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APPROXIMATING CELLULAR MAPS BY HOMEOMORPHISMS

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OUR AIM expressed for closed manifolds, is to show that (with a few possible exceptions) a continuous map $f: M \to N$ is a limit of homeomorphisms if and only if each point preimage $f^{-1}(y)$, $y \in N$, is compact and contractible. Contractibility is here meant in the weak sense of Borsuk [10], and is equivalent to the terms *shape of a point* or *cell-like* or UV_∞.

The space $H(M, N)$ of homeomorphisms M to N of homeomorphic manifolds that are compact without boundary (i.e. closed), and of dimension $m > 0$, is never closed in the space of continuous maps M to N for the compact-open topology. In fact M. Brown has proved [12] that if X is any compactum in M which is *cellular* in M (i.e. X has small open neighbourhoods that are open m-cells), then M/X ($=M$ with X collapsed to a point) is a manifold, call it N, and the quotient map $M \to N$ is a limit of homeomorphisms. An example of a compactum cellular in the plane R^2 is the closure of $\{(x, \sin(1/x)) \mid 0 < x \leq 1\}$. On the other hand R. Finney [22] has observed that any map $f: M \to N$, which is a limit of homeomorphisms, is cellular in the sense that $f^{-1}(y)$ is cellular for all y in N. The proof is a pleasant exercise. *One is thus led to conjecture that the limits of homeomorphisms are precisely the cellular maps* (*not less*). And the conjecture is made more significant by the fact from engulfing [35] [42] that, if $m \neq 3, 4$, f is cellular $\Leftrightarrow$ each point preimage $f^{-1}(y)$ has the shape of a point. ($\Rightarrow$ is clear). S. Armentrout has proved the conjecture for $m = 3$; we shall prove it for $m \geq 5$, and give a new proof for $m \leqslant 3$.

To state our result in greater generality, we define CE maps $f: X \to Y$ to be *proper* continuous maps such that, for each y in Y, $f^{-1}(y)$ is compact and *cell-like*.
Proper means that the preimage of each compactum is compact.

Adapting the arguments of [31], we prove:

APPROXIMATION THEOREM A (announced in [46]). *Let* $f: M \to N$ *be a CE map of metric topological m-manifolds,* $m \neq 4$, *with or without boundary, such that* $f \mid \partial M$ *gives a CE map* $\partial M \to \partial N$. *Let* $\varepsilon: M \to (0, \infty)$ *be continuous. Suppose that at least one of the following three conditions holds.*

(a) $m \neq 3, 4, 5$

(b) $m = 5$ *and* $f: \partial M \to \partial N$ *is a homeomorphism*

(c) $m = 3$, *and for each* $y \in N$, $f^{-1}(y)$ *has an open neighbourhood that is prime for connected sum*† (call such an f prime).

Then there exists a homeomorphism $g: M \to N$ *such that* $d(f(x), g(x)) < \varepsilon(x)$ *for all* $x \in M$.

† If X is a 3-manifold possibly with boundary, which is not homeomorphic to an (interior) connected sum $Y \# Z$ where Z is a closed 3-manifold and neither Y nor Z is a 3-sphere, then we agree to call X *prime* (for connected sum).

Using the local contractibility theorem of Černavskii [17] [18] and Edwards and Kirby [19], one can join by isotopies a sequence of g's obtained from Theorem A and converging to f in order to prove

COMPLEMENT TO THEOREM A. *There exists a level preserving CE map* $G: M \times [0, 1] \to N \times [0, 1]$ *such that, if* $G(x, t) = (g_t(x), t)$ *defines* g_t, $0 \leq t \leq 1$, *then* $g_1 = f$ *and, for* $0 \leq t < 1$, g_t *is a homeomorphism with* $d(f(x), g_t(x)) < \varepsilon(x)$.

There is a nearly canonical version of Theorem A and its complement that we will report on elsewhere [47]. It attempts to make g and g_t depend continuously on f for the compact open topologies, and thus gives a foliated version of Theorem A.

We wonder if Theorem A will help to decide whether every triangulation of euclidean space is a combinatorial manifold. A remark of R. D. Edwards in this direction is included (end of §3).

A word about the development of the proof of Theorem A. In 1967, D. Sullivan observed that the geometrical formalism, used by S. P. Novikov to prove that a homeomorphism $h: M \to N$ of manifolds preserves rational Pontrjagin classes, used only the fact that h is proper, and a *hereditary homotopy equivalence* in the sense that for each open $V \subset N$ the restriction $h^{-1}V \to V$ is a homotopy equivalence. Lacher [35] was able to identify such proper equivalences as precisely *CE* maps providing one restricts attention to ENR's (= euclidean neighbourhoods retracts = retracts of open subsets of euclidean space). Sullivan exploited his observation to broaden his result concerning the Hauptvermutung [50] and showed that a *CE* map $h: M \to N$ of *PL* manifolds has zero *PL* normal invariant if

$$H^3(M; Z_2)/\text{Im}H^3(M; Z) = 0. \qquad (*)$$

Hence, if M is for example closed, simply connected, and of dimension ≥ 5, surgery shows that h is *homotopic* to a *PL* homeomorphism. Work of Kirby and Siebenmann [31] showed that (*) is necessary to deform h to a *PL* homeomorphism as counterexamples exist where h is a homeomorphism. Additionally it showed that the *topological* normal invariant of h is always zero.

At this point Theorem A became very plausible. Further, Sullivan's observation seemed equally applicable to the key arguments surrounding the Main Diagram of [31]. And we are grateful to Sullivan for reminding us of this. However close examination reveals that the arguments of [31] break down badly in the dénoument when the Alexander isotopy is introduced. One is faced with a difficult-looking problem of extending a *CE* homotopy. It alone surely accounts for the year elapsed while a proof of Theorem A grew out of [31]. The main new idea of our proof of Theorem A is an inversion device (explained in §2.3 later) that allows one to *divorce* the Alexander isotopy from the arguments surrounding the Main Diagram of [31]. It applies in many situations, e.g. that of local contractibility theorems; hence it may be of further use.

Our own interest in Theorem A was incidental to some unsuccessful attempts to prove the

CONJECTURE (still unsettled). *If M is a closed metric topological manifold and $\delta > 0$ is prescribed, one can find $\varepsilon = \varepsilon(M, \delta) > 0$ so that the following holds: Given any map $f: M \to N$ onto a closed manifold such that, for all y in N, $f^{-1}(y)$ has diameter $< \varepsilon$, there exists a homotopy of f to a homeomorphism, through maps with point preimages of diameter $< \delta$.*

Some comments on Theorem A for $m = 3$. If $\partial N = \emptyset$, condition (c) amounts to insisting that the CE map f be cellular (see [54]). This condition cannot be dispensed with if the classical Poincaré conjecture fails. For, if M^3 is a compact contractible 3-manifold, the quotient map, smashing a spine of M^3 is a CE map to a 3-ball.

For $m = 3$, Theorem A has been proved by S. Armentrout in the last few years (see [2], [3], [4], [5]). Surely Armentrout's arguments yield a proof for $m \le 2$. Coming from [31], the arguments of this article are radically different.

If M and N are PL (=piecewise-linear) manifolds of dimension ≤ 3, then the proof of Theorem A can be strengthened to make g PL. And if f is a PL imbedding near a closed set $C \subset M$, then g can equal f near C. One simply has to work piecewise linearly where possible. See [31] and [45, Remark in §5]. This is not new information, but it is not a bad way to prove Moise's approximation theorems.

In case f is a simplicial map of simplicial homotopy manifolds† a somewhat easier proof of Theorem A has recently been given by M. Cohen [15].‡ It is related to the observation that the (simplicial) mapping cylinder $M(f)$ of f is a simplicial homotopy manifold, hence a topological manifold for $m \ge 6$. (See [44], also Appendix II.2 and [19A].)

Here is a list of material to come: §1 Background; §2 Solution of a handle problem, the inversion device, the main diagram; §3 Proof of Theorem A for $\partial N = \emptyset$, majorant topology, local contractibility, the complement for $\partial N = \emptyset$, Theorem A in general and its complement, why $f: \partial M \to \partial N$ is supposed CE, the meaning of Theorem A for cellular decompositions, Edwards' remark; Appendix I—Identifying $S^{n-1} \times R$; Appendix II—Elementary proofs in dimension ≥ 5.

The preparation of this article was greatly assisted by Russ McMillan's erudition in the realm of decomposition spaces and by Bob Edwards' eagle eye for errors. We sincerely thank them both.

§1. DEFINITIONS AND BACKGROUND MATERIAL

Lacher's articles [35], [36] make excellent preliminary reading. Perhaps the best definition of a *cell-like* compactum C is as follows. C is cell-like iff it can be imbedded into an ENR (=euclidean neighbourhood retract) X so that the following condition holds. UVw: *For each neighbourhood U of C in X there exists a smaller neighbourhood V of C in X such that the inclusion map $V \subset U$ is homotopic to a constant map.* This property is independent of the

† i.e. simplicial complex in which the link of every simplex is homotopy equivalent to a sphere.
‡ The dimension 3 case was first established by Finney [21], 1963.

imbedding. Further Borsuk has emphasized that this property is a homotopy type invariant of C. In fact the compactum C is cell-like iff (=if and only if) C is metrizable of dimension $<\infty$ and equivalent to a point in a new coarser "homotopy" category of shapes [10] whose relation to weak homotopy types explains the relation of Čech homology to singular homology.

A map $f: X \to Y$ is always understood to be a continuous map (=mapping) of Hausdorff topological spaces. Recall that f is called *cell-like* if, for each point $y \in Y, f^{-1}(y)$ is a *cell-like* compactum. The numeral 6 is the image of an injective continuous map of the line. This map is clearly cell-like but not proper.† Cell-like proper maps are called CE maps for brevity. A map $f: X \to Y$ which gives a CE map $X \to f(X)$ but is perhaps not onto, is called CE (into). A map of pairs $f: (X, A) \to (Y, B)$ is CE if both $f: X \to Y$ and its restriction $A \to B$ are CE maps (cf. §3.13 below). Similarly for triples etc. We say that a map $f: X \to Y$ has a certain property *over* a subset $B \subset Y$ if the restriction $f^{-1}B \to B$ of f has this property. This wording is used continually.

Without further mention we will repeatedly use the basic

THEOREM 1.1. *Let $f: X \to Y$ be a map of ENR's. If f is CE, then f is a proper homotopy equivalence.‡ Conversely, if a proper map f is a homotopy equivalence over small neighbourhoods of each point of Y then f is CE.*

The converse part is clear from the definition of cell-like.

This result was first given by Lacher [35]; it has numerous antecedents going back at least as far as Eilenberg and Wilder [20]. It can be rapidly deduced from [20, Theorem 1] using Lacher's observation [36] that $f: X \to Y$ is CE iff, in the mapping cylinder $M(f)$, the subspace $M(f) - Y$ is LC^∞ at Y.

Because of this theorem, a CE map $f: X \to Y$ imposes on Y many of the properties enjoyed by X, at least if Y is supposed (unnecessarily?) to have finite dimension. For one, if X is an ENR so is Y [35]. For another, if X is an open manifold and f is cellular, then Y is at least a homotopy manifold|| (Kwun (1961) and Lacher [31].) Relating to this there are some disturbing counterexamples. First, Y may not be a genuine manifold as Bing [7] showed with his cellular dog-bone decomposition of R^3.

