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THE RECOGNITION PROBLEM: WHAT IS A TOPOLOGICAL MANIFOLD?

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1. The recognition problem for topological manifolds.

1.1. DEFINITION. Let E^n denote the collection of n-tuples $x = (x_1, \ldots, x_n)$, x_i real. Define $d(x, y) = (\Sigma(x_i - y_i)^2)^{1/2}$. Then E^n becomes a metric space with metric d and is called n-dimensional Euclidean space. A (topological) n-manifold M is a separable metric space locally homeomorphic with E^n (see Figure 1).

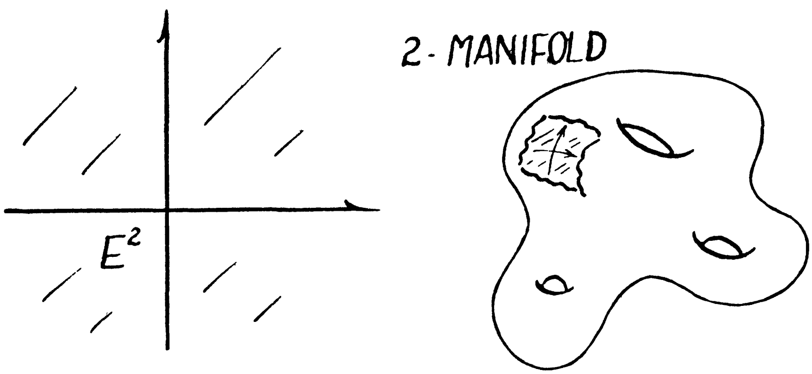

FIGURE 1

The definition of manifold, as just given, is simple. Nevertheless, it is a very difficult matter to determine whether a topological space which appears as the result of some construction in the midst of a mathematical argument is or is not a topological manifold. (See Supplement 1 for low dimensional illustrations of this difficulty.) We are thus led to the recognition problem for topological manifolds.

1.2. RECOGNITION PROBLEM. Find a short list of topological properties, reasonably easy to check, that characterize topological manifolds among topological spaces.

Recent work in geometric topology suggests that a satisfactory solution to the recognition problem for topological manifolds is imminent. The purpose of this paper is to report on that work.

A good solution to the recognition problem should make no mention of homeomorphisms as part of the hypotheses since homeomorphisms are terribly difficult to construct. A good solution probably should not involve an induction on dimension since nice submanifolds of a manifold are, in an

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abstract setting, difficult to come by. A good solution probably should not involve the notion of homogeneity (see Supplement 5) since, in applications, the spaces constructed which are to be checked are obviously manifolds at some points, so that recognizing homogeneity is precisely the difficulty. Finally, a satisfactory solution should allow one to solve problems of independent interest.

Before spring of this year (1977) no conjectured characterization of topological manifolds seemed to have a clear cut advantage over any other. But the situation has changed rapidly so that we can make the following conjecture with some confidence.

1.3. CONJECTURE. A topological n-manifold may be characterized as a generalized n-manifold satisfying a minimal amount of general position. (Definitions follow.)

Prerequisites for understanding the conjecture in particular and the paper in general include a knowledge of basic homology theory (as presented, for example, in [33]) and basic PL topology (as presented, for example, in [57]). A good introduction to the particular point of view that we shall pursue (concerning tameness and wildness) appears in [**19**]. Nevertheless, even without those prerequisites the reader will probably understand and enjoy some of the historical material in §§2 and 3 and the discussion of Antoine's necklace in §§5 and 6.

1.4. DEFINITION. A generalized n-manifold M is a Euclidean neighborhood retract (ENR) (= retract of an open subset of some Euclidean space E^k) with the local homology groups at each point of Euclidean n-space E^n: $H_*(M, M - \{x\};Z) \cong H_*(E^n, E^n - \{0\};Z)$ (for each $x \in M$).

The probable appropriate general position condition for $n \geqslant 5$ is the disjoint disk property.

1.4′. DEFINITION. A space M satisfies the disjoint disk property if arbitrary maps f, g: $B^2 \to M$ from the 2-dimensional disk B^2 into M can be approximated by maps f', g': $B^2 \to M$ with $f'(B^2) \cap g'(B^2) = \varnothing$.

The conjecture, as completed by Definitions 1.4 and 1.4′, was proved during the spring of this year (1977) for a large class of generalized manifolds by J. W. Cannon [**22**] and Cannon, J. L. Bryant, and R. C. Lacher [**23**] (see Supplement 4). The fertile source of generalized manifolds supplied by cell-like upper semicontinuous decompositions of manifolds was then, for $n \geqslant 5$, completely mastered by R. D. Edwards [**36**] (see Supplement 4); his result confirmed the conjecture for all cell-like decompositions of n-manifolds, $n \geqslant 5$. An infinite dimensional analogue of the conjecture for Q manifolds was proved early in the year by H. Torunczyk [65] (see Supplement 3).

