ is bilinear, and (11.13) holds. If we take (Γ_2) with $\Phi(h', g)$ and apply h, using (11.13) gives a relation with terms acting on g. It being true for all g, we can drop it, and find (Γ_2) with $\Psi(h, h')$, i.e. (P_2). (Q_1) gives (P_1). By (Q_3'), i.e. the second relation in (11.10), and (5.4) (which follows from (11.13)),

$$(hI \smile h'\sigma_j^q)\cdot g\sigma_j^q = hI\cdot(h'\sigma_j^q \frown g\sigma_j^q) = h\cdot\gamma(\frown; h', g).$$

For fixed h and h', the last term is a map of G into itself, defining an element of H, which we call $\gamma(\smile; h, h')$; thus

$$\gamma(\smile; h, h')\cdot g = h\cdot\gamma(\frown; h', g). \tag{11.14}$$

Clearly $\gamma(\smile; h, h')$ is bilinear. If h^* is the coefficient of $hI \smile h'\sigma_j^q$ in σ_j^q, then the above relations give

$$h^*\cdot g = h^*\sigma_j^q\cdot g\sigma_j^q = \gamma(\smile; h, h')\cdot g, \qquad h^* = \gamma(\smile; h, h').$$

Hence, by (Q_1), the first relation in (11.10), i.e. (P_3'), holds.

Let 1 be the identity in H: $1\cdot g = g$ (all g). Consider the following *hypothesis* (H) on H, or on any subring containing 1: The only homomorphism ϕ of H into G (i.e. $\phi(h_1 + h_2) = \phi(h_1) + \phi(h_2)$) such that $\phi(1) = 0$ is $\phi(h) = 0$ (all h). This holds for instance if H is any of the rings given above. If $G = I_0 + I_0 = I_0^2$, we may use, in place of $H \approx I_0^4$, a subring $H' \approx I_0$.

If $\smile$ *is given, and* H *satisfies* (H), *then* $\frown$ *is determined.* For fixed h' and g, the left hand side of (11.13) is a homomorphism ϕ of H into G. Set $g^* = \phi(1)$. Then for all h in H, $\phi(h) = h\cdot g^*$; for if $\phi'(h) = \phi(h) - h\cdot g^*$, then $\phi'(1) = 0$. Set ${}^{pq}\Phi_k^{ij}(h', g) = g^*$; Φ is bilinear, and (11.13) holds. Applying (Γ_2) with $\Psi(h, h')$ to g and using (11.13) gives a relation for all h. Setting $h = 1$ gives (Q_2). (P_1) gives (Q_1). By (5.4) and the first relation in (11.10),

$$hI\cdot(h'\sigma_j^q \frown g\sigma_j^q) = \gamma(\smile; h, h')\sigma_j^q\cdot g\sigma_j^q = \gamma(\smile; h, h')\cdot g.$$

If we set $\gamma(\frown; h', g) = \gamma(\smile; 1, h')\cdot g$, then $\gamma(\frown; h', g)$ is bilinear and (Q_3') holds (as $1\ I\cdot g = I\cdot g$).

THEOREM 11. *Let K be a simplicial complex. Then with coefficient groups G and $H = \mathrm{Ch}_G(G)$, any two $\frown$ products satisfying the new (Q_i), with the same $\gamma(\frown; h, g)$, give the same product in the cohomology and homology groups. The same is true for $\smile$ if $H = \mathrm{Ch}_G(G)$ satisfies the hypothesis (H).*

Recall that (P_3') and (Q_3') are (11.10). First take a fixed vertex x_i^p in each simplex σ_i^p, and define the "join" $x_i^p\sigma_j^q$ for any σ_j^q in $\bar\sigma_i^p$ (which vanishes if σ_j^q contains x_i^p). Set $x_i^p \sum g_j\sigma_j^q = \sum g_j x_i^p\sigma_j^q$. Then if C^q is a q-G-cycle in $\bar\sigma_i^p$, $\partial(x_i^p C^q) = C^q$. (See for instance Lefschetz, [10], p. 111, (9').)

Now given $\frown_1$ and $\frown_2$ with the same γ, set $\frown = \frown_2 - \frown_1$. Then $\gamma(\frown; h, g) = 0$ (all h, g). We shall construct a bilinear product $h\sigma_i^q \wedge g\sigma_j^{p+q}$ as in Theorem 8. The proof there given holds, if we are careful to define the product for all h and g at each step. At a typical step, we have a p-G-cycle

$C^p = h\sigma_i^q \frown g\sigma_j^{p+q} + \cdots$ in $\bar{\sigma}_j^{p+q}$. To make $\partial(h\sigma_i^q \wedge g\sigma_k^{p+q}) = C^p$, we set $h\sigma_i^q \wedge g\sigma_j^{p+q} = x_j^{p+q} C^q$. Because of the new ($R_2$) and ($R_3$), the $\frown$ product vanishes in the cohomology and homology groups; it follows that $\frown_1$ and $\frown_2$ give the same products there.

To prove the last statement, we shall show that if $\gamma(\smile; h, h') = 0$, then $A \smile B \backsim 0$ for cocycles A and B. Construct $\frown$ corresponding to $\smile$; then by (11.14), $\gamma(\frown; h', g) = 0$. Hence, as we saw above, a bilinear $\wedge$ operation may be defined satisfying the revised (R_i). Next we shall construct $\vee$ so that (9.6) will hold. In terms of the coefficients defining $\wedge$ and $\vee$, this takes the form of the relation (11.13). The right hand side being given, we construct the coefficients on the left so the relation is satisfied, exactly as we constructed $\smile$ in terms of $\frown$. We can now show that $(A \smile B)\cdot g\sigma = \delta(A \vee B)\cdot g\sigma$ for cocycles A and B, as in the proof of Theorem 10. As this is true for all g, $A \smile B$ and $\delta(A \vee B)$ have the same coefficient in σ. As this is true for all σ, $A \smile B = \delta(A \vee B) \backsim 0$.

(c) For certain sets of groups G, H and Z, the homology and cohomology groups ${}^p\mathbf{H}^G$ and ${}^p\mathbf{H}_H$ satisfy

$$
{}^p\mathbf{H}^G \approx \mathrm{Ch}_Z({}^p\mathbf{H}_H), \qquad {}^p\mathbf{H}_H \approx \mathrm{Ch}_Z({}^p\mathbf{H}^G). \tag{11.15}
$$

As shown in Whitney, [13], this is so whenever the following conditions are satisfied.

(1) G and H form a group pair with respect to Z.
(2) $G \approx \mathrm{Ch}_Z(H)$, $H \approx \mathrm{Ch}_Z(G)$.
(3) G and H resolve each other completely.
(4) Z is infinitely (better term: completely) divisible.

(11.15) holds for any of the examples (β), (γ), (δ) above, with $Z = G$. We can show this for (β) by first using $Z = R_1$, and noting that in the maps, only a subgroup $\approx G$ of R_1 is used.

THEOREM 12. *Let the products* (11.5) *be defined, and suppose* (11.15) *is satisfied. Then, for any* γ, *as soon as one of* $\smile$, $\frown$ *is known, the other is determined by* (11.8).

Recall that for any γ, $\smile$ and $\frown$ are uniquely defined. We shall show that, given $\smile$ or $\frown$, there is a unique corresponding $\frown$ or $\smile$ satisfying (11.8); as the correct products satisfy this relation, the theorem will be proved.

Suppose $\smile$ is defined. For fixed v_j^q and u_k^{p+q}, the left hand side of (11.8) is a homomorphism of ${}^p\mathbf{H}_H$ into Z, and hence, by (11.15), corresponds to a unique element of ${}^p\mathbf{H}^G$; we call this element $v_j^q \frown v_k^{p+q}$. Then $\frown$ is bilinear, and (11.8) is satisfied. We find $\smile$ in terms of $\frown$ in the same manner.

12. Construction of the products in low dimensional complexes

We shall construct all products $\sigma^q \frown \sigma^{p+q}$ for $p \leqq 2$ in a particular fashion; then $\sigma^p \smile \sigma^q$ is determined for $p \leqq 2$. We can then determine the $\smile$ products in the cohomology and homology groups of complexes of dimensions $\leqq 5$, with the help of Theorem 12 and (5.12); the remaining $\frown$ products are then given with the aid of Theorem 12.

We begin by ordering the vertices of K in a fixed fashion, $x_1, x_2, \cdots$. For each σ_i^p, let V_i^p be its first vertex. For each vertex x_h of σ_i^p, let $W_i^p x_h$ be a 1-chain in $\bar{\sigma}_i^p$ whose boundary is $x_h - V_i^p$. Set (using $\gamma = 1$)

$$\sigma_i^p \frown \sigma_i^p = V_i^p, \tag{12.1}$$

$$\sigma_j^{p-1} \frown \sigma_i^p = {}^p\partial_i^j W_i^p V_j^{p-1} \qquad (\sigma_j^{p-1} \text{ in } \bar{\sigma}_i^p). \tag{12.2}$$

Finally, take any σ_j^{p-2} of $\bar{\sigma}_i^p$; say

$$\partial\sigma_i^p = \sum \alpha_k \sigma_k^{p-1}, \qquad \delta\sigma_j^{p-2} = \sum \beta_k \sigma_k^{p-1} \text{ in } \bar{\sigma}_i^p.$$

As $\partial\partial\sigma_i^p = 0$, $\sum \alpha_k\beta_k = 0$. Now (see §8)

$$C^1 = \delta\sigma_j^{p-2} \frown \sigma_i^p + \sigma_j^{p-2} \frown \partial\sigma_i^p = \sum \beta_k W_i^p V_k^{p-1} + \sum \alpha_k W_k^{p-1} V_j^{p-2}$$

is a cycle in $\bar{\sigma}_i^p$, and we may choose a 2-chain $\sigma_j^{p-2} \frown \sigma_i^p$ bounded by it.