An example showing that Y may not be a homotopy manifold if $f: E^3 \to Y$ is CE but not cellular is obtained by collapsing a non-cellular arc in E^3 [9]. Bing [9] gives an easy

† However a cell-like (continuous) map $f: M^n \to N^n$ of an open manifold onto an open manifold of the same dimension is known to be proper. Väisälä proves this in [51]. Connell [16] gives a simpler proof for $M^n = R^n$, which can be generalized. R. Solway (thesis, University of Wisconsin) even shows that a map $f: M^n \to N^n$, is proper if it is cell-like over some open subset of N and $f^{-1}(y)$ is a continuum for all $y \in N$, cf. Olnick [0, p. 187].

Similar facts for manifolds with boundary are deduced by doubling.

‡ Homotopy equivalence in the category of proper continuous maps.

|| An ENR Y is a *homotopy n-manifold* if for each $y \in Y$ there exists a basis $N_1 \supset N_2 \supset \dots$ of neighbourhoods of y so that, up to homotopy, $N_1 - y \supset N_2 - y \supset \dots$ is refined by a sequence $N_1 - y \leftarrow S^{n-1} \leftarrow N_{2-y} \leftarrow S^{n-1} \leftarrow \dots$ so that the composed maps $S^{n-1} \leftarrow S^{n-1} \leftarrow \dots$ are up to homotopy the identity. Other definitions demand less.

example of a proper non-*CE* map $f\colon E^3 \to E^3$ where the point preimages are points, circles, and figure eights, all polyhedral (cf. [14]); however the mapping cylinder of f is no manifold in view of Lacher's observation above.

Some conventions. Without some indication to the contrary the term *manifold* will mean finite dimensional metrizable topological manifold, possibly with boundary. All components are assumed to have the same dimension. If M is a manifold, ∂M denotes its boundary; standing alone ∂ is an abbreviation for "the boundary". The symbol $\approx$ stands for homeomorphism.

Some notation. R^n = euclidean n-space with norm $|x| = (x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2)^{1/2}$; $B^n = \{x \in R^n;\ |x| \leq 1\}$ the unit n-ball; $rB^n = \{x \in R^n;\ |x| \leq r\}$ = ball of radius r; $r\mathring{B}^n = \{x \in R^n;\ |x| < r\}$ = open ball of radius r; T^n = n-torus, the n-fold product of circles.

§2. SOLUTION OF A HANDLE PROBLEM

MAIN THEOREM 2.1. *As data consider a CE map* $f\colon V^m \to B^k \times R^n$, $m = k + n$, *of a topological manifold* V^m *onto the product of the standard k-ball* $B^k = \{x \in R^k;\ |x| \leq 1\}$ *with euclidean n-space* R^n, *such that f is a homeomorphism over the neighbourhood* $(B^k - \frac{1}{2}\mathring{B}^k) \times R^n$ *of* $\partial B^k \times R^n$. *If* $m = 3$ *suppose that f is prime.*

For $m \neq 4$, *it is possible to construct a CE map* $f'\colon V^m \to B^k \times R^n$ *such that* f' *is prime if* $m = 3$ *and*

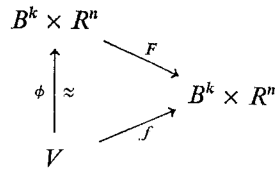
(1) f' *is a homeomorphism over* $B^k \times B^n$.

(2) $f' = f$ *over* $(\partial B^k \times R^n) \cup B^k \times (R^n - r\mathring{B}^n)$ *for some radius r.*

If one regards $B^k \times R^n$ as an open model k-handle with core $B^k \times 0$ it is perhaps suggestive to say that f poses a "*CE* handle problem" that is solved by f'.

Imitating the analysis of the "handle straightening problem" of [31], [32] we will prove:

MAIN LEMMA 2.2. *Given the data of the Main Theorem one can construct a triangle*



in which ϕ *is a homeomorphism, F is cell-like and*

(1) F = *identity over* $(B^k - r\mathring{B}^k) \times R^n \cup B^k \times (R^n - 4\mathring{B}^n)$ *for some* $r < 1$.

(2) $F\phi = f$ *over* $(\partial B^k \times R^n) \cup B^k \times B^n$.

Discussion. In [31] it is shown that if V is a *PL* manifold, f is a homeomorphism *PL* near ∂V and an obstruction $d(f) \in \pi_k(\mathrm{TOP}_m, PL_m)$ vanishes, then in addition F can be a homeomorphism *and* ϕ can be PL. In this situation an Alexander isotopy F_t, $0 \le t \le 1$, of F to the identity fixing the complement of $\text{int}(B^k \times 4B^n)$ yields an explicit isotopy

$$(*) \qquad f_t = f\phi^{-1}F^{-1}F_t\phi$$

of $f_0 = f$ to $f_1 = f\phi^{-1}F^{-1}\phi$. Since $f = F\phi$ over $B^k \times B^n$, one has $f_1\phi^{-1} = f\phi^{-1}F^{-1} =$ identity on $B^k \times B^n$ whence f_1 is PL over $B^k \times B^n$. Thus f_t "straightens" the "handle" $f\colon V^m \to B^k \times R^n$ in the sense of [31] and $f' = f_1$ completes the parallel theorem of [31]. Unfortunately the formula (∗) *is not defined if F is CE with no inverse*. Thus [31] gives no hint how to deduce our Main Theorem from the Main Lemma. It is true that this difficulty can be overcome as soon as a cellular homotopy extension theorem is proved, simply because $F_t\phi | f^{-1}(B^k \times B^n)$ is a cellular homotopy from $f | f^{-1}(B^k \times B^n)$ to an imbedding. Such a theorem can indeed be proved [47] but my proof uses 2.1 and a multi-parameter variant. Frank Quinn intended to prove such an extension theorem directly by proving a version of Browder's $M \times R$ theorem [11] involving smallness conditions.

The following might be called the *Main Idea* of this article. For its sake we postpone the proof of 2.2.

2.3. *Proof of Main Theorem* 2.1 *from Main Lemma* 2.2. Roughly speaking the Main Lemma 2.2 says that f can be altered mod boundary to produce a CE map† (viz. $F\phi$) that is a homeomorphism over a neighbourhood of ∞ and is equal to f over a neighbourhood of $B^k \times 0$. Now the assertion of the Main Theorem is parallel with the roles of ∞ and $B^k \times 0$ interchanged. Hence we are able to deduce the Main Theorem from the Main Lemma and an inversion device as follows. (Compare Fig. 2-a.)

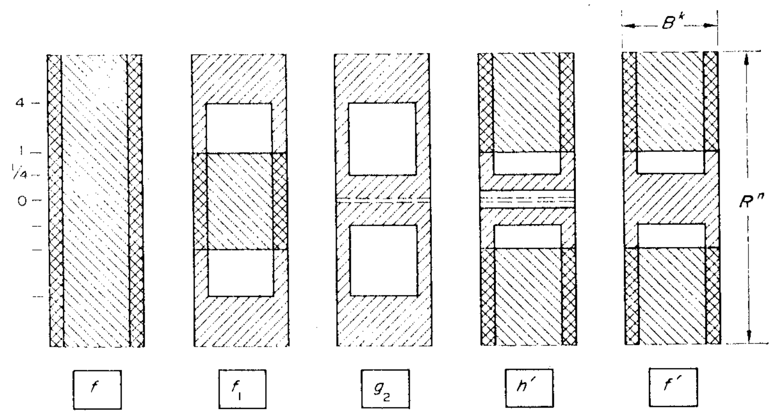


FIG. 2-a. Upsloping hatching as for g_2 indicates parts of $B^k \times R^n$ over which certain mappings are one-to-one. Downsloping hatching indicates some parts over which they equal f.

With the data of the Main Theorem let $f_1 = F\phi\colon V \to B^k \times R^n$ be as given by the Main Lemma. The 1-point compactification $R^n \cup \infty$ is homeomorphic to S^n.

† Throughout 2.3, CE means CE *and* prime if $m = 3$.

Clearly F and ϕ together provide a manifold $\bar{V} \approx B^k \times S^n$ containing V and a CE map $\bar{f}_1: \bar{V} \to B^k \times (R^n \cup \infty)$ extending f_1. We apply the Main Lemma a second time to the restriction g_1 of $\bar{f}_1$ to $\bar{V} - f^{-1}B^k \times 0$

$$\bar{V} - f^{-1}B^k \times 0 \xrightarrow{g_1} B^k \times \{(R^n \mathrm{H}\ \infty) - 0\} \xrightarrow[\approx]{\theta} B^k \times R^n$$

identifying the target to $B^k \times R^n$ by the inversion θ

$$\theta(x, y) = (x, y/|y|^2) \text{ if } y \neq \infty$$
$$\theta(x, \infty) = (x, 0).$$

Note that $\theta^{-1}(B^k \times \partial rB^n) = B^k \times \partial\left(\frac{1}{r}B^n\right)$. Now the Main Lemma changes g_1 fixing boundary to a CE map

$$g_2 : \bar{V} - f^{-1}B^k \times 0 \to B^k \times \{(R^n \cup \infty) - 0\}$$

such that g_2 is a homeomorphism over $B^k \times (\frac{1}{4}B^n - 0)$ and $g_2 = g_1$ over $B^k \times \{(R^n \cup \infty) - \mathring{B}^n\}$ and over $\partial B^k \times \{(R^n \cup \infty) - 0\}$. We can now define

$$h: V - f^{-1}(B^k \times 0) \to B^k \times (R^n - 0)$$

by setting h equal to f over $B^k \times (R^n - \mathring{B}^n)$ and setting $h = g_2$ over $B^k \times (B^n - 0)$. Then h is well-defined and continuous because over $B^k \times \partial B^n$ one has $g_2 = g_1 = \bar{f}_1 = f_1 = f$. Clearly h is CE and equals f over $\partial B^k \times (R^n - 0)$. Thus we can extend h to

$$h': (V - f^{-1}B^k \times 0) \cup \partial V \to B^k \times R^n - \mathring{B}^k \times 0$$

by setting $h' = f$ on ∂V.

Now consider the $(m-1)$-sphere $h'^{-1}\partial(B^k \times \frac{1}{10}B^n)$. It bounds a topological m-ball D in V by the Shoenflies theorem [12] since $V \overset{\phi}{\approx} B^k \times R^n \subset R^{k+n}$. Thus the restriction of h' to the complement of int(D) in V extends by coning to a CE map

$$f': V \to B^k \times R^n.$$

This f' enjoys the properties

(i) f' is a homeomorphism over $B^k \times \frac{1}{4}B^n$ (unfortunately not $B^k \times B^n$).

(ii) $f' = f$ over $\partial B^k \times R^n \cup B^k \times (R^n - rB^n)$ for some r (indeed $r = 1$).

Except for the $\frac{1}{4}$ in (i) these contain the conditions (1), (2) of the Main Theorem 2.1. But the Main Theorem with (i) in place of (1) is clearly equivalent. This completes the proof of the Main Theorem assuming the Main Lemma.

Proof of the Main Lemma 2.2. As in [31] the engine of proof is a procrustean diagram (2-b). In it one constructs in order: e, p, j, the six inclusions at right, α, W_0, β, g_0, W, g, h, F', F, ϕ. A copy of this diagram should be kept at hand throughout the proof.

