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Local formulae for combinatorial Pontryagin classes

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Abstract. Let $p(|K|)$ be the characteristic class of a combinatorial manifold K given by a polynomial p in the rational Pontryagin classes of K. We prove that for any polynomial p there is a function taking each combinatorial manifold K to a cycle $z_p(K)$ in its rational simplicial chains such that: 1) the Poincaré dual of $z_p(K)$ represents the cohomology class $p(|K|)$; 2) the coefficient of each simplex Δ in the cycle $z_p(K)$ is determined solely by the combinatorial type of $\operatorname{link}\Delta$. We explicitly describe all such functions for the first Pontryagin class. We obtain estimates for the denominators of the coefficients of the simplices in the cycles $z_p(K)$.

§ 1. Introduction

The following well-known problem is studied, for instance, in [1]–[5]. Given a triangulation of a manifold, construct a simplicial cycle whose Poincaré dual represents a given Pontryagin class of this manifold. In addition, one usually wants the coefficient of each simplex in this cycle to be determined solely by the structure of the manifold in some neighbourhood of this simplex. First let us discuss the most important results concerning this problem.

Gabrielov, Gelfand and Losik [1], [2] found an explicit formula for the first rational Pontryagin class of a smooth manifold. To apply this formula, one needs the manifold to be endowed with a smooth triangulation satisfying a certain special condition. In their paper [3], Gelfand and MacPherson considered simplicial manifolds endowed with the additional combinatorial structure of a *fixing cycle*. A fixing cycle is a combinatorial analogue of a smooth structure and can be induced by a given smooth structure. For simplicial manifolds with a given fixing cycle, Gelfand and MacPherson constructed rational cycles whose Poincaré duals represent the Pontryagin classes of the manifolds. The coefficients of the simplices in these cycles depend both on the combinatorial structure of a neighbourhood of the simplex and on the restriction of the fixing cycle to this neighbourhood.

Another approach is due to Cheeger. In [4] he obtained explicit formulae for cycles whose Poincaré duals represent real Pontryagin classes. These formulae involve the calculation of the spectra of Laplace operators on pseudo-manifolds

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with a locally flat metric. The coefficients of the simplices in the cycles obtained depend only on the combinatorial type of the link of the simplex. Cheeger's formula can be applied to any pseudo-manifold. It is not known whether the cycles obtained are rational.

Suppose that for any combinatorial manifold K we have a $(\dim K - n)$-dimensional cycle $z(K)$ in its co-oriented simplicial chains (see the definition in § 2.1) such that

$$z(K) = \sum_{\Delta \in K,\, \dim \Delta = \dim K - n} f(\operatorname{link} \Delta)\Delta,$$

where the value $f(L)$ is determined solely by the isomorphism class of the oriented $(n-1)$-dimensional PL sphere L and the function f does not depend on the manifold K. Then we say that z is a *characteristic local cycle* (*c. l. cycle*) of codimension n. The function f is called the *local formula* for this c. l. cycle. We prove that for any rational characteristic class $p \in H^*(\mathrm{BPL}; \mathbb{Q}) = H^*(\mathrm{BO}; \mathbb{Q})$ there is a rational c. l. cycle z_p such that the Poincaré dual of the cycle $z_p(K)$ represents the cohomology class $p(|K|)$ for any combinatorial manifold K. (Here BPL is the classifying space for stable PL bundles.) This improves a theorem of Levitt and Rourke [5]. They obtained a similar result for cycles given by $\sum h(\operatorname{link} \Delta, \dim K)\Delta$, which are not c. l. cycles since the function h depends on the dimension of K. They also proved that for any characteristic class $p \in H^*(\mathrm{BPL}; \mathbb{Z})$ there is a function taking each m-dimensional combinatorial manifold K with a given ordering of the vertices to a cycle $\sum g(\operatorname{star} \Delta, \operatorname{ord})$ whose Poincaré dual represents the cohomology class $p(|K|)$. Here g is a function on the isomorphism classes of oriented ordered triangulations of the m-dimensional disc. The function g does not depend on the given manifold K.

In § 2 we define a cochain complex $\mathcal{T}^*(\mathbb{Q})$ whose elements are functions on the set of isomorphism classes of oriented PL spheres. We prove that a function $f \in \mathcal{T}^*(\mathbb{Q})$ is a cocycle if and only if f is a local formula for some c. l. cycle. If f is a coboundary, then f is a local formula for a c. l. cycle z such that $z(K)$ is a boundary for any combinatorial manifold K. We prove that there is an isomorphism $H^*(\mathcal{T}^*(\mathbb{Q})) \cong H^*(\mathrm{BO}; \mathbb{Q})$. In particular, we see that any rational c. l. cycle represents homology classes dual to some polynomial in Pontryagin classes of manifolds. In § 2.5 we prove that for any $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$ there is a local formula f representing ψ such that the function $f : \mathcal{T}_n \to \mathbb{Q}$ is algorithmically computable.

In § 3 we obtain an explicit formula describing all rational c. l. cycles z such that the Poincaré dual of $z(K)$ represents the first Pontryagin class of a combinatorial manifold K. This result is new because the formulae of [1]–[3] cannot be applied to an arbitrary combinatorial manifold and the formulae of [4] give only *real* c. l. cycles. We use the following approach. First we explicitly find all rational c. l. cycles of codimension 4. Then we notice that any rational c. l. cycle of codimension 4 represents the homology classes dual to the first Pontryagin class multiplied by some rational constant. The use of bistellar moves is very important.

In § 4 we study the denominators of the coefficients of c. l. cycles, that is, the denominators of the values of the local formulae. We estimate these denominators via the number of vertices of the PL sphere. We prove that if f is a local formula for the first Pontryagin class and q is an integer, then there is a PL sphere L

such that the denominator of $f(L)$ is divisible by q. In particular, there are no integer c. l. cycles representing homology classes dual to the first Pontryagin class multiplied by some non-zero integer.

The proof of the isomorphism $H^*(\mathcal{T}^*(\mathbb{Q})) \cong H^*(\mathrm{BO};\mathbb{Q})$ is based on the results of § 5, where we study decompositions of polyhedra into simple cells. A $\mathcal{D}$-structure on a polyhedron is defined as an equivalence class of decompositions of this polyhedron into simple cells with respect to some equivalence relation. We show that $\mathcal{D}$-structures have many properties similar to those of stable bundles. We construct the classifying space $\mathcal{Z}$ for $\mathcal{D}$-structures, which is an analogue of the classifying space BO for stable vector bundles. We also construct a natural map $\mathcal{X}$ taking any block bundle to a $\mathcal{D}$-structure. We prove that the corresponding map $\mathcal{X}\colon \mathrm{BPL} \to \mathcal{Z}$ induces an isomorphism in rational cohomology. This isomorphism is very important for the proof of the main result.

All necessary definitions and results of PL topology can be found in [6]. All manifolds, triangulations, maps, homeomorphisms, and bordisms are supposed to be piecewise linear unless otherwise stated. All bordisms are supposed to be oriented. A PL sphere is a simplicial complex whose geometric realization is PL homeomorphic to the boundary of a simplex. An *m-dimensional combinatorial manifold* is a simplicial complex K such that the link of each vertex $v \in K$ is an $(m-1)$-dimensional PL sphere. Any piecewise-linear triangulation of a manifold is a combinatorial manifold. In §§ 2–4 all manifolds are supposed to be closed. An isomorphism of oriented simplicial complexes is an orientation-preserving isomorphism. An orientation-reversing isomorphism is called an *anti-isomorphism*. Let K be a simplicial complex on a set S. We denote the cone over K by CK. The *full subcomplex* spanned by a subset $V \subset S$ is the subcomplex $L \subset K$ consisting of all simplices $\Delta \in K$ such that all vertices of Δ belong to V. We denote the join of two simplicial complexes K and L by $K * L$.

§ 2. Local formulae

2.1. Main definitions. Let $\mathcal{T}_n$ be the set of all isomorphism classes of oriented $(n-1)$-dimensional PL spheres. (We assume that $\mathcal{T}_0 = \{\varnothing\}$, $\mathcal{T}_{-n} = \varnothing$, $n > 0$.) Usually we do not distinguish between a PL sphere and its isomorphism class. For any $L \in \mathcal{T}_n$ we denote by $-L$ the PL sphere L with the opposite orientation. We say that $L \in \mathcal{T}_n$ is *symmetric* if there is an anti-automorphism of L. Let G be an Abelian group. We denote by $\mathcal{T}^n(G)$ the Abelian group of all functions $f : \mathcal{T}_n \to G$ such that $f(L) = f(-L)$ for every $L \in \mathcal{T}_n$. We assume that $\mathcal{T}^0(G) = G$, $\mathcal{T}^{-n}(G) = 0$, $n > 0$. Let the differential $\delta : \mathcal{T}^n(G) \to \mathcal{T}^{n+1}(G)$ be given by

$$(\delta f)(L) = \sum f(\operatorname{link} v),$$

where the sum is taken over all vertices $v \in L$ and the orientation of $\operatorname{link} v$ is induced by the orientation of L. Evidently, $\delta^2 = 0$. Thus $\mathcal{T}^*(G)$ is a cochain complex.

We denote by $\mathcal{T}_n(\mathbb{Z})$ the Abelian group generated by the set $\mathcal{T}_n$ with relations $L + (-L) = 0$. The boundary operator $\partial\colon \mathcal{T}_{n+1}(\mathbb{Z}) \to \mathcal{T}_n(\mathbb{Z})$ is given by its values on the generators of $\mathcal{T}_{n+1}(\mathbb{Z})$:

$$\partial L = \sum \operatorname{link} v,$$

where the sum is taken over all vertices $v \in L$. Then $\mathcal{T}_*(\mathbb{Z})$ is a chain complex. We have $\mathcal{T}^n(G) = \mathrm{Hom}(\mathcal{T}_n(\mathbb{Z}), G)$ and $(\delta f)(x) = f(\partial x)$ for any $f \in \mathcal{T}^n(G)$, $x \in \mathcal{T}_{n+1}(\mathbb{Z})$. In the same way, we can define the chain complex $\mathcal{T}_*(G)$ for any Abelian group G, but in what follows we shall use only the chain complex $\mathcal{T}_*(\mathbb{Z})$.

Let K be an m-dimensional combinatorial manifold. Let $\widehat{G}$ be the local system on $|K|$ with fibre G and twisting given by the orientation. A *co-orientation of a simplex* $\Delta^n \in K$ is an orientation of $\mathrm{link}\,\Delta^n$. Any m-simplex is supposed to be positively co-oriented. Let $\widehat{C}_n(K;G)$ be the complex of co-oriented simplicial chains of K and let $\widehat{\partial}$ be the boundary operator of this complex. (The incidence coefficient of two co-oriented simplices $\tau^{k-1} \subset \sigma^k$ is equal to $+1$ if the orientation of $\mathrm{link}\,\sigma^k$ is induced by that of $\mathrm{link}\,\tau^{k-1}$. Otherwise it is equal to -1.) The homology of $\widehat{C}_*(K;G)$ is equal to $H_*(|K|;\widehat{G})$. If K is oriented, then we have the augmentation $\varepsilon : \widehat{C}_0(K;G) \to G$.

Suppose that $f \in \mathcal{T}^n(G)$. Let $f_\sharp(K) \in \widehat{C}_{m-n}(K;G)$ be the co-oriented chain given by

$$f_\sharp(K) = \sum_{\Delta^{m-n} \in K} f(\mathrm{link}\,\Delta^{m-n})\Delta^{m-n}.$$

(The summand $f(\mathrm{link}\,\Delta^{m-n})\Delta^{m-n}$ does not depend on the co-orientation of Δ^{m-n}.) We say that $f \in \mathcal{T}^n(G)$ is a *local formula* if the co-oriented chain $f_\sharp(K)$ is a cycle for any combinatorial manifold K. The correspondence $f \mapsto f_\sharp$ obviously provides an isomorphism between the group of local formulae and the group of c. l. cycles.

Proposition 2.1. 1) *f is a local formula if and only if f is a cocycle in the cochain complex $\mathcal{T}^*(G)$.*

2) *If f is a coboundary in $\mathcal{T}^*(G)$, then the cycle $f_\sharp(K)$ is a boundary for any combinatorial manifold K.*

3) *Let K_1 and K_2 be two triangulations of a manifold M^m. If f is a local formula, then the cycles $f_\sharp(K_1)$ and $f_\sharp(K_2)$ are homologous.*

Proof. Notice that $\widehat{\partial} f_\sharp(K) = (\delta f)_\sharp(K)$ for any $f \in \mathcal{T}^n(G)$. This proves the second assertion of the proposition. It also follows that if f is a cocycle, then f is a local formula.

Suppose that $\delta f \neq 0$. Then there is an $L \in \mathcal{T}_{n+1}$ such that $(\delta f)(L) \neq 0$. Consider a combinatorial manifold K such that $\mathrm{link}\,\Delta \cong L$ for some simplex $\Delta \in K$. Then the coefficient of Δ in the chain $\widehat{\partial} f_\sharp(K) = (\delta f)_\sharp(K)$ is non-zero. Hence $f_\sharp(K)$ is not a cycle. Therefore f is not a local formula.

Let us now prove the third assertion of the proposition. The stellar subdivision of a simplex $\Delta \in K$ is the operation transforming K into the simplicial complex

$$(K \setminus (\Delta * \mathrm{link}\,\Delta)) \cup ((C\partial\Delta) * \mathrm{link}\,\Delta).$$

Any two triangulations of a compact polyhedron can be transformed to each other by a finite sequence of stellar subdivisions and inverse stellar subdivisions (see [7]). We can assume without loss of generality that K_1 can be transformed into K_2 by a stellar subdivision of some simplex Δ. Then the support of the cycle $f_\sharp(K_2) - f_\sharp(K_1)$ is contained in the subcomplex $\mathrm{star}\,\Delta$, which is contractible. If $m > n$, then the dimension of the cycle $f_\sharp(K_2) - f_\sharp(K_1)$ is positive. Hence $f_\sharp(K_2) - f_\sharp(K_1)$ is a boundary.

Assume that $n = m$. The case of an orientable manifold M^m will be considered in § 2.2. For a non-orientable manifold M^m, the assertion is proved by taking the two-sheeted orientable covering of M^m.

Given any cohomology class $\psi \in H^n(\mathcal{T}^*(G))$ and any manifold M^m, we denote by $\psi_\sharp(M^m) \in H_{m-n}(M^m; \widehat{G})$ the homology class represented by $f_\sharp(K)$, where K is an arbitrary triangulation of M^m and f is an arbitrary representative of ψ. The homology class $\psi_\sharp(M^m)$ is well defined by Proposition 2.1. We denote the Poincaré dual of $\psi_\sharp(M^m)$ by $\psi^\sharp(M^m) \in H^n(M^m; G)$. If M^m is oriented and $m = n$, then ψ determines an element $\psi^\star(M^n)$ of G by $\psi^\star(M^n) = \langle \psi^\sharp(M^n), [M^n] \rangle$.

The main result of this section is the following theorem.

Theorem 2.1. *For any rational characteristic class $p \in H^n(\mathrm{BPL}; \mathbb{Q})$ there is a unique cohomology class $\varphi_p \in H^n(\mathcal{T}^*(\mathbb{Q}))$ such that $\varphi_p^\sharp(M) = p(M)$ for any manifold M. For any cohomology class $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$ there is a rational characteristic class p such that $\psi = \varphi_p$. Thus,*

$$H^*(\mathcal{T}^*(\mathbb{Q})) \cong H^*(\mathrm{BPL}; \mathbb{Q}) = H^*(\mathrm{BO}; \mathbb{Q}).$$

Corollary 2.1. *There is an additive isomorphism $H^*(\mathcal{T}^*(\mathbb{Q})) \cong \mathbb{Q}[p_1, p_2, \dots]$, $\deg p_j = 4j$.*

By Theorem 2.1, there is a one-to-one correspondence between rational c. l. cycles z such that $z(K)$ is a boundary for any combinatorial manifold K and coboundaries of the complex $\mathcal{T}^*(\mathbb{Q})$.

2.2. Invariance under bordisms.

Proposition 2.2. *Suppose that $\psi \in H^n(\mathcal{T}^*(G))$. Then $\psi^\star(M_1^n) = \psi^\star(M_2^n)$ for any bordant oriented manifolds M_1^n and M_2^n.*

Thus we have a well-defined homomorphism $\star : H^n(\mathcal{T}^*(G)) \to \mathrm{Hom}(\Omega_n, G)$ taking the cohomology class ψ to the homomorphism $\psi^\star$. Here Ω_n is the group of oriented n-dimensional PL bordisms of a point. There is a canonical isomorphism $\mathrm{Hom}(\Omega_n, \mathbb{Q}) \cong H^n(\mathrm{BPL}; \mathbb{Q})$. Hence the homomorphism $\star$ induces a homomorphism $\sharp\colon H^n(\mathcal{T}^*(\mathbb{Q})) \to H^n(\mathrm{BPL}; \mathbb{Q})$.

Suppose that $f \in \mathcal{T}^n(G)$ is a local formula and L is an n-dimensional null-bordant oriented combinatorial manifold. Let us prove that $\varepsilon(f_\sharp(L)) = 0$, where $\varepsilon\colon \widehat{C}_0(L; G) \to G$ is the augmentation. This will prove Proposition 2.2 and the remaining case of Proposition 2.1.

Let K be an oriented combinatorial manifold with boundary such that $\partial K = L$. Suppose that u is a co-oriented vertex of L. Then $\mathrm{link}_K u$ is an oriented triangulation of an n-disc. The co-orientation of the vertex u in K induces the co-orientation of u in L. Hence we have a monomorphism $i\colon \widehat{C}_0(L; G) \to \widehat{C}_0(K; G)$. Obviously, $\partial \,\mathrm{link}_K u = \mathrm{link}_L u$. We denote by $\mathrm{link}_K^* u$ the simplicial complex $\mathrm{link}_K u \cup_{\mathrm{link}_L u} C(\mathrm{link}_L u)$ whose orientation is induced by the orientation of $\mathrm{link}_K u$. Then $\mathrm{link}_K^* u \in \mathcal{T}_{n+1}$. We similarly define a PL sphere $\mathrm{link}_K^* e \in \mathcal{T}_n$ for any co-oriented edge $e \in L$.

We define a 1-chain $f_\sharp(K) \in \widehat{C}_1(K;G)$ by

$$f_\sharp(K) = \sum_{e\in K\setminus L} f(\mathrm{link}_K\, e)e + \sum_{e\in L} f(\mathrm{link}^*_K\, e)e.$$

The support of the chain $\partial f_\sharp(K)$ is obviously contained in the subcomplex L. For any co-oriented vertex $v \in L$ we have $(\delta f)(\mathrm{link}^*_K\, v) = 0$. Therefore,

$$\sum f(\mathrm{link}_K\, e) + \sum f(\mathrm{link}^*_K\, e) = f(\mathrm{link}_L\, v),$$

where the first sum is taken over all co-oriented edges $e \in K\setminus L$ entering the vertex v and the second over all co-oriented edges $e \in L$ entering the vertex v. (We say that a co-oriented edge e enters a co-oriented vertex v if the incidence coefficient of the pair (e, v) is equal to $+1$.) Consequently, $\widehat{\partial} f_\sharp(K) = i(f_\sharp(L))$. Hence, $\varepsilon(f_\sharp(L)) = 0$.

Remark 2.1. The formula $\widehat{\partial} f_\sharp(K) = i(f_\sharp(\partial K))$ actually holds for combinatorial manifolds K of any dimension $m \geqslant n + 1$.

2.3. $\star$ is an epimorphism for rational coefficients.

Theorem 2.2. *The homomorphism* $\star\colon H^n(\mathcal{T}^*(\mathbb{Q})) \to \mathrm{Hom}(\Omega_n, \mathbb{Q})$ *is an isomorphism.*

Corollary 2.2. *The homomorphism* $\sharp\colon H^n(\mathcal{T}^*(\mathbb{Q})) \to H^n(\mathrm{BPL};\mathbb{Q})$ *is an isomorphism.*

In § 5.8 we shall prove that $\star$ is a monomorphism.

To prove that $\star$ is an epimorphism, we need some of the definitions and results of [5]. A *totally ordered simplicial complex* is a simplicial complex together with a total ordering on its set of vertices. A *locally ordered simplicial complex* is a simplicial complex together with a partial ordering on its set of vertices such that the star of each vertex is a totally ordered complex. Let $\mathrm{B}\widetilde{\mathrm{PL}}_m$ be the classifying space for m-dimensional block bundles (see the definition of a block bundle in § 5.7 and [8] for more details). A cohomology class $p \in H^n(\mathrm{B}\widetilde{\mathrm{PL}}_m; G)$ is called a *characteristic class for block bundles.* Given a cohomology class $p \in H^n(\mathrm{B}\widetilde{\mathrm{PL}}_m; G)$, we have a cohomology class $p(\xi) \in H^n(P; G)$ for any m-dimensional block bundle ξ over a polyhedron P. In particular, for any manifold M^m we have the cohomology class $p(M^m) = p(\tau) \in H^n(M^m; G)$, where τ is the tangent block bundle of the manifold M^m.

In their paper [5], Levitt and Rourke studied local formulae for characteristic classes of locally ordered combinatorial manifolds. They proved the following assertion. For any $p \in H^n(\mathrm{B}\widetilde{\mathrm{PL}}_m; G)$, $n \leqslant m$, there is a function g taking each isomorphism class of totally ordered oriented triangulations of an m-disc to an element of the group G such that the cycle $\sum_{\Delta^{m-n}\in K} g(\mathrm{star}\, \Delta^{m-n})\Delta^{m-n}$ represents the Poincaré dual of the cohomology class $p(|K|)$ for any locally ordered oriented combinatorial manifold K.

Consider the case $G = \mathbb{Q}$. Let J be an unordered triangulation of an m-disc. We denote by $h(J)$ the arithmetic mean of the values of g over all different total orderings of J. Levitt and Rourke also proved that, for any (unordered) combinatorial manifold K, the cycle $\sum_{\Delta^{m-n}\in K} h(\operatorname{star}\Delta^{m-n})\Delta^{m-n}$ represents the Poincaré dual of the cohomology class $p(|K|)$.

Let us now prove that $\star$ is an epimorphism. For any n there is a canonical isomorphism

$$\operatorname{Hom}(\Omega_n,\mathbb{Q})\cong H^n(\mathrm{B}\widetilde{\mathrm{PL}}_n;\mathbb{Q}).$$

Suppose that $p\in\operatorname{Hom}(\Omega_n,\mathbb{Q})$. We also denote the corresponding characteristic class of block bundles by p. Now put $m=n$. Let the function $f_1:\mathcal{T}_n\to\mathbb{Q}$ be given by $f_1(L)=h(CL)$, where h is as above. We have

$$\sum_{v\in K} f_1(\operatorname{link} v)=p(|K|)$$

for any n-dimensional oriented combinatorial manifold K. It is possible that $f_1(-L)\neq -f_1(L)$ for some $L\in\mathcal{T}_n$. Therefore it is possible that $f_1\notin\mathcal{T}^n(\mathbb{Q})$. Let the function $f\in\mathcal{T}^n(\mathbb{Q})$ be given by

$$f(L)=\frac{f_1(L)-f_1(-L)}{2}\,.$$

Then $\sum_{v\in K} f(\operatorname{link} v)=p(|K|)$ for any n-dimensional oriented combinatorial manifold K. Obviously, $p(S^n)=0$. Hence $\sum_{v\in K} f(\operatorname{link} v)=0$ for any PL sphere $K\in\mathcal{T}_{n+1}$. Therefore $\delta f=0$, that is, f is a local formula. Let ψ be the cohomology class represented by f. Then $\psi^\star(|K|)=p(|K|)$ for any n-dimensional oriented combinatorial manifold K. Consequently, $\star(\psi)=p$.

The approach of [5] is based on an explicit realization of the classifying space $\mathrm{B}\widetilde{\mathrm{PL}}_n$ as a CW complex. We want to prove that $\star$ is an epimorphism in a more elementary way without using the results of [5] and hence without using the explicit cell decomposition of $\mathrm{B}\widetilde{\mathrm{PL}}_n$.

Let K be an n-dimensional oriented combinatorial manifold. We say that K is *balanced* if, for any $L\in\mathcal{T}_n$, the number of vertices $v\in K$ such that $\operatorname{link} v$ is isomorphic to L is equal to the number for which it is anti-isomorphic to L, that is, if $2\sum_{v\in K}\operatorname{link} v=0$ in the group $\mathcal{T}_n(\mathbb{Z})$. For example, the disjoint union of two anti-isomorphic oriented combinatorial manifolds is balanced.