In the simplest case, we will have

$$\partial\sigma_i^p = \sigma_s^{p-1} + \sigma_t^{p-1} + \cdots, \qquad \delta\sigma_j^{p-2} = \sigma_s^{p-1} - \sigma_t^{p-1} + \cdots,$$

and

$$C^1 = W_i^p V_s^{p-1} - W_i^p V_t^{p-1} + W_s^{p-1} V_j^{p-2} + W_t^{p-1} V_j^{p-2},$$

and we can find a possible $\sigma_j^{p-2} \frown \sigma_i^p$ at once. For most σ_j^{p-2} on σ_i^p (for instance, for all in which $V_j^{p-2} = V_s^{p-1} = V_t^{p-1}$), C^1 will vanish; but there will in general be at least one for which it does not.

III. Invariance Theorems

13. Subdivision and consolidation

Let K' be a complex, satisfying (K_1), (K_2) and (K_3) of §2. Let $\{E_i^p\}$ be a set of distinct closed subcomplexes of K' which cover K', E_i^p being of dimension p. Let $F_i^p = F(E_i^p)$ be the union of all E_j^q ($q < p$) contained in E_i^p. Set $O_i^p = E_i^p - F_i^p$. Assume

(K_1') The common part $E_i^p \cdot E_j^q$ of any two of the subcomplexes is either void or the union of a subset of the subcomplexes.

(K_2') If E_j^q is in E_i^p, $E_j^q \neq E_i^p$, then $q < p$.

(K_3') With integer coefficients, O_i^p is monocyclic or acyclic[21] in the dimension p and is acyclic in all lower dimensions.

[21] O_i^p is *acyclic* in the dimension q if every q-cycle (which need not be boundary-like) is a boundary; O_i^p is *monocyclic* in the dimension p if there is a p-cycle $X^p \neq 0$ such that any p-cycle is a multiple of X^p. (All this is with integer coefficients.) Compare Tucker: "cell-like," "null-like." In [4], Ch. VI, a similar assumption, using any coefficient group G, is made. Note that O_i^p is a subcomplex of K; a chain A in O_i^p is a cycle if, considered as a chain in K, ∂A^p has no part in O_i^p.

Define a complex K as follows. With each O_i^p which is monocyclic we associate a p-cell σ_i^p. Say σ_j^q is a face of σ_i^p if and only if E_j^q is in E_i^p. We shall define the ${}^p\partial_j^i$ in K and prove (K_1), (K_2) and (K_3) for K below. We call K a *consolidation*[22] of K', and K', a *subdivision* of K. If K may be formed from a simplicial K' in this manner, we say K *admits a simplicial subdivision.*

14. Structure of K and K'

Denote the cells of K' by τ_i^p. Let J^p be the union of all E_i^q for $q \leqq p$. We prove:

(α_1) O_i^p contains all the p-cells of E_i^p; for F_i^p is of dimension $<p$.

(α_2) $E_i^p \cdot E_j^p$ is in J^{p-1} if $j \neq i$; for the common part is a union of E_k^q, and as each such E_k^q is in E_i^p, $q < p$, by (K_2').

(α_3) Each τ_i^p is in a unique O_j^q; then E_j^q is the smallest subcomplex containing τ_i^p. To prove this, choose E_j^q as stated. E_j^q is uniquely determined. For if τ_i^p is also in E_k^r, then it is in their common part and hence in an E_l^s contained in both, by (K_1'); hence $E_l^s = E_j^q$, and E_j^q is in E_k^r. As τ_i^p is in no E_k^{q-1}, it is in O_j^q. Suppose τ_i^p were also in $O_k^r \neq O_j^q$. Then E_k^r contains E_j^q, and by (K_2'), $r > q$. Hence E_j^q is in F_k^r, and τ_i^p is not in O_k^r, a contradiction.

In (α_3), we define $E_j^q = E(\tau_i^p)$, $O_j^q = O(\tau_i^p)$. If O_j^q is monocyclic (in the dimension q), we set $\sigma_j^q = \sigma(\tau_i^p)$.

(α_4) If τ is in O_i^p and in E_j^q, then E_j^q contains E_i^p, by (α_3).

For each O_i^p which is monocyclic, let

$$X_i^p = \textstyle\sum_j {}^p a_i^j \tau_j^p \tag{14.1}$$

be the corresponding p-cycle; for other O_i^p, set $X_i^p = 0$. Map $L^p(K)$ into $L^p(K')$ by

$$Sd\sigma_i^p = X_i^p. \tag{14.2}$$

Define $Sd \sum \alpha_i \sigma_i^p$ by linearity. Then

(α_5) $SdA^p = 0$ implies $A^p = 0$. For if $A^p = \sum \alpha_i \sigma_i^p$, then $\sum \alpha_i X_i^p = 0$, and (α_2) shows that each $\alpha_i X_i^p = 0$; as $X_i^p \neq 0$ in this case, $\alpha_i = 0$.

LEMMA 4. *Any cycle A'^p in J^p (with integer coefficients) is SdA^p for a uniquely defined A^p in K.*

First, write $A'^p = \sum A_i'^p$, $A_i'^p$ in E_i^p. As $A'^p - A_i'^p$ is in $\sum_{j\neq i} E_j^p$, by (α_2), so is $-\partial(A'^p - A_i'^p) = \partial A_i'^p$. By ($\alpha_4$), $\partial A_i'^p$ has no part in O_i^p, that is (using (α_1)), $A_i'^p$ is a p-cycle in O_i^p. Hence, if $A_i'^p \neq 0$, then O_i^p is monocyclic, and for some α_i,

$$A_i'^p = \alpha_i X_i^p; \qquad \text{then } A'^p = Sd \textstyle\sum \alpha_i \sigma_i^p.$$

The uniqueness follows from (α_5). We remark that "in J^p" may be replaced by "in $J^p - J^{p-1}$."

[22] "*Zellenzerspaltung*" in [4], Ch. VI. But see the last foot-note.

We now define the boundary relations in K. Given σ_i^p, $\partial Sd\sigma_i^p$ is a cycle in F_i^p, which is in J^{p-1}; hence it may be written uniquely as SdB^{p-1}. *We define* $\partial\sigma_i^p = B^{p-1}$. Then

$$\partial SdA^p = Sd\partial A^p. \tag{14.3}$$

Clearly (K_1) holds for K. Also $\partial Sd\sigma_i^p$ is in E_i^p, hence $\partial\sigma_i^p$ is in $\bar{\sigma}_i^p$, and (K_2) holds. For any σ_i^p, $Sd\partial\partial\sigma_i^p = \partial\partial Sd\sigma_i^p = 0$; hence, by (α_5), $\partial\partial\sigma_i^p = 0$, and (K_3) holds.

THEOREM 13. *If K' is a subdivision of K, then there are homomorphisms ϕ of $L^p(K')$ into $L^p(K)$ and ψ of $L^p(K')$ into $L^{p+1}(K')$ such that*

(a) *$\phi\tau^p$ is in $\bar{\sigma}(\tau^p)$; $\psi\tau^p$ is in $E(\tau^p)$.*

(b) $\partial\phi A'^p = \phi\partial A'^p$.

(c) $\phi SdA^p = A^p$.

(d) $Sd\phi A'^p = A'^p - \psi\partial A'^p - \partial\psi A'^p$.

If we take duals of these relations and (14.3), using:

$\phi' = D(\phi)$, mapping $L^p(K)$ into $L^p(K')$,

$Sd' = D(Sd)$, mapping $L^p(K')$ into $L^p(K)$,

$\psi' = D(\psi)$, mapping $L^{p+1}(K')$ into $L^p(K')$,

we obtain

(b′) $\delta\phi' A^p = \phi'\delta A^p$,

(c′) $Sd'\phi' A^p = A^p$,

(d′) $\phi' Sd' A'^p = A'^p - \psi'\delta A'^p - \delta\psi' A'^p$,

$$\delta Sd' A'^p = Sd'\delta A'^p. \tag{14.3′}$$

We begin by constructing homomorphisms θ_p and ψ_p as follows. θ_p and ψ_p are defined in J^p, and map $L^r(K')$ into $L^r(K')$ and into $L^{r+1}(K')$ respectively. For τ^q in J^{p-1}, set $\theta_p\tau^q = \tau^q$, $\psi_p\tau^q = 0$. If τ^q is in O_i^p, then we shall have:

$$\psi_p\tau^q \text{ is in } O_i^p, \qquad \theta_p\tau^q \text{ is in } \begin{cases} F_i^p \text{ if } q < p, \\ E_i^p \text{ if } q = p. \end{cases}$$

Also

$$\partial\psi_p\tau^q = \tau^q - \psi_p\partial\tau^q - \theta_p\tau^q. \tag{14.4}$$

For τ^0 in J^0, set $\theta_0\tau^0 = \tau^0$, $\psi_0\tau^0 = 0$. Suppose all θ_r and ψ_r are constructed in J^{p-1}; we shall construct them in J^p. We need merely consider θ_p and ψ_p; for (14.4) holds for θ_r and ψ_r, $r > p$, by their definitions. For τ^0 in O_i^p, we may choose $\psi_p\tau^0$ in O_i^p and $\theta_p\tau^0$ in F_i^p so that

$$\partial\psi_p\tau^0 = \tau^0 - \theta_p\tau^0,$$

by (K_3'); then (14.4) holds.

Now suppose ψ_p and θ_p are defined for all cells of dimension $< q$ in O_i^p; we shall define them for τ^q. If $\partial_1\tau^q$ is the part of $\partial\tau^q$ in O_i^p, $\psi_p\partial_1\tau^q$ is in O_i^p; if $\partial_2\tau^q$ is the part in F_i^p, $\psi_p\partial_2\tau^q = 0$. Hence $\psi_p\partial\tau^q$ is in O_i^p. Also, using (14.4) applied to $\partial\tau^q$,

$$\partial(\tau^q - \psi_p\partial\tau^q) = \partial\tau^q - (\partial\tau^q - \theta_p\partial\tau^q) = \theta_p\partial\tau^q, \tag{14.5}$$

which is in F_i^p. Hence $\tau^q - \psi_p\partial\tau^q$, as a chain in the complex O_i^p, is a cycle.