About e and p. Regard T^n as the quotient R^n/Z^n of R^n by the lattice of integer points and define $\bar{e}: R^n \to T^n = R^n/Z^n$ by $\bar{e}(x) =$ (class of $x/8$) in R^n/Z^n. Define $e = (\mathrm{id}|B^k) \times \bar{e}$. Let $p \in T^n$ be a point not in $\bar{e}(2B^n)$.

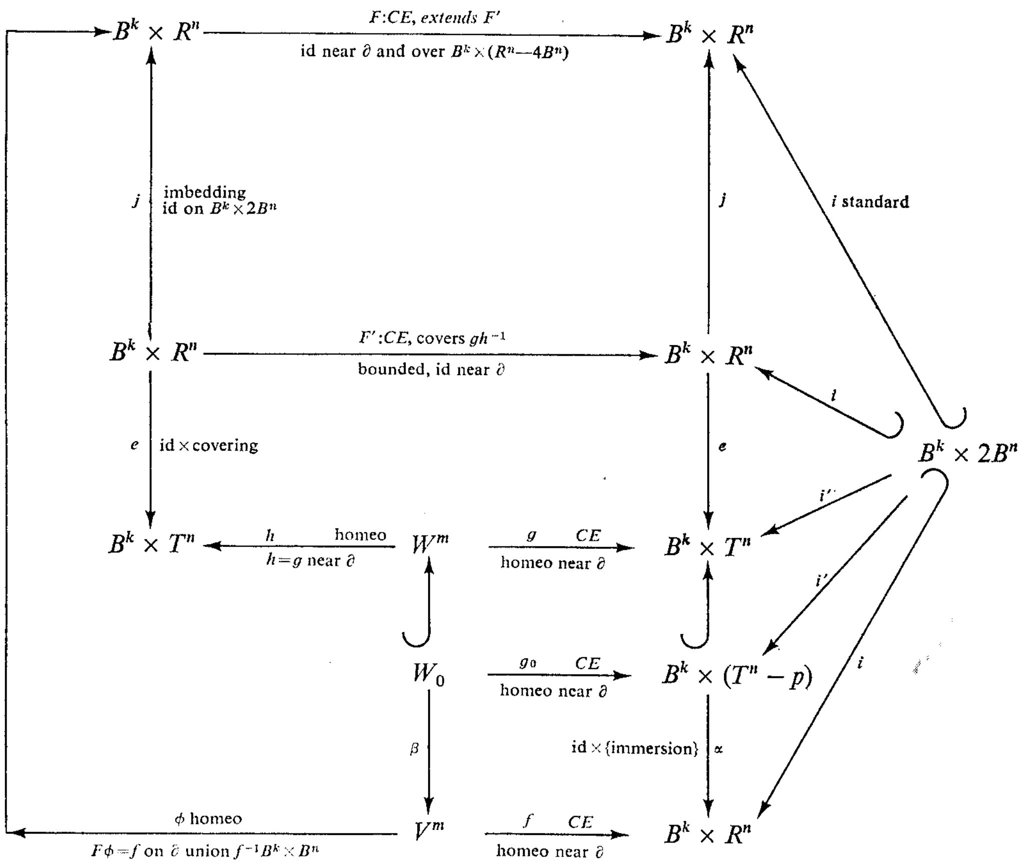


FIG. 2-b.

About j: $B^k \times R^n \to B^k \times R^n$. It is the restriction of the unique ray-preserving embedding $J: R^m \to R^m$ onto $4\mathring{B}^k \times 4\mathring{B}^n$ which fixes precisely $2B^k \times 2B^n$ and on each ray from the origin is linearly conjugate to the embedding γ: $[0, \infty) \to [0, \infty)$ onto $[0, 2)$ with $\gamma(x) = x$, $0 \leq x \leq 1$, and $\gamma(x) = 2 - (1/x)$, $1 \leq x < \infty$.

About i, i', α. Of the imbeddings of $B^k \times 2B^n$ at right, the top two and the last are the natural inclusions. The third i' is the composition $B^k \times 2B^n \subsetneq B^k \times R^n \xrightarrow{e} B^k \times T^n$ and p is chosen outside $e(B^k \times 2B^n)$ to provide the fourth also called i'. An immersion $\bar{\alpha}: (T^n - p) \to R^n$ defining $\alpha = (\mathrm{id}\,|\,B^k) \times \bar{\alpha}$ is so chosen that $\alpha i'$ is the standard inclusion (cf. [32, §3]). *Then the four triangles at right commute.*

About W_0, β *and* g_0. Define W_0 to be the fiber product

$$\{(v, x) \in V^m \times (B^k \times T^n - p)\,|\,f(v) = \alpha(x)\}$$

and set $\beta(v, x) = v$ and $g_0(v, x) = x$. It is straightforward to check that g_0 is CE because f is. Also W_0 is a topological manifold and β is an immersion according to:

LEMMA 2.3. *Let*

$$\begin{array}{ccc} W & \xrightarrow{\ g\ } & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ X & \xrightarrow{\ f\ } & Z \end{array}$$

be a fiber product square such that α is locally a homeomorphism, i.e. each point y_0 of Y has an open neighbourhood U that is mapped homeomorphically by α onto a neighbourhood of $\alpha(y_0)$. Then β is also locally a homeomorphism.

Proof of Lemma. We can assume that $W = \{(x, y) \in X \times Y | f(x) = \alpha(y)\}$ and $\beta(x, y) = x$, $g(x, y) = y$. Given $(x_0, y_0) \in W$, and U chosen for $y_0 \in Y$, the set

$$W \cap \{(x, y) \in X \times Y \,|\, f(x) \in \alpha(U), y \in U\}$$

is an open set that $\beta: (x, y) \mapsto x$ maps bijectively onto $f^{-1}\alpha U$. Since α is an open map, β is too. This completes the lemma.

About W^m and g. We will need two lemmas.

LEMMA 2.4 (D. R. McMillan [41]). *If an open 3-manifold M^3 admits a proper cellular map to R^3 then M^3 is homeomorphic to R^3.*

McMillan's proof is based on Hempel and McMillan [25].

LEMMA 2.5. *Let M be a topological m-manifold $m \neq 4$ without boundary that is properly homotopy equivalent to $S^{m-1} \times R$ and prime in case $m = 3$. Then $M \approx S^{m-1} \times R$.*

A direct proof of this for $m \geq 5$ using only engulfing and formalism is given in Appendix I. For $m = 3$ see [29] and [53]. For $m \leq 2$ the result is trivial.

In applications of 2.5, M will be immersible in R^m hence smoothable. In this form the result is well-known (Browder [11] for $m \geq 6$, Wall [55] for $m = 5$) though not so elementary.

Now the construction. Since f is a homeomorphism over $(B^k - \frac{1}{2}\mathring{B}^k) \times R^n$, g_0 is a homeomorphism over $(B^k - \frac{1}{2}\mathring{B}^k) \times (T^n - p)$. Thus there is a natural extension of $g_0: W_0 \to B^k \times (T^n - p)$ to a CE map $g_1: W_1 \to B^k \times T^n - \frac{1}{2}B^k \times p$ of a manifold W_1, that is a homeomorphism over $(B^k - \frac{1}{2}B^k) \times T^n$. Let $\gamma: B^k \times T^n \to B^k \times T^n$ be a continuous onto map that maps $\frac{1}{2}B^k \times p$ to $0 \times p$, is bijective elsewhere, and is the identity near $e(B^k \times 2B^n)$ (see [12]). Then

$$\gamma g_1: W_1 \longrightarrow B^k \times T^n - 0 \times p$$

is cell-like and $\gamma g_1 = g_0$ near $e(B^k \times 2B^n)$. Next define $g: W \to B^k \times T^n$ to be the 1-point (Alexandroff) compactification of γg_1, viz. $W = W_1 \cup \infty$ and $g(\infty) = 0 \times p$ while $g = \gamma g_1$ on W_1. Clearly g is CE.

ASSERTION. *W is a manifold.*

If $m \neq 3$ this follows directly from Lemma 2.5. If $m = 3$ we must first check that if N is a small open 3-disc containing p then $M^3 = (\gamma g_1)^{-1}(N - p)$ is prime. Now $(N - p) \approx S^2 \times R$ is a union of two open 3-discs. By Lemma 2.4 so is M^3. Hence M^3 is prime by a theorem of Hempel and McMillan [25, Lemma 3].

About $h: W^m \to B^k \times T^n$. It is a homeomorphism such that $h = g$ on a neighbourhood of ∂W and h is homotopic to g.

The case $m = 3$. In this case we can triangulate W^3 so that h is PL near ∂W. We can extend h artificially to a prime CE (hence cellular) map $h': W'^3 \to R^k \times T^n$. Passing to universal coverings we get a *cellular* map $\tilde{h}': \tilde{W}' \to R^k \times R^n = R^3$. By Lemma 2.4, $\tilde{W}' \approx R^3$. Hence W is prime. Similarly V is prime. Then Stallings' theorem of fibration over the circle [49] shows $W^3 \cong B^k \times T^n$ and we easily arrange a piecewise linear isomorphism h with the wanted properties.

The case $m \leq 2$ *is trivial.*

The case $m \geq 5$. Here we present two proofs that h exists. Note that if a homeomorphism $h': W \to B^k \times T^n$ can be found with $h' = g$ on ∂W, there is a standard isotopy of h' to make it agree with g near ∂W (cf. Lemma 2.6 below).

1*st proof that h exists* (topological surgery). The problem of homotoping g fixing ∂ to produce h, has an obvious PL analogue that has been analysed using non-simply-connected surgery [53], [27]. In the PL case an obstruction in $H^3(B^k \times T^n, \partial; Z_2)$ arises. The Z_2 comes from Rochlin's Theorem which is contradicted for topological manifolds [45, §5], [33], at least to the extent that $Z = \pi_4 G/PL \to \pi_4 G/\mathrm{TOP} = Z$ has cokernel $Z_2 = \pi_3 \mathrm{TOP}/PL$. If one carries out the analogous analysis for topological manifolds, as one can using work of Kirby and Siebenmann [33], then no obstruction occurs at all. (See [47A].)