Proposition 2.3. *Suppose that K is a balanced oriented combinatorial manifold. Then a disjoint union of several copies of K is null bordant.*

Proposition 2.3 is an obvious consequence of Theorem 2.2. In § 5.10 we shall prove this proposition using neither Theorem 2.2 nor the results of [5]. Let us now use Proposition 2.3 to show that $\star$ is an epimorphism. We must prove that for any homomorphism $p\in\operatorname{Hom}(\Omega_n,\mathbb{Q})$ there is a local formula $f\in\mathcal{T}^n(\mathbb{Q})$ such that

$$\sum_{v\in K} f(\operatorname{link} v)=p(|K|)$$

for every n-dimensional oriented combinatorial manifold K.

Let $\{L_i\}$ be a sequence of PL spheres in the set $\mathcal{T}_n$ with the following properties.

1) L_i is non-symmetric for every i.

2) If $L \in \mathcal{T}_n$ is non-symmetric, then there is i such that L is either isomorphic or anti-isomorphic to L_i.

3) If $i \neq j$, then L_i is neither isomorphic nor anti-isomorphic to L_j.

The condition $\sum_{v \in K} f(\operatorname{link} v) = p(|K|)$ can be regarded as a linear equation E_K in the values $f(L_i)$. It is sufficient to prove that the system of all equations E_K is consistent. Assume the opposite. Then there are combinatorial manifolds K_j and integers k_j, $j = 1, 2, \dots, q$, such that the linear equation $\sum_{j=1}^{q} k_j E_{K_j}$ is inconsistent, that is, it takes the form $0 = B$, where B is a non-zero number. We reverse the orientation of K_j and the sign of k_j whenever k_j is negative. Thus we obtain that all the k_j are positive. Consider the manifold

$$K = \bigsqcup_{j=1}^{q} K_j^{\sqcup k_j},$$

where $K_j^{\sqcup k_j}$ is the disjoint union of k_j copies of K_j. Then the equation $\sum_{j=1}^{q} k_j E_{K_j}$ coincides with the equation E_K. Hence, E_K is inconsistent. Therefore the combinatorial manifold K is balanced and $p(|K|) \neq 0$. This contradicts Proposition 2.3.

2.4. Cohomology classes $\psi^\sharp(M^m)$ for manifolds of arbitrary dimension. Proposition 2.2 implies that for any cohomology class $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$ there is a rational characteristic class $p = \sharp(\psi)$ such that $\psi^\sharp(M^n) = p(M^n)$ for any n-dimensional manifold M^n. Let us consider the cohomology classes $\psi^\sharp(M^m)$ for manifolds of dimension $m > n$.

Proposition 2.4. *We have $\psi^\sharp(M^m) = p(M^m) = \sharp(\psi)(M^m)$ for any manifold M^m, $m \geqslant n$.*

By Corollary 2.2, $\sharp$ is an isomorphism. Hence Theorem 2.1 follows from Corollary 2.2 and Proposition 2.4.

Suppose that P is a compact polyhedron and $Q \subset P$ is a closed PL subset. We say that P is a *manifold with singularities in Q* if $P \setminus Q$ is a (non-closed) manifold. Such a manifold with singularities is said to be oriented if the manifold $P \setminus Q$ is oriented. We have the Lefschetz duality $H_{m-n}(P, Q; \widehat{G}) \cong H^n(P \setminus Q; G)$ for $n < m$.

Suppose that K is an arbitrary triangulation of P. Let $L \subset K$ be the subcomplex consisting of all closed simplices whose intersection with Q is non-empty. Let $f \in \mathcal{T}^n(G)$ be a local formula with $n < m$. Let the chain $f_\sharp(K, L) \in \widehat{C}_{m-n}(K, L; G)$ be given by $\sum f(\operatorname{link} \Delta^{m-n})\Delta^{m-n}$, where the sum is taken over all $(m-n)$-simplices $\Delta^{m-n} \in K \setminus L$. Arguing as in the proof of Proposition 2.1, one can easily show that $f_\sharp(K, L)$ is a relative cycle whose homology class in the group $H_{m-n}(P, Q; \widehat{G})$ is determined solely by the cohomology class represented by f and does not depend on the choice of the triangulation K. Thus the classes $\psi_\sharp(P, Q) \in H_{m-n}(P, Q; \widehat{G})$ and $\psi^\sharp(P, Q) \in H^n(P \setminus Q; G)$ are well defined for any cohomology class $\psi \in H^n(\mathcal{T}^*(G))$, $n < m$. Let $S \subset P \setminus Q$ be a compact subset. Evidently, the cohomology class $\psi^\sharp(P, Q)\big|_S$ is determined solely by the topology of the pair (U, S), where U is an arbitrarily small neighbourhood of S. Hence we have the following proposition.

Proposition 2.5. *Suppose that N^k is a (closed) oriented manifold, $\psi \in H^n(\mathcal{T}^*(G))$, $n \leqslant k$, and P is an m-dimensional manifold with singularities in Q, $m > k$.*

Let $i : N^k \hookrightarrow P \setminus Q$ be an embedding such that $i(N^k) \subset P \setminus Q$ is a submanifold with trivial normal bundle. We put $\psi^\sharp_m(N^k) = i^*(\psi^\sharp(P,Q))$. Then the cohomology class $\psi^\sharp_m(N^k)$ depends only on the manifold N^k and the number m and not on the choice of the triple (P,Q,i).

Proposition 2.6. *We have $\psi^\sharp_m(N^k) = \psi^\sharp(N^k)$ for any oriented manifold N^k, $k \geqslant n$, and any $m > k$.*

Proof. The join $N^k * \Delta^{m-k-1}$ is an m-dimensional manifold with singularities in $N^k \sqcup \Delta^{m-k-1}$. The points of $N^k * \Delta^{m-k-1}$ are linear combinations

$$t_0 x + \sum_{j=1}^{m-k} t_j y_j,$$

where $x \in N^k$, $y_1, y_2, \ldots, y_{m-k}$ are the vertices of the simplex Δ^{m-k-1}, $t_j \geqslant 0$, $j = 0, 1, \ldots, m-k$, and $\sum_{j=0}^{m-k} t_j = 1$. Let $i : N^k \hookrightarrow N^k * \Delta^{m-k-1}$ be the embedding given by

$$i(x) = \frac{1}{m-k+1}\biggl(x + \sum_{j=1}^{m-k} y_j\biggr).$$

Then $i(N^k)$ is a submanifold with trivial normal bundle. Let K be an arbitrary triangulation of N^k. Then $K * \Delta^{m-k-1}$ is a triangulation of $N^k * \Delta^{m-k-1}$. The submanifold $i(N^k)$ is transversal to the simplices of the triangulation $K * \Delta^{m-k-1}$. We have $|\tau * \Delta^{m-k-1}| \cap i(N^k) = i(|\tau|)$ and $\operatorname{link}_{K*\Delta^{m-k-1}}(\tau * \Delta^{m-k-1}) = \operatorname{link}_K \tau$ for any simplex $\tau \in K$. Hence, for any local formula f, the intersection of the cycles $f_\sharp(K * \Delta^{m-k-1}, K \sqcup \Delta^{m-k-1})$ and $i_*([N^k])$ coincides with the cycle $i_*(f_\sharp(K))$. Therefore,

$$\psi^\sharp_m(N^k) = i^*\bigl(\psi^\sharp(N^k * \Delta^{m-k-1}, N^k \sqcup \Delta^{m-k-1})\bigr) = \psi^\sharp(N^k).$$

Proof of Proposition 2.4. We consider the case $G = \mathbb{Q}$, $n = k$. Let M^m be an orientable manifold with $m > n$. By Propositions 2.5 and 2.6 we have $\psi^\sharp(M^m)\big|_{N^n} = p(N^n)$ for any submanifold $N^n \subset M^m$ with trivial normal bundle. It follows from results of Rokhlin, Schwarz and Thom that $\psi^\sharp(M^m) = p(M^m)$ if $m > 2n+1$.

Assume that $n < m \leqslant 2n+1$ and M^m is orientable. Let $i : M^m \hookrightarrow M^m \times S^{n+1}$ be the standard embedding. By Propositions 2.5 and 2.6 we have

$$i^*\bigl(\psi^\sharp(M^m \times S^{n+1})\bigr) = \psi^\sharp(M^m).$$

On the other hand, $i^*(p(M^m \times S^{n+1})) = p(M^m)$ and $\psi^\sharp(M^m \times S^{n+1}) = p(M^m \times S^{n+1})$ since $\dim(M^m \times S^{n+1}) > 2n+1$. Therefore, $\psi^\sharp(M^m) = p(M^m)$.

Assume now that M^m is not orientable. Let $\pi : \widetilde{M}^m \to M^m$ be the two-sheeted orientable covering. Then $\pi^*(\psi^\sharp(M^m)) = \psi^\sharp(\widetilde{M}^m)$, $\pi^*(p(M^m)) = p(\widetilde{M}^m)$. The proposition follows since π^* is a monomorphism.

2.5. Algorithms for computing the values of local formulae.

Theorem 2.3. *Suppose that $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$. Then there is a local formula f representing ψ such that the function $f\colon \mathcal{T}_n \to \mathbb{Q}$ is algorithmically computable.*

Remark 2.2. S. P. Novikov has proved that for $n \geqslant 6$ there is no algorithm for checking whether a given simplicial complex is an $(n-1)$-dimensional PL sphere. This proof can be found in [9]. Hence a rigorous formulation of Theorem 2.3 is as follows. There is an algorithm that, for a given oriented simplicial complex L, computes the value $f(L)$ if L is an $(n-1)$-dimensional PL sphere and runs for ever if it is not.

Remark 2.3. Obviously, for $n \geqslant 4$ there is an uncomputable coboundary $f \in \mathcal{T}^n(\mathbb{Q})$. Hence there are uncomputable rational local formulae.

In this section we shall consider infinite vectors $b = (b_i)_{i=1}^{\infty}$, $b_i \in \mathbb{Q}$, and infinite matrices $A = (a_{ij})_{i,j=1}^{\infty}$, $a_{ij} \in \mathbb{Q}$. A matrix A is said to be *row-finite* if every row of A contains finitely many non-zero numbers. The product of a row-finite matrix by an infinite vector is well defined. The product of two row-finite matrices is a row-finite matrix. An infinite vector b is said to be computable if there is an algorithm that computes b_i for a given i. A row-finite matrix A is said to be *computable* if there is an algorithm that computes a_{ij} for given i and j and there is an algorithm that for a given i computes a number $j_0(i)$ such that $a_{ij} = 0$ for every $j > j_0(i)$. In what follows all matrices are supposed to be row finite. We denote by $r_i(A)$ the rank of the matrix consisting of i first rows of A. If A is computable, then $r(A)$ is a computable vector.

Proposition 2.7. *Suppose that A is a computable matrix, b is a computable vector, and the linear system $Ax = b$ has a unique solution x^0. Then the vector x^0 is computable.*

Proof. Consider the system $Ax = b$ together with one further equation $x_k = x_k^0 + 1$. This system is inconsistent. Hence it contains a finite inconsistent subsystem. Therefore the value x_k^0 is uniquely determined by some finite subsystem of the system $Ax^0 = b$.

Proposition 2.8. *Suppose that A and B are computable matrices with $AB = 0$, b is a computable vector, the system $Ax = b$ is consistent, and any solution of the system $Ax = 0$ is equal to Bz for some vector z. Then the system $Ax = b$ has a computable solution.*

Proof. We obviously have either $r_i(B) = r_{i-1}(B)$ or $r_i(B) = r_{i-1}(B) + 1$ for any $i > 1$. Let $l_1 < l_2 < l_3 < \dots$ be the sequence consisting of all numbers i such that $r_i(B) = r_{i-1}(B) + 1$. (We assume that $l_1 = 1$ if and only if $r_1(B) = 1$.) This sequence can be either finite or infinite. The vector $l = (l_i)$ is computable in both cases. For any infinite vector v, we denote the vector $(v_{l_1}, v_{l_2}, v_{l_3}, \dots)^{\mathrm{T}}$ by $\widehat{v}$. Let $\widehat{B}$ be the matrix consisting of the rows of B with numbers $l_1, l_2, l_3, \dots$. The system $\widehat{B}z = \widehat{u}$ is consistent for any vector u since any finite subsystem of this system is consistent. Every row of B is a linear combination of rows of $\widehat{B}$. Hence, if $\widehat{B}z = 0$, then $Bz = 0$. Consider the system $Ax = b$ together with the equations

$x_{l_1} = 0$, $x_{l_2} = 0$, $\dots$. (We interleave the equations of the system $Ax = b$ with these new equations.) Let $\widetilde{A}x = \widetilde{b}$ be the resulting system. The matrix $\widetilde{A}$ and the vector $\widetilde{b}$ are computable. To prove the proposition, we must show that the system $\widetilde{A}x = \widetilde{b}$ has a unique solution. Let y be a solution of the system $Ax = b$. Let z^0 be a solution of the system $\widehat{B}z = -\widehat{y}$. Then $x^0 = y + Bz^0$ is a solution of the system $\widetilde{A}x = \widetilde{b}$. Suppose that x^* is a solution of the homogeneous system $\widetilde{A}x = 0$. Then $x^* = Bz^*$ for some vector z^*. But $\widehat{B}z^* = 0$ since $0 = x^*_{l_1} = x^*_{l_2} = \dots$. Therefore $x^* = 0$. Hence the system $\widetilde{A}x = \widetilde{b}$ has a unique solution.

Proof of Theorem 2.3. For every m, the set $\mathcal{T}_m$ is denumerable. To prove this, we note that any PL sphere can be obtained from the boundary of a simplex by finitely many bistellar moves (see § 3.1). Hence there is an algorithm producing a sequence of $(m-1)$-dimensional oriented PL spheres with the following properties.

1) All the PL spheres of this sequence are non-symmetric.

2) If $L \in \mathcal{T}_m$ is non-symmetric, then L is either isomorphic or anti-isomorphic to some PL sphere of this sequence.

3) No two PL spheres of this sequence can be either isomorphic or anti-isomorphic to each other.

We denote such sequences of PL spheres for $m = n+1, n$ and $n-1$ respectively by $(K_1, K_2, \dots)$, $(L_1, L_2, \dots)$ and $(J_1, J_2, \dots)$. Let $Q_1, Q_2, \dots, Q_k$ be oriented combinatorial manifolds whose bordism classes form a basis in $\Omega_n \otimes \mathbb{Q}$. We put $Q_j = K_{j-k}$ for $j > k$. We identify a function $f \in \mathcal{T}^n(\mathbb{Q})$ with the vector $v_f = (f(L_1), f(L_2), \dots)^{\mathrm{T}}$, and a function $g \in \mathcal{T}^{n-1}(\mathbb{Q})$ with the vector $v_g = (g(J_1), g(J_2), \dots)^{\mathrm{T}}$. Let B be the matrix of the linear operator $\delta\colon \mathcal{T}^{n-1}(\mathbb{Q}) \to \mathcal{T}^n(\mathbb{Q})$. Obviously, B is row finite. To any $f \in \mathcal{T}^n(\mathbb{Q})$ we assign the vector $w_f = (\varepsilon(f_\sharp(Q_1)), \varepsilon(f_\sharp(Q_2)), \dots)^{\mathrm{T}}$. We consider the linear operator taking v_f to w_f for every $f \in \mathcal{T}^n(\mathbb{Q})$ and denote the matrix of this operator by A. Evidently, A and B are computable and $AB = 0$.

We put $p = \star(\psi)$. It follows from Theorem 2.2 that $f \in \mathcal{T}^n(\mathbb{Q})$ is a local formula representing ψ if and only if $Av_f = b$, where $b_j = p(|Q_j|)$. The vector b is computable since $b_j = 0$ for $j > k$. Any solution of the system $Ax = 0$ is equal to Bv_g for some g. Hence the system $Ax = b$ satisfies all the conditions of Proposition 2.8. Therefore this system has a computable solution x^0. The required local formula f is given by $f(L_i) = x^0_i$.

Remark 2.4. This proof actually contains an explicit algorithm for computing the values $f(L)$.

§ 3. An explicit local formula for the first Pontryagin class

An explicit local formula for the first Pontryagin class is given in § 3.3. In § 3.1 we construct some objects needed for this formula. In particular, we define graphs Γ_n and a homomorphism $s\colon \mathcal{T}^4(\mathbb{Q}) \to C^1_{\mathbb{Z}_2}(\Gamma_2; \mathbb{Q})$, where $C^1_{\mathbb{Z}_2}(\Gamma_2; \mathbb{Q})$ is the group of one-dimensional equivariant cochains of Γ_2. In § 3.2 we find generators for $H_1(\Gamma_2; \mathbb{Z})$. A sketch of the proof of the main theorem is given in § 3.3. Some parts of the proof are postponed until §§ 3.4 and 3.5.

3.1. Bistellar moves. Let K be a combinatorial manifold. Suppose that there is a simplex $\Delta_1 \in K$ such that $\operatorname{link}\Delta_1 = \partial\Delta_2$ is the boundary of a simplex.

(Obviously, Δ_2 does not belong to K.) Then $\Delta_1 * \partial\Delta_2$ is a full subcomplex of K. The *bistellar move* associated with the simplex Δ_1 is the operation transforming K to the simplicial complex

$$\beta(K) = (K \setminus (\Delta_1 * \partial\Delta_2)) \cup (\partial\Delta_1 * \Delta_2).$$

If $\dim \Delta = 0$, then we assume that $\partial\Delta = \varnothing$. We assume that $\Delta * \varnothing = \Delta$ for any simplex Δ. Then stellar subdivisions of maximal simplices and inverse stellar subdivisions of maximal simplices are bistellar moves. The combinatorial manifolds K and $\beta(K)$ are PL homeomorphic for any bistellar move β. Pachner [10] has proved that two combinatorial manifolds are PL homeomorphic if and only if there is a sequence of bistellar moves transforming the first into the second (see also [11]). Specifically, for any two PL spheres of the same dimension there is a sequence of bistellar moves transforming the first into the second.

For any positive integer n we define a graph Γ_n in the following way. The vertex set of Γ_n is the set $\mathcal{T}_{n+1}$. Suppose that $L_1, L_2 \in \mathcal{T}_{n+1}$. Let β_1 and β_2 be bistellar moves transforming L_1 into L_2 that are associated with the simplices Δ_1 and Δ_2 respectively. We say that β_1 and β_2 are *equivalent* if there is an automorphism of L_1 taking Δ_1 to Δ_2. Let L_1 and L_2 be two different vertices of Γ_n. The edges of Γ_n with endpoints L_1 and L_2 are in one-to-one correspondence with the equivalence classes of bistellar moves transforming L_1 into L_2. Let us now describe the set of edges both of whose endpoints coincide with some vertex L of Γ_n. For any bistellar move β we denote the inverse bistellar move by β^{-1}. A bistellar move β transforming L into itself is said to be *inessential* if β is equivalent to β^{-1}, and *essential* otherwise. We assign no edges of Γ_n to equivalence classes of inessential bistellar moves. The equivalence classes of essential bistellar moves transforming L into itself can be divided into pairs of mutually inverse equivalence classes. The edges of Γ_n both of whose endpoints coincide with L are in one-to-one correspondence with such pairs of equivalence classes. The graph Γ_n is connected by Pachner's theorem. For any essential bistellar move β, we denote the corresponding edge of Γ_n by e_β. Then the edges e_β and $e_{\beta^{-1}}$ coincide but have opposite orientations.

Let $C_*(\Gamma_n;\mathbb{Z})$ be the cellular chain complex of Γ_n. The group $\mathbb{Z}_2$ acts on Γ_n by reversing the orientations of all PL spheres. The group $\mathbb{Z}_2$ acts on the group $\mathbb{Q}$ by reversing sign. Let $C^*_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Z}_2}(C_*(\Gamma_n;\mathbb{Z}),\mathbb{Q})$ be the equivariant cochain complex of Γ_n. (This means that the action of $\mathbb{Z}_2$ on the group $\mathbb{Z}$ is trivial.) We denote the differential of the complex $C^*_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q})$ by d. Let $H^*_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}) = H^*(C^*_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}))$ be the equivariant cohomology of Γ_n. We have a canonical isomorphism

$$H^1_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Z}_2}(H_1(\Gamma_n;\mathbb{Z}),\mathbb{Q}).$$

Evidently, $C^0_{\mathbb{Z}_2}(\Gamma_{n-1};\mathbb{Q}) = \mathcal{T}^n(\mathbb{Q})$. Hence we have the differential

$$\delta\colon C^0_{\mathbb{Z}_2}(\Gamma_{n-1};\mathbb{Q}) \to C^0_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q}).$$

Suppose that β is a bistellar move transforming L_1 into L_2, where $L_1, L_2 \in \mathcal{T}_{n+1}$. We may assume that L_1 and L_2 are simplicial complexes on the same vertex set V.

Indeed, this is true whenever β is associated with a simplex whose dimension is neither 0 nor n. Otherwise we assume that one of the two simplicial complexes contains a vertex v_0 which is not a simplex of this complex. For any vertex $v \in V$ the bistellar move β either does not change the link of v or induces a bistellar move β_v transforming $\operatorname{link}_{L_1} v$ into $\operatorname{link}_{L_2} v$. Let $W \subset V$ be the set of all vertices v such that the bistellar move β_v is essential. (We assume that $v_0 \notin W$.) The differential $\delta : C^1_{\mathbb{Z}_2}(\Gamma_{n-1};\mathbb{Q}) \to C^1_{\mathbb{Z}_2}(\Gamma_n;\mathbb{Q})$ is given by

$$(\delta h)(e_\beta) = \sum_{v\in W} h(e_{\beta_v}).$$

It is easy to show that $\delta^2 = 0$ and $\delta d = d\delta$.

We put $C^{j,n} = C^j_{\mathbb{Z}_2}(\Gamma_{n-1};\mathbb{Q})$. Then $C^{*,*}$ is a bigraded complex. We have $\operatorname{bideg} d = (1,0)$ and $\operatorname{bideg}\delta = (0,1)$. Let $Z^{*,*}_d$, $B^{*,*}_d$, and $H^{*,*}_d$ be respectively the cocycle group, the coboundary group, and the cohomology group of the complex $C^{*,*}$ with respect to the differential d. Let $Z^{*,*}_\delta$, $B^{*,*}_\delta$, and $H^{*,*}_\delta$ be respectively the cocycle group, the coboundary group, and the cohomology group of the complex $C^{*,*}$ with respect to the differential δ. The graph Γ_{n-1} is connected. Hence $H^{0,n}_d = 0$. Therefore $d : C^{0,n} \to C^{1,n}$ is a monomorphism.

Suppose that $L_1, L_2 \in \mathcal{T}_n$. Let β, V and W be as above. Suppose that the bistellar move β replaces the subcomplex $\Delta_1 * \partial\Delta_2 \subset L_1$ by the subcomplex $\partial\Delta_1 * \Delta_2 \subset L_2$. We consider the cone CL_1 with vertex u_1 and the cone CL_2 with vertex u_2. Let L_β be the simplicial complex on the set $V \cup \{u_1, u_2\}$ given by $L_\beta = CL_1 \cup CL_2 \cup (\Delta_1 * \Delta_2)$. Then L_β is an n-dimensional PL sphere. We choose the orientation of L_β in such a way that the induced orientation of $\operatorname{link} u_2$ coincides with the given orientation of L_2. Then $L_\beta \in \mathcal{T}_{n+1}$. If β_1 and β_2 are equivalent bistellar moves, then the PL spheres L_{β_1} and L_{β_2} are isomorphic. The PL spheres L_β and $L_{\beta^{-1}}$ are anti-isomorphic. If β is inessential, then L_β is symmetric. Let the homomorphism $s : C^{0,n+1} \to C^{1,n}$ be given by $s(f)(e_\beta) = f(L_\beta)$.

Since $\delta d = d\delta$, we see that $d : C^{0,*} \to C^{1,*}$ is a chain homomorphism of complexes with differential δ.

Proposition 3.1. *The homomorphism s is a chain homotopy between the chain homomorphisms d and 0 of $C^{0,*}$ to $C^{1,*}$, that is, $d = \delta s + s\delta$.*

Proof. Suppose that $f \in C^{0,n}$, $L_1, L_2 \in \mathcal{T}_n$, and β is a bistellar move transforming L_1 into L_2. For any $v \in V\backslash W$, the link of v in L_β is symmetric. For any $v \in W$, the link of v in L_β is isomorphic to $-L_{\beta_v}$. The links of u_1 and u_2 are isomorphic to $-L_1$ and L_2 respectively. Hence,

$$\begin{aligned} s(\delta f)(e_\beta) = (\delta f)(L_\beta) &= -\sum_{v\in W} f(L_{\beta_v}) + f(L_2) - f(L_1) \\ &= -\sum_{v\in W} s(f)(e_{\beta_v}) + f(\partial e_\beta) = -\delta s(f)(e_\beta) + df(e_\beta). \end{aligned}$$

Consequently, $df = \delta s(f) + s(\delta f)$.

We denote by A^n the subgroup of $C^{1,n}$ such that $h \in A^n$ if and only if $\delta h \in B_d^{1,n+1}$.

