Proceedings of the International Congress of Mathematicians
Helsinki, 1978

The Topology of Manifolds and Cell-Like Maps

Robert D. Edwards

1. Introduction. The focus of this expository article will be on the notion of a cell-like set and a cell-like map (definitions below). It will be discussed how these notions arise naturally in the study of certain problems in topology, and how some solutions to these problems have been achieved. It should be emphasized at the outset that the problems discussed here are all *topological* in nature, and so in particular there will be a minimum of extra global structure on the various spaces at hand.

The principal questions to be discussed (and motivated) are the following:

I. POINT-LIKE QUESTION. Which compact subsets of the m-sphere have the property that their complements are homeomorphic to euclidean m-space R^m? (Such subsets are called *point-like,* for the natural reason.)

II. POLYHEDRAL MANIFOLD QUESTION. When is a polyhedron a topological manifold? In particular, are there any "unexpected" examples of such polyhedra? (i.e., examples which do not locally polyhedrally embed in the euclidean space of the same dimension.)

III. MANIFOLD FACTOR QUESTION. When is a space X a factor of a manifold, i.e., when is it the case that $X \times Y$ is a manifold for some space Y? (Usually Y is taken itself to be some euclidean space.)

2. Definitions. All spaces throughout are locally compact separable metric (except in §11, where local compactness is dropped). A manifold will always be understood to be a topological manifold, either finite dimensional, or else modelled on the hilbert cube I^∞ (which is the countably infinite topological product of the interval $[-1, 1]$ with itself). Precisely stated, then, a *(topological) manifold* is a separable metric space each point of which has a neighborhood homeomorphic either to the m-cell I^m or to I^∞. Our manifolds will always be connected.

Later we will be talking about the notion of an *absolute neighborhood retract* (ANR) which we will take to mean a (locally compact separable metric) space which can be embedded as a closed subset of $I^\infty \times [0, \infty)$ (recall any locally compact separable metric space can be so embedded) in such a manner that some neighborhood U of the image retracts to the image, i.e., there is a map $r: U \to$image such that $r|$image=identity. A basic fact about ANR's is that this retraction property is independent of the embedding chosen: if it holds for one closed embedding, it holds for any closed embedding (see e.g. [Hu]; the embeddings even need not be closed). A finite dimensional ANR is called a *euclidean neighborhood retract* (ENR) because it can be embedded as a closed subset of euclidean space so as to have this retraction property.

The fundamental notion in this article is that of *cell-likeness* which, as we will see, broadens a bit the notion of contractibility. A *cell-like* space is a compact metric space C having the following property of a cell: there exists an embedding of C into the hilbert cube I^∞ such that

(*) for any neighborhood U of C in I^∞, C is null-homotopic in U

(examples and properties are given in the next section).

A *proper* map is a map such that the preimage of each compact subset is compact. A *cell-like* map is a proper surjection such that each point-inverse is cell-like. A *near-homeomorphism* is a proper surjection which can be *approximated arbitrarily closely by homeomorphisms*. On compact spaces, this means ordinary uniform approximation. On non-compact, locally compact spaces (which are only of secondary interest in this article) we take this to mean "majorant approximable" by homeomorphisms, i.e., given $f: X \to Y$ and given any majorant map $\varepsilon: X \to (0, \infty)$, there should exist a homeomorphism $h: X \to Y$ such that for each $x \in X$, $\operatorname{dist}(f(x), h(x)) < \varepsilon(x)$. Finally, $X \approx Y$ denotes that X is homeomorphic to Y.

3. Examples and properties of cell-like compacta. There are two basic comments concerning the definition of *cell-like* which should be recalled at this time.

REMARK. (1) Cell-likeness is an intrinsic property of the compact metric space C. That is, if property (*) holds for one embedding $i: C \hookrightarrow I^\infty$, then it holds for any embedding $j: C \hookrightarrow I^\infty$.

(2) In the definition of cell-like, and in (1), if the hilbert cube I^∞ is replaced by any ANR, the statements remain true.

Both remarks are a simple consequence of the map extension property of ANR's. For example, if $i: C \hookrightarrow I^\infty$ is a given embedding satisfying (*), and if $j: C \hookrightarrow W$ is any embedding of C into an ANR W, and U is any neighborhood of $j(C)$ in W, then the fact that W is an ANR implies that there exists some neighborhood V of $i(C)$ in I^∞ and a map $f: V \to U$ extending $ji^{-1}: i(C) \to j(C)$. So if $\alpha_t: C \to V$,

$0 \leq t \leq 1$, is a homotopy provided by $(*)$ such that $\alpha_0 = i$ and $\alpha_1(C)$=point, then $f\alpha_t: C \to U$ provides the desired null-homotopy of C in U.

The simplest examples of cell-like spaces are of course cells, that is, spaces which are homeomorphic to the closed unit ball in some euclidean space. More generally, any contractible compactum is cell-like. In fact, we have

REMARK. Suppose C is a compact ANR. Then C is cell-like $\Leftrightarrow$ C is contractible.

PROOF. To establish the implication $\Rightarrow$, suppose that $C \hookrightarrow I^\infty$ and let $r: U \to C$ be a retraction of a neighborhood, and let $\alpha_t: C \to U$ be a null-homotopy of C in U. Then $r\alpha_t: C \to C$ provides a contraction of C.

Thus the notion of cell-likeness can be regarded as a generalization of the notion of contractibility, and this notion is most useful for non-ANR's. An example of a noncontractible (hence non-ANR) cell-like compactum is the following planar *wedge* ($\equiv$one-point-union) of two cones on cantor sets.

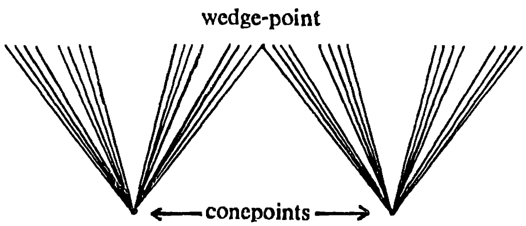

Figure 1. The wedge of two cones on cantor sets

One can construct many more interesting examples using the following

REMARK (Operations preserving cell-likeness).

(1) A countable null ($\equiv$ diameters tending to 0) wedge of cell-like spaces is cell-like.

(2) A product (finite or countable) of cell-like spaces is cell-like.