2*nd proof that h exists.* In [28], Hsiang and Wall give an argument† showing that the tangent bundle reduction obstruction, $\rho(W, \partial)$ is zero in $H^4(B^k \times T^n, \partial; \pi_3\mathrm{TOP}/PL)$ for the problem of introducing a PL structure Σ on W extending the one on ∂W given by $g|\partial W$ (see [31], [32]). Their argument is explicitly given only if $k = 0$. In fact, this case suffices because the obstruction to finding Σ is clearly the pull-back by the quotient map

$$g': B^k \times T^n \approx [0, 1]^k \times T^n \to T^{k+n}$$

of the obstruction to introducing a PL structure on the manifold $(W/\sim) \simeq T^{k+n}$ where $\sim$ denotes the identifications induced by $\partial W \xrightarrow{g|\partial} \partial B^k \times T^n \xrightarrow{g'|\partial} T^{k+n}$. Now $g: W_\Sigma \to B^k \times T^n$ is a homotopy triangulation of $B^k \times T^n$ rel boundary. Another homotopy triangulation g': $W' \to B^k \times T^n$ rel ∂ is said to be equivalent if one can find a PL isomorphism $W \cong W'$ so that g is homotopic to g' rel ∂. The equivalence classes form a set $\mathscr{S}'(B^k \times T^n, \partial)$ in (1–1) correspondence with $H^3(B^k \times T^n, \partial; Z_2)$ [33]. Clearly the set $\mathscr{S}(B^k \times T^n, \partial) \cong H^3(B^k \times T^n; Z_2)$ of isotopy classes fixing ∂ of PL structures on $B^k \times T^n$ standard on ∂ maps

† *Warning*: The homotopy invariance of ρ asserted in [28] is false in general, but valid (trivially) for spheres, and hence for tori as required here.

naturally to $\mathscr{S}'(B^k \times T^n, \partial)$. Now an argument in [30A] using finite coverings of odd degree and local contractibility of the homeomorphism group shows that this map

$$\mathscr{S}(B^k \times T^n, \partial) \to \mathscr{S}'(B^k \times T^n, \partial)$$

is injective and hence a bijection of sets having the same finite number of elements.

Remarks towards a third proof that h exists. (i) For the purposes of the proof of the Main Lemma, it would suffice (as in [31]) to pass first to a finite covering $\bar{g}$ of g. $\bar{g}:\overline{W} \to B^k \times T^n$ and then find a homeomorphism $\bar{h}:\overline{W} \to B^k \times T^n$ equal to $\bar{g}$ on ∂.

ASSERTION. *For $k + n \geq 5$ and $k > 0$ this can be done for any large covering $\bar{g}$ using topological engulfing and local contractibility of the homeomorphism group—but no surgery or handlebody theory.* See Appendix II for details. This reduces Theorem A for $m \geq 5$ and $\partial N = \varnothing$ to pure geometry provided $f: M \to N$ is a homeomorphism over some open set in each component of N.

Remark (ii). One can eliminate from the second proof that h exists the argument [28] of Hsiang and Wall. Indeed one can show directly that W admits a PL structure Σ so that $g|\partial W$ is a PL—as follows. If $k = 0$, V^m and W_0 admit PL structures and as $m \geq 5$ there is no further obstruction to W admitting one. If $k > 0$ and $m = k + n \geq 5$ remark (i) asserts something more, at least after passage to a large covering of odd degree. But this doesn't change the obstruction $\rho(W, \partial)$ in $H^4(W, \partial; Z_2)$ to finding Σ.

About F' and F. Let F': $B^k \times R^n \to B^k \times R^n$ be a covering of gh^{-1} (i.e. $eF' = (gh^{-1})e$), fixing $\partial B^k \times R^n$, and extend to $F_0' : R^m \to R^m$ fixing $(R^k - \mathring{B}^k) \times R^n$. Since gh^{-1} fixes fundamental group, F_0' commutes with the translations by $Z^n = 0 \times Z^n \subset R^m$. Hence F_0' is bounded, in fact $|F_0'(x) - x|$ attains its maximum in $B^k \times [0, 8]^n$. Then we get a continuous map $F_0 : R^m \to R^m$ by setting $F_0(x) = JF_0'J^{-1}(x)$ for $x \in 4\mathring{B}^k \times 4\mathring{B}^n$ and $F_0(x) = x$ otherwise, see [32, Section 3]. (Recall J was defined with j.) Being continuous, F_0 is obviously proper and CE. F: $B^k \times R^n \to B^k \times R^n$ is by definition the CE restriction of F_0. Since gh^{-1} is the identity near ∂, so is F; this gives condition (1) of 2.2. Definition makes $F = F'$ over $B^k \times 2B^n$ and $F =$ identity over $B^k \times (R^n - 4\mathring{B}^n)$.

About ϕ. The homeomorphism ϕ is constructed in two steps.

Step 1. *Finding a natural homeomorphism ψ making the following triangle commute:*

$$\begin{array}{ccc} F^{-1}(B^k \times 2B^n) & & \\ \uparrow \psi \approx & \overset{F}{\searrow} & \\ & & B^k \times 2B^n. \\ & \overset{f}{\nearrow} & \\ f^{-1}(B^k \times 2B^n) & & \end{array}$$

The commutative triangles on the right of Main Diagram were arranged to make this possible. ψ can be expressed as the composition $\psi = \psi_1\psi_2\psi_3$ of three homeomorphisms lying in the commutative diagram:

$$\begin{array}{ccc}
F^{-1}B^k \times 2B^n = jF'^{-1}B^k \times 2B^n & \xrightarrow{F} & B^k \times 2B^n \\
\Big\downarrow {\scriptstyle \psi_1{}^{-1}\ \text{equals } ej^{-1}} & & \Big\downarrow {\scriptstyle i' \text{ equals } e} \\
B^k \times T^n \supset hg^{-1}i'B^k \times 2B^n & \xrightarrow{gh^{-1}} & i'B^k \times 2B \subset B^k \times T^n \\
\Big\uparrow {\scriptstyle \psi_2\ \text{equals } h} & & \Big\| \\
W_0 \supset g_0{}^{-1}i'B^k \times 2B^n & \xrightarrow{g_0} & i'B^k \times 2B^n \\
\Big\uparrow {\scriptstyle \psi_3} & & \Big\uparrow {\scriptstyle i'} \\
V \supset f^{-1}B^k \times 2B^n & \xrightarrow{f} & B^k \times 2B^n.
\end{array}$$

In it the horizontal arrows are CE restrictions of maps previously constructed. Noting that $e|F'^{-1}B^k \times 2B^n$ is locally a homeomorphism and that $(gh^{-1})e = eF'$ is CE on $F'^{-1}B^k \times 2B^n$ we see that $e|F'^{-1}B^k \times 2B^n$ is a homeomorphism onto its image $hg^{-1}i'B^k \times 2B^n$. Define ψ_1 by $\psi_1{}^{-1}(x) = ej^{-1}(x)$. Define ψ_2 to be the restriction of h. To define ψ_3 recall that

$$W_0 = \{(x, y) \in V \times [B^k \times (T^n - p)] \,|\, f(x) = \alpha(y)\}$$

and for points $x \in f^{-1}B^k \times 2B^n$ set

$$\psi_3(x) = (x, i'f(x))$$

which lies in W_0 since $\alpha i' =$ identity. This ψ_3 is a continuous bijective map onto $g_0{}^{-1}i'B^k \times 2B^n = \{(x, y) \in W_0 \,|\, i'f(x) = y \in i'B^k \times 2B^n\}$. Its inverse is $\beta\colon (x, y) \to x$. We have constructed $\psi = \psi_1\psi_2\psi_3$.

Step 2. *Obtaining ϕ from ψ and engulfing.* If $M \geq 5$ the engulfing method of Stallings [47], [41] shows that there exist "engulfing" homeomorphisms $E\colon f^{-1}B^k \times 2\mathring{B}^n \to V^m$ and $E'\colon F^{-1}B^k \times 2\mathring{B}^n \to B^k \times R^n$ fixing respectively $f^{-1}B^k \times B^n$ and $F^{-1}B^k \times B^n$. Piecewise linear engulfing suffices. Indeed the obstruction theory of [31] (or topological immersion theory [37]) shows that V admits a piecewise linear structure extending the one near $f^{-1}B^k \times B^n \simeq V$ given by $f^{-1}B^k \times 2\mathring{B}^n \xrightarrow[\approx]{\psi} F^{-1}B^k \times 2\mathring{B}^n \subset B^k \times R^n$. Essentially all the details of argument needed to obtain E, E' from the engulfining theorem are repeated in [42, §4]. All the homotopy theoretic hypotheses needed (and more) are guaranteed by the fact that f and F are CE.

If $m = 3$, the engulfing is possible because for each positive λ, the preimages under f and F of $B^k \times (R^n - \lambda B^n)$ are homeomorphic to $B^k \times (R^n - \lambda B^n)$ by the result of [3] or [41]. Here we need to know that V (like W) is prime (see the construction of h).

Consider now the composition $\phi' = E'\psi E^{-1}\colon V^m \to B^k \times R^n$. It satisfies the condition

(2′) $F\phi' = f$ *over* $B^k \times B^n$.

To complete the construction of ϕ we alter ϕ' to a homeomorphism ϕ satisfying the stronger condition of the Main Lemma.

(2) $F\phi = f$ *over* $(\partial B^k \times R^n) \cup B^k \times B^n$.

To do this consider

$$\theta: \partial B^k \times R^n \xrightarrow{\phi' f^{-1}} \partial B^k \times R^n.$$

By (2′), θ fixes $(\partial B^k) \times B^n$ pointwise. Thus a standard lemma provides an "Alexander" isotopy of θ fixing $\partial B^k \times B^n$ to the identity—namely:

LEMMA 2.6. *Let $f: M \times R \to M \times R$ be a self-homeomorphism with $f \mid M \times (-\infty, 0] =$ identity where M is a topological space. Then if f^1 and f^2 are the components of f on M and R, a standard isotopy of f to the identity fixing $M \times (-\infty, 0]$ is given by*

$$f_t(m, u) = \left(f^1\left(m, u - \frac{t}{1-t}\right), f^2\left(m, u - \frac{t}{1-t}\right) + \frac{t}{1-t}\right), \textit{ for } 0 \le t < 1, \textit{ and } f_1 = \textit{identity}.$$

This isotopy of θ to the identity together with a standard collaring of $\partial B^k \times R^n$ provides the change of ϕ' to ϕ satisfying (2). This completes the construction of ϕ and with it the proof of the Main Lemma.

§3. PROOF OF THEOREM A AND SOME COROLLARIES

Anyone who has read §2 will understand that progress in converting a CE map to a homeomorphism is inevitably measured in the target space. Hence a lemma is needed to convert smallness measured in the source to smallness measured in the target.

LEMMA 3.1. *Let $f: X \to Y$ be a proper continuous onto map of metric spaces and let $\delta: X \to (0, \infty)$ be a continuous map. Then there exists a continuous map $\delta': Y \to (0, \infty)$ such that $\delta(x) > \delta'(f(x))$ for all $x \in X$.*

Proof of 3.1. For each point $y \in Y, f^{-1}(y)$ is compact and we can find an open neighbourhood U_y of $f^{-1}(y)$ and a constant C_y such that $\delta(x) > C_y$ for all $x \in U_y$. The set of points $y' \in Y$ such that $f^{-1}(y') \subset U_y$ is an open neighbourhood V_y of y. (Prove this by contradiction exploiting the hypothesis that f is proper and onto.) For the cover $\{V_y \mid y \in Y\}$ we can build a locally finite refinement $\{W_\alpha\}$ covering Y. Then for each W_α there is a constant C_α such that $\delta(x) > C_\alpha$ if $f(x) \in W_\alpha$. Build a continuous map $\delta': Y \to (0, \infty)$ such that $\delta'(y) < C_\alpha$ if $y \in W_\alpha$. Then $\delta > \delta' f$ as required.

3.2. *Proof of Theorem A when $\partial N = \varnothing$.* It is a straightforward consequence of the Main Theorem 2.1. A similar passage from the solution of a handle problem to a global theorem is given in detail for the Concordance Implies Isotopy Theorem of [32, §4]. So here we are content with an outline.