Proposition 3.2. *The homomorphism $s|_{Z_\delta^{0,n}}$ is a monomorphism and $s(Z_\delta^{0,n}) \subset A^{n-1}$.*

Proof. It follows from Proposition 3.1 that

$$d\big|_{Z_\delta^{0,n}} = \delta s\big|_{Z_\delta^{0,n}}.$$

Since d is a monomorphism, we see that $s\big|_{Z_\delta^{0,n}}$ is a monomorphism. If $f \in Z_\delta^{0,n}$, then $\delta s(f) = df$. Hence $s(f) \in A^{n-1}$.

Corollary 3.1. *We have $Z_\delta^{0,3} = 0$. Hence $H_\delta^{0,3} = 0$.*

Proof. Γ_1 is isomorphic to the graph with vertex set $\{3, 4, 5, 6, \dots\}$ such that for any k there is a unique edge with endpoints k and $k+1$. The action of $\mathbb{Z}_2$ is trivial. Therefore $C^{1,2} = 0$. By Proposition 3.2, there is a monomorphism from $Z_\delta^{0,3}$ to $C^{1,2}$. Hence $Z_\delta^{0,3} = 0$.

Proposition 3.3. *The homomorphism $s\big|_{B_\delta^{0,4}}$ is an isomorphism of $B_\delta^{0,4}$ onto $B_d^{1,3}$.*

Proof. We have $C^{1,2} = 0$. Therefore, $dg = s(\delta g)$ for every $g \in C^{0,3}$. The proposition follows.

Thus s induces a monomorphism

$$s^* : H_\delta^{0,4} \to H_d^{1,3} = H^1_{\mathbb{Z}_2}(\Gamma_2; \mathbb{Q}).$$

Let $\widetilde{A}^3$ be the kernel of the homomorphism $\delta^* : H_d^{1,3} \to H_d^{1,4}$ induced by the chain homomorphism $\delta : C^{*,3} \to C^{*,4}$. Then $s^*(H_\delta^{0,4}) \subset \widetilde{A}^3$.

3.2. Generators of $H_1(\Gamma_2; \mathbb{Z})$. In what follows, if we say that $\{u_1, u_2, \dots, u_l\}$ is a simplex of an $(l-1)$-dimensional oriented simplicial complex L, we mean that the sequence of vertices $u_1, u_2, \dots, u_l$ provides the given orientation of L. If we show a 2-dimensional simplicial complex in a figure, then we mean that the orientation is clockwise. For any cycle $\gamma \in Z_1(\Gamma_n; \mathbb{Z})$ let $\bar{\gamma} \in H_1(\Gamma_n; \mathbb{Z})$ be the homology class represented by γ. Suppose that L is an oriented 2-dimensional PL sphere. An edge $e \in L$ is said to be *admissible* if there is a bistellar move associated with e.

Let $\Delta_1, \Delta_2 \in L$ be two different triangles. We apply to L the bistellar move associated with Δ_1 and denote by v_1 the new vertex created. Then we apply to the resulting PL sphere the bistellar move associated with Δ_2 and denote by v_2 the new vertex created. Then we apply to the resulting PL sphere the bistellar move associated with v_1. Finally, we apply to the resulting PL sphere the bistellar move associated with v_2. Let $\alpha_1(L, \Delta_1, \Delta_2)$ be the resulting cycle in the graph Γ_2 (see Fig. 1, *a*, *b*, *c*).

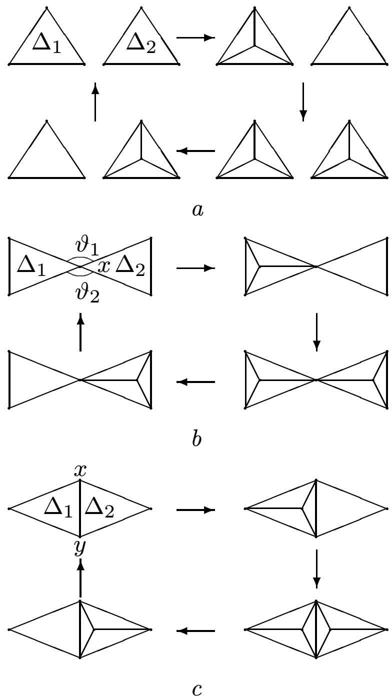

Figure 1

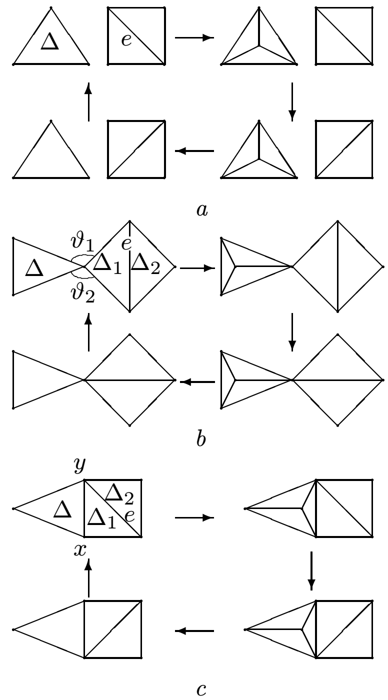

Figure 2

There are three possibilities: Δ_1 and Δ_2 have 0, 1 or 2 common vertices. Let $\mathcal{S}_1^0$ be the set of all homology classes $\bar{\alpha}_1(L,\Delta_1,\Delta_2)\in H_1(\Gamma_2,\mathbb{Z})$ such that the triangles Δ_1 and Δ_2 have no common vertices (see Fig. 1, *a*). We denote by $\mathcal{S}_1^1(p,q)$ the set of all homology classes $\bar{\alpha}_1(L,\Delta_1,\Delta_2)$ such that

1) the triangles Δ_1 and Δ_2 have a unique common vertex x (see Fig. 1, *b*),

2) there are exactly p triangles containing x and situated in the angle ϑ_1,

3) there are exactly q triangles containing x and situated in the angle ϑ_2.

We denote by $\mathcal{S}_1^2(p,q)$ the set of all homology classes $\bar{\alpha}_1(L,\Delta_1,\Delta_2)$ such that

1) the triangles Δ_1 and Δ_2 have a common edge e with endpoints x and y such that the triangle Δ_1 is on the right when we pass along e from x to y (see Fig. 1, *c*),

2) there are exactly p triangles that contain x and coincide with neither Δ_1 nor Δ_2,

3) there are exactly q triangles that contain y and coincide with neither Δ_1 nor Δ_2.

Suppose that $\Delta\in L$ is a triangle, $e\in L$ is an admissible edge, and $e\not\subset\Delta$. We denote by $\alpha_2(L,\Delta,e)$ the cycle shown in Fig. 2. Let $\mathcal{S}_2^0$ be the set of all homology classes $\bar{\alpha}_2(L,\Delta,e)$ such that the triangle Δ has no vertices in common with Δ_1 or Δ_2, where Δ_1 and Δ_2 are the two triangles containing e (see Fig. 2, *a*). We denote by $\mathcal{S}_2^1(p,q)$ the set of all homology classes $\bar{\alpha}_2(L,\Delta,e)$ such that

1) the triangles Δ and Δ_1 have a unique common vertex x (see Fig. 2, *b*),

2) there are exactly p triangles containing x and situated in the angle ϑ_1,

3) there are exactly q triangles containing x and situated in the angle ϑ_2.

We denote by $\mathcal{S}_2^2(p,q)$ the set of all homology classes $\bar{\alpha}_2(L,\Delta,e)$ such that

1) the triangles Δ and Δ_1 have a common edge e_1 with endpoints x and y (see Fig. 2, c),

2) there are exactly p triangles that contain x and coincide with neither Δ nor Δ_1,

3) there are exactly q triangles that contain y and coincide with none of the triangles Δ, Δ_1, Δ_2.

Suppose that e_1 and e_2 are admissible edges such that the bistellar move associated with e_1 takes e_2 to an admissible edge and there is no triangle containing both e_1 and e_2. We denote by $\alpha_3(L,e_1,e_2)$ the cycle shown in Fig. 3. Let $\mathcal{S}_3^0$ be the set of all homology classes $\bar{\alpha}_3(L,e_1,e_2)$ such that any triangle containing the edge e_1 has no vertices in common with any triangle containing the edge e_2 (see Fig. 3, a). We denote by $\mathcal{S}_3^1(p,q)$ the set of all homology classes $\bar{\alpha}_3(L,e_1,e_2)$ such that

1) the triangles Δ_1 and Δ_2 have a unique common vertex x (see Fig. 3, b),

2) there are exactly p triangles containing x and situated in the angle ϑ_1,

3) there are exactly q triangles containing x and situated in the angle ϑ_2.

We denote by $\mathcal{S}_3^2(p,q)$ the set of all homology classes $\bar{\alpha}_3(L,e_1,e_2)$ such that

1) the triangles Δ_1 and Δ_2 have a common edge with endpoints x and y (see Fig. 3, c),

2) there are exactly p triangles that contain x and coincide with none of the triangles Δ_1, Δ_2, and Δ_4,

3) there are exactly q triangles that contain y and coincide with none of the triangles Δ_1, Δ_2, Δ_3.

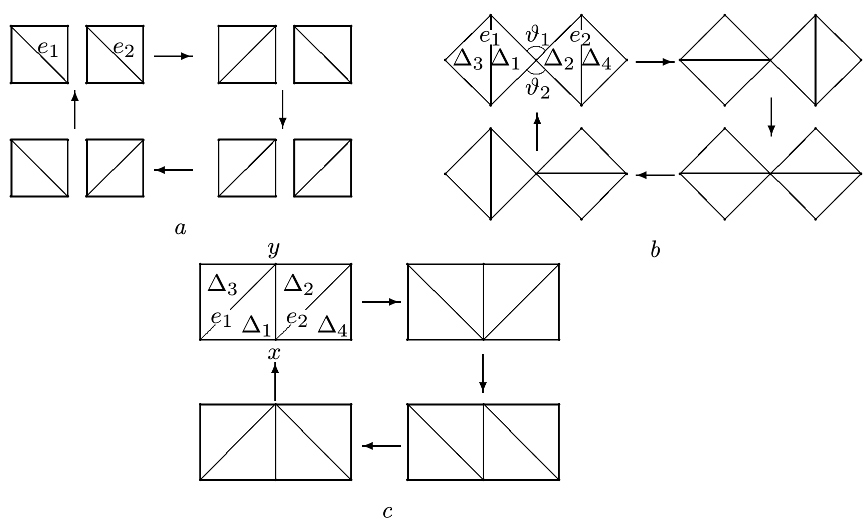

Figure 3

Suppose that x, y, z are vertices of L and there is a (unique) vertex u such that $\{u,x,y\}$, $\{u,y,z\}$, and $\{u,z,x\}$ are (oriented) triangles of L. Then we denote by $\alpha_4(L,x,y,z)$ the cycle shown in Fig. 4. Let $\mathcal{S}_4(p,q,r)$ be the set of all homology classes $\bar{\alpha}_4(L,x,y,z)$ such that there are exactly p, q, and r triangles that contain the vertices x, y, and z respectively and coincide with none of the triangles $\{u,x,y\}$, $\{u,y,z\}$, $\{u,z,x\}$.

Suppose that x,y,z,u are vertices of L such that the full subcomplex spanned by the set $\{x,y,z,u\}$ consists of the triangles $\{x,y,z\}$ and $\{x,z,u\}$, and their edges and vertices. Then we denote by $\alpha_5(L,x,y,z,u)$ the cycle shown in Fig. 5. Let $\mathcal{S}_5(p,q,r,k)$ be the set of all homology classes $\bar\alpha_5(L,x,y,z,u)$ such that there are exactly p, q, r, and k triangles that contain the vertices x, y, z, and u respectively and coincide with neither $\{x,y,z\}$ nor $\{x,z,u\}$.

Suppose that x,y,z,u,v are vertices of L such that the full subcomplex spanned by the set $\{x,y,z,u,v\}$ consists of the triangles $\{x,y,z\}$, $\{x,z,u\}$, and $\{x,u,v\}$, and their edges and vertices. Then we denote by $\alpha_6(L,x,y,z,u,v)$ the cycle shown in Fig. 6. Let $\mathcal{S}_6(p,q,r,k,l)$ be the set of all homology classes $\bar\alpha_6(L,x,y,z,u,v)$ such that there are exactly p, q, r, k, and l triangles that contain the vertices x, y, z, u, and v respectively and coincide with none of the triangles $\{x,y,z\}$, $\{x,z,u\}$, $\{x,u,v\}$.

We denote by $\mathcal{S}$ the union of all sets $\mathcal{S}_1^0$, $\mathcal{S}_1^1(p,q)$, $\mathcal{S}_1^2(p,q)$, $\mathcal{S}_2^0$, $\mathcal{S}_2^1(p,q)$, $\mathcal{S}_2^2(p,q)$, $\mathcal{S}_3^0$, $\mathcal{S}_3^1(p,q)$, $\mathcal{S}_3^2(p,q)$, $\mathcal{S}_4(p,q,r)$, $\mathcal{S}_5(p,q,r,k)$, and $\mathcal{S}_6(p,q,r,k,l)$.

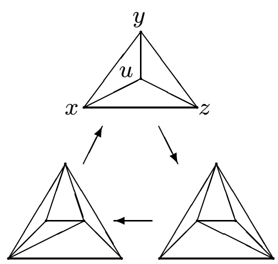

Figure 4

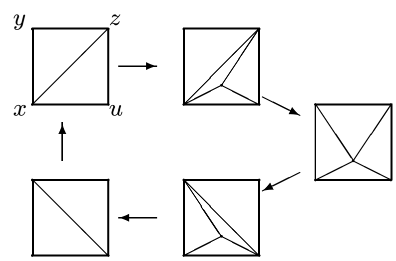

Figure 5

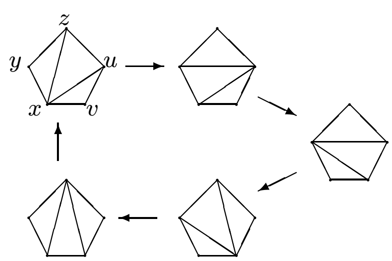

Figure 6

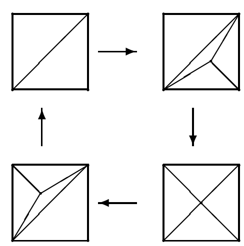

Figure 7

Proposition 3.4. *The set $\mathcal{S}$ generates the group $H_1(\Gamma_2,\mathbb{Z})$.*

Proof. The proof of the proposition is based on the following statements.

1) Any 2-dimensional PL sphere can be realized as the boundary of a convex simplicial polytope in $\mathbb{R}^3$.

2) Suppose that P_0 and P_1 are two convex simplicial polytopes in $\mathbb{R}^3$ with isomorphic boundaries. Then there is a continuous deformation P_t, $t\in[0,1]$, such that P_t is a convex simplicial polytope isomorphic to P_0 for every $t\in[0,1]$.

3) Suppose that L is a 2-dimensional PL sphere and $e\in L$ is an admissible edge. Then there is a convex polytope $P\subset\mathbb{R}^3$ containing a quadrilateral face F and such

that all other faces of P are triangular, and L is isomorphic to the boundary of P with F decomposed into two triangles.

These statements are proved in [12] (see also [13]).

We denote by $\mathcal{T}_{3,l}$ the set of all oriented 2-dimensional PL spheres with at most l vertices. Let $\Gamma_{2,l}$ be the full subgraph of Γ_2 spanned by the set $\mathcal{T}_{3,l}$.

Let us consider an affine space $\mathbb{R}^{3l}$, $l \geqslant 12$. The points of this space are regarded as rows $y = (y_1, y_2, \dots, y_l)$, $y_j \in \mathbb{R}^3$. The points y_j are called the *coordinates* of y. We denote by $P(y)$ the convex hull of the points $y_1, y_2, \dots, y_l$. Let $V(y)$ be obtained from the set $y_1, y_2 \dots, y_l$ by deleting all points that belong to the interior of $P(y)$. (The set $V(y)$ may contain multiple points.) We denote by $\Theta^{3l} \subset \mathbb{R}^{3l}$ the subset consisting of all y such that $P(y)$ is not contained in a 2-dimensional plane. We denote by $\Theta^{3l}_0 \subset \Theta^{3l}$ the subset consisting of all y such that the set $V(y)$ is in general position, that is, $V(y)$ does not contain four points belonging to a 2-dimensional plane. We denote by $\Theta^{3l-1}_1 \subset \Theta^{3l} \backslash \Theta^{3l}_0$ the subset consisting of all y such that $V(y)$ contains exactly one quadruple of points belonging to a 2-dimensional plane. We denote by $\Theta^{3l-2}_{2,1} \subset \Theta^{3l} \backslash \bigl(\Theta^{3l}_0 \cup \Theta^{3l-1}_1\bigr)$ the subset consisting of all y such that $V(y)$ contains exactly two quadruples of points belonging to a 2-dimensional plane. We denote by $\Theta^{3l-2}_{2,2} \subset \Theta^{3l} \backslash \bigl(\Theta^{3l}_0 \cup \Theta^{3l-1}_1\bigr)$ the subset consisting of all y such that $V(y)$ contains exactly one triple of points belonging to a line and such that four points of $V(y)$ belong to a 2-dimensional plane if and only if three of them belong to a line. We denote by $\Theta^{3l-2}_{2,3} \subset \Theta^{3l} \setminus \bigl(\Theta^{3l} \cup \Theta^{3l-1}\bigr)$ the set of all y such that $V(y)$ contains exactly one 5-tuple of points belonging to a 2-dimensional plane and such that four points of $V(y)$ belong to a 2-dimensional plane if and only if they are contained in this 5-tuple. By definition, we put $\Theta^{3l-2}_2 = \Theta^{3l-2}_{2,1} \cup \Theta^{3l-2}_{2,2} \cup \Theta^{3l-2}_{2,3}$. Then

$$\dim \Theta^{3l}_0 = 3l, \qquad \dim \Theta^{3l-1}_1 = 3l-1, \qquad \dim \Theta^{3l-2}_2 = 3l-2$$
$$\dim\bigl(\mathbb{R}^{3l} \setminus \bigl(\Theta^{3l}_0 \cup \Theta^{3l-1}_1 \cup \Theta^{3l-2}_2\bigr)\bigr) = 3l-3.$$

It follows from statements 1)–3) that the set $\Theta^{3l}_0 \cup \Theta^{3l-1}_1$ is connected. The group S_l acts on Θ^{3l} by permuting the coordinates.

To each point $y \in \Theta^{3l}_0$ we assign the vertex of $\Gamma_{2,l}$ corresponding to the isomorphism class of the boundary of $P(y)$. A sequence of bistellar moves (that is, a path in $\Gamma_{2,l}$) corresponds to each smooth curve $y\colon [0,1] \to \Theta^{3l}_0 \cup \Theta^{3l-1}_1$ transversal to Θ^{3l-1}_1. This correspondence can obviously be extended to a well-defined homomorphism

$$j\colon H_1(\Theta^{3l}_0 \cup \Theta^{3l-1}_1; \mathbb{Z}) \to H_1(\Gamma_{2,l}; \mathbb{Z}).$$

The following proposition is a consequence of statements 1)–3).

Proposition 3.5. *Suppose that γ is a closed path in $\Gamma_{2,l}$ starting at the vertex corresponding to the boundary of a tetrahedron. Then there is a curve $y\colon [0,1] \to \Theta^{3l}_0 \cup \Theta^{3l-1}_1$ transversal to Θ^{3l-1}_1 such that y induces the path γ. The curve y can be chosen in such a way that $y(1) = \nu y(0)$ for some permutation $\nu \in S_l$.*

Proposition 3.6. *Suppose that $y^0, y^1 \in \Theta^{3l}_0$ are points such that $P(y^0)$ and $P(y^1)$ are tetrahedra and $\nu y^0 = y^1$ for some permutation $\nu \in S_l$. Then there is a smooth* *curve* $y\colon [0,1] \to \Theta_0^{3l} \cup \Theta_1^{3l-1}$ *transversal to* Θ_1^{3l-1} *such that* $y(0) = y^0$, $y(1) = y^1$ *and the curve* y *induces a cycle homologous to zero in* $\Gamma_{2,l}$.

Proof. We may assume without loss of generality that each coordinate of the point y^i either belongs to $V(y^i)$ or coincides with the barycentre of the tetrahedron $P(y^0) = P(y^1)$. Since $l \geqslant 12$, we may assume that the permutation ν has at least four fixed elements. Then there is a point $y^{\frac{1}{2}} \in \Theta_0^{3l}$ such that $\nu y^{\frac{1}{2}} = y^{\frac{1}{2}}$.

Let $y\colon \left[0, \frac{1}{2}\right] \to \Theta_0^{3l} \cup \Theta_1^{3l-1}$ be an arbitrary smooth curve transversal to Θ_1^{3l-1} such that $y(0) = y^0$ and $y\left(\frac{1}{2}\right) = y^{\frac{1}{2}}$. We put $y(t) = \nu y(1-t)$ for every $t \in \left[\frac{1}{2}, 1\right]$. Smoothing the curve y in the neighbourhood of $y\left(\frac{1}{2}\right)$, we obtain the required curve.

It follows from Propositions 3.5 and 3.6 that j is an epimorphism. The group $H_1(\Theta_0^{3l} \cup \Theta_1^{3l-1}; \mathbb{Z})$ is generated by circuits ω_X around the connected components X of the set Θ_2^{3l-2}. If X is a connected component of $\Theta_{2,1}^{3l-2}$ or $\Theta_{2,3}^{3l-2}$, then either $j(\omega_X) \in \mathcal{S}$ or $-j(\omega_X) \in \mathcal{S}$. If X is a connected component of $\Theta_{2,2}^{3l-2}$, then $j(\omega_X)$ can be represented by a cycle as shown in Fig. 7. Hence $j(\omega_X)$ is the sum of two generators belonging to $\mathcal{S}$.

3.3. The formula. It follows from Theorem 2.1 that there is a unique generator $\varphi \in H^4(\mathcal{T}^*(\mathbb{Q})) \cong \mathbb{Q}$ such that $\varphi^\sharp(M) = p_1(M)$ for any manifold M. The following theorem gives an explicit description of the generator φ.

Theorem 3.1. *Suppose that* $c_0 : \mathcal{S} \to \mathbb{Q}$ *is the function given by*

$$c_0(\bar{\alpha}) = 0, \qquad \bar{\alpha} \in \mathcal{S}_1^0 \cup \mathcal{S}_2^0 \cup \mathcal{S}_3^0,$$

$$c_0(\bar{\alpha}) = \frac{q-p}{(p+q+2)(p+q+3)(p+q+4)}, \qquad \bar{\alpha} \in \mathcal{S}_1^1(p,q) \cup \mathcal{S}_2^1(p,q) \cup \mathcal{S}_3^1(p,q),$$

$$c_0(\bar{\alpha}) = \frac{q}{(q+2)(q+3)(q+4)} - \frac{p}{(p+2)(p+3)(p+4)}, \qquad \bar{\alpha} \in \mathcal{S}_1^2(p,q) \cup \mathcal{S}_3^2(p,q),$$

$$c_0(\bar{\alpha}) = \frac{q}{(q+2)(q+3)(q+4)} + \frac{p}{(p+2)(p+3)(p+4)}, \qquad \bar{\alpha} \in \mathcal{S}_2^2(p,q),$$

$$c_0(\bar{\alpha}) = \frac{1}{(p+2)(p+3)} - \frac{1}{(q+2)(q+3)} + \frac{1}{(r+2)(r+3)} - \frac{1}{12}, \qquad \bar{\alpha} \in \mathcal{S}_4(p,q,r),$$

$$c_0(\bar{\alpha}) = \frac{1}{(p+2)(p+3)} - \frac{1}{(q+2)(q+3)} - \frac{1}{(r+2)(r+3)} + \frac{1}{(k+2)(k+3)}, \qquad \bar{\alpha} \in \mathcal{S}_5(p,q,r,k),$$

$$c_0(\bar{\alpha}) = \frac{1}{(p+2)(p+3)} + \frac{1}{(q+2)(q+3)} + \frac{1}{(r+2)(r+3)} + \frac{1}{(k+2)(k+3)} + \frac{1}{(l+2)(l+3)} - \frac{1}{12}, \qquad \bar{\alpha} \in \mathcal{S}_6(p,q,r,k,l).$$

Then there is a unique linear extension of c_0 *to* $H_1(\Gamma_2; \mathbb{Z})$. *This extension is also denoted by* c_0 *and belongs to* $H^1_{\mathbb{Z}_2}(\Gamma_2; \mathbb{Q}) = \operatorname{Hom}_{\mathbb{Z}_2}(H_1(\Gamma_2; \mathbb{Z}), \mathbb{Q})$. *Then we have*

$c_0 = s^*(\phi)$. *Thus s maps the affine space of all local formulae for the first Pontryagin class isomorphically onto the affine space of all cocycles $\widehat{c}_0 \in C^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$ representing c_0.*

Remark 3.1. Similar numbers appear in the solution of a quite different problem in [14] (see also [15]), where Kazarian obtains a formula for the Chern-Euler class of a circle bundle via singularities of the restrictions of a Morse function on the total space to the fibres.