Suppose first that $q < p$. Then, by (K_3'), we may find a chain $\psi_p\tau^q$ in O_i^p and a chain $\theta_p\tau^q$ in F_i^p so that (14.4) holds. Suppose next that $q = p$. Then if $\tau^q = \tau_j^p$,

$$\tau_j^p - \psi_p\partial\tau_j^p = \alpha_j X_i^p, \tag{14.6}$$

by (K_3'). If $X_i^p = 0$, we take $\alpha_j = 0$. Set $\psi_p\tau_j^p = 0$, $\theta_p\tau_j^p = \alpha_j X_i^p$; again (14.4) follows.

We prove some properties of ψ_p and θ_p. Taking the boundary of (14.4) gives

$$\begin{aligned} 0 &= \partial\tau^q - (\partial\tau^q - \psi_p\partial\partial\tau^q - \theta_p\partial\tau^q) - \partial\theta_p\tau^q, \\ \partial\theta_p\tau^q &= \theta_p\partial\tau^q. \end{aligned} \tag{14.7}$$

Next, by (14.1) and (14.6), summing over all p-cells in O_i^p,

$$\textstyle\sum_j \alpha_j {}^pa_i^j X_i^p = \sum_j {}^pa_i^j(\tau_j^p - \psi_p\partial\tau_j^p) = X_i^p - \psi_p\partial X_i^p.$$

But ∂X_i^p is in F_i^p, and hence $\psi_p\partial X_i^p = 0$. Therefore

$$\textstyle\sum_j \alpha_j {}^pa_i^j = 1 \qquad \text{if } X_i^p \neq 0. \tag{14.8}$$

It follows that, even if $X_i^p = 0$,

$$\theta_p X_i^p = \textstyle\sum_j {}^pa_i^j\theta_p\tau_j^p = \sum_j {}^pa_i^j\alpha_j X_i^p = X_i^p. \tag{14.9}$$

Now define a homomorphism θ of $L^q(K')$ into $L^q(J^q)$ by

$$\theta\tau^q = \theta_q\theta_{q+1}\cdots\theta_n\tau^q, \tag{14.10}$$

supposing K' is of dimension n. By the definition of $\theta_p\tau^q$ for $p = q$, we may define ϕ in $L^q(K')$ so that

$$\theta\tau^q = Sd\phi\tau^q. \tag{14.11}$$

By (14.3), (14.9) and (14.7),

$$\partial\theta\tau^q = Sd\partial\phi\tau^q = \theta_{q-1}Sd\partial\phi\tau^q = \theta_{q-1}\partial\theta\tau^q = \theta_{q-1}\theta_q\cdots\theta_n\partial\tau^q = \theta\partial\tau^q; \tag{14.12}$$

hence

$$Sd\partial\phi\tau^q = \partial Sd\phi\tau^q = \partial\theta\tau^q = Sd\phi\partial\tau^q,$$

and (b) follows from (α_5).

Finally, define ψ by

$$\psi\tau^q = \textstyle\sum_{k=q}^n \psi_k\theta_{k+1}\cdots\theta_n\tau^q, \tag{14.13}$$

using $\theta_{n+1} \cdots \theta_n \tau^q = \tau^q$. As $k + 1 > q$ in each term of the sum, $\theta_{k+1} \cdots \theta_n \tau^q$ is in J^k, so that $\psi_k \theta_{k+1} \cdots \theta_n \tau^q$ is defined. Note that $\psi_{q-1}\theta_q \cdots \theta_n \partial\tau^q = 0$, as $\theta_q \cdots \theta_n \partial \tau^q$ is of dimension $q - 1$. Hence, by (14.4) and (14.7),

$$\begin{aligned} \partial\psi\tau^q &= \textstyle\sum_{k=q}^n \partial\psi_k \theta_{k+1} \cdots \theta_n \tau^q \\ &= \textstyle\sum_{k=q}^n \theta_{k+1} \cdots \theta_n \tau^q - \sum_{k=q}^n \psi_k \partial\theta_{k+1} \cdots \theta_n \tau^q \\ &\quad - \textstyle\sum_{k=q}^n \theta_k \theta_{k+1} \cdots \theta_n \tau^q = \tau^q - \psi\partial\tau^q - \theta\tau^q; \end{aligned}$$

this, with (14.11), gives (d).

To prove (c), (14.9) gives

$$Sd\phi Sd\sigma_i^p = Sd\phi X_i^p = \theta X_i^p = \theta_p X_i^p = X_i^p = Sd\sigma_i^p;$$

(c) now follows from (α_5).

15. Combinatorial invariance

We can now prove (compare Theorem 13)

THEOREM 14. *Let K' be a subdivision of K, and let G be an abelian group. Then both ϕ and Sd induce isomorphisms between ${}^p\mathbf{H}^G(K)$ and ${}^p\mathbf{H}^G(K')$, and both ϕ' and Sd' induce isomorphisms between ${}^p\mathbf{H}_G(K)$ and ${}^p\mathbf{H}_G(K')$. If K and K' each admits a product theory, then these isomorphisms preserve products.*[23]

The meaning of the last phrase is seen from (15.4), (15.5), (15.7) and (15.8) below.

Set $Sd \sum g_i\sigma_i^p = \sum g_i Sd\sigma_i^p$ etc. Using u and v for homology and cohomology classes as in §11, set

$$\text{(15.1)} \qquad \phi u' A'^p = u\phi A'^p, \qquad SduA^p = u' SdA^p,$$

$$\text{(15.2)} \qquad \phi' vA^p = v'\phi' A^p, \qquad Sd'v'A'^p = vSd'A'^p,$$

the chains being cycles in (15.1) and cocycles in (15.2). The proof that these are isomorphisms follows at once from Theorem 13. Consider for instance ϕ'. To show that $\phi' v$ is uniquely determined, suppose $vA^p = vB^p$. Then $A^p - B^p = \delta C^{p-1}$, and

$$v'\phi' A^p - v'\phi' B^p = v'\phi'\delta C^{p-1} = v'\delta\phi' C^{p-1} = 0.$$

Suppose $\phi' vA^p = \phi' vB^p$. Then $\phi' A^p - \phi' B^p = \delta C'^{p-1}$, and

$$A^p - B^p = Sd'\phi'(A^p - B^p) = Sd'\delta C'^{p-1} = \delta Sd' C'^{p-1},$$

so that $vA^p = vB^p$. Given a $v'A'^p$, to find a vA^p mapping into it, set $A^p = Sd'A'^p$. Then $\delta A^p = Sd'\delta A'^p = 0$, and

$$\phi' vA^p = v'\phi' Sd' A'^p = v'(A'^p - \psi'\delta A'^p - \delta\psi' A'^p) = v' A'^p.$$

[23] If topological coefficient groups are used, the isomorphisms are continuous. The part of this theorem relating to the homology groups has been proved by Tucker, [12], for integral coefficients, and by Alexandroff-Hopf, [4], Ch. VI, using the stronger condition noted in footnote 21.

To prove that the $\smile$ products agree in the cohomology groups of K and K' (if they are defined), consider first integer coefficients. Take a definite $\smile$ product in K', and define a new one in K by

$$A \smile B = Sd'(\phi' A \smile \phi' B). \tag{15.3}$$

We must prove (P_1) through (P_3). To prove (P_1), say

$$\sigma_i^p \smile \sigma_j^q = \alpha\sigma_k^r + \cdots, \qquad \alpha \neq 0 \qquad (r = p + q).$$

Then

$$\phi'\sigma_i^p \smile \phi'\sigma_j^q = \beta\tau^r + \cdots, \qquad Sd'\tau^r = \gamma\sigma_k^r, \qquad \beta \neq 0, \gamma \neq 0.$$

As then $Sd\sigma_k^r = \gamma\tau^r + \cdots$, τ^r is in O_k^r, and $\sigma(\tau^r) = \sigma_k^r$. Say

$$\phi'\sigma_i^p = \epsilon\tau^p + \cdots, \qquad \phi'\sigma_j^q = \zeta\tau^q + \cdots, \qquad \tau^p \smile \tau^q = \eta\tau^r + \cdots$$
$$(\epsilon, \zeta, \eta \neq 0).$$

As then $\phi\tau^p = \epsilon\sigma_i^p + \cdots$, σ_i^p is in $\bar{\sigma}(\tau^p)$. As τ^p is in $\bar{\tau}^r$, $E(\tau^p)$ is in $E(\tau^r)$, and σ_i^p is in $\bar{\sigma}(\tau^r) = \bar{\sigma}_k^r$. Similarly σ_j^q is in $\bar{\sigma}_k^r$.