Referring to the statement of Theorem A in §0, note first that the case $\partial N = \varnothing$ is implied by inductive application of the case where the target N is an open subset of R^m. Recall that any m-manifold is a union of $m + 1$ open sets imbeddable in R^m.

Suppose now that N is an open subset of R^m. Using 3.1 choose a continuous function $\varepsilon': N \to (0, \infty)$ so small that $\varepsilon(x) > \varepsilon'(f(x))$ for all $x \in M$. Find a linear triangulation T of N so fine that the diameter of the star $\mathrm{St}(v, T)$ of each vertex of T has diameter $< \sup\{\varepsilon'(x) \mid x \in \mathrm{St}(v, T)\}$. Associated to T one has a standard handle decomposition $\{H_\sigma \mid \sigma \in T\}$ of N with one k-handle H_σ for each k-simplex σ of T. H_σ is by definition the star of the barycenter of

σ in the second barycentric subdivision T'' of T; its core is $\sigma \cap H_\sigma$. One applies the Main Theorem 2.1 first to the 0-handles then to 1-handles, 2-handles etc. In applying it to a handle H_σ, $\dim \sigma = k$ one identifies $B^k \times R^n$ with H_σ union a collar lying in $\mathrm{St}(\sigma, T')$ of its frontier in the union of handles of dimension $\geq k$ so that $B^k \times 1/2B^n$ corresponds to H_σ.

After $k + 1$ steps, f has been changed to a homeomorphism over handles of dimensions $\leq k$. After $m + 1$ steps one has a homeomorphism $g: M \to N$ and one easily checks that, for all x, $f(x)$ and $g(x)$ lie in $\mathrm{St}(v, T)$ for some vertex v. Hence $d(f(x),g(x)) < \varepsilon'(f(x)) < \varepsilon(x)$ for all $x \in M$, which completes Theorem A when $\partial N = \varnothing$.

The case of Theorem A where $\partial N \neq \varnothing$ will depend on the Complement of Theorem A for $\partial N \neq \varnothing$. So we direct attention to the latter, which involves the local contractibility theorems for homeomorphisms, and requires discussion of the *majorant* topology.

3.3. *Definitions.* Let M be a metric space. $C(M)$ will henceforth denote the set of continuous maps M to M.

A basis for what is called the *majorant* topology on $C(M)$ can be described in any of the following three ways.

(1) Call positive continuous functions $\delta: M \to (0, \infty)$ *majorants*, and for f in $C(M)$ define

$$N_\delta(f) = \{g \in C(M) \mid d(f(x), g(x)) < \delta(f(x)), \forall x \in M\}.$$

The sets $N_\delta(f)$ for varying f and δ are the first basis.

(2) If U is any neighbourhood of the diagonal in $M \times M$ and $f \in C(M)$, define

$$N(f; U) = \{g \in C(M) \mid (f(x), g(x)) \in U, \forall x \in M\}.$$

Such $N(f; U)$ form the second.

(3) If $\mathscr{U}$ is any covering of M by open sets and $f \in C(M)$ define $N(f; \mathscr{U})$ to be the set of all g in $C(M)$ such that $f(x)$ and $g(x)$ belong both to some one set in $\mathscr{U}$. Such $N(f; \mathscr{U})$ form the third basis.

Note that if M is compact the majorant and the compact open topologies on $C(M)$ agree.

If we restrict attention to proper maps, then, by 3.1, one could replace $\delta(f(x))$ in (1) by $\delta(x)$.

When M is non-compact but locally compact, the proper maps $M \to M$ constitute an open and closed subset for the majorant topology.

(*Proof.* If $\mathscr{U}$ is a locally finite covering of M by relatively compact, open sets and $g \in N(f; \mathscr{U})$, then f proper $\Leftrightarrow g$ proper.)

The rule of composition $(f, g) \mapsto fg$ is a majorant continuous map $C(M)^2 \to C(M)$, as one easily sees using (2), and the existence, for given U, of V with $V \circ V \subset U$ [29A, p. 157]. Here $V \circ V$ is the set of all (x, y) in $M \times M$ so that for some $z \in M$, (x, z) and (z, y) belong to V.

To a neighbourhood U of the diagonal in $M \times M$, there corresponds a neighbourhood of the diagonal in $C(M) \times C(M)$

$$\mathscr{D}(U) = \{(f, g) \mid (f(x), g(x)) \in U \text{ for all } x \text{ in } M\}.$$

With the inherited majorant topology, $H(M) \subset C(M)$ is a topological group. The continuity of $f \mapsto f^{-1}$ is easy using (2). Indeed $(g^{-1}, f^{-1}) \in \mathscr{D}(U) \Leftrightarrow (\mathrm{id}, f^{-1}g) \in \mathscr{D}(U) \Leftrightarrow (f, g) \in \mathscr{D}(fU)$.

The majorant topology is needed to formulate a local contractibility theorem for a *noncompact* topological manifold M.

Warning. If f_t, $0 \le t \le 1$, is a path in $C(M)$ continuous for the majorant topology where M is a manifold, then there exists a compactum $K \subset M$ such that $f_t = f_0$ on $M - K$ for all t. Hence $H(M)$ is not even locally path connected for the majorant topology.

Warning. If M is a disjoint union of infinitely many circles, then $H(M)$ is not locally connected in the compact-open topology.

Let $H(M)^I$ be the set of homeomorphisms $M \times I \to M \times I$ respecting $M \times t$ for all $t \in I$. On it consider two topologies.

(a) The *majorant* topology inherited from the majorant topology of $H(M \times I)$.

(b) The *compact open* topology inherited from the compact-open topology on $H(M \times I)$. $H(M)^I$ is the set of all compact open paths in $H(M)$ or isotopies of homeomorphisms of M.

THEOREM 3.4. Černavskii (proofs in [18] [19] also [46A]). *Consider a manifold M and a majorant neighbourhood N_1 of* $(\mathrm{id} \mid M)$ *in $H(M)$. There exists a majorant neighbourhood $N_2 \subset N_1$ of* $(\mathrm{id} \mid M)$ *and a function*

$$\phi: H(M) \to H(M)^I$$

which is continuous for the c.o. (=compact-open) topologies such that for each f in N_2, the isotopy $\phi(f)$ is a path from f to $(\mathrm{id} \mid M)$ *in N_1.*

It is more usual to observe that ϕ is continuous for the majorant topologies, as in [18], [19]. But see [46A]. The compact-open continuity will be needed in [47]. Here is an equivalent restatement that better suits our purposes.

THEOREM 3.5. *Consider a manifold M and a neighbourhood U_1 of the diagonal in $M \times M$. Then there exists a smaller such neighbourhood U_2 and a function*

$$\Phi: \mathscr{D}(U_2) \cap \{H(M) \times H(M)\} \to H(M)^I$$

continuous for c.o. topologies, such that $\Phi(f, g)$ is an isotopy f to g and $(f, \Phi_t(f, g)) \in \mathscr{D}(U_1)$, throughout the isotopy, $0 \le t \le 1$.

Proof of 3.5 *from* 3.4. Let N_i in 3.4 be $N(\mathrm{id}; U_i)$ to obtain ϕ. Then set $\Phi_t(f, g) = \phi_t(gf^{-1})f$, $0 \le t \le 1$. The continuity of Φ is then clear. It remains only to check that $(f, g) \in \mathscr{D}(U_2)$ implies $(f, \phi_t(gf^{-1})f) \in \mathscr{D}(U_1)$ for all t. But $(f, g) \in \mathscr{D}(U_2) \Leftrightarrow (\mathrm{id}, gf^{-1}) \in \mathscr{D}(U_2) \Leftrightarrow gf^{-1} \in N(\mathrm{id}; U_2) \Rightarrow \phi_t(gf^{-1}) \in N(\mathrm{id}; U_1) \Leftrightarrow (f, \phi_t(gf^{-1})f) \in \mathscr{D}(U_1)$.

Consider a map $\rho: \Lambda \to C(M)$ of a parameter space Λ to $C(M)$, continuous for the c.o. topology. (Think at first of $\Lambda =$ point; the generality is for [47].) Suppose that ρ can be majorant approximated by such maps into $H(M)$ in the following sense: Given any neighbourhood U of the diagonal in $M \times M$, there exists a c.o. continuous map $\rho' : \Lambda \to H(M)$ such that $(\rho(\lambda), \rho'(\lambda)) \in \mathscr{D}(U)$ for all $\lambda \in \Lambda$.

PROPOSITION 3.6. *In this situation, for any neighbourhood V of the diagonal in $M \times M$ there exists $\sigma : \Lambda \times [0, 1] \to C(M)$ continuous for the c.o. topology on $C(M)$ so that $\sigma(\Lambda \times [0, 1[) \subset H(M)$ while $\sigma(\lambda, 1) = \rho(\lambda)$ and $(\sigma(\lambda, t), \rho(\lambda)) \in \mathscr{D}(U)$ for all λ in Λ, t in* $[0, 1]$.

Proof of 3.6. Theorem 3.5 provides neighbourhoods $V = V_0 \supset V_1 \supset V_2 \supset \ldots$ of the diagonal in $M \times M$ and c.o. continuous maps $\Phi^i : \mathscr{D}(V_{i+1}) \cap \{H(M) \times H(M)\} \to H(M)^I$ so that $\Phi^i(f, g)$ is an isotopy f to g with $(f, \Phi_t{}^i(f, g)) \in \mathscr{D}(V_i)$, $0 \leq t \leq 1$. We arrange inductively that $\{\mathscr{D}(V_i)\}$ be a basis of neighbourhoods of the diagonal of $C(M) \times C(M)$ for the c.o. topology and that V_i is symmetric with $V_i \circ V_i \subset V_{i-1}$.

Choose a sequence $\rho_i : \Lambda \to H(M)$, $i = 1, 2, \ldots$ of maps with $(\rho_i(\lambda), \rho(\lambda)) \in \mathscr{D}(V_i)$, for all λ in Λ. Then define σ on $\Lambda \times [a_i, a_{i+1}]$, $a_i = 1 - 2^{-i}$, $i \geq 0$, by setting $\sigma(\lambda, t) = \Phi^i_{\theta_i(t)}(\rho_{i+1}(\lambda), \rho_{i+2}(\lambda))$ where θ_i is the oriented linear map of $[a_i, a_{i+1}]$ onto $[0, 1]$. This makes sense as $V_{i+1} \circ V_{i+2} \subset V_i$. Clearly $\sigma(\lambda, a_i) = \rho_i(\lambda)$, $i \geq 1$, and if we set $\sigma(\lambda, 1) = \rho(\lambda)$, then σ becomes the c.o. continuous map $\Lambda \times [0, 1] \to C(M)$ asserted.

Remark 3.7. For compact M, 3.6 shows that $H(M)$ is LC^∞ in $C(M)$. *Then* [20, Theorem 1] *assures that* 3.6 *holds with two alterations*:

(i) Assume $\rho(\Lambda) \subset \overline{H(M)}$ (= closure of $H(M)$ in $C(M)$ but suppose no approximability of ρ.

(ii) Assume Λ is separable metrizable of finite dimension.

Hence, letting Λ be a simplex we find that *the inclusion $H(M) \hookrightarrow \overline{H(M)}$ is a weak homotopy equivalence* for compact manifolds M (Haver [24]). Whether this inclusion is a genuine homotopy equivalence seems to be a technically difficult question.