(3) The intersection of a countable nested collection of cell-like spaces is cell-like.

Regarding (3), note that the cell-like set pictured above is an intersection of 2-cell neighborhoods (when regarded as a subset of the plane).

In general, a cell-like space embedded in R^m (or S^m or any manifold) is said to be *cellularly embedded* (or *cellular*) if it has arbitrarily small neighborhoods homeomorphic to cells. This notion of cellularity definitely depends on the embedding. For example, the following picture (from [F—A]) shows an arc in R^3 which is not cellularly embedded there.

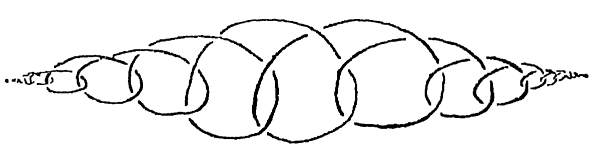

Figure 2. The Artin-Fox wild arc in R^3

Another example of a cell-like, noncellular subset of R^3 is the familiar horned ball of Antoine and Alexander, pictured here (the closed bounded region really is homeomorphic to a ball).

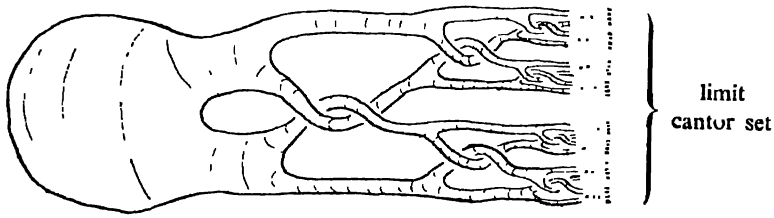

Figure 3. The Antoine-Alexander horned ball in $\mathbf{R}^3$

These examples show that cells can admit noncellular embeddings in euclidean space. In fact, any finite dimensional cell-like set (except a point) can be noncellularly embedded in any euclidean space of greater than twice its dimension. But more importantly, every finite dimensional cell-like set admits a cellular embedding in some euclidean space. In fact, if C is cell-like and $C \hookrightarrow R^m$, then $C \hookrightarrow R^{m+1}$ is cellular (deducible from [McM]).

This discussion leads to an answer to Introductory Question I (provided by M. Brown).

THEOREM. *A compact subset C of S^m is point-like (that is, $S^m - C \approx R^m$) $\Leftrightarrow$ C is cellular in S^m.*

SKETCH OF PROOF. The easier implication is $\Rightarrow$, for letting rB^m be an arbitrarily large ball in R^m, then $S^m - h(\operatorname{int} rB^m)$ is an arbitrarily small ball neighborhood of C in S^m (where $h: R^m \underset{\approx}{\to} S^m - C$ is the hypothesized homeomorphism). (Note: It is far from obvious that $S^m - h(\operatorname{int} rB^m)$ is a ball; in fact, this amounts to the Annulus Conjecture, which has been established in almost all dimensions through the efforts of many, especially R. Kirby, but remains unresolved for $m=4$. However, the above proof can be modified so as to circumvent this issue.) To prove the implication $\Leftarrow$, one first establishes, using the converse of the preceding argument, that any compact subset of $S^m - C$ lies in the interior of an m-cell in $S^m - C$. Then one shows that any space having this property is in fact homeomorphic to R^m. Details of these arguments are in [$\mathbf{Br}_1$] and [$\mathbf{Br}_2$].

Still, this theorem begs the question somewhat. How does one recognize a cellular subset of S^m? In the 2-sphere (or the plane) this is comparatively simple. A compact subset C of S^2 is cellular (and hence point-like) $\Leftrightarrow$ both C and $S^2 - C$ are nonempty and connected (i.e., C is a nonseparating continuum). This is an instructive exercise in plane topology. In higher dimensions, since being cellular always implies being cell-like, let us assume we can recognize when a subset $C \subset S^m$

is cell-like (inasmuch as this property is basically a homotopy property, C is often presented as a cell-like subset by the problem at hand). Given that C is cell-like, how does one tell whether it is cellular? (Remember the preceding examples.) A natural and useful condition to verify is whether $S^m - C$ is "simply-connected at infinity", i.e., whether given any neighborhood U of C in S^m, there exists a smaller neighborhood V of C such that the homomorphism $\pi_1(V-C)\to\pi_1(U-C)$ is trivial. This condition clearly is necessary when $m\geq 3$. It turns out that this *cellularity criterion* is also sufficient, at least when $m\neq 4$ **[McM]**.

This result may be regarded as one of the prototypal theorems in the study of tame versus wild embeddings of compacta in manifolds, a subject which has developed into a very coherent theory during the last twenty years (e.g. see the brief surveys in **[La$_2$]** and **[Ed$_1$]**).

4. Examples and properties of cell-like maps. We now move on to cell-like maps. Probably the most useful characterization of a cell-like map is the following.

PROPOSITION (Homotopy characterization of cell-like maps). *Suppose $f: X\to Y$ is a proper surjection of* ANR's. *Then f is cell-like $\Leftrightarrow$ for each open subset U of Y, the restriction $f|: f^{-1}(U)\to U$ is a (proper) homotopy equivalence.*

Note: The parenthetical word (proper) can be inserted for $\Rightarrow$, and deleted for $\Leftarrow$, to provide the strongest statements.

To understand this proposition one should initially assume that X and Y are as nice as possible, e.g. manifolds. Consider first the implication $\Rightarrow$. As a special argument, consider establishing that $f_*: \pi_1(X)\to\pi_1(Y)$ is onto (which clearly it must be, if f is to be a homotopy equivalence). To verify this surjectivity, one takes a loop $\alpha: S^1\to Y$, and attempts to find a loop $\beta: S^1\to X$ such that $f\beta$ is homotopic to α (basepoints suppressed here). This is achieved by partitioning S^1 very finely, say by $\theta_1, \ldots, \theta_n$, and then arbitrarily choosing a point $\beta(\theta_i)\in f^{-1}(\theta_i)$ for each i. Now one attempts to join adjacent $\beta(\theta_i)$'s with paths which map under f to paths of small diameter. If the partition $\{\theta_i\}$ was chosen sufficiently fine, then each adjacent pair $\{\beta(\theta_i), \beta(\theta_{i+1})\}$ lies very close to some point-inverse $f^{-1}(y)$. Hence, by the definition of cell-likeness, $\beta(\theta_i)$ and $\beta(\theta_{i+1})$ can be joined by a path lying near $f^{-1}(y)$, since the null-homotopy of $f^{-1}(y)$ can be assumed to carry along a nearby neighborhood. Stringing these paths together gives a map $\beta: S^1\to X$ with the property that $f\beta$ is (pointwise) close to α, and hence homotopic to α. Thus f_* is surjective on π_1. This argument is a classical lifting argument which occurs over and over again topology. The full implication $\Rightarrow$ is a straightforward generalization.