To prove (P_2), (b′) and (14.3′) with (P_2) for $\smile$ in K' give

$$\delta(A \smile B) = Sd'(\phi'\delta A \smile \phi' B \pm \phi' A \smile \phi'\delta B) = \delta A \smile B \pm A \smile \delta B.$$

To prove (P_3), note that $\phi' I = I'$; hence, using (P_3) in K',

$$I \smile \sigma_j^q = Sd'(\phi' I \smile \phi'\sigma_j^q) = Sd'\phi'\sigma_j^q = \sigma_j^q.$$

Therefore we may use the product of (15.3) in K. To prove that ϕ' and Sd' preserve products in the cohomology groups, note that (for cocycles), using (d′) and (5.6), (5.7),

$$\phi'(A \smile B) = \phi' Sd'(\phi' A \smile \phi' B) \backsim \phi' A \smile \phi' B, \tag{15.4}$$

$$Sd'A' \smile Sd'B' = Sd'(\phi' Sd'A' \smile \phi' Sd'B') \backsim Sd'(A' \smile B'). \tag{15.5}$$

We may similarly define $\frown$ in K in terms of $\frown$ in K' by

$$A \frown B = \phi(\phi' A \frown SdB). \tag{15.6}$$

This corresponds to the $\smile$ product in K, and hence is a $\frown$ product. To show this, (4.6) gives

$$(A \smile B)\cdot C = Sd'(\phi' A \smile \phi' B)\cdot C = (\phi' A \smile \phi' B)\cdot SdC,$$
$$A\cdot(B \frown C) = A\cdot\phi(\phi' B \frown SdC) = \phi' A\cdot(\phi' B \frown SdC).$$

Applying (5.4) in K' shows that these are equal; hence $\smile$ and $\frown$ correspond in K, by (5.4).

We map cycles [cocycles] from K into K' and from K' into K with Sd and ϕ [with ϕ' and Sd']. The invariance of the $\frown$ product is given by

$$(15.7) \qquad Sd(A \frown B) = Sd\phi(\phi'A \frown SdB) \sim \phi'A \frown SdB,$$

$$(15.8) \qquad Sd'A' \frown \phi B' = \phi(\phi'Sd'A' \frown Sd\phi B') \sim \phi(A' \frown B'),$$

where A, A' are cocycles, B, B' are cycles.

Now consider the coefficient group G as in §11, (a). The $\smile$ products in the cohomology groups in both K and K' are formed by (11.3). As (15.3) holds with A and B replaced by ${}^ph\sigma_i^p$ and ${}^qh\sigma_j^q$, (15.3) etc. hold with any coefficient group. (15.4) and (15.5) prove the invariance. Similarly for the $\frown$ product.

16. Topological invariance

We shall show how to associate homology and cohomology groups and a product theory with a polyhedron P by means of any simplicial subdivision. By Theorem 14 we may find these groups and products, using any complex K which admits, as a subdivision, a simplicial triangulation of P, and admits a product theory.

The theorem and proof extend at once to prove the existence of groups and products in a bicompact space; compare Steenrod (see footnote 8), §9.

The proof is based on Theorem 13. However, if we restrict ourselves to simplicial complexes, Theorem 13 becomes much more simple. Hence we give it again for this case, as a lemma. We may then prove, as before, the first part of Theorem 14.

LEMMA 6. *Let K' be a simplicial subdivision of the simplicial complex K, and let ϕ be a pseudo-identical map*[24] *of K' into K. Then there is a map ψ as in Theorem* 13 *such that the conclusions of Theorem* 13 *hold.*

The statements about ϕ and Sd are well-known; we shall construct ψ. For each vertex x' of K', let $\psi x'$ be any 1-chain in the subdivision of the smallest cell $\sigma(x')$ of K containing x', which is bounded by $x' - \phi x'$. Suppose ψ is constructed in $L^0(K'), \cdots, L^{p-1}(K')$. Then applying (d) of Theorem 13 to the $(p-1)$-chain $\partial\tau^p$, we find

$$\partial(Sd\phi\tau^p - \tau^p + \psi\partial\tau^p) = \partial Sd\phi\tau^p - \partial\tau^p + \partial\tau^p - Sd\phi\partial\tau^p = 0.$$

Hence $Sd\phi\tau^p - \tau^p + \psi\partial\tau^p$ is a p-cycle in the subdivision of $\sigma(\tau^p)$ $(p > 0)$, and therefore, as is well-known, we can find a chain $-\psi\tau^p$ there bounded by it. The ψ as thus constructed clearly has the required properties.

THEOREM 15. *Let K and K' be simplicial triangulations of homeomorphic polyhedra P and P'. Then there are isomorphisms between ${}^p\mathbf{H}^G(K)$ and ${}^p\mathbf{H}^G(K')$ and between ${}^p\mathbf{H}_G(K)$ and ${}^p\mathbf{H}_G(K')$ which preserve the $\smile$ and $\frown$ products.*

[24] That is, a simplicial map such that each vertex x' of K' goes into a vertex of a cell of K containing it.

By identifying corresponding points of P and P', we may suppose K and K' are triangulations of P. We may subdivide K' into K_1', then K into K_1, and then K_1' into K_2', so that the following conditions hold.[25] There are simplicial maps ψ_2, ϕ_1, ψ_1, of K_2' into K_1, K_1 into K_1', and K_1' into K, such that $\Psi = \phi_1\psi_2$ and $\Phi = \psi_1\phi_1$ are pseudo-identical. The duals ψ_1', ϕ_1', ψ_2' of ψ_1, ϕ_1, ψ_2 induce homomorphisms of the cohomology groups of K, K_1', K_1 into those of K_1', K_1, K_2'. Combining pairs of these homomorphisms gives homomorphisms of ${}^p\mathbf{H}_G(K)$ into ${}^p\mathbf{H}_G(K_1)$ and of ${}^p\mathbf{H}_G(K_1')$ into ${}^p\mathbf{H}_G(K_2')$. These are induced by the duals Φ' and Ψ' of Φ and Ψ, and are isomorphisms, by Theorem 14.[26] Set $\psi^* = (\Phi')^{-1}\phi_1'$; then ψ_1' and ψ^* map ${}^p\mathbf{H}_G(K)$ into ${}^p\mathbf{H}_G(K_1')$ and vice versa. Further, using E for the identity, we have, in the cohomology groups,

$$\begin{aligned}\psi^*\psi_1' &= (\phi_1'\psi_1')^{-1}\phi_1'\psi_1' = E,\\ \psi_1'\psi^* &= (\Psi'^{-1}\Psi')\psi_1'(\phi_1'\psi_1')^{-1}\phi_1'\\ &= (\psi_2'\phi_1')^{-1}\psi_2'(\phi_1'\psi_1')(\phi_1'\psi_1')^{-1}\phi_1' = E.\end{aligned}$$

It follows that ψ_1' and ψ^* are isomorphisms (see for instance [4], p. 558). Combining this isomorphism with an isomorphism χ between ${}^p\mathbf{H}_G(K_1')$ and ${}^p\mathbf{H}_G(K')$ (Theorem 14) gives an isomorphism θ between ${}^p\mathbf{H}_G(K)$ and ${}^p\mathbf{H}_G(K')$. By the same process it is seen that ψ_1 induces an isomorphism between ${}^p\mathbf{H}^G(K_1')$ and ${}^p\mathbf{H}^G(K)$. As ψ_1 is simplicial, the isomorphisms induced by ψ_1' and ψ_1 preserve products (see §7). The same is true of χ and its dual, and hence of θ and its dual. This completes the proof.

IV. Manifolds

17. Dual complexes in a manifold

Let K be a subdivision of a closed oriented combinatorial manifold,[27] and let K' be the first derived (simplicial) subdivision of K. Order the vertices of K' by choosing first the vertices of K, next the centers of 1-cells of K, etc. Say K is of dimension n. For each cell σ_i^p of K, let E_i^{n-p} be the subcomplex of K' containing all $(n-p)$-cells of K' which have the center of σ_i^p as their first vertex, and all faces of these cells. The hypotheses on K show that the complex K^* thus formed, the "dual" of K, admits K' as a simplicial subdivision (see §13), and admits a product theory. The maps Sd and Sd^* of $L^p(K)$ and $L^p(K^*)$ into $L^p(K')$ are defined in the natural manner; for the latter, see (19.4). Define ϕ and ϕ^* as in Part III.

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[25] See J. W. Alexander, *Combinatorial Analysis Situs*, Trans. Am. Math. Soc., vol. 28 (1926), pp. 308–310.

[26] Hence $\Phi' = \phi_1'\psi_1'$ has an inverse; but this alone does not imply that ϕ_1' and ψ_1' have inverses.

[27] Compare Seifert-Threlfall, *Topologie*, Ch. X, or Lefschetz, [10], Ch. III.

18. The Poincaré duality theorem

Using K and K^*, we may find at once the form of the duality theorem given by Kolmogoroff, [8], and Čech, [5]. Let $\tau_i^{n-p} = \mathfrak{D}(\sigma_i^p)$ be the cell of K^* dual to σ_i^p ; set $\sigma_i^p = \mathfrak{D}^*(\tau_i^{n-p})$. Recall that

$$\partial\mathfrak{D}(\sigma^{p-1}) = (-1)^p\mathfrak{D}\sigma^p + \cdots \text{ if and only if } \partial\sigma_i^p = \sigma^{p-1} + \cdots ; \tag{18.1}$$

it follows that

$$\partial\mathfrak{D}(A^p) = (-1)^{p+1}\mathfrak{D}(\delta A^p), \qquad \delta\mathfrak{D}(A^p) = (-1)^p\mathfrak{D}(\partial A^p). \tag{18.2}$$

Similar relations hold for $\mathfrak{D}^*$. From (18.2) we may conclude at once that $\mathfrak{D}$ establishes an isomorphism between ${}^p\mathbf{H}_G(K)$ and ${}^{n-p}\mathbf{H}^G(K^*)$. But ϕSd^* establishes an isomorphism between ${}^{n-p}\mathbf{H}^G(K^*)$ and ${}^{n-p}\mathbf{H}^G(K) = {}^{n-p}\mathbf{H}^G$ (see Theorem 13); hence

$${}^p\mathbf{H}_G \approx {}^{n-p}\mathbf{H}^G. \tag{18.3}$$

Suppose that, as in §11, (c), ${}^p\mathbf{H}_G(K) \approx \mathrm{Ch}_Z({}^p\mathbf{H}^H(K))$; then

$${}^{n-p}\mathbf{H}^G \approx \mathrm{Ch}_Z({}^p\mathbf{H}^H). \tag{18.4}$$

19. Products and intersections[28]

Supposing K *is simplicial*, we shall (a) define intersections in terms of the $\frown$ product, (b) give a relation defining $\mathfrak{D}$, or rather $Sd^*\mathfrak{D}$, (c) find the relation between $\smile$ and $\frown$ in the cohomology and homology groups, and (d) relate $\smile$ and intersections.