An easy consequence of the work, and one of its great motivations, is the famous double suspension conjecture:

1.5. THEOREM. *The double suspension* $\Sigma^2 H^n$ *of any homology n-sphere is homeomorphic with the* $(n + 2)$*-sphere* S^{n+2}. (*A homology n-sphere* H^n *is an n-manifold satisfying* $H_*(H^n;Z) \cong H_*(S^n;Z)$; *the kth suspension of a space is the join of that space with the* $(k - 1)$*-sphere* S^{k-1}; *see Definition* 1.6 *for the definition of a sphere.*)

The connection of Theorem 1.5 with Conjecture 1.3 is as follows. The theorem is obviously true for $n \leq 2$ since the notions of sphere and homology sphere coincide in those dimensions. Suppose however that $n \geq 3$. Then it may be that H^n is not simply connected ($\pi_1(H^n) \neq 1$; see Supplement 10). In that case the single suspension $\Sigma^1 H^n$ does not have the requisite general position properties of an $(n + 1)$-manifold at the two suspension points, hence is not a manifold. Any suspension of a homology sphere is a generalized manifold. Since the double suspension has, in addition, the appropriate general position property, the disjoint disk property, it is a manifold by the special cases of Conjecture 1.3 proved in [22] and [23]. Since the double suspension has the homotopy type of S^{n+2}, it follows from the high dimensional Poincaré conjecture that the double suspension is homeomorphic with S^{n+2}.

Theorem 1.5 was first proved for most double suspensions and all triple suspensions by Edwards [34], [35]. Cannon [22] proved the general case. Edwards' latest results [36] give an alternative proof. This paper illustrates the ideas surrounding Conjecture 1.3 by outlining one of the author's proofs of Theorem 1.5.

All of the proofs of Theorem 1.5 involve directly or indirectly the notions of wildness (limit misbehavior) and other infinite processes (see §§4, 5, 6 and [19] for an introduction to the notion of wildness). That the proofs should involve these notions is, on the face of it, surprising since the hypothesis and conclusion of Theorem 1.5 make explicit mention only of finite triangulable objects. (Homology spheres are triangulable in all dimensions with the possible exception of 4 [50] and [44].) Nevertheless, our discussion will establish the fact that wildness is not only inherent in Theorem 1.5 but even in the group theoretic notion of the perfect group. The connection of homology spheres with wildness thus proceeds as follows. If H^n is a nonsimply connected homology sphere, then the fundamental group π of H^n is a nontrivial perfect group (π equals its own commutator subgroup $[\pi, \pi]$) (see Supplement 11). Alexander's wild crumpled cube (Definition 1.8 and §5) and its immediate generalizations are the geometric realizations of the elements of a perfect group (see Supplement 13). The suspension circle in $\Sigma^2 H^n$ may be thought of as a wild simple closed curve in $\Sigma^2 H^n$ (§4). The wildness can be destroyed by searching out certain perfect group elements geometrically realized in wild crumpled cubes and replacing those crumpled cubes by real cubes (see §7). Results from taming theory and cell-like decomposition space theory then yield the theorem easily (§7). One can visualize the entire proof in remarkable detail by visually analyzing the wildness of Antoine's wild Cantor set in the 3-sphere S^3 (§§5 and 6).

In a series of supplements following the main discussion, we discuss in more depth a number of topics hinted at in the main exposition and raise a number of important unanswered questions. The results and methods which we discuss lead to new proofs of a number of recent important results in the topology of manifolds; we outline some of those proofs in the supplements.

We will use throughout the paper the notions of sphere, ball or cell or disk, and crumpled cube:

1.6. DEFINITION. The standard $(n - 1)$-sphere S^{n-1} is the set $\{x \in$

$E^n | d(x, 0) = 1\}$. Any space homeomorphic with S^{n-1} is also called an $(n-1)$-sphere.

1.7. DEFINITION. The standard n-ball (or cell or disk or cube) B^n is the set $\{x \in E^n | d(x, 0) \leq 1\}$. Again, any homeomorph of B^n is also called an n-ball.

1.8. DEFINITION. Let S be an $(n-1)$-sphere in S^n and U a component of $S^n - S$. Then $C = \text{Closure}(U)$ is called a crumpled cube with boundary S. Any homeomorph of C is also called a crumpled n-cube. A crumpled n-cube may or may not be an n-cube.

2. The classical view of manifolds. Early topologists took a restrictive view of manifolds in order to obtain results (see Definition 2.1). Nevertheless, they came remarkably close to capturing what we now consider the essence of the topological manifold for all of their restrictions. We discuss their views in this section and the next.