To prove the implication $\Leftarrow$, we use the hypothesis that $y\in Y$ has a (arbitrarily small) contractible neighborhood U (assuming still for simplicity that Y is a manifold). Hence by the hypothesis, $f^{-1}(U)$ is a (arbitrarily small) contractible neighborhood of $f^{-1}(y)$. The general case, where Y is merely an ANR, is only slightly more complicated.

Cell-like maps are to be regarded as generalizations of homeomorphisms. This is a recurring theme. One important advantage that cell-like maps have over homeomorphisms is that (unlike homeomorphisms) they are closed under the operation of taking limits.

PROPOSITION (Operations with cell-like maps of ANR's). (1) *If a proper map* $f: X \to Y$ *of ANR's is approximable* (*as in* §2) *by cell-like maps, then* f *is cell-like.*
(2) *The composition of cell-like maps of ANR's is cell-like.*

The proof is an interesting exercise, using the preceding proposition (one should assume X and Y are manifolds, at least initially). See [La$_1$] and [La$_2$].

We close this section by mentioning a classical theorem of R. L. Moore (as refined by Roberts–Steenrod and Youngs). The theorem essentially describes all possible cell-like maps defined on surfaces. (Those defined on 1-manifolds clearly are just those maps having point and interval point-inverses.)

THEOREM ([Mo], [R–S] and [Yo]). *Suppose* $f: M^2 \to Y$ *is a cell-like map defined on a closed surface* M^2 (*i.e. each* $f^{-1}(y)$ *is connected, and* $M^2 - f^{-1}(y)$ *is connected and has genus equal that of* M^2). *Then* Y *is also a surface, and* f *is approximable by homeomorphisms.*

The proof is a tour de force in plane topology.

5. Cell-like maps as limits of homeomorphisms. The preceding proposition implies that a near-homeomorphism is a cell-like map. When is the converse true? (Certainly not always, e.g. the map interval $\to$ point.)

It turns out to be natural to restrict attention to the case where the source is a manifold-without-boundary (possibly even a hilbert cube manifold). If in addition the target is also assumed to be a manifold, then we have the following fundamental answer.

THEOREM (for $m \leqq 2$ see above; m=3 Armentrout [Ar$_2$]; $5 \leq m < \infty$ Siebenmann [Si]; $m = \infty$ Chapman [Ch$_1$]; $m=4$ unknown). *Suppose* $f: M^m \to N^m$ *is a cell-like map* (*read cellular if* $m=3$) *of m-manifolds-without-boundary,* $m \neq 4$. *Then* f *is approximable by homeomorphisms.*

We now arrive at the focal point of this article, which is: What happens in the above theorem if N is not at the outset assumed to be a manifold (but M is)? Does the rest of the data (namely, that N is a cell-like image of a manifold) necessarily imply that N is a manifold? If $\dim M \leq 2$, then N is necessarily a manifold, by the Moore-et-al Theorem. But if $\dim M \geq 3$, then in fact N need not be a manifold, e.g., let $N = S^3$/Fox–Artin arc (see §3). So the problem becomes that of finding good conditions which ensure that N is a manifold.

Starting in the 1950's, a great deal of energy was put into understanding various special but important cases of this question. Most of the energy and insight was provided by R. H. Bing, whose pioneering work opened up the area and established a viable theory. Progress in the area was steady and remarkable, mostly at first in dimension 3, and later in higher dimensions. Here we will pass over all of these efforts, to concentrate in the next few sections on only the most recent developments.

6. The Approximation Problem. We cast our problem in the following form.

APPROXIMATION PROBLEM: *Suppose $f: M \to X$ is a cell-like map from a topological m-manifold-without-boundary onto an ANR X (possibly $m=\infty$ here, i.e., M may be a hilbert cube manifold). Find natural and useful conditions on X which guarantee that f be approximable by homeomorphisms.*

In light of the Moore–Armentrout–Siebenmann–Chapman Theorem above, this can be regarded as asking for conditions which guarantee that X be a manifold, at least if $\dim X \neq 4$. However, the reason for formulating it as an approximation problem will become clear, especially in §9.

The assumption here that X be an ANR is one of unfortunate necessity. If $m=\infty$, there exists an example of a cell-like map of the hilbert cube onto a non-ANR **[Ta]**. If $m<\infty$, it is not known whether such an X as in this problem need be an ANR. (This is a significant unresolved question; it is known to be equivalent to whether X is finite dimensional. In fact, this question is known to be equivalent to that of whether a compact metric space of finite cohomological dimension necessarily has finite (covering) dimension **[Ed$_5$]**.) At any rate, in most of the interesting situations which arise X is independently known to be an ANR.

7. The Approximation theorem in finite dimensions ≥ 5. In the next three sections, we restrict attention to finite dimensions.

In past years it was most often a certain special case of the Approximation Problem which was examined (as a rule), namely the stabilized case, where one asks whether $f: M \times R^1 \to X \times R^1$ is approximable by homeomorphisms. Progress on this special question was slow but steady, working in general on f's with increasingly pathological singularities. For example, one of the cases here which took a long time to resolve, eventually affirmatively, was the case where f has only one single nontrivial point-inverse.

A key step in recent years was made by J. Cannon, who turned the focus back to the pure, unstabilized question, by introducing in **[Ca$_2$]** as a workable X-condition the disjoint disc property. A space X has the *disjoint disc property* if, given any two maps $f_1, f_2: B^2 \to X$, there are arbitrarily close maps $g_1, g_2: B^2 \to X$ which have disjoint images. That is, two maps of the 2-disc into X can be general positioned apart. (Interestingly, a version of the disjoint disc property was used by Bing in his fundamental paper **[Bi$_3$]**.)