3.8. *Proof of Complement to Theorem A when $\partial N = \varnothing$*. This follows immediately from 3.6 since A) has been proved in this case.

3.9. *Proof of A when $\partial N \neq \varnothing$*. Adjoin $M \times [0, 1]$ to M making $M \times 1 = \partial M$ and call the result M'. Similarly form N'. We extend $f : M \to N$ to $f : M' \to N'$ so that $f | M \times [0, 1]$ is of the form $f(x, t) = (f_t(x), t)$ where $f_1 = f | \partial M$ and $f_t : \partial M \to \partial N$ is a homeomorphism for $t < 1$ with $d(f_t(x), f_1(x) < \frac{1}{4}\varepsilon(x)$. (This applies the complement to $f | \partial M$.) Clearly $f : M' \to N'$ is then well-defined and a homeomorphism over $N \times [0, 1) \subset N'$.

Given collarings of ∂M and ∂N [13], we have standard homeomorphisms $\phi : M \to M'$, $\psi : N \to N'$ so that $g = \psi^{-1} f \phi : M \to N$ is a homeomorphism over ∂N. The reader will check that for suitable collarings one can arrange that $d(f, g) < \frac{1}{2}\varepsilon$. Then apply the case of Theorem A with no boundary to $\text{int} M \xrightarrow{g} \text{int} N$ to find a homeomorphism $f' : \text{int } M \to \text{int } N$ with $d(g, f') < \frac{1}{2}\varepsilon$ so near $g | \text{int } M$ that it extends to $f' : M \to N$ by $f' | \partial M = g | \partial M$. Then f' is a homeomorphism and $d(f, f') < \varepsilon$ as required.

This completes the proof of Theorem A. The complement to Theorem A now follows from 3.6.

To round off the discussion of Theorem A when $\partial N \neq \varnothing$, we explain why Theorem A supposes that $f : \partial M \to \partial N$ be *CE*. A *CE* map $f : M \to N$ of n-manifolds with boundary necessarily maps ∂M onto ∂N (use Theorem 1.1). However, $f^{-1}\partial N$ can, of course, contain more than ∂M. In this case $\partial M \xrightarrow{f} \partial N$ may not be CE. Hence this assumption cannot be deleted from Theorem A.

Example. Let X^m, $m \geq 4$, be any compact contractible manifold with $\pi_1 \partial X \neq 1$. Let $M = X \perp\!\!\!\perp N$ be a boundary connected sum with any m-manifold N with $\partial N \neq \varnothing$. Let f: $M \to N$ collapse X to a point and be bijective elsewhere. The induced map $\pi_1 \partial M \to \pi_1 \partial N$ has kernel containing $\pi_1 \partial X$.

All such examples involve fundamental group. Indeed:

PROPOSITION 3.10. *Let $f: M \to N$ be a CE map of manifolds. If the restriction $(f|\partial M)$: $\partial M \to \partial N$ is $1 - UV$ in the sense of* [6], [35] *then $(f|\partial M)$ is CE.*

Proof of 3.10.

First suppose $N = R^{m-1} \times [0, \infty)$.

For integral homology one has $H_*(M) = H_*(N) = 0$. Also $H_*(M, \partial M) \cong H_c^*(M) \cong H_c^*(N) = 0$, where H_c^* is cohomology with compact supports, by Theorem 1.1. Hence $H_*(\partial M) = 0$.

Since $(f|\partial M): \partial M \to \partial N$ is 1-UV, $\pi_1 \partial M = \pi_1 \partial N = 0$ [6], [35]. Thus $\pi_* \partial M = 0$ and ∂M is contractible (using theorems of Hurewicz and Whitehead).

Applied to the general case, this discussion shows that for each open subset U of ∂N homeomorphic to R^{m-1}, $(f|\partial M)^{-1}(U) \subset \partial M$ is contractible. Hence $(f|\partial M): \partial M \to \partial N$ is CE as 3.13 asserts.

This section will now taper off with some discussion of the meaning of Theorem A.

It is well known that the study of proper onto maps $f: X \to Y$ of *metric spaces* is essentially equivalent to the study of u.s.c. (= upper semi-continuous) decompositions of such spaces into compacta. The decomposition for f is $\mathcal{D} = \{f^{-1}(y) \mid y \in Y\}$, and its decomposition space $X/\mathcal{D}$ is naturally homeomorphic to Y. See [26], [19A, appendix].

Suppose for the sake of argument that $f: X \to Y = X/\mathcal{D}$ can be majorant approximated by homeomorphisms. Theorem A gives conditions that assure this.

STATEMENT 1. *Let $\mathcal{U}$ be an open covering of X by $\mathcal{D}$-saturated open sets. There exists a proper onto self-map $g: X \to X$ such that $\mathcal{D} = \{g^{-1}(x) \mid x \in X\}$ and for each compactum $A \in \mathcal{D}$ there exists a $U \in \mathcal{U}$ such that $A \cup g(A) \subset U$.* Indeed g can be $f'^{-1}f$ where f' is a homeomorphism approximating f. Now g can itself be majorant approximated by homeomorphisms (cf. Lemma 3.1). So one can prove:

STATEMENT 2. *If $\varepsilon: X \to (0, \infty)$ is a positive continuous function, there exists a homeomorphism $h: X \to X$ so that for all $A \in \mathcal{D}$ diameter $h(A) < \inf\{\varepsilon(x) \mid x \in A\}$ and $A \cup h(A)$ lies in some $U \in \mathcal{U}$.*

The last statement shows that the decomposition $\mathcal{D}$ of X is *shrinkable* in the sense of L. F. McAuley [39, p. 24] with respect to any metric on X. An argument of Bing and McAuley (see [39, p. 24] and [38, p. 454]) shows that, given *any* u.s.c. decomposition $\mathcal{D}'$

shrinkable for a complete metric, the quotient map X $X/\mathscr{D}'$ is majorant *approximable* by homeomorphisms. For proof see [19A], noting that the isotopies there are optional.

For certain X and $\mathscr{D}$ there is a much used criterion of Bing which assures that $(X/\mathscr{D}) \times R$ is homeomorphic to $X \times R$. To fix ideas suppose $X = R^n$ and $\mathscr{D}$ consists of points and one cell-like compactum C. Bob Edwards pointed out to me that if $n + 1 \geqslant 5$, then Theorem A implies that Bing's criterion is necessary as well as sufficient for a homeomorphism $(R^n/C) \times R \approx R^{n+1}$. Here Bing's criterion [8], [1] reads: *For each $\varepsilon > 0$ there exists a uniformly continuous isotopy μ_t, $0 \leq t \leq 1$, of* id : $R^n \times R \to R^n \times R$ *fixing points at a distance $> \varepsilon$ from $C \times R$ such that μ_t changes the last co-ordinate $< \varepsilon$ and $\mu_1(X \times u)$ has diameter $< \varepsilon$ for each $u \in R^1$.*

If $(R^n/C) \times R \approx R^{n+1}$ and $n + 1 \geq 5$, Bing's criterion can be verified by applying Theorem A and its complement to the restriction of the quotient map $g : R^n \times S^1 \to (R^n/C) \times S^1$ over a neighbourhood of $g(C \times S^1)$ with preimage in the $\varepsilon/2$ neighbourhood of $C \times S^1$. This lets us construct an isotopy μ_t' of $\mathrm{id}|R^n \times S^1$ of which the wanted μ_t is the covering. Uniform continuity of μ_t and μ_t' is clear since μ_t' fixes points outside a compactum. To be more specific, μ_t' can be $g_0^{-1} g_{t_\tau}$ where g_t, $0 \leqslant t \leqslant 1$, with $g_1 = g$, comes from the complement applied as suggested with a small majorant, and $\tau < 1$ is near 1.

It is amusing to note that, by a general principle [32, §6] μ_t' and μ_t can be diffeotopies or *PL* isotopies.

APPENDIX I

Here we give a proof avoiding surgery of:

LEMMA 2.5. *If M^n, $n \geq 5$ is a (metrizable) topological n-manifold without boundary that is homotopy equivalent to $S^{n-1} \times R$ and $1 - LC$ at ∞. Then M^n is homeomorphic to $S^{n-1} \times R$.*

Proof of Lemma. By a method of Homma (see Gluck [23]) one can imbed locally flatly and properly a line $L \approx R^1$ that joins the two ends of M. Of course one could use the recent results of [31], [32] to do this.

ASSERTION. *Since M is $1 - LC$ at ∞, $M - L$ is also $1 - LC$ at ∞.*

Proof of assertion. By an elementary stretching argument of Stallings [48], L has a closed product neighbourhood in M that we identify with $L \times B^{n-1}$. Note that any compactum in M is contained in one of the form $C = A \cup [-r, r] \times \frac{1}{r} B^{n-1}$, where r is positive, A is a compactum so that $A \cap \left(L \times \frac{1}{r} B^{n-1}\right)$ lies in $[-r, r] \times \frac{1}{r} B^{n-1}$, and $M - C$ has two unbounded components $\{M - C\}_+$, $\{M - C\}_-$ neighbourhoods of the positive and negative ends of M. Then compacta of the form $C_0 = C - L \times \frac{1}{2r} B^{n-1}$ contain any compactum in $M - L$. By Van Kampen's theorem,

$$(*) \qquad \pi_1(M - L - C_0) = \pi_1 M - C_0 = \pi_1\{M - C\}_+ \; * \; \pi_1\{M - C\}_-$$

the injection of free factors being given by inclusion $M - C \subsetneq M - C_0$. Choose a nest of such compacta C, say $C^0 \subset C^1 \subset C^2 \ldots$, with $\bigcup_i\{\text{interior } C^i\} = M$, so that each map of

$$\pi_1\{M - C^0\}_+ \leftarrow \pi_1\{M - C^1\}_+ \leftarrow \pi_1\{M - C^2\}_+ \leftarrow \cdots$$

is zero, and likewise with $-$ in place of $+$. This is possible because M is $1 - LC$ at ∞. Then in view of (*) the corresponding sequence

$$\pi_1(M - L - C_0{}^0) \leftarrow \pi_1(M - L - C_0{}^1) \leftarrow \pi_1(M - L - C_0{}^2) \leftarrow \cdots$$

also consists of zero maps. This means $M - L$ is $1 - LC$ at ∞ as asserted.

By a homological argument with Poincaré-Lefschetz duality, $H_*(M - L; Z) = 0$. Also $\pi_1\,(M - L) \cong \pi_1\,M = 0$.

So $M - L$ is contractible as well as $1 - LC$ at ∞. Thus by engulfing [48] [42] $M - L \approx R^n$.

In M consider the meridian disc $D = 0 \times B^{n-1}$ if L. We can enlarge D to a locally flat $(n - 1)$-sphere $S \supset D$ separating the ends of M as follows. For the argument compactify $M - L$ with one point ∞. Then $(M - L) \cup \infty \approx S^n$ naturally contains a copy $0 \times (B^{n-1} - 0) \cup \infty$ of D locally flat except at its center point ∞, hence locally flat by [30]. Thus (trivially) D is flat, and so is part of a locally flat $(n - 1)$-sphere $S^{n-1} \subset (M - L) \cup \infty$ which clearly gives the wanted $S \subset M$.