The basic definitions and results of manifold theory first appeared in a series of beautiful papers by Henri Poincaré [54]. Poincaré explained the motivation for his topological studies in an analysis of his scientific work which he prepared in 1901 (see [54, p. 183]). He wrote:

"A method which allows us to recognize qualitative relations in spaces of more than three dimensions can, to some extent, render service analogous to that rendered in low dimensions by pictures. This method is none other than the topology of more than three dimensions. Unfortunately this branch of science has been but little cultivated. After Riemann came Betti, who introduced some fundamental ideas; but no one has followed Betti. As for me, all of the diverse paths which I have successively followed have led me to topology. I have needed the gifts of this science to pursue my studies of the curves defined by differential equations and for the generalization to differential equations of higher order, and, in particular, to those of the three body problem. I have needed topology for the study of nonuniform functions of two variables. I have needed it for the study of the periods of multiple integrals and for the application of that study to the expansion of perturbed functions. Finally, I have glimpsed in topology a means to attack an important problem in the theory of groups, the search for discrete or finite groups contained in a given continuous group." (Approximate translation from the French; see [54, p. 183] for the original.)

Poincaré originally defined manifolds via the implicit function theorem of advanced calculus. He defined an $(n-p)$-dimensional submanifold of Euclidean space E^n as the set of points x in E^n satisfying p equations $F_1(x) = 0, \ldots, F_p(x) = 0$, where the functions $F_1, \ldots, F_p$ are required to have continuous partial derivatives and the matrix $(\partial_j F_i)_{i,j}$ of partial derivatives is required to have rank p at each point x. (See [54, p. 196].)

Mathematicians immediately recognized the difficulty of working directly with the definition of manifold. Poincaré himself, in order to obtain results, assumed that the manifolds considered could be given the structure of a simplicial complex or polyhedron. Under that assumption he enunciated and proved the fundamental duality relationship between the homology and cohomology of a manifold, Poincaré duality. If we recall that a completely successful treatment of singular homology first appeared in the 1940s [68], then we find not at all surprising the following attitude toward manifolds expressed during the 1930s in the beautiful book by Seifert and Threlfall [60, p. 236]:

"In two and three dimensions we have defined a manifold to be a homogeneous complex, that is, a complex such that each point has a neighborhood homeomorphic with the interior of a two or three dimensional ball. For the current state of topology this definition is inappropriate for more than three dimensions because it cannot be formulated combinatorially. One has no procedure for deciding whether a simplicial complex of more than three dimensions given in terms of its incidence scheme is homogeneous or not. It is indeed unknown whether one can conclude from the homogeneity of an n-dimensional complex K^n that the $(n - 1)$-dimensional link of a vertex is homeomorphic to a simplicial subdivision of the $(n - 1)$-sphere. But even if that were the case, the question would still remain whether a given $(n - 1)$-dimensional simplicial complex is an $(n - 1)$-sphere or not. This 'sphere problem' is unsolved for more than two dimensions. However, one can prove a great number of theorems that deal with homology properties (and not homotopy properties) of homogeneous complexes without fully using the homogeneity of the complex. These theorems are valid without change for an arbitrary complex which behaves like a homogeneous complex with respect to its homology properties in the neighborhood of each point. It suffices to require that the homology groups at each point be the same as those of the $(n - 1)$-dimensional sphere.

"Accordingly we define: A (closed) n-dimensional manifold M^n $(n > 0)$ is a connected, finite n-dimensional complex that at each point has the same homology groups as the $(n - 1)$-dimensional sphere." (Translated from the German.)

On p. 323 of [60] we further read:

"The idea to base the definition of manifold on the homology properties at a point rather than homogeneity occurred to several authors independently: Alexander, Pontrjagin (unpublished), Vietoris, Weyl. Van Kampen is to be thanked for the first complete treatment; Pontrjagin named these manifolds h-manifolds." (Translated from the German.)

In summary, many of the great topologists of the first quarter of this century followed the maxim of Pólya's "traditional mathematics professor" [55, p. 208]: "My method to overcome a difficulty is to go round it." Unable to completely pin down the properties characteristic of manifolds, they generalized, picking out those properties with which they knew how to work. They obtained thereby the notion of (polyhedral) generalized manifold.

2.1. DEFINITION. A polyhedral generalized manifold is a space M satisfying

(1) M is a polyhedron and

(2) $H_*(M, M - \{x\}; Z) \cong H_*(E^n, E^n - \{0\}; Z)$ for each $x \in M$.

We shall discuss in the next section how remarkably close these classical topologists came to capturing the essential properties of a manifold in their definition.