Let us describe how to make calculations using this theorem. We first choose a representative $\widehat{c}_0$ of the class c_0 to fix a local formula f for the first Pontryagin class. Suppose that we want to calculate $f(L)$, where L is an oriented 3-dimensional PL sphere. Let $\beta_1, \beta_2, \dots, \beta_l$ be a sequence of bistellar moves transforming the boundary of a 4-simplex into L. We denote by L_j the PL sphere obtained from $\partial\Delta^4$ by applying the bistellar moves $\beta_1, \beta_2, \dots, \beta_{j-1}$. Let W_j be the set of all vertices $v \in L_j$ such that the bistellar move β_j induces an essential bistellar move β_{jv} of $\operatorname{link} v$.

Corollary 3.2. $f(L) = \sum_{j=1}^{l} \sum_{v\in W_j} \widehat{c}_0(e_{\beta_{jv}})$.

Proof. We have

$$f(L_{j+1}) - f(L_j) = df(e_{\beta_j}) = \delta\widehat{c}_0(e_{\beta_j}) = \sum_{v\in W_j} \widehat{c}_0(e_{\beta_{jv}}).$$

To find a cycle whose Poincaré dual represents the first Pontryagin class of a compact combinatorial manifold, we do not actually need to choose a cocycle $\widehat{c}_0$ on the whole graph Γ_2. It is sufficient to choose a cocycle representing the restriction of c_0 to some finite subgraph of Γ_2.

To calculate the first Pontryagin number of an oriented 4-dimensional combinatorial manifold K, we do not need to choose $\widehat{c}_0$ at all. Let $L_1, L_2, \dots, L_k$ be the links of all vertices of K. Let $\beta_{i1}, \beta_{i2}, \dots, \beta_{il_i}$ be a sequence of bistellar moves transforming the boundary of a 4-simplex into L_i. We denote by L_{ij} the PL sphere obtained from $\partial\Delta^4$ by applying the bistellar moves $\beta_{i1}, \beta_{i2}, \dots, \beta_{i,j-1}$. Let W_{ij} be the set of all vertices $v \in L_{ij}$ such that the bistellar move β_{ij} induces an essential bistellar move β_{ijv} of $\operatorname{link} v$. We put

$$\gamma = \sum_{i=1}^{k} \sum_{j=1}^{l_i} \sum_{v\in W_{ij}} (e_{\beta_{ijv}} - \widetilde{e}_{\beta_{ijv}}) \in C_1(\Gamma_2;\mathbb{Z}),$$

where $\widetilde{e}$ is the edge of Γ_2 such that the action of $\mathbb{Z}_2$ takes e to $\widetilde{e}$. It is easy to show that γ is a cycle.

Corollary 3.3. *The first Pontryagin number of K is equal to $\frac{1}{2}\, c_0(\bar{\gamma})$.*

Proof. Suppose that $\widehat{c}_0 \in C^1_{\mathbb{Z}_2}(\Gamma_2;\mathbb{Q})$ is a cocycle representing c_0. Then the first Pontryagin number of K is equal to

$$\begin{aligned}\sum_{i=1}^{k} f(L_i) &= \sum_{i=1}^{k} \sum_{j=1}^{l_i} \sum_{v\in W_{ij}} \widehat{c}_0(e_{\beta_{ijv}}) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{l_i} \sum_{v\in W_{ij}} \left(\frac{1}{2}\,\widehat{c}_0(e_{\beta_{ijv}}) - \frac{1}{2}\,\widehat{c}_0(\widetilde{e}_{\beta_{ijv}})\right) = \frac{1}{2}\,\widehat{c}_0(\gamma) = \frac{1}{2}\, c_0(\bar{\gamma}).\end{aligned}$$

To use this formula in actual calculations, we must represent the cycle $\bar{\gamma}$ as a linear combination of elements of $\mathcal{S}$.

Remark 3.2. Unlike the formula of Corollary 3.2, the formula of Corollary 3.3 is non-local.

Sketch of proof of Theorem 3.1. Since

$$\star\colon H^*(\mathcal{T}^*(\mathbb{Q})) \to \operatorname{Hom}(\Omega_*, \mathbb{Q})$$

is an epimorphism, we see that $\dim H^4(\mathcal{T}^*(\mathbb{Q})) \geqslant 1$. On the other hand, we shall prove in § 3.4 that for any cohomology class $c \in \widetilde{A}^3$ there is a $\lambda \in \mathbb{Q}$ such that $c(\bar{\alpha}) = \lambda c_0(\bar{\alpha})$ for any $\bar{\alpha} \in \mathcal{S}$. Therefore $\dim \widetilde{A}^3 \leqslant 1$. But $s^* : H^4(\mathcal{T}^*(\mathbb{Q})) \to \widetilde{A}^3$ is a monomorphism. Hence,

$$\dim H^4(\mathcal{T}^*(\mathbb{Q})) = \dim \widetilde{A}^3 = 1.$$

Therefore the cohomology class c_0 is well defined, $c_0 \in \widetilde{A}^3$ and $s^*(\varphi) = \lambda c_0$ for some rational $\lambda \neq 0$. We shall prove in § 3.5 that $\lambda = 1$.

Remark 3.3. This proof does not use the fact that $\star$ is a monomorphism. However, it follows from the proof that $\star$ is a monomorphism in dimension 4.

3.4. The group $\widetilde{A}^3$. Let c be an arbitrary element of $\widetilde{A}^3 \subset \operatorname{Hom}_{\mathbb{Z}_2}(H_1(\Gamma_2;\mathbb{Z}),\mathbb{Q})$.

Suppose that $\alpha = \alpha_i(L, \dots)$ is a cycle as shown in one of Figs. 1–6. Let $X(\alpha)$ be the set of all vertices denoted by x, y, z, u, v in the corresponding figure. (Some of these letters are absent from some figures.) A generator $\bar{\alpha} \in \mathcal{S}$ is said to be *regular* if the following conditions hold.

1) $\bigcup_{a \in X(\alpha)} \operatorname{star} a$ is a full subcomplex of L.

2) If $w \notin X(\alpha)$, $a, b \in X(\alpha)$ and $\{w,a\}, \{w,b\} \in L$, then $\{w,a,b\} \in L$.

Suppose that $\bar{\alpha}_1(L, \Delta_1, \Delta_2) \in \mathcal{S}_1^0$. Let $K \in \mathcal{T}_4$ be a PL sphere containing a vertex u such that $\operatorname{link} u \cong L$ and $\operatorname{star} u$ is a full subcomplex of K. We identify the simplicial complexes $\operatorname{link} u$ and L. We put $\widetilde{\Delta}_1 = \Delta_1 \cup \{u\}$ and $\widetilde{\Delta}_2 = \Delta_2 \cup \{u\}$. We apply to K the bistellar move associated with the tetrahedron $\widetilde{\Delta}_1$ and denote by z_1 the new vertex created. Then we apply to the resulting PL sphere the bistellar move associated with $\widetilde{\Delta}_2$ and denote by z_2 the new vertex created. Then we apply to the resulting PL sphere the bistellar move associated with z_1. Finally, we apply to the resulting PL sphere the bistellar move associated with z_2. We denote the resulting cycle in Γ_3 by γ. For every vertex v of K, a sequence of bistellar moves transforming K into itself induces a sequence of bistellar moves transforming $\operatorname{link}_K v$ to itself. Hence the cycle γ induces a cycle γ_v in Γ_2. Then

$$\delta^*(c)(\bar{\gamma}) = \sum_{v \in K} c(\bar{\gamma}_v).$$

We have $\delta^*(c) = 0$ since $c \in \widetilde{A}^3$. The cycle γ_v is homologous to zero for any vertex $v \in K$ different from u, and γ_u coincides with $\alpha_1(L, \Delta_1, \Delta_2)$. Hence $c(\bar{\alpha}_1(L, \Delta_1, \Delta_2)) = 0$. We similarly have $c(\bar{\alpha}) = 0$ for any $\bar{\alpha} \in \mathcal{S}_2^0 \cup \mathcal{S}_3^0$.

Proposition 3.7. *For any $p, q > 0$ the restriction of c to $\mathcal{S}_1^1(p,q)$ is a constant function.*

Proof. Suppose that $\bar{\alpha}_1(L^{(i)}, \Delta_1^{(i)}, \Delta_2^{(i)}) \in \mathcal{S}_1^1(p,q)$, $i = 1, 2$. Let $x^{(i)}$ be the common vertex of the triangles $\Delta_1^{(i)}$ and $\Delta_2^{(i)}$. We assume that $\bar{\alpha}_1(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)})$ is regular. Let us show that there is a 3-dimensional oriented PL sphere K containing an edge e with endpoints u and w such that

1) the link of e is a $(p+q+2)$-gon containing two edges denoted by e_1 and e_2,

2) the link of u is isomorphic to $L^{(1)}$, and this isomorphism takes the triangle spanned by the vertex w and the edge e_j to the triangle $\Delta_j^{(1)}$, $j = 1, 2$,

3) the link of w is isomorphic to $-L^{(2)}$, and this isomorphism takes the triangle spanned by the vertex u and the edge e_j to the triangle $\Delta_j^{(2)}$, $j = 1, 2$.

We consider a cone $CL^{(1)}$ with vertex u and a cone $CL^{(2)}$ with vertex w. We identify the full subcomplex $\operatorname{star} x^{(1)} \subset CL^{(1)}$ with the subcomplex $\operatorname{star} x^{(2)} \subset CL^{(2)}$ in such a way that $x^{(1)}$ is identified with w, u is identified with $x^{(2)}$, and the tetrahedron spanned by u and $\Delta_j^{(1)}$ is identified with the tetrahedron spanned by w and $\Delta_j^{(2)}$, $j = 1, 2$. Gluing $CL^{(1)}$ and $CL^{(2)}$ along this identification of their subcomplexes, we obtain a triangulation J of a 3-disc. Then $K = J \cup_{\partial J} C(\partial J)$ is a PL sphere satisfying conditions 1)–3). (We have $K \in \mathcal{T}_4$ because any triangulation of a 3-dimensional sphere is a PL sphere.) In this proof we need $\operatorname{star} x^{(1)} \subset CL^{(1)}$ to be a full subcomplex (otherwise it may happen that J is not a simplicial complex). Hence it is essential that $\bar{\alpha}_1\bigl(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)}\bigr)$ be regular. We shall omit proofs of this kind in what follows.

We denote by $\widetilde{\Delta}_j$ the tetrahedron of K spanned by the edges e and e_j. Let the cycle γ be as above. Then the induced cycle γ_v is homologous to zero for any vertex $v \in L$ different from u and w. The cycle γ_u coincides with the cycle $\alpha_1\bigl(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)}\bigr)$, and γ_v coincides with $\alpha_1\bigl(-L^{(2)}, \Delta_1^{(2)}, \Delta_2^{(2)}\bigr)$. Hence $c\bigl(\bar{\alpha}_1\bigl(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)}\bigr)\bigr) = c\bigl(\bar{\alpha}_1\bigl(L^{(2)}, \Delta_1^{(2)}, \Delta_2^{(2)}\bigr)\bigr)$. To complete the proof, we note that each set $\mathcal{S}_1^1(p,q)$ contains a regular generator.

We denote the value of the function c on the set $\mathcal{S}_1^1(p,q)$ by $\rho(p,q)$.

Proposition 3.8. *For any $p, q > 0$, the restriction of c to $\mathcal{S}_1^2(p,q)$ is a constant function. Let $\tau(p,q)$ be the value of c on $\mathcal{S}_1^2(p,q)$. Then $\tau(p,q)+\tau(q,r)+\tau(r,p) = 0$ for any $p, q, r > 0$.*

Proof. Suppose that $\bar{\alpha}_i(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)}) \in \mathcal{S}_1^2(p,q)$, $r > 0$. Let $\bar{\alpha}_1(L^{(2)}, \Delta_1^{(2)}, \Delta_2^{(2)}) \in \mathcal{S}_1^2(q,r)$ and $\bar{\alpha}_1(L^{(3)}, \Delta_1^{(3)}, \Delta_2^{(3)}) \in \mathcal{S}_1^2(r,p)$ be regular generators. There is a 3-dimensional oriented PL sphere K containing a triangle Δ_0 with vertices $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$ such that

1) the link of $u^{(i)}$ is isomorphic to $L^{(i)}$,

2) the isomorphism $L^{(i)} \to \operatorname{link} u^{(i)}$ takes $\Delta_j^{(i)}$ to the 2-dimensional face of $\widetilde{\Delta}_j$ opposite to $u^{(i)}$, $j = 1, 2$ (here $\widetilde{\Delta}_1, \widetilde{\Delta}_2 \in L$ are the two tetrahedra containing Δ_0).

Arguing as in the proof of Proposition 3.7, we obtain

$$c\bigl(\bar{\alpha}_1\bigl(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)}\bigr)\bigr) + c\bigl(\bar{\alpha}_1\bigl(L^{(2)}, \Delta_1^{(2)}, \Delta_2^{(2)}\bigr)\bigr) + c\bigl(\bar{\alpha}_1\bigl(L^{(3)}, \Delta_1^{(3)}, \Delta_2^{(3)}\bigr)\bigr) = 0.$$

We can replace the generator $\bar{\alpha}_1\big(L^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)}\big)$ by any generator belonging to $\mathcal{S}_1^2(p,q)$. Hence the function c is constant on $\mathcal{S}_1^2(p,q)$. The equality $\tau(p,q) + \tau(q,r) + \tau(r,p) = 0$ follows.

Obviously, $\tau(p,q) = -\tau(q,p)$. Therefore there is a function $\chi : \mathbb{Z}_{>0} \to \mathbb{Q}$ such that $\tau(p,q) = \chi(q) - \chi(p)$ for any $p, q > 0$. The function χ is unique up to a rational constant. The function ρ maps $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ to $\mathbb{Q}$. We extend ρ to $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ by putting $\rho(0,p) = \chi(p)$, $\rho(p,0) = -\chi(p)$, and $\rho(0,0) = 0$.

Proposition 3.9. *The function ρ satisfies the equations*
(i) $\rho(p,q) = -\rho(q,p)$,
(ii) $\rho(p,q+r+2) + \rho(q,r+p+2) + \rho(r,p+q+2) = \rho(p,q+r+1) + \rho(q,r+p+1) + \rho(r,p+q+1)$.

Proof. Since $\alpha_1(L,\Delta_1,\Delta_2) = -\alpha_1(L,\Delta_2,\Delta_1)$, we see that equation (i) holds for $p, q > 0$. If $p = 0$ or $q = 0$, then equation (i) follows immediately from the definition of ρ. Let L be an oriented simplicial 2-sphere containing a vertex x such that there are exactly $(p+q+r+3)$ triangles containing x. Let Δ_1, Δ_2, and Δ_3 be triangles containing x such that a clockwise circuit around x passes successively through Δ_1, r other triangles, Δ_2, p other triangles, Δ_3, q other triangles, and again through Δ_1. We denote by L_j the PL sphere obtained from L by applying the bistellar move associated with Δ_j. It is easy to show that

$$\sum_{j=0}^{2} \alpha_1(L_j, \Delta_{j+1}, \Delta_{j+2}) = \sum_{j=0}^{2} \alpha_1(L, \Delta_{j+1}, \Delta_{j+2}),$$

where the sums of subscripts are understood modulo 3. Applying c to the homology classes of both sides of this equality, we obtain equation (ii).

Proposition 3.10. *Suppose that the function $\rho : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Q}$ satisfies equations (i) and (ii). Then there are $b_1 \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}$ such that $\rho(p,q) = \frac{\lambda(q-p)}{(p+q+2)(p+q+3)(p+q+4)}$ for every $p, q > 0$ and $\rho(0,q) = \frac{\lambda q}{(q+2)(q+3)(q+4)} + b_1$ for every $q > 0$.*

Proof. For any $b_1, \lambda \in \mathbb{Q}$, we easily see that the function given by these formulae satisfies equations (i) and (ii). Thus it remains to prove that the function ρ is uniquely determined by equations (i) and (ii) and the values $\rho(0,1)$ and $\rho(1,2)$. Putting $p = q = r = 0$ and $p = 1, q = r = 0$ in (ii), we get $\rho(0,2) = \rho(0,1)$ and $2\rho(0,3) + \rho(1,2) = 2\rho(0,2)$ respectively. Hence the values $\rho(k,l)$ with $k + l \leqslant 3$ are uniquely determined by $\rho(0,1)$ and $\rho(1,2)$. Let us prove that the values $\rho(k,l)$ with $k + l = m$, $m \geqslant 4$, are uniquely determined by the values $\rho(k,l)$ with $k + l = m - 1$. We consider equations (ii) for all triples (p,q,r) such that $p + q + r + 2 = m$. On the left-hand side of every such equation, we replace all terms $\rho(k,l)$ with $k > l$ by $-\rho(l,k)$ and all terms $\rho(k,k)$ by 0. The resulting system of equations can be regarded as a system of linear equations in the variables $\rho(0,m), \rho(1,m-1), \dots, \rho(n-1,n+1)$ if $m = 2n$ and in the variables $\rho(0,m), \rho(1,m-1), \dots, \rho(n-1,n)$ if $m = 2n-1$. We must prove that this system has a unique solution for any value of the right-hand side, that is, the rank of the system is equal to n in both cases.

To prove this, we shall find a subsystem of n equations such that the matrix of this subsystem is non-singular. If $m = 2n$, then we consider the subsystem consisting of the equations corresponding to the triples $(p,q,r) = (m-2-k,k,0)$, $k = 0,1,\dots,n-1$. If $m = 2n-1$, then we consider the subsystem consisting of the equations corresponding to the triples $(p,q,r) = (m-2-k,k,0)$, $k = 0,1,\dots,n-2$ and the equation corresponding to the triple $(p,q,r) = (m-4,1,1)$. The matrices of these subsystems are easily seen to be non-singular.

Since the function χ was defined up to a constant, we may assume that $\rho(p,q) = \frac{\lambda(q-p)}{(p+q+2)(p+q+3)(p+q+4)}$ for every $p,q \geqslant 0$.

Suppose that $\bar{\alpha}_2(L,\Delta,e) \in \mathcal{S}_2^1(p,q)$. Let Δ_1, Δ_2 and x be as in § 3.2. Let K be an oriented 3-dimensional PL sphere containing a vertex u such that $\operatorname{star} u$ is a full subcomplex of K and $\operatorname{link} u \cong L$. We identify the complexes $\operatorname{link} u$ and L. We denote by $\widetilde{e}$, $\widetilde{\Delta}$, $\widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$ the simplices of K spanned by the vertex u and the simplices e, Δ, Δ_1, and Δ_2 respectively. We apply to K the bistellar move associated with $\widetilde{\Delta}$. Then we apply to the resulting complex the bistellar move associated with $\widetilde{e}$. Then we restore the tetrahedron $\widetilde{\Delta}$. Finally, we restore the tetrahedra $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$. We denote the resulting cycle in Γ_3 by γ. The cycles γ_v are homologous to zero for all vertices v except u and x. Since $\bar{\gamma}_x \in \mathcal{S}_1^1(q,p)$, we have $c(\bar{\gamma}_x) = \rho(q,p)$. Therefore,

$$c(\bar{\alpha}_2(L,\Delta,e)) = c(\bar{\gamma}_u) = -c(\bar{\gamma}_x) = \rho(p,q).$$

Similarly, $c(\bar{\alpha}) = \rho(p,q)$ for any $\bar{\alpha} \in \mathcal{S}_3^1(p,q)$.

Proposition 3.11. *If $\bar{\alpha}_2(L,\Delta,e) \in \mathcal{S}_2^2(p,q)$, then*

$$c(\bar{\alpha}_2(L,\Delta,e)) = \rho(0,p) + \rho(0,q) + b_2,$$

where the number $b_2 \in \mathbb{Q}$ is independent of p and q.

Proof. Suppose that $\bar{\alpha}_2(L^{(0)},\Delta^{(0)},e^{(0)}) \in \mathcal{S}_2^2(r,q)$ is a regular generator. We define the simplices $\Delta_1^{(0)}$, $\Delta_2^{(0)}$, $e^{(0)}$, $x^{(0)}$, and $y^{(0)}$ in the same way as the simplices Δ_1, Δ_2, e, x and y were defined in § 3.2 (see Fig. 2, *c*). There is an oriented 3-dimensional PL sphere K containing tetrahedra $\widetilde{\Delta}$, $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ with the following properties.

1) The tetrahedra $\widetilde{\Delta}$ and $\widetilde{\Delta}_1$ have a common 2-dimensional face. The tetrahedra $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ have a common 2-dimensional face, which is denoted by $\widetilde{e}$. The tetrahedra $\widetilde{\Delta}$, $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ have a common edge, which is denoted by ε. We denote the endpoints of ε by u and v.

2) There is no edge connecting the vertices of $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ that are opposite to the face $\widetilde{e}$.

3) The link of u is isomorphic to L. The isomorphism $L \to \operatorname{link} u$ takes the triangles Δ, Δ_1 and Δ_2 to the 2-dimensional faces of $\widetilde{\Delta}$, $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ (respectively) that are opposite to u.

4) The link of v is isomorphic to $-L^{(0)}$. The anti-isomorphism $L^{(0)} \to \operatorname{link} v$ takes the triangles $\Delta^{(0)}$, $\Delta_1^{(0)}$ and $\Delta_2^{(0)}$ to the 2-dimensional faces of $\widetilde{\Delta}$, $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ (respectively) that are opposite to v.

Let the cycle γ be as in the previous case. We denote by w the vertex that belongs to $\widetilde{\Delta}$ and $\widetilde{\Delta}_1$ and coincides with neither u nor v. We have $c(\bar{\gamma}_u)+c(\bar{\gamma}_v)+c(\bar{\gamma}_w)=0$ since γ_t is homologous to zero for every vertex t except for u, v and w. But $c(\bar{\gamma}_w)=-\rho(0,p)+\rho(0,r)$ since $\bar{\gamma}_w\in S_1^2(p,r)$. The cycle γ_u coincides with $\alpha_2(L,\Delta,e)$, and the cycle γ_v coincides with $\alpha_2\bigl(-L^{(0)},\Delta^{(0)},e^{(0)}\bigr)$. Hence,

$$c(\bar{\alpha}_2(L,\Delta,e))-c\bigl(\bar{\alpha}_2\bigl(L^{(0)},\Delta^{(0)},e^{(0)}\bigr)\bigr)=\rho(0,p)-\rho(0,r).$$

Therefore the restriction of c to $\mathcal{S}_2^2(p,q)$ is a constant function. We denote the value of c on $\mathcal{S}_2^2(p,q)$ by $\xi(p,q)$. Then

$$\xi(p,q)-\xi(r,q)=\rho(0,p)-\rho(0,r).$$

If $\bar{\alpha}_2(L,\Delta,e)\in\mathcal{S}_2^2(p,q)$, then $-\bar{\alpha}_2(-L,\Delta,e)\in S_2^2(q,p)$. Hence $\xi(p,q)=\xi(q,p)$. The proposition follows.

Suppose that $\bar{\alpha}_3(L,e_1,e_2)\in\mathcal{S}_3^2(p,q)$. Let x, y, e, Δ_1, Δ_2, Δ_3, Δ_4 be as in Fig. 3, *c*. Let K be an oriented 3-dimensional PL sphere containing a vertex u such that $\operatorname{star} u$ is a full subcomplex of K and $\operatorname{link} u\cong L$. We identify the complexes $\operatorname{link} u$ and L. We denote by $\widetilde{e}_j$, $\widetilde{\Delta}_j$ the simplices spanned by the vertex u and the simplices e_j, Δ_j respectively. We assume that there are exactly r triangles that contain e and are different from $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$. We apply to K the bistellar move associated with $\widetilde{e}_1$. Then we apply to the resulting complex the bistellar move associated with $\widetilde{e}_2$. Then we restore the tetrahedra $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_3$. Finally, we restore the tetrahedra $\widetilde{\Delta}_2$ and $\widetilde{\Delta}_4$. We denote the resulting cycle in Γ_3 by γ. The cycle γ_v is homologous to zero for any vertex v except u, x and y. The cycle γ_u coincides with $\alpha_3(L,e_1,e_2)$. But $\bar{\gamma}_x\in\mathcal{S}_2^2(r,p)$ and $(-\bar{\gamma}_y)\in\mathcal{S}_2^2(r,q)$. Therefore $c(\bar{\alpha}_3(L,e_1,e_2))=\rho(0,q)-\rho(0,p)$.