Cannon showed in **[Ca$_2$]** that, with regard to the Approximation Problem, if X has the disjoint disc property, and if X is already known to be a manifold except on a codimension ≥ 3 subset, then in fact f is approximable by homeomorphisms. This in turn inspired the following generalization.

Approximation Theorem [$\mathbf{Ed_3}$]. *Suppose $f: M^m \to X$ is a cell-like map from a topological m-manifold-without-boundary M onto an ANR X, and suppose $5 \leqslant m < \infty$. Then f is approximable by homeomorphisms $\Leftrightarrow$ X has the disjoint disc property.*

The proof is outlined in §9. For one thing, this theorem provides another proof of the Siebenmann Approximation Theorem (§5), inasmuch as that is not an ingredient.

8. Applications of the Approximation Theorem to the introductory questions II and III. The Polyhedral Manifold Question naturally arose in early attempts to understand triangulations of topological manifolds (e.g. see the historical discussions in [$\mathbf{Ca_1}$, §2] and [$\mathbf{Ed_2}$, Introduction]). Given a simplicial complex K which is topologically a manifold-without-boundary (i.e., K is topologically homogeneous), it is not too hard to establish via basic algebraic topology that K has the following two properties.

(1) For any (closed) simplex σ of K, the homology groups of the link of σ in K $\left(=lk(\sigma, K) \equiv \text{the collection } \{\tau\} \text{ of all closed simplexes such that } \tau \cap \sigma = \emptyset\right.$ and the span $\tau * \sigma$ is a simplex of $\left.K\right)$ coincide with the homology groups of some sphere (in fact a sphere of dimension $\dim K - \dim \sigma - 1$), and

(2) In addition, if σ is a vertex and $\dim K > 2$, then $lk(\sigma, K)$ is simply connected.

Are these conditions sufficient to guarantee that K be a topological manifold? Yes, if $\dim K \leqslant 3$, but the situation becomes less clear in higher dimensions. The essence of this question turns out to be the:

Multiple Suspension Question. Suppose H^m is a homology m-sphere (defined below). For some $l \geqslant 2$, is it true that the lth suspension of H^m, $\Sigma^l H^m$ (which is the same as the join $S^{l-1} * H^m$), is topologically a manifold?

(If so, it is known to be a $(m+l)$-sphere, since it is necessarily covered by two coordinate patches. Hence, if the answer is yes for some l, it remains yes for any greater l.)

A *homology m-sphere* can be taken to be a topological m-manifold-without-boundary whose homology groups coincide with those of S^m. The $l=1$ case is passed over, because if H^m is not simply-connected (e.g., Poincaré's famous homology 3-sphere which has for fundamental group the 120-element binary dodecahedral group), then $\Sigma^1 H^m$ cannot possibly be a manifold at the two suspension points. The relation of this question to the preceding question is that H^m can be taken to be the link of some simplex σ in K, and $l = \dim \sigma + 1$, in which case a neighborhood of the open simplex $\mathring{\sigma}$ in K is homeomorphic to an open subset of $\Sigma^l H^m$ containing part of the suspension $(l-1)$-sphere.

An affirmative answer to the Multiple Suspension Question for any nonsimply-connected triangulated homology sphere would provide a non-combinatorial triangulation of a sphere, i.e., a triangulation which cannot be locally polyhedrally embedded in the euclidean space of the same dimension (c.f. Introductory Question II).

How does this tie in with the earlier discussion of cell-like maps? The connection is that it can be shown (without too much trouble, at least in most cases) that given any homology m-sphere H^m, and any $l \geqslant 2$, then there is a cell-like map $f: S^{m+l} \to \to \Sigma^l H^m$ from the $(m+l)$-sphere onto $\Sigma^l H^m$. Hence, in view of the earlier discussion, the Multiple Suspension Question boils down to whether this map f is approximable by homeomorphisms.

The Approximation Theorem of the preceding section provides an answer. The point is, the target space $\Sigma^l H^m$ has the disjoint disc property whenever $l \geqslant 2$ (recall without loss $m \geqslant 3$). One way to verify this is to show that given any map $f: B^2 \to \Sigma^l H^m$, there is an arbitrarily close map such that $f^{-1}(\Sigma^{l-1})$ has all of its components of arbitrarily small diameter, where Σ^{l-1} denotes the suspension $(l-1)$-sphere. Hence, if one started with two maps $f_1, f_2: B^2 \to \Sigma^l H^m$, then one could find nearby maps $g_1, g_2: B^2 \to \Sigma^l H^m$ such that $g_1(B^2) \cap g_2(B^2) \cap \Sigma^{l-1} = \emptyset$, by arranging the images of these components to be points and moving them to be disjoint. Then it is merely a matter of applying general position in the manifold $\Sigma^l H^m - \Sigma^{l-1} \approx H^m \times R^l$ to achieve that $g_1(B^2) \cap g_2(B^2) = \emptyset$.

Historically, the Multiple Suspension Question was answered affirmatively in almost all cases by the work described in [**Ed$_2$**], and it was completely settled by the subsequent work in [**Ca$_2$**]. The Approximation Theorem [**Ed$_3$**] came later. For the Multiple Suspension question itself, proofs have been improved now to the point where they are quite succinct (e.g. see [**Ca$_3$**]).

Regarding Introductory Question II, there is now a very satisfactory answer which follows as a consequence of the preceding work (excluding the unknown dimension 4).

POLYHEDRAL-TOPOLOGICAL MANIFOLD CHARACTERIZATION THEOREM. *A simplicial complex K of dimension $\neq 4$ is topologically a manifold-without-boundary if and only if conditions* (1) *and* (2) *above hold.*

Note that this theorem includes as a special case an affirmative answer to the Multiple Suspension Question, for all $l \geqslant 2$

It is worth pointing out that the related question of whether a given topological manifold is homeomorphic to some simplicial complex (i.e., the triangulation problem, in its broader form) is now known, as a consequence of the preceding work and the work of Galewski–Stern and Matumoto, to rest entirely on the question of whether certain homology 3-spheres bound acyclic 4-dimensional manifolds (see [**G–S**] or [**Ma**]). This problem is smack in the middle of an active area of research in low dimensional manifold topology.

Regarding Introductory Question III, the following corollary to the Approximation Theorem offers some insight.