We note in passing that the affirmative solution to the double suspension conjecture answers one of the questions raised in the quotation from Seifert and Threlfall: one cannot conclude from the homogeneity of an n-dimensional complex K^n that the $(n-1)$-dimensional link of a vertex is homeomorphic to a simplicial subdivision of the $(n-1)$-sphere. The "sphere problem" is also essentially answered by the same considerations (see Theorem 3.5).

3. Critique of the classical view of manifolds. Throughout this section we let M denote a polyhedral generalized manifold (satisfying conditions (1) and (2) of Definition 2.1.).

3.1. Condition (1) of Definition 2.1 requires that M be a polyhedron. It is not at all obvious that a topological manifold, abstractly defined by Definition 1.1 is homeomorphic with a simplicial complex. In fact, L. C. Siebenmann proved [44] that there exist topological manifolds of dimension $n \geqslant 5$ which admit no simplicial structure of that particularly nice kind called "combinatorial." However, Edwards' solution to the triple suspension problem [35] showed that at least one such bad manifold, while admitting no combinatorial triangulation, does admit a noncombinatorial triangulation as a simplicial complex. The current state of affairs is summarized in the following theorem.

3.2. THEOREM (GALEWSKI-STERN [40]; MATUMOTO [49]). *Every topological manifold (without boundary) of dimension $n \geqslant 5$ is homeomorphic with a polyhedron if and only if there is at least one homology 3-sphere D^3 satisfying*

(1) *μD^3 is nontrivial (μD^3 is the Rohlin invariant of D^3 [40]);*

(2) *The double suspension $\Sigma^2 D^3$ of D^3 is homeomorphic with S^5; and*

(3) *The connected sum $D^3 \# D^3$ of D^3 with itself bounds a PL acyclic 4-manifold.*

Thus it is still conceivable that the restriction that M be a polyhedron in Definition 2.1 is no restriction at all in the study of manifolds. Possibly the most important problem in the study of topological manifolds is the search for a homology 3-sphere D^3 satisfying conditions (1)–(3) of Theorem 3.2:

3.3. *Question.* Does there exist a homology 3-sphere D^3 satisfying the three conditions of Theorem 3.2?

3.4. Condition (2) of Definition 2.1 requires that the local homology of a polyhedral generalized manifold agree with the local homology of Euclidean space. This homology condition was chosen as precisely that property which enabled one to prove Poincaré duality. Early topologists were well aware that the class of polyhedral generalized manifolds contained spaces that were not true topological manifolds in the sense of Definition 1.1. The standard example consisted in taking a nonsimply connected homology n-sphere D^n $(n \geqslant 3)$, such as Poincaré's homology sphere (Supplement 10) and suspending once. The single suspension is obviously a nonmanifold at the two suspension points; nevertheless, it is a polyhedral generalized manifold. Presumably, one thought, the double suspension would be a nonmanifold on the suspension circle, the triple suspension would be a nonmanifold on the suspension 2-sphere, etc. The actual state of affairs is summarized in the following theorem, which is an easy consequence of our work on the double suspension problem and was pointed out to us by Edwards:

3.5. THEOREM. *Suppose M is a polyhedral generalized n-manifold. Then M is locally an n-manifold except possibly at the vertices. If $n \leq 3$ or if $n \geq 5$ and* $\pi_1(\mathrm{Lk}(x, M)) = 1$, *then M is locally an n-manifold at x.*

Obvious corollaries are the following:

3.6. COROLLARY. *$M \times E^1$ is a manifold.*

PROOF. If $n \leq 3$, M is an n-manifold so that $M \times E^1$ is an $(n + 1)$-manifold. If $n \geq 4$, then $M \times E^1$ is a polyhedral generalized manifold of dimension ≥ 5 having only simply connected vertex-links. The corollary thus follows from the theorem.

3.7. COROLLARY. *$\Sigma^2 D^n = S^{n+2}$ for each homology n-sphere D^n.*

PROOF. The result is clear for $n \leq 2$. Suppose $n \geq 3$. Let S^1 denote the suspension circle. Clearly $\Sigma^2 D^n - S^1$ is a topological manifold. The circle S^1 has a neighborhood in $\Sigma^2 D^n$ of the form $\mathrm{OC}(D^n) \times S^1$, where $\mathrm{OC}(D^n)$ denotes the open cone on D^n. The set $\mathrm{OC}(D^n) \times S^1$ is a manifold by Corollary 3.6 ($\mathrm{OC}(D^n) = M$). Thus $\Sigma^2 D^n$ is a manifold. By the Poincaré conjecture [28], [62], $\Sigma^2 D^n \cong S^{n+2}$ $(n + 2 \geq 5)$ because $\Sigma^2 D^n$ has the homotopy type of S^{n+2}.

We conclude our critique of the classical view by noting then that classical topologists, with their polyhedral generalized manifolds, may only have missed the class of topological manifolds by a locally finite collection of nonmanifold points, namely the vertices in some triangulation.