(a) The intersection of a chain A^p of K and a chain B^{*q} of K^* is the following chain[29] of K':

$$A^p \circ B^{*q} = \phi^{*\prime}\mathfrak{D}A^p \frown Sd^*B^{*q} \quad (\frown \text{ product from §6}). \tag{19.1}$$

We may deduce the ordinary boundary relation:

$$\begin{aligned}\partial(A^p\circ B^{*q}) &= (-1)^{q-(n-p)}\delta\phi^{*\prime}\mathfrak{D}A^p \frown Sd^*B^{*q} + \phi^{*\prime}\mathfrak{D}A^p \frown \partial Sd^*B^{*q}\\ &= (-1)^{n-q}\phi^{*\prime}\mathfrak{D}\partial A^p \frown Sd^*B^{*q} + \phi^{*\prime}\mathfrak{D}A^p \frown Sd^*\partial B^{*q}\\ &= (-1)^{n-q}\partial A^p\circ B^{*q} + A^p\circ\partial B^{*q}.\end{aligned} \tag{19.2}$$

Note that, by (7.2) and Theorem 13, (c),

$$\phi^*(A^p\circ B^{*q}) = \mathfrak{D}A^p \frown B^{*q}. \tag{19.3}$$

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[28] Compare Čech, [5]; Freudenthal, [6].

[29] We wish $A\circ B^*$ to be a cycle if A and B^* are cycles. To apply the $\smile$ or $\frown$ product, we must turn at least one of them into a cocycle; we use $\mathfrak{D}A$. It is best to use a fixed $\frown$ product, which we may do in K'. We map a cocycle of K^* into a cocycle of K' with $\phi^{*\prime}$, and a cycle of K^* into a cycle of K' with Sd^*. In this manner the form of (19.1) is determined.

(b) Order the vertices of K' in the *opposite* manner from that used above. Define ϕ by mapping each vertex of K' into the last vertex of the simplex of K containing it. If Z^n is the fundamental n-cycle of K, then $Z'^n = SdZ^n$ is the fundamental n-cycle of K'. The map $Sd^*\mathfrak{D}$ is given by

$$(19.4) \qquad Sd^*\mathfrak{D}A^p = \phi' A^p \frown Z'^n \qquad (\frown \text{ from §6}).$$

Further, as $\phi Z'^n = \phi SdZ^n = Z^n$, (7.2) gives

$$(19.5) \qquad \phi Sd^*\mathfrak{D}A^p = A^p \frown \phi Z'^n = A^p \frown Z^n;$$

it is this map that Čech uses in place of $\mathfrak{D}$.

(c) Set $\theta A^p = A^p \frown Z^n$. Then, by (6.2),

$$(19.6) \qquad \theta(A \smile B) = A \frown \theta B.$$

As Sd^* and ϕ induce isomorphisms in the homology groups, (19.5) shows that θ induces the same isomorphism of ${}^p\mathbf{H}_G$ into ${}^{n-p}\mathbf{H}^G$ that $\mathfrak{D}$ does. Hence θ^{-1} exists in these groups, and (19.6) gives

$$(19.7) \qquad v_1 \smile v_2 = \theta^{-1}(v_1 \frown \theta v_2), \qquad v_1 \frown u_1 = \theta(v_1 \smile \theta^{-1} u_1).$$

(d) By (19.1), (19.4) and (6.2),

$$\mathfrak{D}^* A^* \circ \mathfrak{D}B = \phi^{*\prime} A^* \frown Sd^*\mathfrak{D}B = (\phi^{*\prime} A^* \smile \phi' B) \frown Z'^n.$$

By (c), applying this to the cohomology groups gives

$$(19.8) \qquad \theta v_1 \circ \theta v_2 = \theta(v_1 \smile v_2),$$

and hence

$$(19.9) \qquad u_1 \circ u_2 = \theta(\theta^{-1} u_1 \smile \theta^{-1} u_2).$$

A final remark. For a positively oriented σ^n, $\sigma^n \frown Z^n$ is the first vertex of σ^n. Hence

$$(19.10) \qquad I \cdot (A^n \frown Z^n) = A^n \cdot Z^n.$$

20. On intersections of chains and complexes

Let M^n be a manifold, and let K and K' be singular complexes (i.e. continuous maps of complexes into M^n) in "general position" in M^n; that is, so that no σ^p intersects any σ'^{n-p-1}. (A slight deformation of K' will bring this about.) Then all Kronecker indices $(\sigma_i^p \circ \sigma_j'^{n-p})$ have meaning, and $(A^p \circ \partial B'^{n-p+1}) = (-1)^p(\partial A^p \circ B'^{n-p+1})$. (See for instance Lefschetz, [10], p. 169, (20).) Set

$$(20.1) \qquad g\sigma_i^p = \textstyle\sum_j (\sigma_i^p \circ \sigma_j'^{n-p})\sigma_j'^{n-p}, \qquad g'\sigma_i'^{n-p} = \textstyle\sum_j (\sigma_j^p \circ \sigma_i'^{n-p})\sigma_j^p,$$

and hence define gA^p, $g'A'^p$. These are dual. We may call the chain gA^p of K' *the intersection of the chain* A^p *of* K *with* K'.

For A^p in K and B'^{n-p} in K', we have the Kronecker index

$$(A^p \circ B'^{n-p}) = gA^p \cdot B'^{n-p} = A^p \cdot g'B'^{n-p}. \tag{20.2}$$

As

$$\delta g A^p \cdot \sigma' = gA^p \cdot \partial\sigma' = (A^p \circ \partial\sigma') = (-1)^p(\partial A^p \circ \sigma') = (-1)^p g\partial A^p \cdot \sigma',$$

we have $\delta g = (-1)^p g\partial$. Taking duals also, we have[30]

$$\delta g A^p = (-1)^p g\partial A^p, \qquad \delta g' A'^p = (-1)^p g' \partial A'^p. \tag{20.3}$$

In particular, cycles of K or K' map into cocycles of K' or K.

We may let K be a subdivision of M^n and let K' be a deformed position of K; then chains of K are mapped into chains of K', which may be considered as chains of K again. Then g takes the place of the $\mathfrak{D}$ in §18.

21. Dual bases

In this section, *we shall use the group R of real or of rational numbers* as coefficient group. Say a set of p-cycles $X_1^p, \cdots, X_s^p$ forms a *base* if they are linearly independent with homology, i.e.

$$\alpha_1 X_1^p + \cdots + \alpha_s X_s^p \sim 0 \qquad \text{implies} \qquad \alpha_1 = \cdots = \alpha_s = 0,$$

and if any p-cycle is homologous to a linear combination of them with real or rational coefficients. In other words, their homology classes form a base for $\mathbf{H}^p = {}^p\mathbf{H}^R$. Define a base for p-cocycles similarly. *Bases exist in any complex.* To show this, note first that $\mathbf{H}^p$ is a vector group; for it has a finite number of generators, (using elements of R as coefficients), and no elements of finite order. (If $kX^p \sim 0$, $k \neq 0$, then $kX^p = \partial Y^{p+1}$, $X^p = \partial(Y^{p+1}/k) \sim 0$.) Hence we may choose independent generators $u_1, \cdots, u_s$. Let X_i^p be a cycle in the class u_i; then $X_1^p, \cdots, X_s^p$ form a base. Say a base $X_1^p, \cdots, X_s^p$ for p-cycles and a base $C_1^p, \cdots, C_t^p$ for p-cocycles (then $t = s$) are *dual* if

$$C_i^p \cdot X_j^p = \delta_{ij} \quad (= 1 \text{ if } i = j, \text{ and } = 0 \text{ if } i \neq j). \tag{21.1}$$

Dual bases exist in any complex. First, let $X_1^p, \cdots, X_s^p$ be a base for p-cycles. As $\mathbf{H}_p = {}^p\mathbf{H}_R \approx \mathrm{Ch}_R\mathbf{H}^p$, the group of characters of $\mathbf{H}^p$ into R, (see for instance Whitney, [13], Theorems 7 and 8), we may choose cocycles $C_1^p, \cdots, C_s^p$ such that (21.1) holds. Let v_i be the cohomology class of C_i^p. Clearly any character of $\mathbf{H}^p$ may be expressed uniquely as a linear combination of $v_1, \cdots, v_s$; hence $C_1^p, \cdots, C_s^p$ form a base for p-cocycles.

Now consider a closed orientable manifold M^n. Dual bases for n-cycles and n-cocycles are formed by the fundamental n-cycle Z^n and a single n-cell σ^n, oriented so that $\sigma^n \cdot Z = 1$; similarly a vertex x and the cocycle I form dual

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[30] The converse relations $\partial g = \pm g\delta$ etc. are false in general.

bases. Say bases $C_1^p, \cdots, C_s^p$ and $D_1^{n-p}, \cdots, D_t^{n-p}$ form *dual bases* for p-cocycles and $(n - p)$-cocycles (then $t = s$) if

$$(21.2) \qquad C_i^p \smile D_j^{n-p} \backsim \delta_{ij}\sigma^n, \qquad \text{i.e.} \qquad (C_i^p \smile D_j^{n-p})\cdot Z^n = \delta_{ij}.$$

We may have $n - p = p$. Then if

$$(21.3) \qquad X_i^{n-p} = C_i^p \frown Z^n, \qquad Y_i^p = D_i^{n-p} \frown Z^n,$$

we find, using (5.4) and (5.12),

$$C_i^p \cdot Y_i^p = \delta_{ij}, \qquad D_i^{n-p}\cdot X_j^{n-p} = (-1)^{p(n-p)}\delta_{ij}.$$

Hence the C_i^p and Y_i^p, also the $(-1)^{p(n-p)}D_i^{n-p}$ and X_i^{n-p}, form dual bases.