To complete the lemma, bicollar S in M [12]; check that the inclusion of S into M or into either complementary domain is a homotopy equivalence; and finally [42] engulf M in the bicollar.

APPENDIX II. ELEMENTARY PROOFS IN DIMENSION $\geqslant 5$.

Here we explain *how Theorem A in dimension* ≥ 5 *can be proved by pure geometry—provided that* $\partial N = \phi$† *and* $f: M \to N$ *is a homeomorphism over some open set in each component of* N. The geometric tools we use are engulfing and deformation of homeomorphisms; we shall avoid the surgery and handlebody theory used in §2 with their attendant quadratic forms and algebraic K-theory.

Recalling the proof of Theorem A we need only check that one can solve as in 2.1 any CE handle problem $V^m \to B^k \times R^n$, $m = k + n$, obtained by restricting such an $f: M \to N$ over a handle $B^k \times R^n$ in N. We deal first with index $k > 0$ (which will not involve the proviso) and second with index $k = 0$.

For index $k > 0$ we have already observed (§2, Remarks towards a third proof that h exists) that it will suffice to prove the

PROPOSITION II.1. *Let* $g : W^m \to B^k \times T^n$, $m = n + k \geq 5$, $k \geq 1$, *be a homotopy equivalence of a topological manifold giving a homeomorphism* $\partial W \to \partial B^k \times T^n$. *A large standard covering* $\bar{g} : \overline{W} \to B^k \times T^n$ *of* g *is homotopic rel boundary to a homeomorphism.*

For each integer $d > 0$ we define a so-called standard covering map $\mu_d : B^k \times T^n \to B^k \times T^n$ by $\mu_d(x, y) = (x, dy)$ using multiplication by d in the abelian Lie group T^n.

† Or that $f: \partial M \to \partial N$ can be approximated by homeomorphisms—see §3.9, §3.6.

The corresponding $\bar{g} = \bar{g}_d$ comes from forming the fiber product of g and μ_d

$$\begin{array}{ccc} \overline{W} & \xrightarrow{\bar{g} = \mu_d{}^* g} & B^k \times T^n \\ {\scriptstyle g^*\mu_d}\downarrow & & \downarrow{\scriptstyle \mu_d} \\ W & \xrightarrow{\quad g \quad} & B^k \times T^n. \end{array}$$

Corollary II.2 (see discussion in [44] [19A]). *Let M^3 be a compact contractible 3-manifold (fake 3-disc), and $f: M^3 \to B^3$ any map giving a homeomorphism $\partial M^3 \to \partial B^3$. Then $f \times T^2 : M^3 \times T^2 \to B^3 \times T^2$ is homotopic rel boundary to a homeomorphism.*

Adding a circle at infinity to the R^2-bounded covering homeomorphism $M^3 \times R^2 \to B^3 \times R^2$ we get a homeomorphism of double suspensions $S^1 * M^3 \to S^1 * B^3 \approx B^5$ sending S^1 to S^1. (See [44, Theorem A] for details.)

Proof of II.1. This is an extension of the disproof of the Hauptvermutung in [31] permitting replacement of the s-cobordism theorem by engulfing and a now popular wrapping argument. An earlier version of this proof was announced by Kirby and Siebenmann in [44] and [45, §5]—see the exposition of Glaser [22A] based on indications of Kirby.

We identify B^k topologically with $B^s \times [0, 1]$, $s = k - 1$. Then W gives a cobordism (triad) $c = (W; V_0, V_1)$ where $V_i = g^{-1}(B^s \times i \times T^n)$, $i = 0, 1$. Recall from [31, p.748] that it will suffice to prove that c is a product cobordism, i.e. $(W; V_0, V_1) \approx V_0 \times ([0, 1], 0, 1)$, $\approx$ indicating homeomorphism of triads. In the present context the argument of [31] runs as follows. Supposing c is a product, we have a homeomorphism $h: (W; V_0, V_1) \to B^s \times ([0, 1]; 0, 1) \times T^n$. We can easily arrange that $h = g$ on V_0 and also on $\partial W - V_1$. If h happens to equal g on all of ∂W, then it is necessarily homotopic rel ∂W to g (— except if $s = 0$, in which case we can at least arrange that $h \simeq g$ rel ∂ by composing h with a self-homeomorphism of $[0, 1] \times T^n$ fixing boundary). To check this recall $T^n = K(Z^n, 1)$. In general $h|\partial W$ differs from $g|\partial W$ by a self homeomorphism $\theta = (h|V_1) \circ (g|V_1)^{-1}$ of $B^s \times 1 \times T^n$ fixing ∂ and homotopic rel ∂ to the identity. Passing to coverings, $\bar{g}$, $i\bar{h}$, $\bar{\theta} = (\bar{h}|\bar{V}_1) \circ (\bar{g}|\bar{V}_1)^{-1}$ of g, h, θ for μ_d, we can get $\bar{\theta}$ as small as we please (at least after a standard isotopy recalled below). Then $\bar{\theta}$ is isotopic rel ∂ to the identity by a deformation principle for homeomorphisms [19] and we can use this isotopy to adjust $\bar{h}$ to equal $\bar{g}$ on ∂ thus proving the proposition.

We now give a similar argument to show that any large standard covering $\bar{c}$ of c is a product cobordism. Begin with the fact that c is an h-cobordism giving a product cobordism from ∂V_0 to ∂V_1 in ∂W. Then by topological engulfing (see Connell, Newman, Stallings references in [42]), c is an invertible cobordism, i.e. $W = \mathring{C}_0 \cup \mathring{C}_1$ where $C_0 \approx V_0 \times [0, 1]$ is a closed collar of V_0 in $W - V_1$ and C_1 is described similarly. Consider Fig. II-a (see next page) in which $W' = C_1 \cap C_2$ and $W_i = W - \mathring{C}_i$, $i = 1, 2$, are indicated.

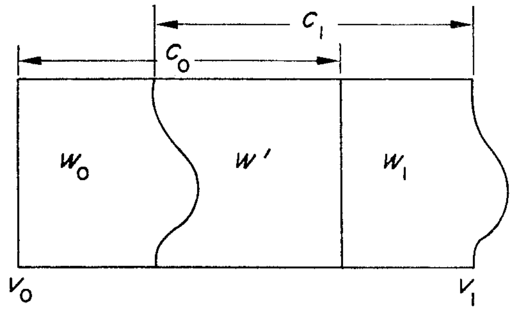


FIG. II-a

The construction of [42A, Fig. 5] provides a homeomorphism $h: V_0 \times T^1 \to V_1 \times T^1$ as follows. Identifying $V_i \times [0, 1]$ to C_i under a collaring homeomorphism α_i, with $\alpha_i(x, i) = x$ for $x \in V_i$, and identifying $T^1 = R^1/Z$, we can regard $V_i \times T^1$ as a quotient of C_i. Then $V_i \times T^1$ is the union of a copy of W' and a copy of W_i. We let $h|W'$ be the identity map onto $W' \subset V_1 \times T^1$. Then complete the definition of h by letting $h|W_0$ be the homeomorphism to $W_1 \subset V_1 \times T^1$ given as a composition of two homeomorphisms $W_0 \approx W \approx W_1$. The first uses $W = W_0 \cup C_1$, the second $W = C_0 \cup W_1$. In order that h be well-defined check that these two homeomorphisms need (only) to map $V_1 \times 0$ and $V_0 \times 0$ in the most obvious way.

Since $V_i = B^s \times i \times T^n$, we can (and do) interpret h as a self-homeomorphism of $B^s \times T^1 \times T^n = (B^s \times i \times T^n) \times T^1$. On the boundary, we arrange that h be a product $\partial B^s \times \phi \times T^n$ where ϕ is a homeomorphism of T^1. This occurs naturally if we choose α_i so that $g\alpha_i|\partial V_i \times [0, 1]$ is a product with an imbedding $[0, 1] \to [0, 1]$.

h necessarily fixes the fundamental group. Clearly so if $\partial B^s \neq \emptyset$. If $s = 0$, at least $h|T^n$ is (visibly) homotopic to the inclusion: We let the reader check the same for $h|T^1$ as there would be no harm in *altering* h on intW' to make h fix $\pi_1 T^1$, and so the whole fundamental group.

Next we show how the isotopy class of h determines c; the same considerations apply to the standard coverings of h and c.

Here is a simple process (cf. Notices AMS 15 (1968) p. 811) to extract an invariant from any homeomorphism $H : V \times R \to V \times R$ where V is a connected compact manifold and H respects the ends of $V \times R$ (i.e. does not interchange them). Consider real numbers λ, μ such that $H(V \times (-\infty, \lambda] \subset V \times (-\infty, \mu)$. It is clear that the *homeomorphism class* of the cobordism (triad) $c(\lambda, \mu) = (W_\lambda; H(V \times \lambda), V \times \mu)$, where $W_\lambda = H(V \times [\lambda, \infty)) \cap V \times (-\infty, \mu]$, is independent of λ and μ—call it $\tau(H)$. (A better invariant is the class of $c(\lambda, \mu)$ as a cobordism of V to itself, see [42A].) If H' is another such homeomorphism of $V \times R$ and $H' = H$ on some $V \times \lambda$, then clearly $\tau(H') = \tau(H)$. It follows that $\tau(H)$ is an isotopy invariant of H; for if $H \sim H''$ ($\sim$ indicates isotopy) the isotopy extension principle [19] yields H' so that $H = H'$ on $V \times O$ and $H' = H''$ outside a compactum.

The example we have in mind is an infinite cyclic covering $H : B^s \times R \times T^n \to B^s \times R \times T^n$ of h (here $V = B^s \times T^n$). Our construction of h makes it clear that $\tau(H)$ is the homeomorphism class of c. Further, if $\bar{h}$ is any standard covering of h, and $\bar{H}$ such an infinite cyclic cover of $\bar{h}$, then $\tau(\bar{H})$ is the class of the standard covering $\bar{c}$ of c for the same integer d.

Next we assert that, since h fixes π_1, and h is so simple on ∂, any standard covering $\bar{h} = \bar{h}_d$ of h for μ_d, is necessarily isotopic to the identity if d is large. It is trite that the component of $\bar{h}$ on $T^1 \times T^n$ is small† for d large, if $\bar{h}$ is suitably chosen (recall that $\bar{h}$ is in general determined only up to covering translations, here "rotations" isotopic to the identity). An Alexander-type isotopy of $\bar{h}$ involving only B^s co-ordinates then makes the component of $\bar{h}$ on B^s as small as we please (For this see [45, p.66–7] and recall that the component on B^s of $h|\partial$ and $\bar{h}|\partial$ is projection.) Thus for d large, $\bar{h}$ is up to isotopy as near as we please to the identity. So the assertion follows from the local contractibility of the homeomorphism group of $B^s \times T^1 \times T^n$ [19].

Over $\bar{h}$ choose an infinite cyclic covering $\bar{H} : B^s \times R \times T^n \to B^s \times R \times T^n$. Now $\bar{h} \sim \text{id} \Rightarrow \bar{H} \sim \text{id} \Rightarrow \tau(\bar{H}) = \tau(\text{id}) \Rightarrow \bar{c}$ is a product cobordism. Thus $\bar{c}$ is a product cobordism for all large standard coverings as required to complete II.1‡.