Let p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{25}, p_{34}, p_{35}, p_{36}, p_{45}, p_{46}, p_{56} be integers greater than 2. Suppose that $\omega_j\in Z_1(\Gamma_2;\mathbb{Z})$, $j=1,2,\dots,6$ are cycles such that the following conditions hold:

1) $\omega_1=\alpha_4(L_1,x_1,y_1,z_1)$, $\bar{\omega}_1\in\mathcal{S}_4(p_{13},p_{14},p_{12})$,
2) $\omega_2=\alpha_5(L_2,x_2,y_2,z_2,u_2)$, $\bar{\omega}_2\in\mathcal{S}_5(p_{23},p_{12},p_{24},p_{25})$,
3) $\omega_3=\alpha_6(L_3,x_3,y_3,z_3,u_3,v_3)$, $\bar{\omega}_3\in\mathcal{S}_6(p_{34},p_{13},p_{23},p_{35},p_{36})$,
4) $\omega_4=\alpha_6(L_4,x_4,y_4,z_4,u_4,v_4)$, $\bar{\omega}_4\in\mathcal{S}_6(p_{34},p_{46},p_{45},p_{24},p_{14})$,
5) $\omega_5=\alpha_5(L_5,x_5,y_5,z_5,u_5)$, $\bar{\omega}_5\in\mathcal{S}_5(p_{45},p_{56},p_{35},p_{25})$,
6) $\omega_6=\alpha_4(L_6,x_6,y_6,z_6)$, $\bar{\omega}_6\in\mathcal{S}_4(p_{46},p_{36},p_{56})$,
7) at least 5 of the generators $\bar{\omega}_j$ are regular.

We denote by $L_j^{(1)}$, $j=1,6$ the PL sphere obtained from L_j by the bistellar move associated with u_j. Here the vertex u_j is defined as the vertex u in Fig. 4. There is a 3-dimensional oriented PL sphere K containing vertices w_j, $j=1,2,\dots,6$ such that the following conditions hold.

1) The full subcomplex spanned by the set $\{w_j,\ j=1,\dots,6\}$ consists of the tetrahedra $\{w_1,w_2,w_3,w_4\}$, $\{w_2,w_3,w_4,w_5\}$, and $\{w_3,w_4,w_5,w_6\}$ and all their faces.

2) For any edge $\{w_i,w_j\}\in K$, $i<j$, there are exactly p_{ij} tetrahedra that contain the edge $\{w_i,w_j\}$ and coincide with none of the tetrahedra $\{w_1,w_2,w_3,w_4\}$, $\{w_2,w_3,w_4,w_5\}$, $\{w_3,w_4,w_5,w_6\}$.

3) The links of the vertices w_j can be identified with the complexes L_j for $1 < j < 6$ and the complexes $L_j^{(1)}$ for $j = 1, 6$ so that the vertices are identified by the formulae $w_1 = y_2 = y_3 = v_4$, $w_2 = z_1 = z_3 = u_4 = u_5$, $w_3 = x_1 = x_2 = x_4 = z_5 = y_6$, $w_4 = y_1 = z_2 = x_3 = x_5 = x_6$, $w_5 = u_2 = u_3 = z_4 = z_6$, $w_6 = v_3 = y_4 = y_5$.

We apply to K the following sequence of bistellar moves.

1) We replace the tetrahedra $\{w_1, w_2, w_3, w_4\}$ and $\{w_2, w_3, w_4, w_5\}$ by the tetrahedra $\{w_1, w_2, w_3, w_5\}$, $\{w_1, w_3, w_4, w_5\}$ and $\{w_1, w_4, w_2, w_5\}$.

2) We replace the tetrahedra $\{w_1, w_3, w_4, w_5\}$ and $\{w_3, w_4, w_5, w_6\}$ by the tetrahedra $\{w_1, w_3, w_4, w_6\}$, $\{w_1, w_4, w_5, w_6\}$ and $\{w_1, w_5, w_3, w_6\}$.

3) We replace the tetrahedra $\{w_1, w_2, w_3, w_5\}$ and $\{w_1, w_5, w_3, w_6\}$ by the tetrahedra $\{w_2, w_3, w_1, w_6\}$, $\{w_2, w_1, w_5, w_6\}$ and $\{w_2, w_5, w_3, w_6\}$.

4) We replace the tetrahedra $\{w_1, w_4, w_2, w_5\}$, $\{w_1, w_6, w_4, w_5\}$ and $\{w_1, w_2, w_6, w_5\}$ by the tetrahedra $\{w_1, w_2, w_6, w_4\}$ and $\{w_2, w_6, w_4, w_5\}$.

5) We replace the tetrahedra $\{w_1, w_2, w_3, w_6\}$, $\{w_1, w_3, w_4, w_6\}$ and $\{w_1, w_4, w_2, w_6\}$ by the tetrahedra $\{w_1, w_2, w_3, w_4\}$ and $\{w_2, w_3, w_4, w_6\}$.

6) We replace the tetrahedra $\{w_2, w_3, w_4, w_6\}$, $\{w_2, w_4, w_5, w_6\}$ and $\{w_2, w_5, w_3, w_6\}$ by the tetrahedra $\{w_2, w_3, w_4, w_5\}$ and $\{w_3, w_4, w_5, w_6\}$.

This sequence of bistellar moves transforms K into itself. Let γ be the resulting cycle in Γ_3. The cycle γ_{w_j} is homologous to ω_j for $j = 3, 5, 6$. The cycle γ_{w_j} is homologous to $-\omega_j$ for $j = 1, 2, 4$. Therefore,

$$c(\bar{\omega}_1) + c(\bar{\omega}_2) - c(\bar{\omega}_3) + c(\bar{\omega}_4) - c(\bar{\omega}_5) - c(\bar{\omega}_6) = 0.$$

Consequently the restriction of c to each of the sets $\mathcal{S}_4(p, q, r)$, $\mathcal{S}_5(p, q, r, k)$, and $\mathcal{S}_6(p, q, r, k, l)$, $p, q, r, k, l \geqslant 3$, is a constant function. We denote the values of c on these sets by $\eta(p, q, r)$, $\zeta(p, q, r, k)$, and $\theta(p, q, r, k, l)$ respectively.

Proposition 3.12. *There is a constant $b_5 \in \mathbb{Q}$ such that*

$$\theta(p, q, r, k, l) = \frac{\lambda}{(p+2)(p+3)} + \frac{\lambda}{(q+2)(q+3)} + \frac{\lambda}{(r+2)(r+3)} + \frac{\lambda}{(k+2)(k+3)} + \frac{\lambda}{(l+2)(l+3)} + b_5$$

for any $p, q, r, k, l \geqslant 3$.

Proof. Suppose that

$$\bar{\alpha}_6(L, x, y, z, u, v) \in \mathcal{S}_6(p, q, r, k, l), \qquad p, q, r, k, l \geqslant 3.$$

Let $\Delta \in L$ be a triangle such that x is a vertex of Δ but y, z, u, v are not. Let us go clockwise round the vertex x. Suppose that we pass successively through Δ, through p' other triangles, through the triangles $\{x, y, z\}$, $\{x, z, u\}$, $\{x, u, v\}$, through p'' other triangles, and then again through Δ. Then $p' + p'' = p - 1$. The cycle $\alpha_6(L, x, y, z, u, v)$ is a sequence of five bistellar moves. Let L_j $(j = 0, 1, 2, 3, 4)$ be the PL sphere obtained from L by the first j of these moves. In particular, $L_0 = L$. We denote by $L_j^{(1)}$ the simplicial complex obtained from L_j by the bistellar move associated with Δ. Let us define a graph G in the following way. The vertex set of G

is the set $\{L_0,\dots,L_4,L_0^{(1)},\dots,L_4^{(1)}\}$. There are 15 edges in G. For each j, an edge with endpoints L_j and $L_j^{(1)}$ corresponds to the bistellar move that transforms L_j into $L_j^{(1)}$ and is associated with Δ. For each j, an edge with endpoints L_j and L_{j+1} corresponds to the $(j+1)$st bistellar move in the cycle $\alpha_6(L,x,y,z,u,v)$. Similarly, for each j, an edge with endpoints $L_j^{(1)}$ and $L_{j+1}^{(1)}$ corresponds to the $(j+1)$st bistellar move in the cycle $\alpha_6(L_0^{(1)},x,y,z,u,v)$. Sums of subscripts are understood modulo 5. There is a canonical map from G to Γ_2. (This map is not necessarily injective.) The graph G is isomorphic to the 1-skeleton of a pentagonal prism. The circuit around a 2-dimensional face of this prism yields a cycle in Γ_2. In this way, we obtain the cycles $\alpha_6(L,x,y,z,u,v)$, $-\alpha_6(L_0^{(1)},x,y,z,u,v)$, $\alpha_2(L_0,\Delta,\{x,z\})$, $\alpha_2(L_1,\Delta,\{u,x\})$, $\alpha_2(L_2,\Delta,\{y,u\})$, $\alpha_2(L_3,\Delta,\{v,y\})$, $\alpha_2(L_4,\Delta,\{z,v\})$. The sum of all these cycles is equal to zero. Consequently,

$$\theta(p,q,r,k,l)-\theta(p+1,q,r,k,l)-\rho(p',p''+1)+\rho(p'+1,p'')=0.$$

Hence,

$$\theta(p+1,q,r,k,l)-\theta(p,q,r,k,l)=-\frac{2\lambda}{(p+2)(p+3)(p+4)}\,.$$

In addition, the function θ is cyclically symmetric. The proposition follows from the equality

$$\sum_{i=1}^{j}\frac{1}{i(i+1)(i+2)}=\frac{1}{4}-\frac{1}{2(j+1)(j+2)}\,.$$

Proposition 3.13. *The constant b_2 in Proposition* 3.11 *is equal to zero.*

Proof. Suppose that

$$\bar{\alpha}_6(L,x,y,z,u,v)\in\mathcal{S}_6(p,q,r,k,l),\qquad p,q,r,k,l\geqslant 3.$$

Let Δ be the triangle that contains the edge $\{x,y\}$ and does not coincide with $\{x,y,z\}$. Arguing as in the proof of Proposition 3.12, we see that

$$\begin{aligned}&\theta(p+1,q+1,r,k,l)-\theta(p,q,r,k,l)\\&\qquad=-\frac{2\lambda}{(p+2)(p+3)(p+4)}-\frac{2\lambda}{(q+2)(q+3)(q+4)}-b_2.\end{aligned}$$

Hence $b_2=0$.

The following two propositions are proved in the same way as Proposition 3.12.

Proposition 3.14. *There is a constant $b_3\in\mathbb{Q}$ such that*

$$\eta(p,q,r)=\frac{\lambda}{(p+2)(p+3)}-\frac{\lambda}{(q+2)(q+3)}+\frac{\lambda}{(r+2)(r+3)}+b_3.$$

for any $p,q,r\geqslant 3$.

Proposition 3.15. *There is a constant $b_4 \in \mathbb{Q}$ such that*

$$\zeta(p,q,r,k) = \frac{\lambda}{(p+2)(p+3)} - \frac{\lambda}{(q+2)(q+3)} - \frac{\lambda}{(r+2)(r+3)} + \frac{\lambda}{(k+2)(k+3)} + b_4.$$

for any $p,q,r,k \geqslant 3$.

Suppose that

$$\bar{\alpha}_6(L,x,y,z,u,v) \in \mathcal{S}_6(p,q,r,k,l), \qquad p,q,r,k,l \geqslant 2.$$

Let Δ be the triangle that contains the edge $\{x,y\}$ and does not coincide with $\{x,y,z\}$. Let $L^{(1)}$ be the PL sphere obtained from L by the bistellar move associated with Δ. Arguing as in the proof of Proposition 3.13, we see that

$$c\bigl(\bar{\alpha}_6(L,x,y,z,u,v)\bigr) - c\bigl(\bar{\alpha}_6(L^{(1)},x,y,z,u,v)\bigr) = \frac{2\lambda}{(p+2)(p+3)(p+4)} + \frac{2\lambda}{(q+2)(q+3)(q+4)}\,.$$

Consequently the restriction of c to $S_6(p,q,r,k,l)$ is a constant function and the formula of Proposition 3.12 holds for any $p,q,r,k,l \geqslant 2$. Similarly the formulae of Propositions 3.14 and 3.15 hold for any $p,q,r \geqslant 2$ and $p,q,r,k \geqslant 2$ respectively. The formula of Proposition 3.14 also holds for $p=q=r=1$.

Obviously, $\eta(1,1,1)=0$ and $\zeta(2,2,2,2)=0$. Hence, $b_3 = -\frac{\lambda}{12}$ and $b_4 = 0$. It is easy to show that $\theta(2,2,2,2,2) = -5\eta(2,2,2) = \frac{\lambda}{6}$. Therefore, $b_5 = -\frac{\lambda}{12}$.

Thus, $c(\bar{\alpha}) = \lambda c_0(\bar{\alpha})$ for every $\bar{\alpha} \in \mathcal{S}$.

3.5. The constant $\boldsymbol{\lambda}$. We have $s^*(\varphi) = \lambda c_0$ for some $\lambda \in \mathbb{Q}$. To prove that $\lambda = 1$, we must show that the formula of Corollary 3.3 holds for at least one oriented 4-dimensional combinatorial manifold with non-zero first Pontryagin number.

Kühnel and Banchoff [16] constructed a triangulation K of $\mathbb{C}P^2$ with 9 vertices (see also [17]). The links of all vertices of this triangulation are isomorphic to the same 3-dimensional oriented PL sphere L. L is one of the two 3-dimensional PL spheres with 8 vertices that are not polytopal spheres (see [18]). One can number the vertices of L in such a way that the 3-dimensional simplices are given by

1243	3476	5386	7165
1237	3465	4285	1785
1276	4576	4875	1586
2354	2385	4817	1682
2376	2368	4371	1284

(The order of the vertices in each simplex is chosen in such a way that the signature of $\mathbb{C}P^2$ with the corresponding orientation is equal to $+1$.)

We consider the following sequence of nine bistellar moves.

1) We replace the simplices 1243, 1237, and 4371 by the simplices 1247 and 3274.

2) We replace the simplices 2354, 2385, and 4285 by the simplices 2384 and 3584.
3) We replace the simplices 7165, 1785, and 1586 by the simplices 1786 and 5687.
4) We replace the simplices 1786, 1682, and 1276 by the simplices 1278 and 6287.
5) We replace the simplices 1247, 1278, 1284, and 4817 by the simplex 2487.
6) We replace the simplices 3274, 2384, and 2487 by the simplices 2387 and 3487.
7) We replace the simplices 2387, 6287, 2376, and 2368 by the simplex 6387.
8) We replace the simplices 3465, 5386, and 3584 by the simplices 4386 and 5486.
9) We replace the simplices 4386, 6387, 3476, and 3487 by the simplex 6487.

This sequence transforms L into the boundary of a 4-dimensional simplex. Using this sequence, one can check the formula of Corollary 3.3 by direct calculation.

Corollary 3.4. *Suppose that f is a local formula representing φ. Then $f(L)=\frac{1}{3}$.*

§ 4. The denominators of the values of the local formulae

Suppose that $f\in\mathcal{T}^n(\mathbb{Q})$ is a local formula. Let us estimate the denominator of $f(L)$ via the number of vertices of L. As in § 3.2, let $\mathcal{T}_{n,l}$ be the set of all oriented $(n-1)$-dimensional PL spheres with at most l vertices. We denote by $\operatorname{den}_l(f)$ the least common denominator of all the $f(L)$, $L\in\mathcal{T}_{n,l}$.

4.1. An upper bound. Let us prove the following theorem.

Theorem 4.1. *Suppose that $\psi\in H^n(\mathcal{T}^*(\mathbb{Q}))$ is an arbitrary cohomology class. Then there are a local formula f representing ψ and an integer constant $b\neq 0$ such that the number $\operatorname{den}_l(f)$ is a divisor of $b(l+1)!$ for every $l\geqslant n$.*

Proof. We put $p=\star(\psi)$ and denote the corresponding characteristic class of block bundles also by p. We define functions g,h,f_1 and a local formula f representing ψ as in § 2.3. There is a positive integer b_1 such that the characteristic class b_1p belongs to the image of the natural homomorphism $H^n(\mathrm{B}\widetilde{\mathrm{PL}}_n;\mathbb{Z})\to H^n(\mathrm{B}\widetilde{\mathrm{PL}}_n;\mathbb{Q})$. Then the function g can be chosen in such a way that $b_1g(J)$ is an integer for any totally ordered triangulation J of an n-disc. Hence the denominator of the value $f(L)=\frac{h(CL)-h(-CL)}{2}$ is a divisor of $2b_1(l+1)!$ for every $L\in\mathcal{T}_{n,l}$.

4.2. A lower bound. Let us prove the following theorem.

Theorem 4.2. *Let f be an arbitrary local formula representing the generator φ of the group $H^4(\mathcal{T}^*(\mathbb{Q}))$. Then the number $\operatorname{den}_l(f)$ is divisible by the least common multiple of the numbers $1,2,3,\dots,l-3$ for any even $l\geqslant 10$.*

Proof. We put $l=2k$. We consider a convex $(l-5)$-gon with vertices $v_1,v_2,\dots,v_{l-5}$. Suppose that L_0 is an arbitrary triangulation of this $(l-5)$-gon such that the vertex set of L_0 coincides with the set $\{v_1,v_2,\dots,v_{l-5}\}$. We add to L_0 a vertex v_0 and the triangles $\{v_0,v_1,v_2\}$, $\{v_0,v_2,v_3\}$, …, $\{v_0,v_{l-5},v_1\}$. Thus we obtain a two-dimensional PL sphere L. We orient this PL sphere in such a way that the triangle $\{v_0,v_1,v_2\}$ is positively oriented. We put $\alpha=\alpha_1(L,\{v_0,v_1,v_2\},\{v_0,v_{k-2},v_{k-1}\})$. Then $\bar\alpha\in\mathcal{S}_1^1(k-4,k-3)$. Therefore,

$$c_0(\bar\alpha)=\frac{1}{(l-5)(l-4)(l-3)}\,.$$

We put $\widehat{c}_0 = s(f)$. Let the cycle α consist of the bistellar moves β_1, β_2, β_3, and β_4. Then
$$\sum_{i=1}^{4} f(L_{\beta_i}) = \sum_{i=1}^{4} \widehat{c}_0(e_{\beta_i}) = c_0(\bar{\alpha}).$$
Hence the least common denominator of the values $f(L_{\beta_i})$ is divisible by $(l-5)(l-4)(l-3)$. Evidently, $L_{\beta_i} \in \mathcal{T}_{4,l}$. Therefore $\operatorname{den}_l(f)$ is divisible by $(l-5)(l-4)(l-3)$ for any even $l \geqslant 10$. But $\operatorname{den}_l(f)$ is divisible by $\operatorname{den}_m(f)$ for any $m < l$. Consequently $\operatorname{den}_l(f)$ is divisible by the least common multiple of the numbers $5, 6, \dots, l-3$ for any even $l \geqslant 10$. It remains to prove that $\operatorname{den}_{10}(f)$ is divisible by 4.

Suppose that L is a two-dimensional PL sphere on the set $\{v_0, v_1, \dots, v_6\}$ such that L consists of the triangles $\{v_0, v_1, v_6\}$, $\{v_1, v_2, v_6\}$, $\{v_2, v_0, v_6\}$, $\{v_0, v_2, v_3\}$, $\{v_0, v_3, v_4\}$, $\{v_0, v_4, v_5\}$, $\{v_0, v_5, v_1\}$, $\{v_1, v_5, v_4\}$, $\{v_2, v_1, v_4\}$, $\{v_3, v_2, v_4\}$, and their edges and vertices. Then $c_0(\bar{\alpha}_4(L, v_0, v_1, v_2)) = \eta(4,3,3) = -\frac{5}{84}$. Let the cycle $\alpha_4(L, v_0, v_1, v_2)$ consist of the bistellar moves β_1, β_2 and β_3. Then the least common multiple of the values $f(L_{\beta_i})$, $i = 1,2,3$ is divisible by 4. To complete the proof, we note that $L_{\beta_i} \in \mathcal{T}_{4,10}$, $i = 1,2,3$.

4.3. Local formulae with coefficients in subgroups of $\mathbb{Q}$. We obviously have $\mathcal{T}^1(G) = 0$ and $\mathcal{T}^2(G) = 0$ for any Abelian group G that contains no elements of order 2. By Corollary 3.1, the group $\mathcal{T}^3(\mathbb{Q})$ does not contain non-zero local formulae. Hence the group $\mathcal{T}^3(G)$ does not contain non-zero local formulae for any subgroup $G \subset \mathbb{Q}$.

Theorem 4.3. *We have $H^4(\mathcal{T}^*(G)) = 0$ for any proper subgroup $G \subset \mathbb{Q}$.*

Proof. Suppose that $f \in \mathcal{T}^4(\mathbb{Q})$ is a local formula which is not a coboundary. It follows from Theorem 4.2 that for any positive integer q there is a positive integer l such that $\operatorname{den}_l(f)$ is divisible by q. Therefore, if $f \in \mathcal{T}^4(G)$ is a local formula, then there is a $g \in \mathcal{T}^3(\mathbb{Q})$ such that $f = \delta g$. It remains to prove that $g \in \mathcal{T}^3(G)$. Recall that $s(f) = dg \in C^1_{\mathbb{Z}_2}(\Gamma_2; \mathbb{Q})$. The value of the cochain $s(f) = dg$ on every edge of Γ_2 belongs to G since $f \in \mathcal{T}^4(G)$. Hence $g \in \mathcal{T}^3(G)$.

§ 5. Decompositions into simple cells

In this section all polyhedra are supposed to be compact and all manifolds are supposed to be compact manifolds with or without boundary. We denote by L' the barycentric subdivision of a simplicial complex L.

5.1. $\mathcal{D}$-structures. Suppose that L is an $(n-1)$-dimensional PL sphere and $\tau \neq \varnothing$ is a simplex of L. Let S_τ be the set of barycentres of all simplices $\sigma \supseteq \tau$, $\sigma \in L$. We denote by Q_τ the full subcomplex of L' spanned by S_τ. We assume that $Q_\varnothing = CL'$. Then $\dim Q_\tau = n - 1 - \dim \tau$. (We assume that $\dim \varnothing = -1$.) The *simple cell* Q dual to L is the simplicial complex CL' with the decomposition into closed subsets Q_τ. The subsets Q_τ are called *faces* of the simple cell Q. Faces of codimension one are called *facets*. The vertex of the cone CL' is called the *barycentre* of Q. The *barycentre of the face Q_τ* is the barycentre of the simplex $\tau \in L$. The triangulation CL' is called the *barycentric subdivision of the simple cell Q*. Obviously, Q_τ is a simple cell dual to $\operatorname{link} \tau$. If L is the boundary of a convex simplicial polytope, then Q is the dual simple polytope. In particular, the simplex Δ^n and the cube I^n are simple cells.

A simple cell is always the simple cell dual to some PL sphere. An *isomorphism of two simple cells* is an isomorphism of their barycentric subdivisions taking each face of the first onto some face of the second. Evidently, two simple cells dual to PL spheres L_1 and L_2 are isomorphic if and only if L_1 and L_2 are isomorphic. Let us consider several simple cells and glue them together along isomorphisms of their faces. We allow two simple cells to be glued along several faces, but we do not allow two different vertices of a simple cell to become identified by this gluing. Then we obtain a *complex of simple cells*. It is easy to define the notions of barycentric subdivision and isomorphism for complexes of simple cells.

Definition 5.1. A *decomposition of a polyhedron P into simple cells* is a pair (K, h), where K is a complex of simple cells and $h : P \to K$ is a homeomorphism. (We usually identify P and K.) Two decompositions (K_1, h_1) and (K_2, h_2) of P into simple cells are said to be *isomorphic* if there is an isomorphism $i : K_1 \to K_2$ such that $i \circ h_1 = h_2$. In what follows, decompositions are always understood as decompositions into simple cells unless otherwise stated.

Suppose that Y is a decomposition of a polyhedron P into simple cells. Let us consider a subset $R \subset P$ such that R is the union of several (closed) cells of Y. The corresponding decomposition of R into simple cells is called the *restriction of Y to R*.

Definition 5.2. Two decompositions Y_1 and Y_2 of P into simple cells are said to be *equivalent* if there is a decomposition X of $P \times [0, 1]$ into simple cells such that the restriction of X to $P \times \{0\}$ is isomorphic to Y_1 and the restriction of X to $P \times \{1\}$ is isomorphic to Y_2.

An equivalence class of decompositions of P into simple cells is called a *$\mathcal{D}$-structure* on P. We denote by $D(P)$ the set of all decompositions of P into simple cells. Let $\mathcal{D}(P)$ be the set of all $\mathcal{D}$-structures on P.

We denote by $\dim_x P$ the local dimension of P at a point $x \in P$. The following proposition will be proved in § 5.2.