The next four sections illustrate the ideas behind Theorem 3.5 by outlining a proof of Corollary 3.7 = Theorem 1.5.

4. Wildness and the double suspension problem. A good introduction to the notions of tameness and wildness in low dimensions is [19].

4.1. The setting in which there is a unified theory of tameness and wildness is the following:

X denotes either an uncountable compact metric space C^j of dimension j or a sphere S^k of dimension k.

E' denotes the n-sphere.

j is in the trivial range $2j + 2 \leq n$.

k is neither 0 nor n and avoids the global knotting dimension $n - k = 2$.

4.2. BASIC TAMING THEOREM. (1) *For $n \geq 3$ there exist both a simplest (tame) and many nonstandard and nonequivalent (wild) topological embeddings of X in E'.* (2) *For $n \neq 4$ the tame embeddings, all of which are topologically equivalent, can be distinguished by the fact that $E' - X$ is 1-ULC = uniformly locally simply connected (= for each $\varepsilon > 0$ there is a $\delta > 0$ such that loops in $E' - X$ of diameter less than δ bound singular disks in $E' - X$ of diameter less than ε).*

The basic taming theorem has a long history; for much of that history one might consult [19] and [58]. The two basic classical examples of wild sets are Antoine's necklace (a wild Cantor set in S^3) and Alexander's horned sphere (a wild 2-sphere in S^3); see §§5 and 6 for a discussion of these two basic examples; Supplements 12 and 13 also provide additional insight into the nature of Alexander's horned sphere.

One of the insights leading to our solution of the double suspension problem is the following observation. Suppose D^n is a nonsimply connected homology sphere, E is its double suspension, and X is the suspension circle (so that E is the PL join of D^n and X); if E is homeomorphic with $E' = S^{n+2}$, then the image of X in E' must be wild in E', for the complement of X in E is not 1-ULC; in fact the link of X or any small simplex in X has the homotopy type of D^n, hence is not simply connected as it would be if X were tame. One recognizes immediately from these considerations that, although E and E' are objects with natural finite triangulations, no homeomorphism between them can be finite (PL). Therefore, if the double suspension theorem is to be true, it must involve wildness and infinite processes. In fact, our proof of the theorem will consist in destroying the "wildness" of the embedding of the suspension circle X in the double suspension E. The process alters E until it is easily recognizable as $E' = S^{n+2}$. An easy argument then recovers the original embedding of X in such a way that one recognizes $E = \Sigma^2 D^n$ and $E' = S^{n+2}$ as the same topological type.

We illustrate the ideas of the proof in an analogous analysis of the wildness of Antoine's necklace (§§5 and 6).

5. Analysis of the wildness of Antoine's necklace: an illustration of the ideas in the proof of the double suspension theorem. Our proof of the double suspension theorem suggests that essentially all wildness is Alexander crumpled cube wildness. We illustrate this assertion by analyzing the wildness of Antoine's necklace, a wild Cantor set in $E = S^3$. Antoine's necklace may be described as the intersection $X = \bigcap X_i$ of compact sets $X_0 \supset X_1 \supset X_2 \supset \dots$ as pictured in Figure 2. The set X_0 consists of a single unknotted solid torus in E; the set X_1 is the union of four unknotted solid tori linked in $\operatorname{Int} X_0$ as pictured; each component A of X_1 contains four solid tori of X_2 linked in A just as the four components of X_1 are linked in X_0; etc. The Cantor set X is homeomorphic with the standard middle-thirds Cantor set $X' \subset [0, 1] \subset S^1 \subset S^3 = E$. But no homeomorphism $h: E \to E$ can take the wild Cantor set X to the tame Cantor set X'. This is easily seen from the fact that the simple closed curve $J \subset \operatorname{Bd} X_0$ represents a nontrivial element of $\pi_1(E - X)$ while $\pi_1(E - X')$ is trivial. M. L. Antoine is one of the two great names in the classical study of wild embeddings. He described his Cantor set in papers which appeared in 1920 [7], [8]. J. W. Alexander is the other great name in the classical theory. Alexander announced a theorem in 1922 [3] that implied that no such example as Antoine's could exist. Shortly afterward, however, Antoine's papers came to his attention and Alexander realized his mistake. In three two page papers which appeared back to back in the 1924 volume of the Proceedings of the National Academy of Science (U.S.) [4], [5], [6], Alexander proved that every PL embedding of the boundary of a tetrahedron into E is topologically equivalent to the standard embedding, he advertised Antoine's constructions of wild Cantor sets and wild 2-spheres, and he described his own horned sphere.