COROLLARY. *Suppose an ANR X is a cell-like image of some topological manifold-without-boundary. Then $X \times R^2$ is itself a topological manifold.*

This follows by an argument of R. Daverman, who shows that $X \times R^2$ has the disjoint disc property (assuming without loss that $\dim X \geqslant 3$; observe that trivially $X \times R^5$ has the disjoint disc property, by applying general position in the R^5 coordinate). The corollary remains unresolved with R^2 replaced by R^1 (even if $\dim X \geqslant 4$, i.e., it is unknown whether $X \times R^1$ has the disjoint disc property).

Actually, with regard to Question II, there is a very natural and appealing:

Conjecture: A space X is a finite-dimensional-manifold factor $\Leftrightarrow$ X is an ENR homology manifold.

Being a homology m-manifold means that $H^*(X, X-x; \mathbf{Z}) \approx H^*(R^m, R^m - 0; \mathbf{Z})$ for each point $x \in X$. That a manifold factor X has this property is a straightforward consequence of Alexander duality.

There is further discussion of this conjecture in §12.

9. Sketch of the proof of the Approximation Theorem. The purpose of this section is to give some indication of the ideas, and their history, that go into the proof. We confine ourselves to the case where M and X are compact.

The first important point to make is that the proof uses the Bing Shrinking Criterion, which is a tool introduced by Bing almost three decades ago in [Bi$_1$] for showing that a map is approximable by homeomorphisms. We discuss it only for compact spaces. This theorem was first stated in the following form by L. McAuley.

Shrinking Theorem. *A surjective map $f: X \to Y$ of compact metric spaces is approximable by homeomorphisms $\Leftrightarrow$ the following Bing Shrinking Criterion holds. Given any $\varepsilon > 0$, there is a homeomorphism $h: X \to X$ such that*

(1) $\operatorname{dist}(fh, f) < \varepsilon$ *and*

(2) *for each* $y \in Y$, $\operatorname{diam} h(f^{-1}(y)) < \varepsilon$.

Our proof will make use of the implication $\Leftarrow$. The reverse implication is mentioned only for completeness; it is quickly proved by letting $h = g_0^{-1} g_1$ for two successively chosen homeomorphisms g_0, g_1 approximating f. Concerning the implication $\Leftarrow$, it is worth presenting here a slick baire category proof (which is not the way the proof was originally discovered and developed). In the baire space $\mathscr{C}(M, X)$ of maps from M to X, with the uniform metric topology, let $\mathscr{E}$ be the closure of the set $\{fh \mid h: M \to M$ is a homeomorphism$\}$. The Bing Shrinking Criterion amounts to saying that for any $\varepsilon > 0$, the open subset of ε-maps in $\mathscr{E}$ ($\equiv$ maps having all point-inverses of diameter $< \varepsilon$), denoted $\mathscr{E}_\varepsilon$, is dense in $\mathscr{E}$. Hence $\mathscr{E}_0 \equiv \bigcap_{\varepsilon > 0} \mathscr{E}_\varepsilon$ is dense in $\mathscr{E}$, since $\mathscr{E}$ is a baire space. Since $\mathscr{E}_0$ consists of homeomorphisms, this shows that $f \in \mathscr{E}$ is approximable by homeomorphisms.

As a consequence of this discussion we see that, in order to prove the Approximation Theorem, it suffices to construct, for any given $\varepsilon > 0$, a homeomorphism $h: M \to M$ as described in the Shrinking Theorem.

The next basic point about the proof is that it proceeds more or less by induction. The idea is to filter the target X as

$$X = X^{(m)} \supset X^{(m-1)} \supset \ldots \supset X^{(1)} \supset X^{(0)} \supset X^{(-1)} = \emptyset,$$

where each $X^{(i)}$ is σ-compact ($\equiv$ a countable union of compacta) and $\dim(X^{(i)}-X^{(i-1)})=0$ (hence $\dim X^{(i)}=i$). Such a filtration is easy to find, and is common in dimension theory arguments. J. Cannon is probably most responsible for introducing the filtration method of argument into "shrinking theory".

Given the filtration, the idea is to take the given cell-like map f, and to approximate it by successively better cell-like maps $\{f_i\}$. Each f_i will have the property that it is 1-1 *over* $X^{(i)}$ (that is, the restriction $f_i|f_i^{-1}(X^{(i)})$ is 1-1). Of course, when we reach $i=m$ we are done.

Given a map $f\colon M\to X$, the *singular set* is the set $S(f)=\cup\{x\in X|f^{-1}(x)$ contains more than one point$\}$. Observe that $S(f)$ is σ-compact, because $S(f)= =\bigcup_{\varepsilon>0}\{x\in X|\operatorname{diam} f^{-1}(x)\geqq\varepsilon\}$.

In going from $X^{(i-1)}$ to $X^{(i)}$, we make use of the following:

0-Dimensional Approximation Proposition. *Suppose $f\colon M\to Y$ is a cell-like map such that $S(f)$ is 0-dimensional (i.e., each compact subset of $S(f)$ is totally disconnected) and $S(f)$ is π_1-negligible in Y, that is, for each open set U in Y, $\pi_1(U-S(f))\to\pi_1(U)$ is an isomorphism. Then f is approximable by homeomorphisms.*

Discussion of Proof. One should consider first the simplest possible case, where $S(f)$ is a single point, say y. In this case $f^{-1}(y)$ satisfies the "cellularity criterion" (see §3), and so is cellular in M. Hence Y $(\approx M/\{f^{-1}(y)\sim\text{point}\})$ is a manifold, and f is approximable by homeomorphisms.