Proposition 5.1. *Suppose that P_1, P_2 are polyhedra with $\dim_y P_2 \geqslant \dim P_1$ for every $y \in P_2$, $R \subset P_1$ is a closed piecewise-linear subset, and $Y_2 \in D(P_2)$, $X \in D(R)$. Let $h\colon P_1 \to P_2$ be a continuous map taking each cell of X isomorphically onto some cell of Y_2. Then one can find a decomposition $Y_1 \in D(P_1)$ and a piecewise-linear map $g\colon P_1 \to P_2$ such that*

1) *X is the restriction of Y_1 to R,*
2) *g is homotopic to h, the homotopy being constant on R,*
3) *g maps each cell of Y_1 isomorphically onto some cell of Y_2.*

The *direct product* of two decompositions $Y_1 \in D(P_1)$ and $Y_2 \in D(P_2)$ is the decomposition $Y_1 \times Y_2 \in D(P_1 \times P_2)$ whose cells are given by $Q_1 \times Q_2$, where Q_1 is a cell of Y_1 and Q_2 is a cell of Y_2. If Y_1 and $\widetilde{Y}_1$ are equivalent, then $Y_1 \times Y_2$ and $\widetilde{Y}_1 \times Y_2$ are equivalent. Hence the direct product $\mathcal{Y}_1 \times \mathcal{Y}_2 \in \mathcal{D}(P_1 \times P_2)$ of $\mathcal{D}$-structures $\mathcal{Y}_1 \in \mathcal{D}(P_1)$ and $\mathcal{Y}_2 \in \mathcal{D}(P_2)$ is well defined.

Let $h\colon P_1 \to P_2$ be a continuous map. Let us define the pullback $h^*\colon \mathcal{D}(P_2) \to \mathcal{D}(P_1)$ in the following way. We consider an arbitrary $\mathcal{D}$-structure $\mathcal{Y}_2 \in \mathcal{D}(P_2)$. Suppose that n is a positive integer with $\dim_y(P_2 \times I^n) \geqslant \dim P_1$ for every $y \in P_2 \times I^n$, and Y_2 is an arbitrary decomposition in the equivalence class $\mathcal{Y}_2$. By Proposition 5.1, one can find a decomposition $Y_1 \in D(P_1)$ and a map $g\colon P_1 \to P_2 \times I^n$ homotopic to $h \times \mathrm{pt}\colon P_1 \to P_2 \times I^n$ such that g maps each cell of Y_1 isomorphically onto some cell of $Y_2 \times I^n$. We denote the equivalence class of Y_1 by $h^*\mathcal{Y}_2$ and call it the *pullback* of $\mathcal{Y}_2$.

Proposition 5.2. *The $\mathcal{D}$-structure $h^*\mathcal{Y}_2$ depends on $\mathcal{Y}_2$ and h only and not on Y_2, n or g.*

Proof. Let $\widetilde{Y}_1 \in D(P_1)$ be the decomposition obtained by taking $\widetilde{Y}_2$, $\widetilde{n}$ and $\widetilde{g}$ instead of Y_2, n and g respectively. We claim that Y_1 and $\widetilde{Y}_1$ are equivalent. Indeed, first of all, there is a standard embedding of $P_2 \times I^n$ in $P_2 \times I^m$ with $m > n$, and Y_1 is not changed if we replace n by m and g by the composite of g and this standard embedding. Therefore we can assume that $n = \widetilde{n} \geqslant \dim P_1 + 1$.

Since Y_2 and $\widetilde{Y}_2$ are equivalent, we see that there is an $X_2 \in D(P_2 \times I^n \times [0,1])$ whose restrictions to the subsets $P_2 \times I^n \times \{0\}$ and $P_2 \times I^n \times \{1\}$ are isomorphic to Y_2 and $\widetilde{Y}_2$ respectively. Let $G\colon P_1 \times [0,1] \to P_2 \times I^n$ be a homotopy between g and $\widetilde{g}$. We define a map $\widehat{G}\colon P_1 \times [0,1] \to P_2 \times I^n \times [0,1]$ by $\widehat{G}(x,t) = (G(x,t),t)$. We consider the decomposition Y_1 of $P_1 \times \{0\}$ and the decomposition $\widetilde{Y}_1$ of $P_1 \times \{1\}$. The polyhedron $P_1 \times [0,1]$, its subset $P_1 \times \{0\} \sqcup P_1 \times \{1\}$ and the map $\widehat{G}$ satisfy all conditions of Proposition 5.1. Therefore there is a decomposition $X_1 \in D(P_1 \times [0,1])$ such that the restrictions of X_1 to $P_1 \times \{0\}$ and $P_1 \times \{1\}$ are isomorphic to Y_1 and $\widetilde{Y}_1$ respectively. Hence Y_1 and $\widetilde{Y}_1$ are equivalent, as required.

Obviously, the pullback h^* remains unchanged under a homotopy of h. It is easy to show that $(h_2 \circ h_1)^* = h_1^* \circ h_2^*$ for any continuous maps $h_1\colon P_1 \to P_2$ and $h_2\colon P_2 \to P_3$. The *direct sum* of two $\mathcal{D}$-structures $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{D}(P)$ is the $\mathcal{D}$-structure

$$\mathcal{Y}_1 \oplus \mathcal{Y}_2 = d^*(\mathcal{Y}_1 \times \mathcal{Y}_2),$$

where $d\colon P \hookrightarrow P \times P$ is the diagonal map.

Any triangulation of a polyhedron P is a decomposition into simple cells. Any two triangulations of P are equivalent since any triangulation of $P \times \{0\} \sqcup P \times \{1\}$ can be extended to a triangulation of $P \times [0,1]$. Thus the set $\mathcal{D}(P)$ contains a distinguished $\mathcal{D}$-structure $\mathcal{E}_P$ corresponding to triangulations of P. Obviously, $h^*\mathcal{E}_{P_2} = \mathcal{E}_{P_1}$ for any map $h\colon P_1 \to P_2$. We shall show in § 5.3 that $\mathcal{Y} \oplus \mathcal{E}_P = \mathcal{Y}$ for any $\mathcal{Y} \in \mathcal{D}(P)$. The set $\mathcal{D}(P)$ is a commutative semigroup with respect to the direct sum operation. The $\mathcal{D}$-structure $\mathcal{E}_P$ is the zero of this semigroup. A decomposition of a polyhedron is said to be *trivial* if it is equivalent to a triangulation.

5.2. Proof of Proposition 5.1. We first consider the case $(P_1, R) \cong (D^n, S^{n-1})$. Since $\dim_y P_2 \geqslant \dim P_1$ for every $y \in P_2$, we see that there is a piecewise-linear map $h_1 : P_1 \to P_2$ such that:

1) h_1 is homotopic to h, the homotopy being constant on R,
2) $h_1(P_1)$ is contained in the n-skeleton of Y_2,

3) there is an open dense subset $U \subset P_1$ such that the restriction of h_1 to U is a local homeomorphism onto its image.

Let J_1 and J_2 be triangulations of P_1 and P_2 such that the map $h_1 : |J_1| \to |J_2|$ is simplicial. We consider a triangulation K_2 of P_2 such that

1) K_2 is a common (rectilinear) subdivision of J_2 and Y_2',

2) the restriction K_Q of K_2 to any n-dimensional cell Q of Y_2 is isomorphic to some (rectilinear) subdivision of an n-simplex.

Let K_1 be a subdivision of J_1 such that the map $h_1 : |K_1| \to |K_2|$ is simplicial. Then the restriction of h_1 to any simplex of K_1 is a linear homeomorphism onto some simplex of K_2.

Given an n-cell Q of the decomposition Y_2, we denote the $(n-1)$-skeleton of K_Q by L_Q. We realize Q as a geometric n-simplex $\Delta^n \subset \mathbb{R}^n$ in such a way that all the simplices of K_Q are realized as geometric simplices. Take a point o in the interior of Δ^n such that o belongs to none of the planes containing simplices of L_Q. We denote the radial projection from o by $\pi\colon |L_Q| \to \partial\Delta^n$. Let $\widetilde{L}_Q$ be a subdivision of L_Q such that the image of each simplex of $\widetilde{L}_Q$ under π is contained in some facet of Δ^n. We denote by $\widetilde{\pi}\colon |\widetilde{L}_Q| \to \partial\Delta^n$ the pseudo-radial projection from the point o (see [6] for a definition). Unlike π, the map $\widetilde{\pi}$ is piecewise linear.

Let us define a piecewise-linear map $t\colon \Delta^n \to \Delta^n$ such that $t|_{|L_Q|} = \widetilde{\pi}$. Let τ_0 be the n-simplex of K_Q such that o belongs to the interior of τ_0. We extend the homeomorphism $\widetilde{\pi}|_{\partial\tau_0}\colon \partial\tau_0 \to \partial\Delta^n$ to a homeomorphism $\tau_0 \to \Delta^n$. This yields a realization of the restriction of t to τ_0. Suppose that τ is an n-simplex of K_Q, $\tau \neq \tau_0$. We denote by $(\partial\tau)_-$ the set of all points $x \in \partial\tau$ such that the interval with endpoints x and o is disjoint from the interior of τ. We denote the closure of $\partial\tau\backslash(\partial\tau)_-$ by $(\partial\tau)_+$. The quadruple $(\partial\tau, (\partial\tau)_-, (\partial\tau)_+, (\partial\tau)_- \cap (\partial\tau)_+)$ is homeomorphic to the standard quadruple $(S^{n-1}, D_-^{n-1}, D_+^{n-1}, S^{n-2})$. The projection $\widetilde{\pi}$ maps each of the sets $(\partial\tau)_-$ and $(\partial\tau)_+$ homeomorphically onto some closed disc $B^n \subset \partial\Delta^n$. Let b_τ be the barycentre of τ. We denote by τ_- and τ_+ the cones with vertex b_τ over $(\partial\tau)_-$ and $(\partial\tau)_+$ respectively. We put $\tau_m = \tau_+ \cap \tau_-$. Let the restriction of t to τ_m be realized by an arbitrary homeomorphism $\tau_m \to \overline{\partial\Delta^n\backslash B}$ coinciding with $\widetilde{\pi}$ on $\partial\tau_m = (\partial\tau)_- \cap (\partial\tau)_+$. Then the restriction of t to either of the sets $\partial\tau_-$ and $\partial\tau_+$ is a homeomorphism of this set onto $\partial\Delta^n$. Extending these homeomorphisms to homeomorphisms $\tau_- \to \Delta^n$ and $\tau_+ \to \Delta^n$ respectively, we obtain the restriction of t to τ.

The map $t\colon Q \to Q$ is constant on ∂Q. We see that t maps the simplex τ_0 and all the sets τ_-, τ_+ (for every τ) homeomorphically onto Q. We consider such maps for all n-cells Q. We obtain a map $t\colon \operatorname{Sk}^n Y_2 \to \operatorname{Sk}^n Y_2$ which is homotopic to the identity map and coincides with the identity map on the $(n-1)$-skeleton of Y_2. Let ρ be any n-simplex of K_1. Then h_1 maps ρ homeomorphically onto some n-simplex $\tau \in K_2$. Consider the n-cell Q of Y_2 such that $\tau \subset Q$. If $\tau \neq \tau_0$, then the partition $\tau = \tau_- \cup \tau_+$ induces a partition $\rho = \rho_- \cup \rho_+$. The sets ρ_- and ρ_+ are cells. Thus we obtain a cell decomposition Y_1 of P_1. The composite $g = t \circ h_1$ maps each cell of Y_1 homeomorphically onto some cell of Y_2. Therefore Y_1 can be regarded as a decomposition into simple cells. It is easy to show that the decomposition Y_1 and the map g satisfy all conditions of Proposition 5.1.

For an arbitrary polyhedral pair (P_1, R), the proof is by induction over the cells of P_1.

5.3. The classifying space $\mathcal{Z}$. A *numbering of the vertices of a simple cell Q* is an injection of the vertex set of Q into $\mathbb{Z}$. A *numbering of the vertices of a decomposition Y into simple cells* is a map of the vertex set of Y to $\mathbb{Z}$ whose restriction to the vertex set of each cell $Q \in Y$ is injective. An *isomorphism between two simple cells with numbered vertices* is a numbering-preserving isomorphism. Sometimes we do not distinguish between a simple cell and its isomorphism class.

We denote by $\mathcal{P}_n$ the set of all isomorphism classes of n-dimensional simple cells with numbered vertices (here simple cells are not supposed to be oriented). Let us construct the space $\mathcal{Z}$ by induction over skeletons. The zero-dimensional skeleton of $\mathcal{Z}$ is the set $\mathbb{Z}$. The set of n-dimensional cells of $\mathcal{Z}$ coincides with $\mathcal{P}_n$. Each simple cell $Q \in \mathcal{P}_n$ is attached to the $(n-1)$-skeleton of $\mathcal{Z}$ in such a way that each facet of Q is mapped isomorphically onto the corresponding cell of $\mathcal{Z}$. The resulting decomposition of $\mathcal{Z}$ into simple cells is also denoted by $\mathcal{Z}$.

In the definition of a pullback in § 5.1, it is immaterial whether or not P_2 is a compact polyhedron. Hence for any continuous map $h\colon P \to \mathcal{Z}$ there is a well-defined $\mathcal{D}$-structure $h^*\mathcal{Z} \in \mathcal{D}(P)$, which remains unchanged under homotopies of h. Thus we have a natural map $i_P\colon [P, \mathcal{Z}] \to \mathcal{D}(P)$.

Theorem 5.1. *The natural map i_P is a bijection.*

Proof. Suppose that $Y \in D(P)$. Choose a numbering of vertices of Y. Let $\overline{Y}$ be the resulting decomposition with numbered vertices. We map each cell of $\overline{Y}$ isomorphically onto the corresponding cell of $\mathcal{Z}$ and denote the resulting map $P \to \mathcal{Z}$ by $g_{\overline{Y}}$. The equivalence class $g^*_{\overline{Y}}\mathcal{Z}$ contains Y. Hence i_P is a surjection.

Consider two maps $h_0, h_1\colon P \to \mathcal{Z}$ such that $h_0^*\mathcal{Z} = h_1^*\mathcal{Z} = \mathcal{Y}$. By Proposition 5.1, there are decompositions $\overline{Y}_0, \overline{Y}_1 \in \mathcal{Y}$ with numbered vertices such that h_0 is homotopic to $g_{\overline{Y}_0}$ and h_1 is homotopic to $g_{\overline{Y}_1}$. Suppose that the numbers of all vertices of $\overline{Y}_0$ are different from the numbers of all vertices of $\overline{Y}_1$. Then there is a decomposition $\overline{X} \in D(P \times [0,1])$ with numbered vertices such that the restrictions of $\overline{X}$ to $P \times \{0\}$ and $P \times \{1\}$ are isomorphic to $\overline{Y}_0$ and $\overline{Y}_1$ respectively. The map $g_{\overline{X}}\colon P \times [0,1] \to \mathcal{Z}$ is a homotopy between $g_{\overline{Y}_0}$ and $g_{\overline{Y}_1}$. Now assume that the number of some vertex of $\overline{Y}_0$ coincides with that of some vertex of $\overline{Y}_1$. We denote by $\overline{Y}_2$ a decomposition $\overline{Y}_0$ with another numbering of vertices such that the numbers of all vertices of $\overline{Y}_2$ are different from those of all vertices of $\overline{Y}_0$ and $\overline{Y}_1$. Arguing as above, we see that $g_{\overline{Y}_2}$ is homotopic to both $g_{\overline{Y}_0}$ and $g_{\overline{Y}_1}$. Thus h_0 and h_1 are homotopic. Therefore i_P is an injection.

For any $\mathcal{Y} \in \mathcal{D}(P)$ we denote by $g_{\mathcal{Y}}$ an arbitrary map representing $i_P^{-1}(\mathcal{Y})$. The map $g_{\mathcal{E}_P}$ is obviously homotopic to a constant map for any polyhedron P.

Given any $\psi \in H^*(\mathcal{Z}; G)$, we have a function that takes every $\mathcal{Y} \in \mathcal{D}(P)$ to $\psi(\mathcal{Y}) = g^*_{\mathcal{Y}}(\psi) \in H^*(P; G)$. This function is called a *characteristic class* of decompositions into simple cells. We shall also say that the cohomology classes $\psi \in H^*(\mathcal{Z}; G)$ and $\psi(\mathcal{Y}) \in H^*(P; G)$ are *characteristic classes* of decompositions into simple cells.

Suppose that $\chi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is an injection. We denote by μ the map $\mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ that coincides with χ on the zero-dimensional skeleton of $\mathcal{Z}$ and maps cells of $\mathcal{Z} \times \mathcal{Z}$

isomorphically onto cells of $\mathcal{Z}$. It is easy to show that the homotopy class of μ is independent of χ. If $\mathcal{Y}_1 \in \mathcal{D}(P_1)$ and $\mathcal{Y}_2 \in \mathcal{D}(P_2)$, then we obviously have

$$g_{\mathcal{Y}_1 \times \mathcal{Y}_2} \simeq \mu \circ (g_{\mathcal{Y}_1} \times g_{\mathcal{Y}_2}).$$

Hence if $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{D}(P)$, then

$$g_{\mathcal{Y}_1 \oplus \mathcal{Y}_2} \simeq \mu \circ (g_{\mathcal{Y}_1} \times g_{\mathcal{Y}_2}) \circ d.$$

Therefore $\mathcal{Y} \oplus \mathcal{E}_P = \mathcal{Y}$ for any $\mathcal{Y} \in \mathcal{D}(P)$.

5.4. The cohomology of $\mathcal{Z}$. Suppose that Q is an oriented cell of $\mathcal{Z}$. Then Q is dual to some PL sphere $L \in \mathcal{T}_n$. The correspondence $Q \mapsto L$ induces a homomorphism $\varkappa\colon H^*(\mathcal{T}^*(G)) \to H^*(\mathcal{Z}; G)$.

Proposition 5.3. *If $G = \mathbb{Q}$, then $\varkappa$ is an isomorphism.*

Proof. By a *factorization* of $Q \in \mathcal{P}_n$ we always mean a factorization of the form $Q = Q_1 \times \Delta^k$, where Q_1 is a simple cell of dimension $n-k$. Let $\widehat{\mathcal{P}}_n$ be the set of all isomorphism classes of n-dimensional simple cells with a given factorization and numbering of vertices. (Here all isomorphisms are supposed to preserve the factorization and the numbering of vertices.) The *rank* of $Q \in \widehat{\mathcal{P}}_n$ is the dimension of the corresponding cell Q_1. We construct a space $\widehat{\mathcal{Z}}$ whose set of cells is $\widehat{\mathcal{P}}_*$ using the same construction as for the space $\mathcal{Z}$ whose set of cells is $\mathcal{P}_*$. There is an embedding $i\colon \mathcal{Z} \hookrightarrow \widehat{\mathcal{Z}}$ that maps each cell Q isomorphically onto the cell Q with the same numbering of vertices and factorization $Q = Q \times \Delta^0$. There is a retraction $r\colon \widehat{\mathcal{Z}} \to \mathcal{Z}$ that maps each cell $Q = Q_1 \times \Delta^k$ isomorphically onto the cell Q with the same numbering of vertices.

Let Z^p be the union of all cells $Q \in \widehat{\mathcal{P}}_*$ whose rank does not exceed p. The sets Z^p form a filtration in $\widehat{\mathcal{Z}}$. Let $E_*^{*,*}$ be the cohomology spectral sequence of this filtration with rational coefficients.

The set $Z^p \setminus Z^{p-1}$ contains no cells of dimension less than p. Therefore $E_1^{p,q} = 0$ for $q < 0$. Each p-cell of $Z^p \setminus Z^{p-1}$ has the factorization $Q = Q \times \Delta^0$ and is a relative cycle in $C_*(Z^p, Z^{p-1}; \mathbb{Z})$. It is easy to show that such cycles are homologous if and only if the corresponding cells are isomorphic. (Here the isomorphism preserves orientation but not the numbering of vertices.) Hence there is an isomorphism $E_1^{p,0} = \mathcal{T}^p(\mathbb{Q})$. The differential δ_1 coincides with the differential δ of $\mathcal{T}^*(\mathbb{Q})$. Hence $E_2^{p,0} = H^p(\mathcal{T}^*(\mathbb{Q}))$.

Consider a relative cycle $\alpha \in C_{p+q}(Z^p, Z^{p-1}; \mathbb{Z})$, $q > 0$. We claim that there is an integer $m \neq 0$ such that $m\alpha$ is a relative boundary. Indeed, α may be written as

$$\alpha = \sum_{j=1}^{n} l_j (Q_j \times \Delta_j^q),$$

where $l_j \in \mathbb{Z}$ and the $Q_j \times \Delta_j^q$ are oriented cells of $\widehat{\mathcal{Z}}$ with $\dim Q_j = p$. We group together all the terms of this sum with the same combinatorial type as Q_j. Then $\alpha = \alpha_1 + \alpha_1 + \ldots + \alpha_k$, where each α_s is the sum of the $l_j(Q_j \times \Delta_j^q)$ with the same combinatorial type as Q_j. Obviously, α_s is a relative cycle for any s. Hence we may assume that all the Q_j have the same combinatorial type. Let $Q = Q \times \Delta^0$ be a cell of $\widehat{\mathcal{Z}}$ isomorphic to the cells Q_j. We denote by m the order of the group of all automorphisms of Q. (Here the automorphisms may change the orientation and numbering of vertices.) The cells $\widetilde{Q}_{j,h} \cong Q \times \Delta^{q+1}$ that contain $Q_j \times \Delta^q_j$ and Q are in one-to-one correspondence with the isomorphisms $h\colon Q_j \cong Q$. We choose the orientation of $\widetilde{Q}_{j,h}$ in such a way that the incidence coefficient of $\widetilde{Q}_{j,h}$ and $Q_j \times \Delta^q_j$ is equal to $+1$. Let $\gamma \in C_{p+q+1}(Z^p, Z^{p-1}; \mathbb{Z})$ be the chain

$$\gamma = \sum_{j=1}^{n} \sum_{h} l_j Q_{j,h},$$

where the inner sum is taken over all isomorphisms $h\colon Q_j \cong Q$. It is easy to show that $\partial\gamma = m\alpha$. Hence $H_{p+q}(Z^p, Z^{p-1}; \mathbb{Z})$ is a torsion group for $q > 0$. Therefore the group $E_1^{p,q} = H^{p+q}(Z^p, Z^{p-1}; \mathbb{Q})$ is trivial for $q > 0$.

Consequently, the spectral sequence $E_*^{*,*}$ stabilizes at the E_2 term. Hence $H^*(\widehat{\mathcal{Z}}; \mathbb{Q}) \cong H^*(\mathcal{T}^*(\mathbb{Q}))$. We consider the sequence of homomorphisms

$$H^*(\mathcal{T}^*(\mathbb{Q})) \xrightarrow{\varkappa} H^*(\mathcal{Z}; \mathbb{Q}) \xrightarrow{r^*} H^*(\widehat{\mathcal{Z}}; \mathbb{Q}) \xrightarrow{i^*} H^*(\mathcal{Z}; \mathbb{Q}).$$

We see that $r^* \circ \varkappa$ is an isomorphism. To complete the proof, we note that $i^* \circ r^*$ is the identity homomorphism.

Remark 5.1. The map $\mu\colon \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ induces a coproduct in $H^*(\mathcal{Z}; \mathbb{Q})$. In § 5.7 we shall define a map $\mathcal{X}\colon \mathrm{BPL} \to \mathcal{Z}$ that takes this coproduct to the standard coproduct in $H^*(\mathrm{BPL}; \mathbb{Q})$. The isomorphism $\varkappa$ enables us to define a product and coproduct in $H^*(\mathcal{T}^*(\mathbb{Q}))$. This raises the problem of finding explicit combinatorial formulae for the product and the coproduct in the cohomology of $\mathcal{T}^*(\mathbb{Q})$. The coproduct can even be defined in $\mathcal{T}^*(\mathbb{Q})$ and is given by

$$a(f)(L_1 \otimes L_2) = f(L_1 * L_2).$$

The problem of finding a combinatorial formula for the product is still open.

Remark 5.2. Let Q be a simple cell. Suppose that a point $x \in Q$ belongs to the interior of a simplex $\tau \in Q'$ whose vertices are the barycentres of cells $Q_1 \subset Q_2 \subset \dots \subset Q_k$. Then we say that x is a point with singularity corresponding to the combinatorial type of Q_1. We define a decomposition of $\widehat{\mathcal{Z}}$ into strata by declaring that a point $(x, y) \in Q \times \Delta^l$ has the same singularity as the point $x \in Q$ for any cell $Q \times \Delta^l$ of $\widehat{\mathcal{Z}}$. It is not hard to prove that the stratum corresponding to the combinatorial type of a simple cell Q is homotopy equivalent to $K(G(Q), 1)$, where $G(Q)$ is the automorphism group of Q. One usually obtains a classifying space for a given classification of singularities by gluing together the spaces $K(G, 1)$, where G stands for the symmetry groups of the singularities (see, for instance, [19], [15]). However we do not want to classify all the decompositions into strata corresponding to combinatorial types of simple cells but only those that arise from decompositions into simple cells. Therefore it is unknown whether the embedding $i\colon \mathcal{Z} \hookrightarrow \widehat{\mathcal{Z}}$ is a homotopy equivalence.