Finally, the X_i^{n-p} and Y_i^p form dual bases in the ordinary sense. For, using simple properties of intersections and (19.8) and (19.10),

$$(X_i^{n-p}\circ Y_j^p) = I'\cdot(\theta C_i^p \circ \theta D_j^{n-p}) = I\cdot\theta(C_i^p \smile D_j^{n-p})$$
$$= I\cdot[(C_i^p \smile D_j^{n-p}) \frown Z^n] = (C_i^p \smile D_j^{n-p})\cdot Z^n = \delta_{ij}.$$

V. Products in Product Complexes

22. Definition of the products

Let K_1 and K_2 be two complexes (simplicial or not), with cells σ_i^p and τ_k. Then we have (properly oriented) $(p + r)$-cells $\sigma_i^p \times \tau_k$ and $(p + r)$-chains $A^p \times B^r$ in the product complex $K^* = K_1 \times K_2$. We recall that

$$(22.1) \qquad \partial(\sigma^p \times \tau^r) = (\partial\sigma^p \times \tau^r) + (-1)^p(\sigma^p \times \partial\tau^r).$$

It follows that

$$(22.2) \qquad \delta(\sigma^p \times \tau^r) = (\delta\sigma^p \times \tau^r) + (-1)^p(\sigma^p \times \delta\tau^r).$$

Choose products in K_1 and K_2. We define products in $K_1 \times K_2$ in terms of these by

$$(22.3) \qquad (\sigma^p \times \tau^r) \smile (\sigma^q \times \tau^s) = (-1)^{qr}(\sigma^p \smile \sigma^q) \times (\tau^r \smile \tau^s),$$

$$(22.4) \qquad (\sigma^p \times \tau^r) \frown (\sigma^q \times \tau^s) = (-1)^{p(s-r)}(\sigma^p \frown \sigma^q) \times (\tau^r \frown \tau^s).$$

We shall show that[31] the $\smile$ product has the required properties in $K_1 \times K_2$. (P_1) of §5 clearly holds. Also,

$$I^* \smile I^* = (I_1 \times I_2) \smile (I_1 \times I_2) = (I_1 \smile I_1) \times (I_2 \smile I_2) = I_1 \times I_2 = I^*,$$

[31] Of course $K_1 \times K_2$ as above defined admits a simplicial subdivision, with the proper geometric interpretation. Hence, by Theorem 14, we obtain the correct products from $K_1 \times K_2$.

so that $\gamma = 1$. Now let us calculate the terms in (P_2). We find

$$\begin{aligned}
\delta[(\sigma^p \times \tau^r) \smile (\sigma^q \times \tau^s)] \\
&= (-1)^{qr}[(\delta\sigma^p \smile \sigma^q) \times (\tau^r \smile \tau^s) + (-1)^p(\sigma^p \smile \delta\sigma^q) \times (\tau^r \smile \tau^s)] \\
&\quad + (-1)^{p+q(r+1)}[(\sigma^p \smile \sigma^q) \times (\delta\tau^r \smile \tau^s) + (-1)^r(\sigma^p \smile \sigma^q) \times (\tau^r \smile \delta\tau^s)],
\end{aligned}$$

$$\begin{aligned}
\delta(\sigma^p \times \tau^r) \smile (\sigma^q \times \tau^s) &= (-1)^{qr}(\delta\sigma^p \smile \sigma^q) \times (\tau^r \smile \tau^s) \\
&\quad + (-1)^{p+q(r+1)}(\sigma^p \smile \sigma^q) \times (\delta\tau^r \smile \tau^s),
\end{aligned}$$

$$\begin{aligned}
(\sigma^p \times \tau^r) \smile \delta(\sigma^q \times \tau^s) &= (-1)^{(q+1)r}(\sigma^p \smile \delta\sigma^q) \times (\tau^r \smile \tau^s) \\
&\quad + (-1)^{q(r+1)}(\sigma^p \smile \sigma^q) \times (\tau^r \smile \delta\tau^s).
\end{aligned}$$

From these equations (P_2) follows at once. We may prove similarly that the $\frown$ product has the required properties, or that it corresponds to the $\smile$ product.

In the product K of three complexes, the $\smile$ and $\frown$ products come out the same whether we write K in the form $(K_1 \times K_2) \times K_3$ or $K_1 \times (K_2 \times K_3)$. The signs are as in (23.8) and (23.9) below.

23. The products in Euclidean space

We may subdivide Euclidean n-space $E^n = (u_1, \cdots, u_n)$ by means of the planes $u_i =$ an integer ($i = 1, \cdots, n$). (We could subdivide similarly any small portion of a differentiable manifold.) We shall work out explicitly a $\smile$ product in E^n by using the product of §6 and writing $E^n = E^1 \times \cdots \times E^1$.

First consider E^1. We may denote its cells by

(23.1) $\qquad (\alpha, \beta), \qquad \alpha$ an integer, $\beta = 0$ or 1;

it is either the vertex $u = \alpha$ or the 1-cell $\alpha \leqq u \leqq \alpha + \beta$; its dimension is β. Set $(\alpha, \beta) = 0$ for any other β. If we order the vertices by letting $(\alpha, 0)$ precede $(\alpha', 0)$ if $\alpha < \alpha'$, then the $\smile$ product of §6 may be written

(23.2) $$(\alpha, \beta) \smile (\alpha + \beta, \gamma) = (\alpha, \beta + \gamma),$$

the product vanishing in all other cases.

We turn now to the n-dimensional case. The cells of E^n are

(23.3) $\qquad (\alpha_1, \beta_1; \cdots; \alpha_n, \beta_n), \qquad \alpha_i$ integral, $\beta_i = 0$ or 1;

the dimension of such a cell is $\sum \beta_i$. Any such symbol with some $\beta_i \neq 0$ or 1 is set $= 0$. Set

(23.4) $$S_i(\beta) = S_i(\beta_1, \cdots, \beta_n) = (-1)^{\beta_1 + \cdots + \beta_{i-1}};$$

in particular, $S_1(\beta) = 1$. The boundary relations are

(23.5) $$\begin{aligned}
\partial(\alpha_1, \beta_1; \cdots; \alpha_n, \beta_n) &= \textstyle\sum_i S_i(\beta)(\alpha_1, \beta_1; \cdots; \alpha_i + 1, \beta_i - 1; \cdots) \\
&\quad - \textstyle\sum_i S_i(\beta)(\alpha_1, \beta_1; \cdots; \alpha_i, \beta_i - 1; \cdots),
\end{aligned}$$

$$(23.6)\quad \delta(\alpha_1, \beta_1; \cdots; \alpha_n, \beta_n) = -\textstyle\sum_i S_i(\beta)(\alpha_1, \beta_1; \cdots; \alpha_i, \beta_i + 1; \cdots) + \sum_i S_i(\beta)(\alpha_1, \beta_1; \cdots; \alpha_i - 1, \beta_i + 1; \cdots).$$

Set also

$$(23.7)\quad P(\beta; \gamma) = P(\beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_n) = (-1)^\epsilon, \qquad \epsilon = \sum_{i>j} \beta_i \gamma_j.$$

Suppose each β_i and γ_i is 0 or 1, and $\beta_i + \gamma_i \leqq 1$. Then, dropping out all zeros, $P(\beta; \gamma)$ is the sign of the permutation which carries the β's and γ's from their positions shown to their positions in $(\beta_1 + \gamma_1, \cdots, \beta_n + \gamma_n)$. From (22.3) and (23.2) we now find

$$(23.8)\quad \begin{aligned}(\alpha_1, \beta_1; \cdots; \alpha_n, \beta_n) \smile (\alpha_1 + \beta_1, \gamma_1; \cdots; \alpha_n + \beta_n, \gamma_n)\\ = P(\beta; \gamma)(\alpha_1, \beta_1 + \gamma_1; \cdots; \alpha_n, \beta_n + \gamma_n),\end{aligned}$$

the product vanishes in all other cases. As an example,

$$(\alpha_1, 1; \alpha_2, 0) \smile (\alpha_1 + 1, 0; \alpha_2, 1) = (\alpha_1, 1; \alpha_2, 1),$$
$$(\alpha_1, 0; \alpha_2, 1) \smile (\alpha_1, 1; \alpha_2 + 1, 0) = -(\alpha_1, 1; \alpha_2, 1).$$

By taking the dual of (23.8) as in (5.4), we find at once

$$(23.9)\quad \begin{aligned}(\alpha_1 + \gamma_1 - \beta_1, \beta_1; \cdots; \alpha_n + \gamma_n - \beta_n, \beta_n) \frown (\alpha_1, \gamma_1; \cdots; \alpha_n, \gamma_n)\\ = P(\gamma - \beta, \beta)(\alpha_1, \gamma_1 - \beta_1; \cdots; \alpha_n, \gamma_n - \beta_n).\end{aligned}$$

It is not hard to prove the formulas for $\delta(A \smile B)$ and $\partial(A \frown B)$ from the above formulas, and thus show directly that they are the required products.