The model case of the Proposition, to which the general case is readily reduced, is the "countable null" case, in which $S(f)$ is countable (say $S(f)=\{y_1, y_2, \ldots\}$), and $\operatorname{diam} f^{-1}(y_i)\to 0$ as $i\to\infty$. In this case the $f^{-1}(y_i)$'s can appear to be quite tangled together in M, but in reality they are not (e.g., it turns out that the preimages of any two subsets of $S(f)$ whose closures in Y are disjoint can be separated by a locally smooth $(m-1)$-sphere in $M-f^{-1}(S(f))$. In order to show that this f is approximable by homeomorphisms it suffices, according to the Shrinking Theorem, to find a homeomorphism $h\colon M\to M$, with fh close to f, such that each $h(f^{-1}(y_i))$ has small diameter. Inasmuch as there are only finitely many $f^{-1}(y_i)$'s bigger than any given size, this at first may seem an easy matter, but the difficulty is that in shrinking small a given $f^{-1}(y_i)$, one may inadvertently stretch larger some of the nearby $f^{-1}(y_j)$'s. To find the desired homeomorphism h, we generalize a 1950's argument of Bing, who (implicitly) in **[Bi$_2$]** constructed h for the case where each $f^{-1}(y_i)$ is geometrically a cone in some coordinate patch covering it. With a little bit of work, one can show that our more general $f^{-1}(y_i)$'s have sufficiently good conelike structure (after all, they are almost contractible) so that the Bing program can be made to succeed.

In order to be able to apply this Proposition, we assume an additional condition on the $X^{(i)}$'s, namely, that $X^{(m-3)}$ is π_1-negligible in X, and similarly that $X-X^{(2)}$

is π_1-negligible in X. This is exactly the point where the disjoint disc property of X is used.

Given these tools, the proof of the Approximation Theorem can be summarized quickly as follows. (Note: this is the baire category version of the original argument, and so the Shrinking Theorem appears here only implicitly.) Let $\mathscr{C}(M, X; X_0)$ denote the space of cell-like maps from M onto X which are 1-1 over $X_0 \subset X$, provided with the uniform metric topology (recall M, X are compact). If X_0 is σ-compact, then $\mathscr{C}(M, X; X_0)$ is a G_δ (hence baire) subspace of the baire space $\mathscr{C}(M, X)$, for if X_0 is compact, the set of maps in $\mathscr{C}(M, X)$ which are ε-maps over X_0 is open in $\mathscr{C}(M, X)$.

Our goal is to show that $\mathscr{C}(M, X; X)$ (= the homeomorphisms from M to X) is dense in $\mathscr{C}(M, X; \emptyset) = \mathscr{C}(M, X)$. To achieve this, it suffices to show that each $\mathscr{C}(M, X; X^{(i)})$ is dense in $\mathscr{C}(M, X; X^{(i-1)})$. Write $X^{(i)} = \bigcup_{j=1}^{\infty} X_j^{(i)}$, where each $X_j^{(i)}$ is compact. Then $\mathscr{C}(M, X; X^{(i)}) = \bigcap_{j=1}^{\infty} \mathscr{C}(M, X; X^{(i-1)} \cup X_j^{(i)})$. By the baire property it suffices to show that for each j (and each i), $\mathscr{C}(M, X; X^{(i-1)} \cup X_j^{(i)})$ is dense in $\mathscr{C}(M, X; X^{(i-1)})$. This in turn is a straightforward application of the 0-Dimensional Approximation Proposition, thus: Given $g \in \mathscr{C}(M, X; X^{(i-1)})$, factor g as

$$g: M \underset{g_0}{\rightarrow} Y \underset{g_1}{\rightarrow} X,$$

where Y and g_i are defined by declaring that the nontrivial point-inverses of g_0 are precisely the nontrivial point-inverses of g which lie over $X_j^{(i)}$. That is, Y is the quotient space $Y = M/\{g^{-1}(x) \sim \text{point} \mid x \in X_j^{(i)}\}$. Then the quotient map $g_0: M \to Y$ has 0-dimensional singular set $S(g_0)$ which is π_1-negligible in Y, since either $g_1(S(g_0)) \subset X^{(m-3)}$ or else $g_1(S(g_0)) \subset X - X^{(2)}$. Now by the proposition g_0 is approximable by a homeomorphism, h_0 say. Then $g_1 h_0 \in \mathscr{C}(M, X; X^{(i-1)} \cup X_j^{(i)})$ and it approximates g.

10. Cell-like maps on hilbert cube manifolds. The preceding sections concentrated on cell-like maps of finite dimensional spaces. As already noted, the Approximation Problem (see §6) makes perfectly good sense even in the hilbert cube manifold setting. We repeat it here.

APPROXIMATION PROBLEM: *Suppose $f: M \to X$ is a cell-like map from a hilbert cube manifold M onto an ANR X. Find natural and useful conditions on X which guarantee that f be approximable by homeomorphisms.*

Geometric topologists have recognized for several years now, thanks largely to the work of T. Chapman, that finite dimensional manifold questions often have worthwhile hilbert cube manifold analogues. These analogues are usually more pristine and more tractable, largely because of the homogeneity of the hilbert cube ($\equiv$ for all $x, y \in I^\infty$, there exists a homeomorphism of I^∞ carrying x to y; in particular, the hilbert cube has no "boundary") and the stability of the hilbert cube ($\equiv I^\infty \times I^\infty \approx I^\infty$). A specific example of such a problem is the stabilization problem for cell-like maps (which is a special case of the Approximation Problem): If $f: M\to X$ is a cell-like map from a hilbert cube manifold onto an ANR X, is it true that the stabilized map $f\times\mathrm{id}(I^\infty): M\times I^\infty\to X\times I^\infty$ is approximable by homeomorphisms? This question gained significance after R. Miller established in 1974 **[Mi]** that for any ANR X, the product $X\times[0,\infty)$ is a cell-like image of a hilbert cube manifold (following which J. West **[We]** showed how to eliminate the $[0,\infty)$ factor). Consequently, to establish the longstanding Borsuk conjecture that an ANR crossed with the hilbert cube becomes a hilbert cube manifold, it became sufficient (and necessary, by Chapman's Approximation Theorem (§5)) to establish the above stabilization problem. This was accomplished in 1975 **[Ed$_4$]**, by making use of a Bing Shrinking Criterion argument (theretofore unexploited in infinite dimensional topology).

The following year H. Toruńczyk extended this work in striking fashion, to provide an attractive answer to the Approximation Problem. Completely independently of Cannon, Toruńczyk hit upon the disjoint cells property: A space X has the *disjoint cells property* if, given any two maps from an n-cell (n arbitrary) into X, there are two arbitrarily close maps having disjoint images. (For ANR's, this property has many interesting equivalent formulations, e.g., there exist two maps $i, j: X\to X$, each arbitrarily close to $\mathrm{id}(X)$, such that $i(X)\cap j(X)=\emptyset$.) Clearly for X to be a hilbert cube manifold this is a necessary condition. Toruńczyk established its sufficiency, again using a Bing Shrinking Criterion argument.