5.5. The dual decomposition.

Definition 5.3. A decomposition $Y \in D(P)$ is said to be *good* if the intersection of any two cells of Y is either empty or a cell of Y. Two good decompositions $Y_1, Y_2 \in D(P)$ are said to be *strongly equivalent* if there is a good decomposition $X \in D(P \times [0,1])$ whose restrictions to $P \times \{0\}$ and $P \times \{1\}$ are isomorphic to Y_1 and Y_2 respectively.

A strong equivalence class of good decompositions of P into simple cells is called a $\mathcal{D}_g$-*structure* on P. We denote the set of all good decompositions of P into simple cells by $D_g(P)$ and the set of all $\mathcal{D}_g$-structures on P by $\mathcal{D}_g(P)$.

Suppose that $Y \in D_g(P)$ and Q is a cell of Y. Let $\mathcal{B}(Q)$ be the set of all cells containing Q. This set is partially ordered by inclusion. The partially ordered set $\mathcal{B}(Q)$ is isomorphic to the set of simplices (including the empty simplex) of some simplicial complex by partially ordered inclusion. It is natural to call this complex the link of Q in Y and denote it by $\operatorname{link} Q$ or $\operatorname{link}_Y Q$. If Y is a triangulation of P, then this definition of link is equivalent to the standard one. If P is a manifold without boundary, then $\operatorname{link} Q$ is a $(\dim P - \dim Q - 1)$-dimensional PL sphere.

Suppose that M^m is a manifold without boundary, Y is a good decomposition of M^m into simple cells, and $v_1, v_2, \dots, v_t$ are the vertices of Y. We denote by Q_j^* the closed star of v_j in the triangulation Y'. The sets $Q_1^*, Q_2^*, \dots, Q_t^*$ are closed and their interiors are disjoint. Each Q_j^* can be regarded as a simple cell dual to $\operatorname{link}_Y v_j$. Thus the decomposition of M^m into the sets Q_j^* is a good decomposition into simple cells. This decomposition is called the decomposition *dual* to Y and is denoted by Y^*. Obviously, $(Y^*)' = Y'$. To each k-cell Q of the decomposition Y we assign the $(n-k)$-cell Q^* of Y^* such that

$$Q^* = Q_{j_1}^* \cap Q_{j_2}^* \cap \dots \cap Q_{j_s}^*,$$

where $v_{j_1}, v_{j_2}, \dots, v_{j_s}$ are the vertices of Q. Then Q^* is a simple cell dual to $\operatorname{link} Q$.

We now extend this definition somewhat. Let $\mathbb{R}_+$ be the closed half-line.

Definition 5.4. Let M^m be a compact polyhedron. Let $\mathcal{F}$ be a filtration

$$\varnothing = F^{-1} \subset F^0 \subset F^1 \subset \ldots \subset F^m = M^m$$

in M^m by closed subsets F^k such that for any $y \in F^k \setminus F^{k-1}$ there are a neighbourhood U of y in M^m and a homeomorphism $i \colon U \to \mathbb{R}^k \times \mathbb{R}_+^{m-k}$ that maps $F^l \cap U$ onto the l-skeleton of $\mathbb{R}^k \times \mathbb{R}_+^{m-k}$ for every l. Then the pair $(M^m, \mathcal{F})$ is called an m-dimensional *manifold with corners*. Connected components of $F^k \setminus F^{k-1}$ are called *open k-dimensional faces* of this manifold with corners. The closure of an open face is called a *closed face*.

Let $(M^m, \mathcal{F})$ be a manifold with corners. Let Y be a good decomposition of M^m into simple cells. Suppose that Y is compatible with $\mathcal{F}$, that is, each subset $F^k \subset M^m$ is a subcomplex of Y. As above, let $v_1, v_2, \dots, v_t$ be the vertices of Y. We denote the closed star of v_j in Y' by Q_j^*. Each set Q_j^* can be regarded as a simple cell with the following facets.

1) The set $Q_j^* \cap Q_l^*$ is a facet of Q_j^* if there is an edge of Y with endpoints v_j and v_l.

2) The closure of any connected component of $Q_j^* \cap (F^{m-1} \setminus F^{m-2})$ is a facet of Q_j^*.

It is easy to show that the simple cell Q_j^* is well defined. The cells Q_j^* are glued together along isomorphisms of their faces. Thus we obtain a good decomposition of M^m into simple cells. It is called the decomposition *dual* to Y and denoted by Y^*.

Everywhere except in § 5.6, we shall assume that M^m is a manifold with or without boundary but without corners. Then the dual decomposition Y^* is well defined for any good decomposition $Y \in D_g(M^m)$ since any $Y \in D_g(M^m)$ is compatible with the filtration. In what follows all manifolds are manifolds without corners unless otherwise stated.

Proposition 5.4. *Suppose that M^m is a manifold and $Y_1, Y_2 \in D_g(M^m)$ are strongly equivalent. Then the decompositions Y_1^* and Y_2^* are strongly equivalent.*

Proof. Let X be a good decomposition of $M^m \times [0,1]$ into simple cells such that the restrictions of X to $M^m \times \{0\}$ and $M^m \times \{1\}$ are isomorphic to Y_1 and Y_2 respectively. The cylinder $M^m \times [0,1]$ is a manifold with corners. The decomposition X is compatible with the corresponding filtration. Then X^* is a good decomposition of $M^m \times [0,1]$ into simple cells such that the restrictions of X^* to $M^m \times \{0\}$ and $M^m \times \{1\}$ are isomorphic to Y_1^* and Y_2^* respectively. Therefore the decompositions Y_1^* and Y_2^* are strongly equivalent.

Hence there is a well-defined map $*\colon \mathcal{D}_g(M^m) \to \mathcal{D}_g(M^m)$. The image of a $\mathcal{D}_g$-structure $\mathcal{Y}$ under this map is called the $\mathcal{D}_g$-structure dual to $\mathcal{Y}$ and is denoted by $\mathcal{Y}^*$.

5.6. The canonical subdivision of a decomposition into simple cells. Given a decomposition Y of a polyhedron into simple cells, we want to construct a subdivision $\widehat{Y}$ of Y such that $\widehat{Y}$ is a good decomposition into simple cells and $\widehat{Y}$ is equivalent to Y.

Suppose that Q is a simple cell. Then Q can be regarded as a manifold with corners. We denote by $\widehat{Q}$ the good decomposition of Q dual to the decomposition $Q' \in D(Q)$. We say that $\widehat{Q}$ is the *canonical subdivision* of Q. It is easy to show that the restriction of $\widehat{Q}$ to any face of Q coincides with the canonical subdivision of this face. Let Y be a decomposition of a polyhedron P into simple cells. We replace each simple cell Q of Y by the canonical subdivision of Q. This yields a good decomposition of P into simple cells. It is called the *canonical subdivision* of Y and denoted by $\widehat{Y}$.

Proposition 5.5. *The decompositions Y and $\widehat{Y}$ are equivalent.*

Proof. Let L be an $(n-1)$-dimensional PL sphere. We consider the standard triangulation of the cylinder $S^{n-1} \times [0,1]$ such that the restrictions of this triangulation to the lower and upper ends of the cylinder are isomorphic to L and L' respectively. Attaching the cone over L with vertex u_0 to the lower end of this triangulation and the cone over L' with vertex u_1 to the upper end, we obtain an n-dimensional PL sphere. We denote it by $\widetilde{L}$. Let us introduce the following terminology. The vertex u_0 is called the *lower* vertex. The vertices belonging to the lower end of the cylinder $S^{n-1} \times [0,1]$ are called *middle* vertices. The vertices belonging to the upper end of the cylinder $S^{n-1} \times [0,1]$ and the vertex u_1 are called *upper* vertices. Any vertex $v \in L$ can be regarded as a middle vertex of $\widetilde{L}$. Obviously, $\operatorname{link}_{\widetilde{L}} v = \widetilde{\operatorname{link}_L v}$.

Let Q be a simple cell dual to L. We denote by $\widetilde{Q}$ the simple cell dual to $\widetilde{L}$. The facets of $\widetilde{Q}$ corresponding to the lower, middle, and upper vertices of $\widetilde{L}$ are called *lower*, *lateral*, and *upper* facets respectively. The lower facet of $\widetilde{Q}$ is evidently isomorphic to Q. It is easy to show that the subcomplex of the boundary of $\widetilde{Q}$ consisting of all upper facets is isomorphic to $\widehat{Q}$. The lateral facet of $\widetilde{Q}$ corresponding to a facet Q_0 of Q is isomorphic to $\widetilde{Q}_0$.

Suppose that a decomposition $Y \in D(P)$ consists of simple cells $Q_1, Q_2, \dots, Q_t$. Each cell Q_j is attached to the cells Q_k of dimension $\dim Q_j - 1$ along isomorphisms ι_{jk}, where ι_{jk} maps some facet of Q_j onto Q_k. The isomorphism ι_{jk} induces an isomorphism $\tilde{\iota}_{jk}$ that maps the corresponding lateral facet of $\widetilde{Q}_j$ onto the cell $\widetilde{Q}_k$. We attach each cell $\widetilde{Q}_j$ to the cells $\widetilde{Q}_k$ along the isomorphisms $\tilde{\iota}_{jk}$. Thus we obtain a decomposition $\widetilde{Y} \in D(P \times [0,1])$ whose restrictions to the lower and upper ends of the cylinder $P \times [0,1]$ are isomorphic to Y and $\widehat{Y}$ respectively.

Corollary 5.1. *If Y is a good decomposition, then the decompositions Y and $\widehat{Y}$ are strongly equivalent.*

Proof. To prove this, we notice that the decomposition $\widetilde{Y}$ is good whenever Y is.

Proposition 5.6. *Suppose that Y_1 and Y_2 are equivalent good decompositions of a polyhedron P into simple cells. Then Y_1 and Y_2 are strongly equivalent.*

Proof. Let $X \in D(P \times [0,1])$ be a decomposition whose restrictions to the ends of the cylinder $P \times [0,1]$ are isomorphic to Y_1 and Y_2 respectively. Then the restrictions of the good decomposition $\widehat{X} \in D_g(P \times [0,1])$ to the ends of the cylinder $P \times [0,1]$ are isomorphic to $\widehat{Y}_1$ and $\widehat{Y}_2$ respectively. By Corollary 5.1, Y_j is strongly equivalent to $\widehat{Y}_j$, $\; j = 1, 2$. The proposition follows.

Corollary 5.2. *The natural map $\mathcal{D}_g(P) \to \mathcal{D}(P)$ is a bijection.*

Thus the $\mathcal{D}_g$-structures on a polyhedron are in one-to-one correspondence with the $\mathcal{D}$-structures on the same polyhedron. In what follows we identify the sets $\mathcal{D}_g(P)$ and $\mathcal{D}(P)$.

Proposition 5.7. *Suppose that M^m is a manifold. Then the map $*\colon \mathcal{D}(M^m) \to \mathcal{D}(M^m)$ is an involution.*

Proof. If M^m is a manifold without boundary, then the proposition is obvious since $Y^{**} = Y$ for every $Y \in D_g(M^m)$. Assume that M^m is a manifold with boundary. Suppose that $\mathcal{Y} \in \mathcal{D}(M^m)$. Let $Y \in D(M^m)$ be an arbitrary decomposition representing the $\mathcal{D}$-structure $\mathcal{Y}$. We denote by N the union of all the closed cells Q of the decomposition $\widehat{Y}$ such that $Q \cap \partial M^m = \varnothing$. Obviously, N is a deformation retract of M^m. Hence $i^*\colon \mathcal{D}(M^m) \to \mathcal{D}(N)$ is an isomorphism, where $i\colon N \hookrightarrow M^m$ is the identity embedding. The restrictions of $\widehat{Y}$ and $(\widehat{Y})^{**}$ to N coincide. Therefore $i^*(\mathcal{Y}) = i^*(\mathcal{Y}^{**})$. Hence $\mathcal{Y} = \mathcal{Y}^{**}$.

5.7. The connection with block bundles. We recall some definitions and results from [8].

Let Y be a decomposition of a polyhedron P into simple cells. Let Q_k, $k = 1, 2, \dots, t$ be the cells of Y. A *q-dimensional block bundle* ξ^q/Y over the decomposition Y is a space $E(\xi)$ containing closed balls β_k, $k = 1, 2, \dots, t$ such that $P \subset E(\xi)$ and the following conditions hold.

1) β_k is a $(\dim Q_k + q)$-dimensional ball containing the cell Q_k such that $\partial Q_k = Q_k \cap \partial\beta_k$ and (β_k, Q_k) is an unknotted ball pair. The ball β_k is called the *block* over the cell Q_k.

2) $E(\xi)$ is the union of the blocks β_k.

3) The interiors of the blocks are disjoint.

4) $\beta_k \cap \beta_l$ is the union of the blocks over the cells of the subcomplex $Q_k \cap Q_l$.

Remark 5.3. In [8], Y is an arbitrary decomposition of a polyhedron P into closed cells (not necessarily simple) such that the boundary of any cell and the intersection of any two cells are unions of cells.

Two block bundles ξ_1 and ξ_2 over the decomposition Y are said to be *isomorphic* if there is a homeomorphism $h\colon E(\xi_1) \to E(\xi_2)$ such that the restriction of h to P is the identity homeomorphism and h maps the blocks of ξ_1 onto the corresponding blocks of ξ_2. For any subdivision Y_1 of the decomposition Y there is a unique (up to isomorphism) subdivision ξ_1/Y_1 of the block bundle ξ/Y. Suppose that $Y_1, Y_2 \in D(P)$. Two block bundles ξ_1/Y_1 and ξ_2/Y_2 are said to be *equivalent* if there is a subdivision of ξ_1 isomorphic to some subdivision of ξ_2. The pullback of an equivalence class of block bundles along a continuous map and the direct product and direct sum of two equivalence classes are defined in a standard way. In what follows we do not distinguish between a block bundle and its equivalence class nor between a block bundle and its stable equivalence class. The set of all stable equivalence classes of block bundles over a polyhedron P is denoted by $I(P)$. Then $I(P)$ is an Abelian group with respect to the direct sum operation. The space BPL is a classifying space for stable block bundles, that is, there is the natural isomorphism $I(P) \cong [P, \mathrm{BPL}]$. The trivial block bundle over P is denoted by ε^n_P. Suppose that $N \subset M$ is a locally flat submanifold with $N \cap \partial M = \partial N$. Then there is a unique (up to equivalence) normal block bundle of the submanifold $N \subset M$, that is, a block bundle ν over N such that $E(\nu) \subset M$. The normal block bundle of the diagonal $M \subset M \times M$ is called the *tangent block bundle* of M. If ξ is a block bundle over a manifold M, then $E(\xi)$ is a manifold.

The aim of this subsection is to define a natural map $\mathcal{X}\colon I(P) \to \mathcal{D}(P)$. First we define this map for manifolds.

Suppose that M is a manifold. To avoid ambiguity, we denote the map $*\colon \mathcal{D}(M) \to \mathcal{D}(M)$ by $*_M$. Let ξ be a block bundle over M. Suppose that $i\colon M \to E(\xi)$ is the identity embedding and $r\colon E(\xi) \to M$ is a retraction. We denote the involution $i^* *_{E(\xi)} r^*\colon \mathcal{D}(M) \to \mathcal{D}(M)$ by $*_\xi$ and the map $*_M *_\xi\colon \mathcal{D}(M) \to \mathcal{D}(M)$ by λ_ξ. We denote by $\mathcal{X}(\xi)$ the $\mathcal{D}$-structure

$$\lambda_\xi(\mathcal{E}_M) = *_M i^* *_{E(\xi)} \mathcal{E}_{E(\xi)} \in \mathcal{D}(M).$$

Remark 5.4. This definition can sometimes be formulated more geometrically. Suppose that $\partial M = \varnothing$, $\dim M = m$. We assume that there is a triangulation K of the space $E(\xi)$ such that the submanifold $M \subset E(\xi)$ is contained in the m-skeleton of the dual decomposition K^*. Then the intersection of M with each simplex of K can be regarded as a simple cell. The resulting decomposition $X \in D(M)$ obviously belongs to the equivalence class $\mathcal{X}(\xi)$. Unfortunately, one cannot define the $\mathcal{D}$-structure $\mathcal{X}(\xi)$ as the equivalence class of X in the general case since there may be no suitable triangulation K.

Suppose that M_1, M_2 are manifolds, $Y_1 \in D(M_1)$, and $Y_2 \in D(M_2)$. Then $(Y_1 \times Y_2)^* = Y_1^* \times Y_2^*$. Hence, if ξ_1 and ξ_2 are block bundles over M_1 and M_2 respectively, then

$$*_{\xi_1 \times \xi_2}(\mathcal{Y}_1 \times \mathcal{Y}_2) = *_{\xi_1}(\mathcal{Y}_1) \times *_{\xi_2}(\mathcal{Y}_2) \qquad (*)$$

for every $\mathcal{Y}_1 \in \mathcal{D}(M_1)$ and $\mathcal{Y}_2 \in \mathcal{D}(M_2)$. The involution $*_\xi$ is determined solely by the stable equivalence class of ξ since $\xi \oplus \varepsilon^n_M = \xi \times \varepsilon^n_{\mathrm{pt}}$ for any block bundle ξ over M. Hence the following proposition is a consequence of formula $(*)$.

Proposition 5.8. *For block bundles over manifolds we have*

$$\mathcal{X}(\xi_1 \times \xi_2) = \mathcal{X}(\xi_1) \times \mathcal{X}(\xi_2), \qquad \mathcal{X}(\xi \oplus \varepsilon^n_M) = \mathcal{X}(\xi), \qquad \mathcal{X}(\varepsilon^n_M) = \mathcal{E}_M.$$

Thus the map $\mathcal{X}\colon I(M) \to \mathcal{D}(M)$ is well defined for any manifold M.

Proposition 5.9. *Suppose that M, N are manifolds, $h\colon M \to N$ is a continuous map, and ω is a block bundle over N. Then*

$$h^*\mathcal{X}(\omega) = \mathcal{X}(h^*\omega).$$

To prove this, we need several auxiliary propositions.

Proposition 5.10. *Suppose that M_1, M_2 are manifolds, $\dim M_1 = \dim M_2$, and $i\colon M_1 \hookrightarrow M_2$ is an embedding. Then*

$$i^* *_{M_2} = *_{M_1} i^*.$$

The proof is similar to that of Proposition 5.7.

Corollary 5.3. *Under the hypotheses of the previous proposition we have*

$$i^* *_\xi = *_{i^*\xi} i^*$$

for every $\xi \in I(M_2)$.

Proposition 5.11. *Suppose that ξ and η are block bundles over a manifold M and τ is the tangent block bundle of M. Then*

$$*_{\xi\oplus\eta\oplus\tau} = *_{\xi\oplus\tau} *_\tau *_{\eta\oplus\tau}.$$

Proof. Suppose that $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{D}(M)$. It follows from formula $(*)$ that

$$*_{\xi\times\eta}(\mathcal{Y}_1 \times \mathcal{Y}_2) = *_{\xi\times M} *_{M\times M} *_{M\times\eta}(\mathcal{Y}_1 \times \mathcal{Y}_2).$$

Suppose that $i_1\colon M \hookrightarrow E(\tau)$, $i_2\colon E(\tau) \hookrightarrow M\times M$ are the identity embeddings and $r\colon E(\tau)\to M$ is a retraction. By Corollary 5.3, we have

$$*_{i_2^*(\xi\times\eta)}(i_2^*(\mathcal{Y}_1\times\mathcal{Y}_2)) = *_{i_2^*(\xi\times M)} *_{E(\tau)} *_{i_2^*(M\times\eta)}(i_2^*(\mathcal{Y}_1\times\mathcal{Y}_2)).$$

Obviously, $E(\omega\oplus\tau)=E(r^*\omega)$ for any block bundle ω over M. Hence $*_{r^*\omega} = r^* *_{\omega\oplus\tau} i_1^*$. But

$$i_2^*(\xi\times\eta)=r^*(\xi\oplus\eta), \qquad i_2^*(\xi\times M)=r^*\xi,$$
$$i_2^*(M\times\eta)=r^*\eta, \qquad i_1^*i_2^*(\mathcal{Y}_1\times\mathcal{Y}_2)=\mathcal{Y}_1\oplus\mathcal{Y}_2.$$

Therefore,

$$*_{\xi\oplus\eta\oplus\tau}(\mathcal{Y}_1\oplus\mathcal{Y}_2) = *_{\xi\oplus\tau} *_{\tau} *_{\eta\oplus\tau}(\mathcal{Y}_1\oplus\mathcal{Y}_2).$$

Corollary 5.4. *We have* $\lambda_{\xi\oplus\eta}=\lambda_\xi\lambda_\eta$, *that is*, $*_{\xi\oplus\eta} = *_\xi *_M *_\eta$.

Proof. Suppose that $\nu=-\tau$ in the group $I(M)$, that is, ν is the stable equivalence class of the normal block bundle of an embedding $M\hookrightarrow\mathbb{R}^q$. Substituting ν for ξ in the formula of Proposition 5.11, we obtain $*_\eta = *_M *_\tau *_{\eta\oplus\tau}$, that is, $*_{\eta\oplus\tau} = *_\tau *_M *_\eta$. Similarly, $*_{\xi\oplus\tau} = *_\tau *_M *_\xi$ and $*_{\xi\oplus\eta\oplus\tau} = *_\tau *_M *_{\xi\oplus\eta}$. To complete the proof we substitute these expressions for $*_{\xi\oplus\tau}$, $*_{\eta\oplus\tau}$, and $*_{\xi\oplus\eta\oplus\tau}$ in the formula of Proposition 5.11.

Proof of Proposition 5.9. We may assume that h is piecewise linear and $h(\partial M) = h(M)\cap\partial N$. Then h is a composite $M\hookrightarrow N\times D^q\to N$, where the first map is an embedding and the second is the projection. It was proved by Zeeman that any submanifold of codimension $\geqslant 3$ is locally flat (see [20]). Hence we can choose q in such a way that the embedding $M\hookrightarrow N\times D^q$ has a normal block bundle ξ. Thus h is a composite

$$M \xrightarrow{h_1} E(\xi) \xrightarrow{h_2} N\times D^q \xrightarrow{h_3} N.$$

It suffices to prove the proposition for each of the maps h_1, h_2, h_3. For h_2 and h_3, it follows immediately from Corollary 5.3 and Proposition 5.8 respectively. Let us prove that $h_1^*\mathcal{X}(\omega)=\mathcal{X}(h_1^*\omega)$ for any block bundle $\omega\in I(E(\xi))$. Let $r\colon E(\xi)\to M$ be a retraction. Since r is a homotopy equivalence, we see that there is a block bundle $\eta\in I(M)$ such that $\omega=r^*\eta$. Therefore,

$$h_1^*\mathcal{X}(\omega) = h_1^* *_{E(\xi)} *_{r^*\eta}\mathcal{E}_{E(\xi)} = (h_1^* *_{E(\xi)} r^*)(h_1^* *_{r^*\eta} r^*)\mathcal{E}_M = *_\xi *_{\xi\oplus\eta}\mathcal{E}_M,$$
$$\mathcal{X}(h_1^*\omega) = *_M *_\eta \mathcal{E}_M.$$

Thus the proposition follows from Corollary 5.4.

Proposition 5.12. *Suppose that τ is the tangent block bundle of a manifold M and $\nu=-\tau$. Then $\mathcal{X}(\nu)=\mathcal{E}_M^*$.*

Proof. Let $i\colon M\hookrightarrow\Delta^q$ be an embedding such that $i(M)\cap\partial\Delta^q = i(\partial M)$ and $q\geqslant \dim M+3$. Then ν is the normal block bundle of the submanifold $i(M)\subset\Delta^q$.

The embedding i is the composite of embeddings $i_1\colon M \hookrightarrow E(\nu)$ and $i_2\colon E(\nu)\hookrightarrow \Delta^q$. Hence,

$$\mathcal{X}(\nu) = *_M *_\nu \mathcal{E}_M = *_M i_1^* *_{E(\nu)} \mathcal{E}_{E(\nu)} = *_M i^* *_{\Delta^q} \mathcal{E}_{\Delta^q} = *_M i^* \mathcal{E}_{\Delta^q} = \mathcal{E}_M^*.$$

Here we have used the equation $\mathcal{E}_{\Delta^q}^* = \mathcal{E}_{\Delta^q}$, which holds because the simplex Δ^q is contractible and, therefore, $\mathcal{D}(\Delta^q) = \{\mathcal{E}_{\Delta^q}\}$.