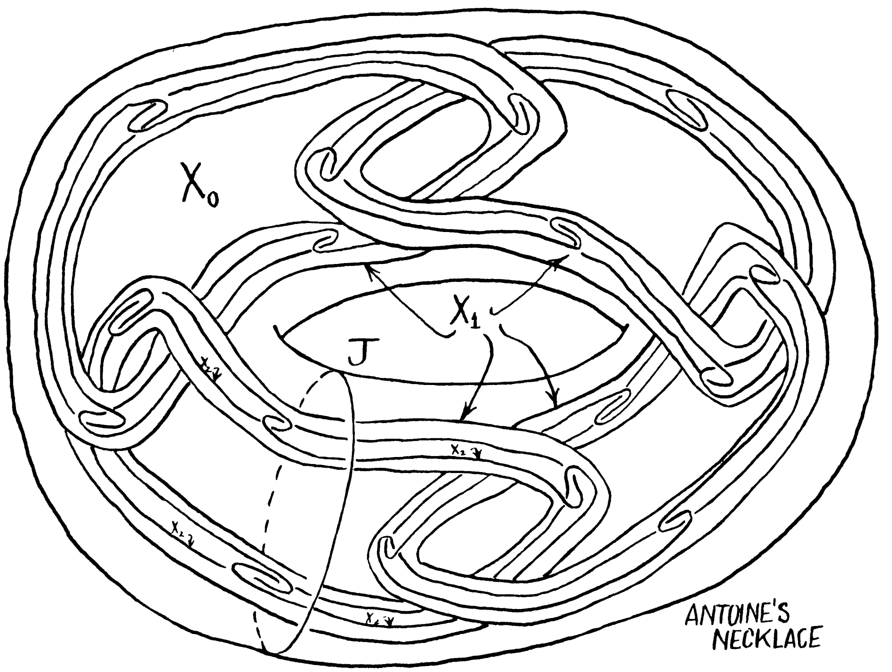


FIGURE 2

Unfortunately, we cannot ask Alexander how he discovered his sphere. However, it may have happened in the following way. He may have drawn a picture of Antoine's necklace as in Figure 2 and attempted to understand its wildness. Realizing that, as Antoine had shown, the curve J on Bd X_0 cannot bound a disk in $X_0 - X$, he may have asked,

5.1 (?) Alexander's (?) first question: "How nearly can a disk in X_0 bounded by J miss X?"

"I can attempt to miss X_1 at least partially," he may have said, "by diverting the obvious disk bounded by J along the surface of X_1 for a time (Figure 3). I must eventually cut into X_1. But then I can avoid X_2 for a time by sliding along its surface Bd X_2 just as I diverted the disk along Bd X_1 for a time." In the limit one does obtain a disk D_0 by this process which is bounded by J and lies essentially in the lower left half of X_0. Of course it must hit X, but, in a sense, it comes as close to missing X as possible.

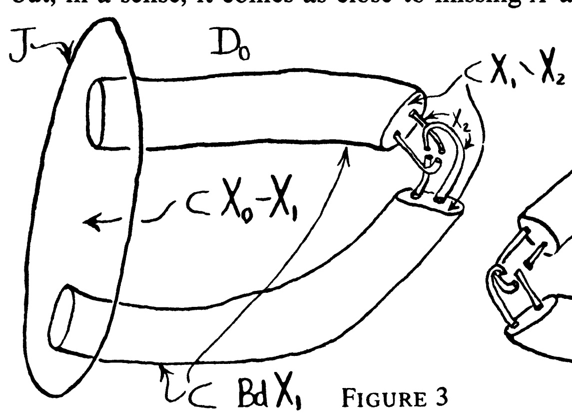


FIGURE 3

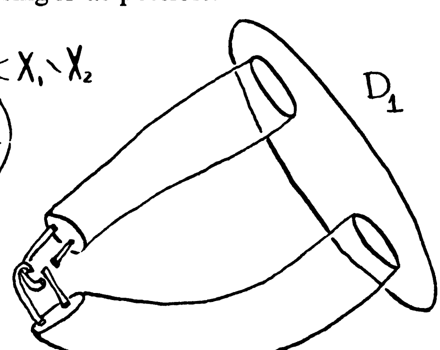


FIGURE 4

Alexander may then have asked himself a second question.

5.2. (?) Alexander's (?) second question: "Is it possible to put two disjoint disks D_0 and D_1 across X_0 with Bd D_0 and Bd D_1 parallel to J in Bd X_0 so that one of the two complementary domains U of $X_0 - (D_0 \cup D_1)$ entirely misses X?"

The answer to this question is yes. One puts in a second disk D_1 (Figure 4) that slides along Bd X_1 for a while, along Bd X_2 for a while, etc., exactly as before but in the lower right half of X_0. The domain U in the lower half of X_0 between D_0 and D_1 misses X entirely. The closure C of U in X_0 is then precisely Alexander's crumpled cube. (See Figure 5.)