24. On the maps of products of manifolds into a complex

We shall use only chains with real or rational coefficients; then dual bases may be defined as in §20. We begin by proving, after Hopf,

THEOREM 16.[32] *Let M^n be a closed orientable manifold with fundamental n-cycle Z^n, and let f be a simplicial map of M^n into a complex K'. Let $C_1^p, \cdots, C_s^p$ and $D_1^q, \cdots, D_s^q$ be dual bases for p-cocycles and q-cocycles in M^n ($p + q = n$; p may $= q$), and let $C_1'^p, \cdots$, and $D_1'^q, \cdots$ be bases in K'. Say*

$$(24.1)\qquad f'C_i'^p \backsim \textstyle\sum_j \lambda_{ij} C_j^p, \qquad f'D_i'^q \backsim \sum_j \mu_{ij} D_j^q.$$

If $fZ^n \sim 0$, then

$$(24.2)\qquad N_{ij} = \sum_{k=1}^{s} \lambda_{ik}\mu_{jk} = 0 \qquad (\text{all } i, j).$$

[32] This theorem was communicated to me by H. Hopf in a letter of September 9, 1937, together with part of Theorem 17. The results (with the complex replaced by a manifold) are corollaries of his paper [7]; we base them on the corresponding relations (7.2).

First we shall find another interpretation of the λ_{ij} and μ_{ij}. Let $\{X_i^p\}$, $\{Y_i^q\}$, $\{X_i'^p\}$, $\{Y_i'^q\}$ be bases for cycles dual to $\{C_i^p\}$, $\{D_i^q\}$, $\{C_i'^p\}$, and $\{D_i'^q\}$ respectively. Say

$$(24.3) \qquad fX_i^p \sim \sum_j \xi_{ij} X_j'^p, \qquad fY_i^q \sim \sum_j \eta_{ij} Y_j'^q;$$

then

$$(24.4) \qquad C_i'^p \cdot fX_j^p = \xi_{ji} = f'C_i'^p \cdot X_j^p = \lambda_{ij},$$

$$(24.5) \qquad D_i'^q \cdot fY_j^q = \eta_{ji} = f'D_i'^q \cdot Y_j^q = \mu_{ij},$$

so that $N_{ij} = \sum \xi_{ki}\eta_{kj}$.

To prove the theorem, we need merely note that

$$\begin{aligned} 0 = (C_i'^p \smile D_j'^q)\cdot fZ^n &= f'(C_i'^p \smile D_j'^q)\cdot Z^n = (f'C_i'^p \smile f'D_j'^q)\cdot Z^n \\ &= \sum_{k,l} \lambda_{ik}\mu_{jl}(C_k^p \smile D_l^q)\cdot Z^n = \sum_{k,l} \lambda_{ik}\mu_{jl}\delta_{kl} = N_{ij}. \end{aligned}$$

THEOREM 17. *Let M^p be a closed orientable p-manifold, and let S^q be a q-sphere, $q \geqq p$. Let $\bar{X}^p$ and $\bar{Y}^q$ be corresponding fundamental cycles. Let M^{p+q} be a simplicial subdivision of $K^{p+q} = M^p \times S^q$, with fundamental cycle Z^{p+q}. If x, y are vertices of M^p, S^q, then*

$$(24.6) \qquad X^p = Sd(\bar{X}^p \times y) \quad \text{and} \quad Y^q = Sd(x \times \bar{Y}^q)$$

are cycles of M^{p+q}. Let f be a simplicial map of M^{p+q} into a complex K', and suppose $fZ^{p+q} \sim 0$.

(a) *If $q > p$ or $q = p$ is even, then either $fX^p \sim 0$ or $fY^q \sim 0$.*

(b) *If $q = p$ is odd, then one of fX^p, fY^q is homologous to a multiple of the other.*

If $q > p$, then as $\mathbf{H}^r(S^q) = 0$ $(r = 1, \cdots, q - 1)$, it follows that X^p and Y^q generate $\mathbf{H}^p(M^{p+q})$ and $\mathbf{H}^q(M^{p+q})$ respectively. Let I and J be the sums of vertices, and let σ^p and τ^q be cells, in M^p and S^q. Let ϕ and Sd be the maps of Theorem 13, and set

$$(24.7) \qquad C^p = \phi'(\sigma^p \times J), \qquad D^q = \phi'(I \times \tau^q), \qquad \sigma^{p+q} = \sigma^p \times \tau^q.$$

These are cocycles. Then, using Theorem 13, (c).

$$(24.8) \qquad C^p \cdot X^p = (\sigma^p \times J)\cdot \phi Sd(\bar{X}^p \times y) = 1, \qquad D^q \cdot Y^q = 1,$$

so that X^p and C^p, also Y^q and D^q, form dual bases. Further, as $\bar{Z}^{p+q} = \phi Z^{p+q}$ is the fundamental cycle of K^{p+q}, (15.4) gives

$$(24.9) \qquad \begin{aligned} (C^p \smile D^q)\cdot Z^{p+q} &= \phi'[(\sigma^p \times J) \smile (I \times \tau^q)]\cdot Z^{p+q} \\ &= (\sigma^p \times \tau^q)\cdot \phi Z^{p+q} = \sigma^{p+q}\cdot \bar{Z}^{p+q} = 1, \end{aligned}$$

so that C^p and D^q form dual bases. Hence, defining the $X_i'^p$ and $Y_j'^q$ as before, and defining ξ_i, η_i by

$$(24.10) \qquad fX^p \sim \sum \xi_i X_i'^p, \qquad fY^q \sim \sum \eta_i Y_i'^q,$$

Theorem 16 gives, with (24.4) and (24.5),

$$\xi_i \eta_j = 0 \qquad \text{(all } i, j). \tag{24.11}$$

It follows that all the $\xi_i = 0$ or all the $\eta_i = 0$, so that $fX^p \sim 0$ or $fY^q \sim 0$.

Now suppose $p = q$. Then X^p and Y^p together generate $\mathbf{H}^p(M^{2p})$ and C^p and D^p together generate $\mathbf{H}_p(M^{2p})$. The relations (24.6) through (24.9) still hold. Also,

$$\begin{gathered} D^p \smile C^p \backsim (-1)^{p^2} C^p \smile D^p, \qquad (D^p \smile C^p)\cdot Z^{2p} = (-1)^p, \\ (C^p \smile C^p)\cdot Z^{2p} = (D^p \smile D^p)\cdot Z^{2p} = 0, \end{gathered} \tag{24.12}$$

so that $\{C^p, D^p\}$ and $\{D^p, (-1)^p C^p\}$ form dual bases for p-cocycles. Further,

$$C^p \cdot Y^p = (\sigma^p \times J)\cdot(x \times \bar{Y}^p) = 0, \qquad D^p \cdot X^p = 0, \tag{24.13}$$

so that $\{X^p, Y^p\}$ and $\{C^p, D^p\}$, and also $\{Y^p, (-1)^p X^p\}$ and $\{D^p, (-1)^p C^p\}$, form dual bases. Comparing (24.10) with (24.3), in which $Y_i'^p = X_i'^p$, we see that (ξ_i, η_i) takes the place of[33] $(\xi_{1i}, \xi_{2i}, \cdots)$, while $(\eta_j, (-1)^p \xi_j)$ takes the place of $(\eta_{1j}, \eta_{2j}, \cdots)$. Hence, by Theorem 16,

$$\xi_i \eta_j + (-1)^p \xi_j \eta_i = 0 \qquad \text{(all } i, j). \tag{24.14}$$

Suppose that $p = q$ is even. If not all the ξ_i are $= 0$, say $\xi_\nu \neq 0$; then setting $i = j = \nu$ in (24.14) shows that $2\xi_\nu\eta_\nu = 0$, $\eta_\nu = 0$, and then setting $i = \nu$, any j, shows that all other $\eta_j = 0$. Thus the theorem is proved for this case.

Finally, if $p = q$ is odd, (24.14) shows that all determinants of the matrix

$$\begin{Vmatrix} \xi_1 & \xi_2 & \cdots \\ \eta_1 & \eta_2 & \cdots \end{Vmatrix}$$

vanish, so the two sets of numbers are proportional. Thus (b) of the theorem holds.

COROLLARY.[34] *If n is even and f is a map of $S^n \times S^n$ into a complex K' of dimension $< 2n$, then one of $S^n \times y$, $x \times S^n$ is mapped into a cycle ~ 0.*

We shall consider briefly what becomes of Theorem 17 when we consider the product $M^{p+q+r} = M^p \times S^q \times S^r$, $p \leqq q \leqq r$. Using the fundamental cycles of the three manifolds, define the three corresponding cycles, X^p, Y^q, Z^r as before. If $p < q < r$, we find by the above methods that $f(Z^{p+q+r}) \sim 0$ implies that one of $f(X^p)$, $f(Y^q)$, $f(Z^r)$ is ~ 0. Consider the case $p = q = r$. Say

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[33] For a direct treatment, compare (24.18) below.

[34] For $K' = S^n$, the theorem was proved by H. Hopf, Fund. Math., vol. 25 (1935), pp. 427–440, part of Satz V. In the present form (with K' a manifold), it was communicated to me by Hopf in the letter referred to.

(24.15) $fX^p \sim \sum \xi_i X_i'^p, \qquad fY^p \sim \sum \eta_i X_i'^p, \qquad fZ^p \sim \sum \zeta_i X_i'^p.$

Let $C_1'^r, \cdots, C_s'^p$ form a base for p-cocycles in K'. Then if $fZ^{3p} \sim 0$, we have

(24.16) $$M_{ijk} = ((C_i'^p \smile C_j'^p) \smile C_k'^p)\cdot fZ^{3p} = 0 \qquad \text{(all } i, j, k).$$

Define C^p, D^p, E^p as before. Then

(24.17) $$C^p \smile C^p \backsim D^p \smile D^p \backsim E^p \smile E^p \backsim 0,$$
$$((C^p \smile D^p) \smile E^p)\cdot Z^{3p} = 1.$$

As $f'C_i'^p \backsim \xi_i C^p + \eta_i D^p + \zeta_i E^p$ etc., the above relations give

(24.18) $$M_{ijk} = \xi_i \eta_j \zeta_k (C^p \smile D^p \smile E^p)\cdot Z + \xi_i \zeta_j \eta_k (C^p \smile E^p \smile D^p)\cdot Z + \cdots$$
$$= \xi_i(\eta_j \zeta_k + \epsilon \eta_k \zeta_j) + \xi_j(\eta_k \zeta_i + \epsilon \eta_i \zeta_k) + \xi_k(\eta_i \zeta_j + \epsilon \eta_j \zeta_i),$$

where $\epsilon = (-1)^p$. Suppose p is even. Then by considering such quantities as M_{iii}, M_{iij}, we see that all the ξ_i or all the η_i or all the ζ_i vanish, so that one of fX^p, fY^p, fZ^p is ~ 0. If p is odd, then all determinants of the matrix of ξ_i, η_i and ζ_i vanish, so that fX^p, fY^p and fZ^p are linearly dependent under homology.