APPROXIMATION THEOREM (H. Toruńczyk **[To$_1$]**). *Suppose $f: M\to X$ is a cell-like map from a hilbert cube manifold M onto an ANR X. Then f is approximable by homeomorphisms $\Leftrightarrow$ X has the disjoint cells property.*

In light of the Miller–West theorem, one can drop the map f from the theorem, and assert the following.

I^∞-MANIFOLD CHARACTERIZATION THEOREM (H. Toruńczyk **[To$_1$]**). *An ANR X is a hilbert cube manifold $\Leftrightarrow$ X has the disjoint cells property.*

The significance of this theorem is in its applications, one of which is a satisfying proof of the following old conjecture.

COROLLARY (Schori–West **[S–W]**, Curtis–Schori **[C–S]**). *Suppose X is a metric continuum ($\equiv$ compact and connected). Let 2^X [resp. $C(X)$] denote the space, provided with the hausdorff metric, of all closed [resp. closed and connected] subsets of X. Then*

(1) *2^X is homeomorphic to the hilbert cube $\Leftrightarrow$ X is locally connected, that is, X is a peano continuum, and*

(2) *$C(X)$ is homeomorphic to the hilbert cube $\Leftrightarrow$ X is a nondegenerate peano continuum and X contains no free arcs.*

The classical case of part (1) of this conjecture, solved by Schori–West, is the case $X=I$. It is a pleasant exercise to verify that 2^I has the disjoint cells property.

11. Analogues in hilbert space topology. In this section we mention briefly some very recent additional work of H. Toruńczyk [To$_2$], which grew out of his preceding work and some earlier work of his on non-locally-compact ANR's. In this section, all spaces are (possibly non-locally-compact) separable complete metric spaces.

The appropriate model manifold in the non-locally-compact infinite dimensional setting is hilbert space l_2 (here we use only its topological structure and so we could as well use its homeomorph $R^\infty \equiv \times_\infty R^1$, as established by R. D. Anderson). Toruńczyk found that in hilbert space topology the appropriate analogue of the disjoint cells property is the following: A space X has the *discrete cells property* if, given any map $f: D \to X$, where $D = +_{n=0}^{\infty} D^n$ is the disjoint union of n-cells $(0 \leq n < \infty)$, then there is an arbitrarily (uniformly) close map $g: D \to X$ so that the images of the D^n's comprise a disjoint, discrete (hence closed) collection of compacta in X. Using this, one has the

APPROXIMATION THEOREM (Toruńczyk). *Suppose $f: M \to X$ is a map from a hilbert space manifold M onto an ANR X. Then f is approximable by homeomorphisms $\Leftrightarrow$ f is a fine homotopy equivalence, and X has the discrete cells property.*

Here "approximable by homeomorphisms" means that given any $\varepsilon: X \to (0, \infty)$, there exists a homeomorphism $h: M \to X$ such that for all $z \in M$, $\operatorname{dist}(f(z), h(z)) < \varepsilon(f(z))$. The phrase "$f$ is a fine homotopy equivalence" means that given any $\varepsilon: X \to (0, \infty)$, there exists a map $g: X \to M$ such that fg is homotopic to $\operatorname{id}(X)$ by a homotopy whose motion is limited by ε, and similarly gf is homotopic to $\operatorname{id}(M)$ by a homotopy whose motion is limited under f by ε. We note that in this theorem f is not assumed to be any special kind of map (e.g. neither proper, nor closed, nor even a quotient map), so that for example the theorem applies to the (known) case of the projection map $l_2 \times l_2 \to l_2$. The above theorem is proved via a Bing shrinking argument.

Combining the above with his earlier proof that an ANR crossed with hilbert space becomes a hilbert space manifold, Toruńczyk obtained the impressive

HILBERT SPACE MANIFOLD CHARACTERIZATION THEOREM (Toruńczyk). *An ANR X is a hilbert space manifold $\Leftrightarrow$ X has the discrete cells property.*

An example of an interesting corollary of this theorem is the following.

COROLLARY. *A countably infinite product of AR's, infinitely many of which are noncompact, is homeomorphic to hilbert space l_2.*

Recall an AR (absolute retract) is nothing more than a contractible ANR. Toruńczyk extended these results to frechet manifolds of higher weights, too.

12. Characterizing topological manifolds. How does one characterize a finite dimensional topological manifold? (compare the nice infinite dimensional characterizations in §§10, 11). This is the kind of question to which there can be many useful answers. One conjecture in particular that is appealing, and ties in very strongly with the material in this article, has been enunciated by J. Cannon.

MANIFOLD CHARACTERIZATION CONJECTURE. Fix $5 \leq m < \infty$. A space X is an m-manifold-without-boundary $\Leftrightarrow X$ is an ENR homology m-manifold having the disjoint disc property.

This conjecture is a bit stronger than the one at the end of §8 (see the definitions there). Again, the forward implication above is well known. Interestingly, an affirmative solution of this conjecture was announced recently by F. Quinn.

13. Dimensions 3 **and** 4. Almost all of the results above exclude dimension 4, and many exclude dimension 3 as well. This is in part due to the lack of an appropriate analogue of the disjoint disc property in these dimensions, and also in part due to our ignorance of the topology of manifolds in these dimensions (particularly dimension 4). What is a good conjecture to make for the Approximation Problem, in either dimensions 3 or 4?

Dimension 4 is particularly bewildering. The difficulties there tie in with the difficulties already encountered by smooth manifold topologists working in that dimension on handlebody-structure related problems (e.g. surgery and s-cobordism theorems). As an example, it is not even known whether a cell-like surjection $f: M^4 \to N^4$ of closed 4-manifolds having exactly one non-trivial point inverse is approximable by homeomorphisms (this amounts to asking whether the nontrivial point-inverse is cellular in M^4). Or whether the cellularity criterion (§3) works in dimension 4. Questions such as these are in need of answers.

References

Some other articles which amplify the material of this article, and provide additional details, history and references, are [Ar_1], [Ca_1], [La_2], [McA, §§11, 12, 13]; [$\text{Ca}_{2,3}$], [$\text{Ed}_{2,3}$], [$\text{To}_{1,2}$].