To conclude this appendix we show how to solve 0-handle problems $V^m \to R^m$ coming from restriction of $f : M \to N$ over a zero handle $R^m \subset N$. If $g : W^m \to T^m$ is the CE map derived from $V^m \to R^m$ in §2 we need only give an elementary construction of a homeomorphism $h : W^m \to T^m$.

Using the proviso about f, we find an isotopy H: $[0, 1] \times R^m \to [0, 1] \times N$ of the inclusion H_0: $R^m \to N$, so that f is a homeomorphism over $H_1(R^m)$. Restricting $[0, 1] \times f$ over $H([0, 1] \times R^m) \subset [0, 1] \times N$ we get a CE map f': $V' \to [0, 1] \times R^m$ such that, over $0 \times R^m$, f' is $V \xrightarrow{f} R^m$, while, over $1 \times R^m$, f' is a homeomorphism. Further $p_1 f'$: $V' \to [0, 1]$ is a submersion. We regard f' as a family of 0-handle problems, one for each point in [0, 1], and begin to solve them all at once forming a fiber product square

$$\begin{array}{ccc} W_0 & \xrightarrow{g_0'} & [0,1] \times (T^m - p) \\ {\scriptstyle \beta' = f'^*\alpha'} \Big\downarrow & & \Big\downarrow {\scriptstyle \alpha'} \quad {\scriptstyle \text{id} \times \text{(immersion)}} \\ V' & \xrightarrow{f' = H^{-1} f} & [0,1] \times R^m. \end{array}$$

Since $p_1 g_0'$ is a submersion $W_0' \to [0, 1]$, and g_0' is CE, it follows that W_0' is a bundle over [0, 1]—by [32, 6.7], [46A, 6.9], cf. [47], a result using engulfing. Now the fiber of W_0' over 0

† Small means near to the projection.

‡ Since we have proved II.1 heuristically in two steps, its conclusion is proved only for $\bar{g} = \bar{g}_d$ where $d = d_1 d_2$ and both d_1, d_2 are large. To prove it for *all* large d, one can easily make do with the 2nd argument only, by using cobordisms rel ∂ from $B^s \times T^n$ *to itself*. Here rel ∂ means that the cobordism of boundaries is given as a product of $\partial B^s \times T^n$ with a 1-dimensional cobordism; equivalence is homeomorphism of triads respecting this extra structure as well as identifications of the "ends" of the triad to $B^s \times T^n$, cf. [42A].

is $W_0 = (W - \text{point})$ as defined in §2, while the fiber over 1 is homeomorphic to $(T^m - p)$ by g_0. Hence there is a homeomorphism $g\colon W \to T^m$ as required in §2 to solve the handle problem $V \xrightarrow{f} R^m$.

REFERENCES

0. *Topology of Manifolds* (Edited by J. C. Cantrell and C. H. Edwards), (Georgia Conference 1969), Markham, Chicago (1970).
1. J. J. Andrews and M. L. Curtis: *n*-space modulo an arc, *Ann. Math.* **75** (1962), 1–7.
2. S. Armentrout: Concerning cellular decompositions of 3-manifolds that yield 3-manifolds, *Trans. Am. math. Soc.* **133** (1968), 307–332.
3. S. Armentrout: Cellular decompositions of 3-manifolds that yield 3-manifolds, *Bull. Am. math. Soc.* **75** (1969), 453–456. Announces (5).
4. S. Armentrout: Concerning cellular decompositions of 3-manifolds with boundary, *Trans. Am. math. Soc.* **137** (1969), 231–236.
5. S. Armentrout: Cellular decompositions of 3-manifolds that yield 3-manifolds *Memoir* 107, *Am. math. Soc.* (1971).
6. S. Armentrout and T. Price: Decompositions into compact sets with *UV* properties, *Trans. Am. math. Soc.* **141** (1969), 433–442.
7. R. H. Bing: A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3, *Ann. Math.* **65** (1957), 484–500.
8. R. H. Bing: The cartesian product of a certain nonmanifold and a line is E^4, *Ann. Math.* **70** (1959), 399–412.
9. R. H. Bing: Decompositions of E^3, *Topology of 3-Manifolds* (Edited by M. K. Fort), pp. 5–21, Prentice-Hall (1962).
10. K. Borsuk: On the concept of shape for metrizable spaces, *Bull. Acad. Pol. Sci.* (*math.*) **18** (1970), 127–132.
11. W. Browder: Structures on $M \times R$, *Proc. Camb. Phil. Soc.* **61** (1965), 337–345.
12. M. Brown: A proof of the generalized Shoenflies theorem, *Bull. Am. math. Soc.* **66** (1960), 74–76.
13. M. Brown: Locally flat embeddings of topological manifolds, *Ann. Math.* **75** (1962), 331–342; also *Topology of 3-Manifolds*, Prentice-Hall (1962).
14. M. Cohen: Simplicial structures and transverse cellularity, *Ann. Math.* **85** (1967), 218–245.
15. M. Cohen: Homeomorphisms between homotopy manifolds and their resolutions, *Invent. Math.* **10** (1970), 239–250.
16. E. H. Connell: Images of E^n under acyclic maps, *Am. J. Math.* **83** (1961).
17. A. V. Černavskii: Local contractibility of the homeomorphism group of a manifold, *Soviet Math. Doklady* **9** (1968), 1171–1174; *Dokl. Akad. Nauk. SSSR*, Tom. **182** (1968), No. 3, 510–513.
18. A. V. Černavskii: Local contractibility of the homeomorphism group of a manifold (Russian) *Math. Sbornik* **79** (121) (1969), 307–356; (English) *Math. U.S.S.R. Sbornik* **8** (1969), No. 3.
19. R. D. Edwards and R. C. Kirby: Deformations of spaces of imbeddings, *Ann. Math.* **93** (1971), 63–88.

19A. R. D. Edwards and L. C. Glaser: A method for shrinking decompositions, *Trans. Am. math. Soc.* (Jan. 1972).

20. S. Eilenberg and R. L. Wilder: Uniform local connectedness and contractibility, *Am. J. Math.* **64** (1942), 613–622.
21. R. Finney: Point-like simplicial mappings of a 3-sphere, *Can. J. Math.* **15** (1963), 591–604.
22. R. Finney: Uniform limits of compact cell-like maps, *Notices Am. math. Soc.* **15** (1968), 942, No. 68TG26.

22A. L. C. Glaser: On suspensions of homology spheres (to appear).

23. H. Gluck: Embeddings in the trivial range, *Bull. Am. math. Soc.* **69** (1963), 824–831; *Ann. Math.* **81** (1965), 195–210.
24. Wm. E. Haver: Cellular mappings on 3-manifolds, *Notices Am. math. Soc.* **16** (1969), 1054.
25. J. P. Hempel and D. R. McMillan, Jr.: Covering 3-manifolds with open cells, *Fundamenta Math.* **64** (1969), 99–104.
26. J. G. Hocking and G. S. Young: *Topology*. Addison-Wesley, Reading, Mass. (1960).
27. W. -C. Hsiang and J. L. Shaneson: Fake tori, *Proceedings of Topology Conference at Athens, Georgia, August* 1969. (To appear).
28. W. -C. Hsiang and C. T. C. Wall: Homotopy Tori II, *Bull. Lond. math. Soc.* **1** (1969).
29. L. S. Hush and T. Price: Finding a boundary for a 3-manifold, *Ann. Math.* **91** (1970), 223–235.
30. R. C. Kirby: The union of flat $(n-1)$-balls is flat in R^n, *Bull. Am. math. Soc.* **74** (1968), 614–617.

30A. R. C. Kirby: *Lectures on triangulation of manifolds*, UCLA (1969).
31. R. C. Kirby and L. C. Siebenmann: On the triangulation of manifolds and the Hauptvermutung, *Bull. Am. math. Soc.* **75** (1969), 742–749.
32. R. C. Kirby and L. C. Siebenmann: Deformation of smooth and piecewise linear manifold structures, *Ann. Math.* (To appear.)
33. R. C. Kirby and L. C. Siebenmann: (to appear).
33A. R. C. Kirby and L. C. Siebenmann: For manifolds the Hauptvermutung and triangulation conjecture are false, *Notices Am. math. Soc.* **16** (1969), 695.
34. J. Kister: Isotopies in 3-manifolds, *Trans. Am. math. Soc.* **97** (1960), 213–224.
35. C. Lacher: Cell-like mappings I, *Pacific J. Math.* **30** (1969), 717–731.
36. C. Lacher: Cell-like mappings II, *Pacific J. Math.* **35** (1970), 649–660.
37. J. Lees: Immersions and surgeries of topological manifolds, *Bull. Am. math. Soc.* **75** (1969), 529–534.
38. L. F. McAuley: Some upper semicontinuous decompositions of E^3 into E^3, *Ann. Math.* **73** (1961), 437–457.
39. L. F. McAuley: Upper semicontinuous decompositions of E^3 into E^3, *Topology of 3-Manifolds* (Edited by M. K. Fort), Prentice-Hall (1962).
40. D. R. McMillan: A criterion for cellularity in a manifold, *Ann. Math.* **79** (1964), 327–337.
41. D. R. McMillan: Cellular sets and a theorem of Armentrout, *Notices Am. math. Soc.* **17** (1970), 840.
42. L. C. Siebenmann: On detecting euclidean space homotopically among topological manifolds, *Invent. Math.* **6** (1968), 245–261.
42A. L. C. Siebenmann: Invertible cobordisms, *Archiv der Math.* **19** (1968), 528–535.
43. L. C. Siebenmann: A total Whitehead torsion obstruction, *Comment. Math. Helvet.* **45** (1970), 1–48.
44. L. C. Siebenmann: Are non-triangulable manifolds triangulable, pp. 77–84 in [0].
45. L. C. Siebenmann: Disruption of low-dimensional handlebody theory by Rohlin's theorem, pp. 57–76 in [0].
46. L. C. Siebenmann: Approximating cellular maps by homeomorphisms, *Notices Am. math. Soc.* **17** (1970), p. 532.
46A. L. C. Siebenmann: Deformation of homeomorphisms on stratified sets, *Comment. Math. Helvet.* (to appear).
47. L. C. Siebenmann: On the closure of the homeomorphism group., (in preparation).
47A. L. C. Siebenmann: *Topological Manifolds*. Proc. ICM Nice 1970, Gauthies-Villars 1971, Vol. 2, pp. 133–163.
48. J. Stallings The piecewise linear structure of Euclidean space, *Proc. Camb. Phil. Soc.* **58** (1962), 481–488.
49. J. Stallings: On fibering certain 3-manifolds, *Topology of 3-Manifolds*, pp. 95–100. Prentice-Hall (1962).
50. D. Sullivan: On the Hauptvermutung for manifolds, *Bull. Am. math. Soc.* **73** (1967), 598–600.
51. J. Väisälä: Invariance of domain under acyclic mappings, *Duke Jour.* **33** (1966), 679–681.
53. C. T. C. Wall: On homotopy tori and the annulus theorem, *Bull. Lond. math. Soc.* **1** (1969).
54. C. T. C. Wall: Open 3-manifolds which are 1-connected at infinity, *Q. Jl. Math. Oxford* **16** (1965), 263–268.
55. C. T. C. Wall: On bundles over a sphere with fibre Euclidean space, *Fundamenta Math.* **61** (1967,) 57–72.

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