Appendix

miscellaneous questions

25. The products in terms of other operations

We shall show how the products of §6 may be defined in terms of two other operations; one acts on single chains, and the other, on two chains of the same dimension.

Writing simplexes in their normal form §6, we define two homomorphisms of L^p into L^q, $q \leqq p$, by means of

(25.1) $\xi_p^q(x_{i_0} \cdots x_{i_p}) = x_{i_0} \cdots x_{i_q}, \qquad \zeta_p^q(x_{i_0} \cdots x_{i_p})\, x_{i_{p-q}} \cdots x_{i_p}.$

Let ξ_q^p and ζ_q^p $(q \leqq p)$ be the dual maps of L^q into L^p. Define also

(25.2) $$(\sum \alpha_i \sigma_i^p)\circ(\sum \beta_i \sigma_i^p) = \sum \alpha_i \beta_i \sigma_i^p.$$

Then (6.1) gives

(25.3) $A^p \smile B^q = \xi_p^{p+q} A^p \circ \zeta_q^{p+q} B^q, \qquad A^q \frown B^{p+q} = \xi_{p+q}^p(\zeta_q^{p+q} A^q \circ B^{p+q}).$

Let I^p be the sum of all (oriented) p-cells: $I^p = \sum x_{i_0} \cdots x_{i_p}$. Then clearly $I^0 = I$. Some relations following at once from the definitions are

(25.4) $$I^p\cdot(A\circ B) = A\cdot B, \qquad (A\circ B)\cdot C = A\cdot(B\circ C),$$

(25.5) $$\xi_q^p A^q \cdot B^p = A^q \cdot \xi_p^q B^p, \qquad (A \smile B)\cdot C = A\cdot(B \frown C).$$

(25.6) $$I^p \circ A^p = A^p \circ I^p = A^p, \qquad \xi_p^q I^p = \zeta_p^q I^p = I^q (q \geqq p),$$

(25.7) $$\delta I^{2p} = 0, \qquad \delta I^{2p+1} = I^{2p+2}.$$

For any subcomplex K' of K, let us interpret K' also as a chain of mixed dimension, namely, the sum of all its cells. Then $K = \sum I^p$, $K' \circ K' = K'$, and

(25.8) $$A^p \circ K' = K' \circ A^p = \text{the "part of } A^p \text{ in } K'\text{"}.$$

A^p is in K' if and only if $A^p \circ K' = A^p$. Set $K'' = K - K'$. Then all the following twelve conditions are equivalent:

K' is closed, K'' is open, $\bar{K}' = K'$, $\mathrm{St}(K'') = K''$,

$$\sigma_i^p \circ K' = \sigma_i^p \quad \text{implies} \quad \partial\sigma_i^p \circ K' = \partial\sigma_i^p,$$

$$\sigma_i^p \circ K'' = \sigma_i^p \quad \text{implies} \quad \delta\sigma_i^p \circ K'' = \delta\sigma_i^p,$$

$$\partial(A \circ K') \circ K'' = 0, \qquad \delta(A \circ K'') \circ K' = 0,$$

$$\partial(A \circ K'') \circ K'' = \partial A \circ K'', \qquad \delta(A \circ K') \circ K' = \delta A \circ K',$$

$$\partial(A \circ K') \circ K' = \partial(A \circ K'), \qquad \delta(A \circ K'') \circ K'' = \delta(A \circ K'').$$

We may obtain certain chains related to vertices:

$$\zeta_0^p x_i \smile \xi_0^q x_i = \sum \text{all } (p+q)\text{-cells with } x_i \text{ as } (p+1)\text{th vertex},$$

$$\sum_k \zeta_0^{p-k} x_i \smile \xi_0^k x_i = \sum \text{all } p\text{-cells with } x_i \text{ as a vertex}.$$

26. Resolution of chains into boundaries and cocycles

By considering the ranks of the incidence matrices, we may prove

THEOREM 18. *Using real or rational numbers as coefficients, any p-chain may be written uniquely as a p-boundary plus a p-cocycle, or as a p-coboundary plus a p-cycle.*

We shall prove the first statement. To prove uniqueness, suppose

$$\partial A_1 + B_1 = \partial A_2 + B_2 \qquad (\delta B_1 = \delta B_2 = 0).$$

Then setting $A = A_2 - A_1$, $B = B_1 - B_2 = \sum \beta_i \sigma_i^p$, (4.7) gives

$$B \cdot B = B \cdot \partial A = \delta B \cdot A = 0, \qquad \sum \beta_i^2 = 0, \qquad \text{each } \beta_i = 0.$$

Hence $B_1 = B_2$, and $\partial A_1 = \partial A_2$, as required. To prove the existence of the decomposition, consider the rank ρ^p and the numbers of rows and columns α^{p-1} and α^p of $\| {}^p\partial_j^i \|$; α^p is the number of p-cells in K. Clearly

(26.1) $$\rho^{p+1} = \text{number of independent } p\text{-boundaries},$$

(26.2) $$\alpha^p - \rho^{p+1} = \text{number of independent } p\text{-cocycles}.$$

With real coefficients, L^p is a linear vector space of dimension α^p. By the above relations, the p-boundaries and p-cocycles form linear subspaces of dimensions ρ^{p+1} and $\alpha^p - \rho^{p+1}$. We saw that the subspaces were orthogonal; hence they generate L^p.

With integral coefficients, the p-boundaries and p-cocycles generate a subgroup M^p of L^p of rank α^p. Hence the difference group $L^p - M^p$ is finite.

and for some integer $m \neq 0$, $m\theta = 0$ for all θ in $L^p - M^p$. This means that *any chain mA^p is (uniquely) a boundary plus a cocycle.* From this the theorem with rational coefficients follows at once.

EXAMPLE. Let K consist of a, b, c, ab, bc, ca. Then using integer coefficients, a 0-chain A^0 is a boundary plus a cocycle if and only if $I \cdot A^0 \equiv 0 \pmod 3$, and a 1-chain A^1 is a coboundary plus a cycle if and only if $I^1 \cdot A^1 \equiv 0 \pmod 3$.

27. Dual maps and changes of base

We shall prove

THEOREM 19. *Let G and H be free groups, and let dual homomorphisms be defined in terms of the bases $x_1, \cdots, x_m$ in G and $y_1, \cdots, y_n$ in H. Then the bases $x_1', \cdots, x_m'$ in G and $y_1', \cdots, y_n'$ in H give the same definition of duality if and only if the new bases are formed from the original ones by permutations and changes of sign.*

Clearly any such changes of base are allowable. To prove the other half of the theorem, suppose

$$x_i' = \sum_j \alpha_{ij} x_j, \qquad x_i = \sum_j \alpha_{ij}^{-1} x_j', \qquad |\alpha_{ij}| = \pm 1,$$
$$y_i' = \sum_j \beta_{ij} y_j, \qquad y_i = \sum_j \beta_{ij}^{-1} y_j', \qquad |\beta_{ij}| = \pm 1.$$

Suppose both pairs of bases give the same definition of dual homomorphisms. Then if ϕ and ψ are dual, we may write

$$\phi x_i = \sum_j \phi_{ij} y_j, \qquad \psi y_i = \sum_j \phi_{ji} x_j,$$
$$\phi x_i' = \sum_j \phi_{ij}' y_j', \qquad \psi y_i' = \sum_j \phi_{ji}' x_j'.$$

These equations with the former ones give

$$\phi_{ij}' = \sum_{k,l} \alpha_{ik} \phi_{kl} \beta_{lj}^{-1}, \qquad \phi_{ji}' = \sum_{k,l} \beta_{ik} \phi_{lk} \alpha_{lj}^{-1}.$$

As these hold for all ϕ_{kl}, we may set $\phi_{kl} = 1$, and all other $\phi_{pq} = 0$. This gives

$$\alpha_{ik} \beta_{lj}^{-1} = \alpha_{kj}^{-1} \beta_{il} \qquad (\text{all } i, j, k, l). \tag{27.1}$$

Multiplying by $\alpha_{pk} \beta_{jq}$ and summing over j and k gives

$$\delta_{lq} \sum_k \alpha_{pk} \alpha_{ik} = \delta_{pi} \sum_j \beta_{jl} \beta_{jq} \qquad (\text{all } i,\, l,\, p,\, q).$$

Giving p and q different values shows that

$$\sum_k \alpha_{pk} \alpha_{ik} = c\delta_{pi}, \qquad \sum_k \beta_{kp} \beta_{ki} = c\delta_{pi}, \tag{27.2}$$

for some number c. The left hand member is an element of the matrix product of $\| \alpha_{ij} \|$ by its transposed; hence its determinant is $| \alpha_{ij} |^2 = 1$. Therefore $c^m = 1$, and $c = \pm 1$; clearly $c = 1$. It follows that $\| \alpha_{ij} \|$ (and also $\| \beta_{ij} \|$) has a ± 1 in each column, the rest of the column being zeros. This proves the theorem.

Suppose we replace *free group* by *linear vector space.* Again we find (27.2); this time we know merely that $c > 0$. Hence *the bases may be altered by applying to each base an orthogonal transformation, and then multiplying each vector of each set by the same constant* $\neq 0$; *these are the only allowable alterations.*

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