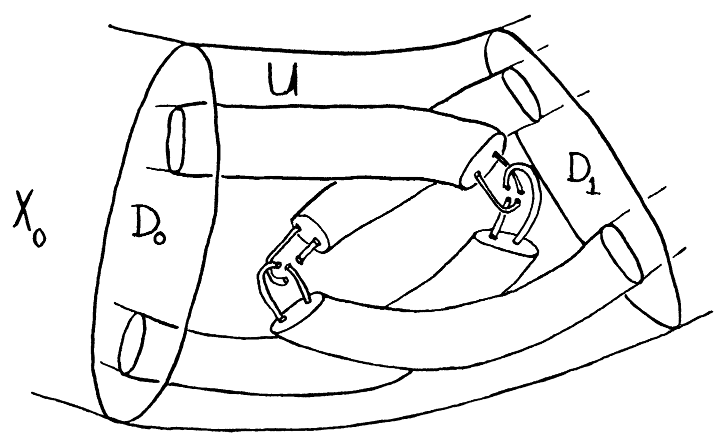


FIGURE 5

Although in retrospect there is an obvious intimate connection between Antoine's necklace and Alexander's horned sphere, as described above, the connection had never been apparent to me. Many topologists with whom I have discussed the matter also had never seen the connection. That there should be such an intimate connection was first suggested to me by my proof of the double suspension theorem. When I then looked for the connection and verified its existence, I was so surprised and delighted that I sat and stared at the picture for an entire afternoon. (Please do not tell the taxpayers of Wisconsin.)

Suppose now that we were Maurits Escher, the graphic artist whom we thank for so many artistic tessellations of the Euclidean and hyperbolic planes [37]. Having found an Alexander crumpled cube in Antoine's necklace and noticing the great regularity in the construction of each, we would feel absolutely compelled to draw more and more Alexander crumpled cubes meshing with Antoine's necklace (Figure 6), getting smaller and ever smaller. Since each of the crumpled cubes contains no points of X in its interior in answer to (?) Alexander's (?) second question, the process of construction can be continued, without obstruction, ad infinitum.

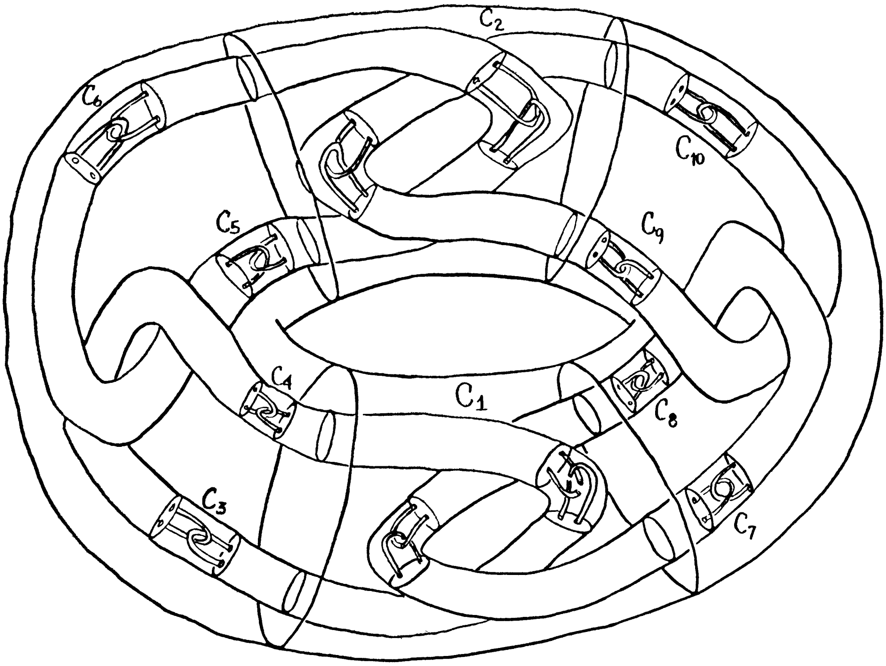


FIGURE 6

Let $C_1, C_2, C_3, \ldots$ denote the null sequence of crumpled cubes constructed as suggested in the previous paragraph and pictured in Figure 6. Note that they break X up into tiny pieces in the precise sense that any two dimensional disk in E can be adjusted slightly so that it hits X only in a subset of the union of the C_i's. That is, the C_i's capture the obstruction in X to the 1-ULC property for $E - X$. In this sense, and in an even more precise sense to be described in the following paragraphs, the crumpled cubes $C_1, C_2, \ldots$ capture the wildness of the wild Cantor set X.