Now let P be an arbitrary polyhedron. We consider an embedding $i\colon P\hookrightarrow M$ such that M is a manifold and $i(P)$ is a deformation retract of M. Let $r\colon M\to P$ be the retraction. By definition, we put $\mathcal{X}(\xi) = i^*\mathcal{X}(r^*\xi)$ for any $\xi\in I(P)$. It follows from Proposition 5.9 that the $\mathcal{D}$-structure $\mathcal{X}(\xi)$ does not depend on the choice of a manifold M and an embedding i.

Corollary 5.5. *The following formulae hold for block bundles over arbitrary polyhedra*:

$$\mathcal{X}(\varepsilon_P^n) = \mathcal{E}_P, \qquad h^*\mathcal{X}(\xi) = \mathcal{X}(h^*\xi),$$
$$\mathcal{X}(\xi\times\eta) = \mathcal{X}(\xi)\times\mathcal{X}(\eta), \qquad \mathcal{X}(\xi_1\oplus\xi_2) = \mathcal{X}(\xi_1)\oplus\mathcal{X}(\xi_2).$$

Thus the map $\mathcal{X}$ is a natural transformation of the functor $I(\,\cdot\,)$ to the functor $\mathcal{D}(\,\cdot\,)$, that is, a natural transformation of the functor $[\,\cdot\,,\mathrm{BPL}]$ to the functor $[\,\cdot\,,\mathcal{Z}]$. Hence the map $\mathcal{X}$ induces a map $\mathrm{BPL}\to\mathcal{Z}$ which is uniquely determined up to homotopy. We also denote this map by $\mathcal{X}$.

Suppose that $\psi\in H^n(\mathcal{T}^*(\mathbb{Q}))$ is an arbitrary cohomology class. Then $\varkappa(\psi)\in H^n(\mathcal{Z};\mathbb{Q})$ is a characteristic class of decompositions into simple cells and $\mathcal{X}^*(\varkappa(\psi))\in H^n(\mathrm{BPL};\mathbb{Q})$ is a characteristic class of stable block bundles. On the other hand, $\sharp(\psi)\in H^n(\mathrm{BPL};\mathbb{Q})$ is a characteristic class of stable block bundles (see § 2.3). We denote by w the automorphism of $H^*(\mathrm{BPL};\mathbb{Q})$ that takes each Pontryagin class p_k to the cohomology class $\widetilde{p}_k\in H^{4k}(\mathrm{BPL};\mathbb{Q})$, where the classes $\widetilde{p}_k$ are determined by the equation

$$(1+p_1+p_2+\dots)\smile(1+\widetilde{p}_1+\widetilde{p}_2+\dots) = 1.$$

It is easy to show that w is an involution. We have $w(p)(\xi) = p(-\xi)$ for any cohomology class $p\in H^*(\mathrm{BPL};\mathbb{Q})$ and any block bundle ξ.

Proposition 5.13. *The following diagram commutes*:

$$\begin{array}{ccc} H^*(\mathcal{T}^*(\mathbb{Q})) & \xrightarrow[\cong]{\varkappa} & H^*(\mathcal{Z};\mathbb{Q}) \\ {\scriptstyle\sharp}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{X}^*} \\ H^*(\mathrm{BPL};\mathbb{Q}) & \xrightarrow[\cong]{w} & H^*(\mathrm{BPL};\mathbb{Q}). \end{array}$$

Proof. Suppose that M^m is a manifold without boundary, τ is the tangent bundle of M^m, K is a triangulation of M^m, f is a rational local formula, and ψ is the cohomology class represented by f. It follows from Proposition 5.12 that

$$\mathcal{X}^*(\varkappa(\psi))(-\tau) = \varkappa(\psi)(\mathcal{E}_{M^m}^*).$$

Therefore the cohomology class $\mathcal{X}^*(\varkappa(\psi))(-\tau) \in H^n(M^m;\mathbb{Q})$ is represented by the cochain c such that $c(Q) = f(L)$ for any simple cell $Q \in K^*$ dual to a PL sphere $L \in \mathcal{T}_n$. On the other hand, the cohomology class $\psi^\sharp(M^m) = \sharp(\psi)(\tau)$ is represented by the same cochain c. Therefore $\sharp(\psi)(\tau) = \mathcal{X}^*(\varkappa(\psi))(-\tau)$. Consequently $\sharp(\psi)(\tau) = w(\mathcal{X}^*(\varkappa(\psi)))(\tau)$. To complete the proof, we note that a rational characteristic class is uniquely determined by its values on the tangent bundles of closed manifolds.

In § 2.3 we proved that $\sharp$ is an epimorphism. Hence $\mathcal{X}^*$ is an epimorphism. In § 5.8 we shall prove that $\mathcal{X}^*$ is an isomorphism. The following corollary enables us to calculate the cohomology class $p(\xi)$ locally if we are given a decomposition $X \in \mathcal{X}(\xi)$.

Corollary 5.6. *Suppose that $f \in \mathcal{T}^n(\mathbb{Q})$ is a local formula representing $\varphi_{w(p)}$. Let c be the cochain such that $c(Q) = f(L)$ for any simple cell $Q \in X$ dual to a PL sphere $L \in \mathcal{T}_n$. Then c represents the cohomology class $p(\xi)$.*

5.8. $\star$ is a monomorphism for rational coefficients. First let us prove the following auxiliary proposition.

Proposition 5.14. *Suppose that P^q is a q-dimensional polyhedron, $K \in D(P^q)$ is a trivial decomposition, M^n is a manifold without boundary, and $Y \in D(M^n)$. Suppose that $i\colon M^n \hookrightarrow P^q$ is an embedding satisfying the following conditions:*

1) i maps the barycentre of each l-cell of the decomposition Y to that of some $(l+q-n)$-cell of the decomposition K,

2) i linearly maps each simplex of Y' onto some simplex of K'.

Then there is a block bundle $\eta \in I(M^n)$ such that the decomposition Y belongs to the equivalence class $\mathcal{X}(\eta)$.

Proof. Replacing P^q by $P^q \times \Delta^3$, we may assume that $q \geqslant n+3$. The embedding i maps the barycentre of each l-cell of the decomposition $\widehat{Y}$ to the barycentre of some $(l+q-n)$-cell of the decomposition $\widehat{K}$ and linearly maps each simplex of $(\widehat{Y})'$ onto some simplex of $(\widehat{K})'$. We denote by N^q the union of all closed cells $Q \in \widehat{K}$ such that $Q \cap i(M^n) \neq \varnothing$. Let K_1 be the restriction of $\widehat{K}$ to N^q. The image of each l-cell Q_k of $\widehat{Y}$ under i is contained in some $(l+q-n)$-cell β_k of $\widehat{K}$. The pair (β_k, Q_k) is an unknotted ball pair since $q \geqslant n+3$. Thus we have a block bundle η over $\widehat{Y}$ with blocks β_k such that $E(\eta) = N^q$. To complete the proof, we note that the decomposition K_1 is trivial and i maps each cell of the decomposition $(\widehat{Y})^*$ isomorphically onto some cell of the decomposition K_1^*.

Proposition 5.15. *The homomorphism $\mathcal{X}^* \circ \varkappa$ is a monomorphism.*

Proof. Suppose that $\psi \in H^n(\mathcal{T}^*(\mathbb{Q}))$ is a non-zero cohomology class and $f \in \mathcal{T}^n(\mathbb{Q})$ is a local formula representing ψ. Since f is not a coboundary, we see that there are PL spheres $L_1, L_2, \dots, L_t \in \mathcal{T}_n$ such that $\sum_{j=1}^t \partial L_j = 0$ in the group $\mathcal{T}_n(\mathbb{Z})$ and $\sum_{j=1}^t f(L_j) = a \neq 0$. Let q be a positive integer such that the number of vertices of each PL sphere L_j does not exceed $q+1$. We denote by Q_j the oriented simple cell dual to L_j. Doubling the set $L_1, L_2, \dots, L_t$ of PL spheres, we may assume that facets of the cells $Q_1, Q_2, \dots, Q_t$ are paired off in such a way that the facets in each pair are anti-isomorphic and belong to different cells Q_j and Q_k.

Let $\Delta^q_1, \Delta^q_2, \dots, \Delta^q_t$ be q-simplices (these simplices are not oriented). We embed the cell Q_j in Δ^q_j in such a way that the barycentre of each l-dimensional face of Q_j is mapped to the barycentre of some $(l+q-n)$-dimensional face of Δ^q_j, and the embedding is linear on the simplices of Q'_j. This embedding is unique up to an automorphism of Δ^q_j. We consider a set of $q+1$ colours and paint the vertices of each simplex Δ^q_j in different colours in all possible ways. This yields a set of simplices $\Delta^q_{j,\alpha}$, $j=1,2,\dots,t$, $\alpha \in S_{q+1}$, with vertices painted in different colours. Each simplex $\Delta^q_{j,\alpha}$ contains the simple cell $Q_{j,\alpha}$.

A facet of $\Delta^q_{j,\alpha}$ is said to be *important* if it contains a facet of $Q_{j,\alpha}$. Suppose that σ_1 and σ_2 are important facets of the simplices $\Delta^q_{j,\alpha}$ and $\Delta^q_{k,\beta}$ respectively. An isomorphism $\iota\colon \sigma_1 \to \sigma_2$ is said to be *admissible* if ι preserves colours of vertices and the restriction of ι to the facet of $Q_{j,\alpha}$ is an anti-isomorphism with the corresponding facet of $Q_{k,\beta}$. Important facets of the simplices $\Delta^q_{j,\alpha}$ can be paired off in such a way that the following conditions hold:

1) the facets in each pair belong to different simplices,

2) for any pair (F_1, F_2) there is an admissible isomorphism $F_1 \cong F_2$.

We glue every such pair of important facets along the admissible isomorphism. Then the simple cells $Q_{j,\alpha} \subset \Delta^q_{j,\alpha}$ are glued together in such a way that we obtain an oriented pseudo-manifold N^n with a fixed decomposition $Y \in D(N^n)$. We have

$$\langle \varkappa(\psi)(\mathcal{Y}), [N^n] \rangle = a(q+1)! \neq 0,$$

where $\mathcal{Y}$ is the equivalence class of Y.

One can find an oriented manifold without boundary M^n and a map $h\colon M^n \to N^n$ such that $h_*([M^n]) = b[N^n]$ for some integer $b \neq 0$. It follows from Proposition 5.1 that there are a map $g \simeq h$ and a decomposition $Y_0 \in D(M^n)$ such that g maps each cell of Y_0 isomorphically onto some cell of Y (here the isomorphism may change the orientation). We denote by Σ the set of all n-cells of the decomposition Y_0. For each n-cell of Y, we are given an embedding of this cell into a q-simplex whose vertices are painted in different colours. Hence, for any n-cell $Q \in \Sigma$, we obtain an embedding of Q into a q-simplex Δ^q_Q with vertices painted in different colours.

We consider the disjoint union of the simplices Δ^q_Q and glue the cells $Q \subset \Delta^q_Q$ together along anti-isomorphisms of their facets to obtain the decomposition Y_0. Simultaneously, we glue the simplices Δ^q_Q along the corresponding admissible isomorphisms of their facets. Thus we obtain a q-dimensional polyhedron $P^q \supset M^n$ with a distinguished decomposition $K \in D(P^q)$ such that all cells of K are simplices. The identity embedding $M^n \hookrightarrow P^q$ maps the barycentre of each l-cell of Y_0 to the barycentre of some $(l+q-n)$-simplex of K and is linear on the simplices of Y'_0. The decomposition K is trivial since the corresponding map $P^q \to \mathcal{Z}$ is homotopic to a constant map. By Proposition 5.14, there is a block bundle $\eta \in I(M^n)$ such that $\mathcal{X}(\eta) = \mathcal{Y}_0$, where $\mathcal{Y}_0$ is the equivalence class of Y_0. Therefore,

$$\langle \mathcal{X}^*(\varkappa(\psi))(\eta), [M^n] \rangle = \langle \varkappa(\psi)(\mathcal{Y}_0), [M^n] \rangle = b\langle \varkappa(\psi)(\mathcal{Y}), [N^n] \rangle \neq 0.$$

Hence $\mathcal{X}^*(\varkappa(\psi)) \neq 0$.

Thus the homomorphism $\star\colon H^*(\mathcal{T}^*(\mathbb{Q})) \to \operatorname{Hom}(\Omega_*, \mathbb{Q})$ is a monomorphism. Theorem 2.2 follows.

5.9. The semigroup $\mathcal{D}(P)$. In §§ 5.1–5.8 we described many properties of $\mathcal{D}$-structures similar to those of stable bundles. The set of all stable bundles over a compact polyhedron is a group.

Question 5.1. *Is the semigroup $\mathcal{D}(P)$ a group for any compact polyhedron P?*

It is sufficient to answer this question for manifolds since the semigroup $\mathcal{D}(P)$ is homotopy invariant. Suppose that M is a manifold and $\mathcal{Y} \in \mathcal{D}(M)$. It is easy to show that the equivalence class $(\mathcal{Y} \oplus \mathcal{Y}^*)^*$ contains a cubical decomposition whose cells are intersections of cells of Y with cells of Y^*, where Y is a decomposition representing the $\mathcal{D}$-structure $\mathcal{Y}$. The following question arises.

Question 5.2. *Is any cubical decomposition trivial?*

If the answer to Question 5.2 is "yes", then $\mathcal{Y} \oplus \mathcal{Y}^* \oplus \mathcal{X}(\tau) = \mathcal{E}_M$ for any $\mathcal{D}$-structure $\mathcal{Y} \in \mathcal{D}(M)$. Hence the answer to Question 5.1 is also "yes". Both questions are open.

5.10. Proof of Proposition 2.3.

Definition 5.5. Suppose that M^n is a manifold without boundary. A *regular decomposition of M^n into manifolds with corners* is a filtration $\mathcal{F}$ in M^n by closed subsets $\varnothing = F^{-1} \subset F^0 \subset \dots \subset F^n = M^n$ satisfying the following conditions:

1) the pair $(\overline{V}, \mathcal{F}|_{\overline{V}})$ is a manifold with corners for any connected component V of the set $F^k \setminus F^{k-1}$,

2) for any $y \in F^k \setminus F^{k-1}$ there are a neighbourhood U of y in M^n and a homeomorphism $i \colon U \to \mathbb{R}^k \times \partial(\mathbb{R}_+^{n-k+1})$ that maps the set $F^l \cap U$ isomorphically onto the l-skeleton of $\mathbb{R}^k \times \partial(\mathbb{R}_+^{n-k+1})$ for every l.

Suppose that V is a connected component of $F^k \setminus F^{k-1}$. Then the closure of V is called a *k-dimensional face of the decomposition* $\mathcal{F}$. Sometimes we say that the closure of V is a face of M^n.

The filtration

$$\varnothing = F^{-1} = F^0 = F^1 = \dots = F^{n-1} \subset F^n = M^n$$

is a regular decomposition of M^n into manifolds with corners. The connected components of M^n are n-dimensional faces of this decomposition.

For any $l \leqslant k \leqslant n$ we have the standard homeomorphism

$$\mathbb{R}^l \times \partial\bigl(\mathbb{R}_+^{k-l+1}\bigr) \times \mathbb{R}_+^{n-k} \cong \mathbb{R}^k \times \mathbb{R}_+^{n-k}.$$

Hence the space $\mathbb{R}^l \times \partial\bigl(\mathbb{R}_+^{k-l+1}\bigr) \times \mathbb{R}_+^{n-k}$ is endowed with the following two filtrations: the filtration $\mathcal{G}_{l,k,n}$ by skeletons of $\mathbb{R}^l \times \partial\bigl(\mathbb{R}_+^{k-l+1}\bigr) \times \mathbb{R}_+^{n-k}$, and the filtration $\mathcal{F}_{l,k,n}$ by skeletons of $\mathbb{R}^k \times \mathbb{R}_+^{n-k}$. Obviously, $F_{l,k,n}^q \subset G_{l,k,n}^q$ for every q.

Definition 5.6. Suppose that $(A, \mathcal{F})$ is a manifold with corners, Y is a decomposition of A into simple cells, and $\mathcal{G}$ is a filtration in A by skeletons of Y. Then Y is said to be *regular* if the following condition holds. If $x \in F^k \setminus F^{k-1}$ and $x \in G^l \setminus G^{l-1}$, then $k \geqslant l$ and there are a neighbourhood U of x in A and a homeomorphism $i\colon U \to \mathbb{R}^l \times \partial\bigl(\mathbb{R}_+^{k-l+1}\bigr) \times \mathbb{R}_+^{n-k}$ that takes the filtrations $\mathcal{F}$ and $\mathcal{G}$ to the filtrations $\mathcal{F}_{l,k,n}$ and $\mathcal{G}_{l,k,n}$ respectively.

It is easy to show that if a triangulation $K \in D(A)$ is compatible with the filtration, then K^* is a regular decomposition.

In this subsection we assume that every manifold with corners is endowed with a fixed regular decomposition into simple cells. An isomorphism of two manifolds with corners is a filtration-preserving isomorphism of the corresponding complexes of simple cells. If we consider a decomposition of a manifold M^n into manifolds with corners, then we assume that these manifolds with corners are glued together along isomorphisms of their faces. We need these assumptions so that we can consider the finite set of isomorphisms of two manifolds with corners instead of the infinite set of homeomorphisms of them.

Suppose that L is a PL sphere and Δ is a simplex of L. We consider the stellar subdivision of Δ. The dual operation for simple cells is called *cutting away* the corresponding face. Suppose that $\mathcal{F}$ is a regular decomposition of M^n into manifolds with corners, A^k is a face of M^n such that $k < n$ and A^k is a closed manifold without corners. Every n-dimensional face of $\mathcal{F}$ is endowed with a regular decomposition into simple cells. All these decompositions are compatible with each other. Thus we obtain a decomposition $Y \in D(M^n)$ such that A^k is a subcomplex of Y. Let $Q_1^k \subset Q_2^n$ be faces of Y such that $Q_1^k \subset A^k$. We cut away the face Q_1^k in the cell Q_2^n for every such pair (Q_1^k, Q_2^n). This operation is called *cutting away* the face A^k. From a topological point of view, we are cutting away the neighbourhood $A^k \times \Delta^{n-k}$ of the face A^k in M^n.

Suppose that A is an oriented manifold with corners. We denote by $-A$ the oppositely oriented manifold with corners. Let $\mathcal{F}$ be a decomposition of a closed oriented manifold M^n into manifolds with corners. The decomposition $\mathcal{F}$ is said to be *balanced* if, for any n-dimensional manifold A with corners, the number of faces of $\mathcal{F}$ that are isomorphic to A is equal to the number of faces of $\mathcal{F}$ that are anti-isomorphic to A.

A decomposition into simple cells which is dual to a triangulation of a closed manifold is a regular decomposition into manifolds with corners. Thus Proposition 2.3 is a consequence of the following proposition.

Proposition 5.16. *Let M^n be an oriented closed manifold admitting a balanced regular decomposition into manifolds with corners. Then a disjoint union of several copies of M^n is null bordant.*

Proof. Let $\mathcal{F}$ be a balanced regular decomposition of M^n into manifolds with corners. Let k be the least positive integer such that $F^k \neq \varnothing$. Let us prove the proposition by reverse induction on k.

Basis of induction. Suppose that $k = n$. Then the n-dimensional faces of $\mathcal{F}$ are connected components of M^n. Suppose that N^n is a connected oriented manifold. The number of connected components A of M^n admitting an orientation-preserving homeomorphism $A \cong N^n$ is equal to the number of connected components A of M^n admitting an orientation-reversing homeomorphism $A \cong N^n$. Hence the manifold $M^n \sqcup M^n$ is null bordant.

Induction step. Replacing M^n by the manifold $M^n \sqcup M^n$, we may assume that the n-dimensional faces of M^n are paired off in such a way that the faces in each pair are anti-isomorphic. Suppose that A_1, A_2 are n-dimensional faces and $i\colon A_1 \to A_2$ is an anti-isomorphism. Then i can be extended to a homeomorphism $\tilde{\imath}$ of a neighbourhood of A_1 to a neighbourhood of A_2 in such a way that $\tilde{\imath}$ preserves the filtration $\mathcal{F}$. We denote the number of n-dimensional faces of M^n by q. We consider a set of q colours. There are $q!$ ways to paint n-dimensional faces of M^n in different colours. We denote by M_1^n the disjoint union of $q!$ copies of the manifold M^n such that the n-dimensional faces of each copy are painted in different colours and each colouring is used exactly once. Let $\mathcal{F}_1$ be the filtration of M_1^n that coincides with the filtration $\mathcal{F}$ in every copy of M^n. Then the n-dimensional faces of M_1^n can be paired off in such a way that for each pair (A_l^-, A_l^+) there is an anti-isomorphism $i_l\colon A_l^- \to A_l^+$ that extends to a filtration- and colour-preserving homeomorphism of a neighbourhood of A_l^- onto a neighbourhood of A_l^+.

Obviously there is an isomorphism $j\colon F_1^k \to F_1^k$ satisfying the following conditions.

1) $j^2 = 1$.

2) If B is a connected component of F_1^k, then $j(B) \cap B = \varnothing$.

3) j extends to a homeomorphism of a neighbourhood of F_1^k to itself that preserves the filtration and colouring and reverses orientation.

Let $j_1, j_2, \dots, j_s$ be all the distinct isomorphisms satisfying these conditions. We denote by M_2^n the disjoint union of s copies of M_1^n. Let $\mathcal{F}_2$ be the filtration in M_2^n coinciding with $\mathcal{F}_1$ in each copy of M_1^n. The points of M_2^n are pairs (y, l), $y \in M_1^n$, $l = 1, 2, \dots, s$. We define an isomorphism $\hat{\jmath}\colon F_2^k \to F_2^k$ by $\hat{\jmath}(y, l) = (j_l(y), l)$. This isomorphism extends to a homeomorphism of a neighbourhood of F_2^k to itself that preserves the filtration and colouring and reverses orientation.

We cut away all the k-dimensional faces of the decomposition $\mathcal{F}_2$. Then we obtain a manifold whose boundary is isomorphic to $F_2^k \times \partial\Delta^{n-k}$. The isomorphism $\hat{\jmath}$ induces a filtration- and colour-preserving involutive anti-isomorphism $F_2^k \times \partial\Delta^{n-k} \to F_2^k \times \partial\Delta^{n-k}$ without fixed points. We glue the set $F_2^k \times \partial\Delta^{n-k}$ to itself along this anti-isomorphism. Thus we obtain a closed manifold M_3^n bordant to M_2^n. The filtration $\mathcal{F}_2$ in M_2^n induces a filtration $\mathcal{F}_3$ in M_3^n. Then $\mathcal{F}_3$ is a regular decomposition into manifolds with corners such that $F_3^k = \varnothing$.

Let us show that the decomposition $\mathcal{F}_3$ is balanced. Take an arbitrary colour c. Let $M_{1,c}^n$, $M_{2,c}^n$, and $M_{3,c}^n$ be the unions of all n-dimensional faces painted in colour c of the manifolds M_1^n, M_2^n, and M_3^n respectively. We denote by $i_c\colon M_{1,c}^n \to M_{1,c}^n$ the anti-isomorphism coinciding with i_l on each face $A_l^- \subset M_{1,c}^n$ and coinciding with i_l^{-1} on each face $A_l^+ \subset M_{1,c}^n$. We extend the isomorphism

$$i_c\big|_{F_1^k \cap M_{1,c}^n}\colon F_1^k \cap M_{1,c}^n \to F_1^k \cap M_{1,c}^n$$

to an isomorphism $\chi_c\colon F_1^k \to F_1^k$ whose restriction to $F_1^k \setminus (F_1^k \cap M_{1,c}^n)$ is the identity isomorphism. We obviously have $\chi_c^2 = 1$, and χ_c extends to a filtration- and colour-preserving homeomorphism of a neighbourhood of F_1^k to itself. The isomorphism $\chi_c \circ j_l \circ \chi_c$ satisfies conditions 1)–3) for every l. Hence there is a positive integer $\kappa_c(l)$ such that $\chi_c \circ j_l \circ \chi_c = j_{\kappa_c(l)}$. Obviously, $\kappa_c\colon \{1, 2, \dots, s\} \to \{1, 2, \dots, s\}$

is an involution. We define an anti-isomorphism $\hat{\imath}_c\colon M^n_{2,c} \to M^n_{2,c}$ by

$$\hat{\imath}_c(y,l) = (i_c(y), \kappa_c(l)).$$

It is easy to show that $\hat{\imath}_c(\hat{\jmath}(y,l)) = \hat{\jmath}(\hat{\imath}_c(y,l))$ for every $(y,l) \in M^n_{2,c} \cap F^k_2$. Therefore the anti-isomorphism $\hat{\imath}_c$ induces an anti-isomorphism $M^n_{3,c} \cong M^n_{3,c}$. Hence the restriction of $\mathcal{F}_3$ to $M^n_{3,c}$ is balanced. Consequently the decomposition $\mathcal{F}_3$ of the manifold M^n_3 is balanced. By the induction hypothesis, a disjoint union of several copies of the manifold M^n_3 is null bordant, as required.

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