[An] R. D. Anderson, *Homeomorphisms on infinite-dimensional manifolds,* Actes, Congrès International des Mathématiciens, Nice 1970, Tome 2, 13—18.

[Ar_1] S. Armentrout, *Monotone decompositions of E^3*, Topology Seminar (Wisconsin, 1965), Ann. of Math. Studies, no. 60, Princeton Univ. Press, Princeton, N. J., 1966. pp. 1-25.

[Ar_2] ——, *Cellular decompositions of 3-manifolds that yield 3-manifolds,* Mem. Amer. Math. Soc. **107**, (1971).

[Bi_1] R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres,* Ann. of Math. **56** (1952), 354—362.

[Bi_2] ——, *Upper semicontinuous decompositions of E^3*, Ann. of Math. **65** (1957), 363—374.

[Bi_3] ——, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math. **65** (1957), 484—500.

[Bi_4] ——, *The cartesian product of a certain non-manifold and a line is E^4*, Ann. of Math. **70** (1959), 399—412.

[Br$_1$] M. Brown, *A proof of the generalized Schoenflies theorem,* Bull. Amer. Math. Soc. **66** (1960) 74—76.

[Br$_2$] M. Brown, *The monotone union of open n-cells is an open n-cell,* Proc. Amer. Math. Soc. **12** (1961), 812—814.

[Ca$_1$] J. W. Cannon, *The recognition problem: What is a topological manifold?* Bull. Amer. Math. Soc. **84** (1978), 832—866.

[Ca$_2$] J. W. Cannon, *Shrinking cell-like decompositions of manifolds: codimension three,* Ann. of Math. **110** (1979), 83—112.

[Ca$_3$] *The characterization of topological manifolds of dimension $n \geq 5$,* these Proceedings Helsinki, 1978.

[Ch$_1$] T. Chapman, *Cell-like mappings of Hilbert cube manifolds: applications to simple homotopy theory,* Bull. Amer. Math. Soc. **79** (1973), 1286—1291.

[Ch$_2$] *Topological classification of simple homotopy equivalences,* Proceedings of the International Congress of Mathematicians, Vancouver, 1974, Vol. 1, 509—513.

[C-S] D. W. Curtis and R. M. Schori, *Hyperspaces of polyhedra are Hilbert cubes,* Fund. Math. **99** (1978), 189—197. Announceg in Bull. Amer. Math. Soc. **80** (1974), 927—931.

[Ed$_1$] R. D. Edwards, *Demension theory,* Geometric Topology (Proceedings of the Park City Conference), L. C. Glaser and T. B. Rushing, eds., Lecture Notes in Math., no. 438, Springer-Verlag, Berlin and New York, 1975.

[Ed$_2$] *Suspensions of homology spheres,* Annals of Math., (to appear.)

[Ed$_3$] *Approximating certain cell-like maps by homeomorphisms,* preprint.

[Ed$_4$] *The ANR Theorem,* Chapter 14 in T. A. Chapman, Lectures on Hilbert Cube Manifolds, Regional Conference Series in Mathematics no 28, Amer. Math. Soc. 1976.

[Ed$_5$] *On the relation between cohomological dimension and covering dimension,* preprint.

[F-A] R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space,* Ann. of Math. **49** (1948), 979—990.

[G-S] D. E. Galewski and R. J. Stern, *Classification of simplicial triangulations of topological manifolds,* Ann. of Math. (to appear). (Announced in Bull. Amer. Math. Soc. **82** (1976), 916—918.)

[Hu] S.-T. Hu, *Theory of retracts,* Wayne State Univ. Press, 1965.

[La$_1$] R. C. Lacher, *Cell-like mappings,* I and II, Pacific J. Math. **30** (1969), 717—731; **35** (1970), 649—660.

[La$_2$] R. C. Lacher, *Cell-like mappings and their generalizations,* Bull. Amer. Math. Soc. **83** (1977), 495—552.

[Ma] T. Matumoto, *Variétés simpliciales d'homologie et variétés topologiques métrisables,* Thesis, University of Paris, Orsay, 1976.

[McA] L. F. McAuley, ed. *Some fundamental theorems and problems related to monotone mappings,* Proc. First Conf. on Monotone Mappings and Open Mappings, SUNY at Binghamton, Binghamton, N. Y., 1971. pp. 1—36.

[McM] D. R. Mc Millan, Jr., *A criterion for cellularity in a manifold,* Ann. of Math. **79** (1964), 327—337.

[Mi] R. T. Miller *Mapping cylinder neighborhoods of some ANR's,* Ann. of Math. **103** (1976), 417—427. Announced in Bull. Amer. Math. Soc. **79** (1973), 1286—1291.

[Mo] R. L. Moore, *Concerning upper semi-continuous collections of continua,* Trans. Amer. Math Soc. **27** (1925), 416—428.

[R-S] J. H. Roberts and N. E. Steenrod, *Monotone transformations of 2-dimensional manifolds* Ann. of Math. **39** (1938), 851—862.

[S-W] R. M. Schori and J. E. West, *The hyperspace of the closed unit interval is a Hilbert cube* Trans. Amer. Math. Soc. **213** (1975), 217—235. Announced in Bull. Amer. Math. Soc. **81** (1975), 187—188.

[Si] L. C. Siebenmann, *Approximating cellular maps by homeomorphisms,* Topology **11** (1973), 271—294.

[Ta] J. L. Taylor, *A counterexample in shape theory*, Bull. Amer. Math. Soc. **81** (1975), 629—632.

[To$_1$] H. Toruńczyk, *On CE-images of the Hilbert cube and characterization of Q-manifolds*, preprint.

[To$_2$] ——, *Characterizing Hilbert space topology*, Institute of Mathematics, Polish Academy of Sciences, Preprint 143, 1978.

[We$_1$] J. E. West, *Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk*, Ann. of Math. **106** (1977), 1—18. Announced in Bull. Amer. Math. Soc. **81** (1975), 163—165.

[We$_2$] J. E. West, *Hilbert cube manifolds—meeting ground of geometric topology and absolute neighborhood retracts*, these Proceedings, Helsinki, 1978.

[Yo] J. W. T. Youngs, *Homeomorphic approximations to monotone mappings*, Duke Math. J. **15** (1948), 87—94.

University of California at Los Angeles
Los Angeles, California 90024