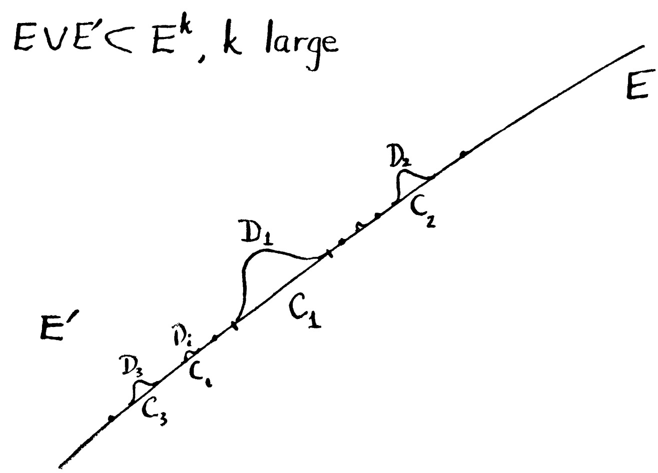


FIGURE 7

Consider E as a slice in some high dimensional Euclidean space or sphere. We wish to destroy the wildness of X in E by physically replacing each of the crumpled cubes C_i by a real cube D_i in that high dimensional space, D_i having the same 2-sphere boundary and approximately the same diameter as C_i. This replacement is possible by the classical embedding theorem which states that maps of three dimensional objects into a high dimensional manifold can be approximated by embeddings. Let E' denote the new space $(E - \bigcup C_i) \cup (\bigcup D_i)$ (Figure 7). It follows by an argument which we shall not give here that E' is homeomorphic with S^3.

The Cantor set X lies in $E \cap E'$ since $X \cap \text{Int } C_i = \emptyset$ for each i. But X is tame in the three-manifold E' ($E' - X$ is 1-ULC) even though it is wild in the three-manifold E. We thus see that we have indeed captured the wildness of X in the wild crumpled cubes $C_1, C_2, \ldots$ so that their replacement by real cells $D_1, D_2, \ldots$ tames X. This result justifies in a precise way the assertion that the wilderness of Antoine's necklace is Alexander crumpled cube wildness.

6. Analysis of the wildness of Antoine's necklace (summary and conclusion). To this point we have completed two of four major steps in our analysis of the wildness of Antoine's necklace:

6.1. Find a null sequence of crumpled cubes $C_1, C_2, \ldots$ in E whose interiors miss X such that $C_1, C_2, \ldots$ capture, in a precise way, the wildness of X.

6.2. Replace the crumpled cubes $C_1, C_2, \ldots$ by real cubes $D_1, D_2, \ldots$ having the same boundaries.

We have also stated without proof the third major step of the analysis:

6.3. Prove that the resulting space $E' = (E - \bigcup C_i) \cup (\bigcup D_i)$ is homeomorphic with S^3.

The fourth major step is, strictly speaking, irrelevant to our analysis of Antoine's necklace, but it will be important in the proof of the double suspension theorem. Suppose for the moment that we did not know at the start of our analysis that E was S^3. Suppose that nevertheless we were able to complete steps 6.1, 6.2, and 6.3. Could we then prove that our original space E was homeomorphic with E'? Showing that we could so conclude is the content of step 6.4, which we summarize as follows:

6.4. Recover the original wild embedding of X in E from the tame embedding $X \subset E'$ via decomposition space techniques (see **[9]**, **[61]**, **[45]** for a survey of decomposition space theory) in such a way as to conclude that E and E' are homeomorphic.

The technique for step 6.4 can be explained as follows. For each i there is a very nice surjective map p_i: $D_i \to C_i$ which is the identity when restricted to Bd D_i = Bd C_i and whose nondegenerate point preimages form a Cantor set of collar arcs in a (wild) interior collar on Bd D_i (see Figure 8 and **[22]**). The function $p = [\text{id}|E' - \bigcup D_i] \cup [\bigcup p_i]$: $E' \to E$ is a closed surjection with cellular arcs as nondegenerate point preimages. It would follow that E' and E are homeomorphic if one could construct a closed surjection q: $E' \to E'$ having precisely the same point preimages as p, for then pq^{-1}: $E' \to E$ would be a homomorphism. Since, as one may easily prove, the 3-cells D_i are tame in E' and form a null sequence, it is one of the most trivial and standard of decomposition space problems to construct the surjection q by shrinking, one after another, the Cantor sets of collar arcs mentioned as nondegenerate point preimages in the maps p_i above; see [22, §8] for a detailed outline of the construction of q; one simply begins from the tame end of the interior boundary collar of D_i, that is at the tame sphere Bd D_i, and pushes inward, shortening the appropriate collar arcs a little bit at a time, taking care that the other arcs not be pulled long. Thus one concludes that E' and E are homeomorphic and that the embedding $X \subset E$ can be obtained as the image of a tame Cantor set under a pseudoisotopy of E.

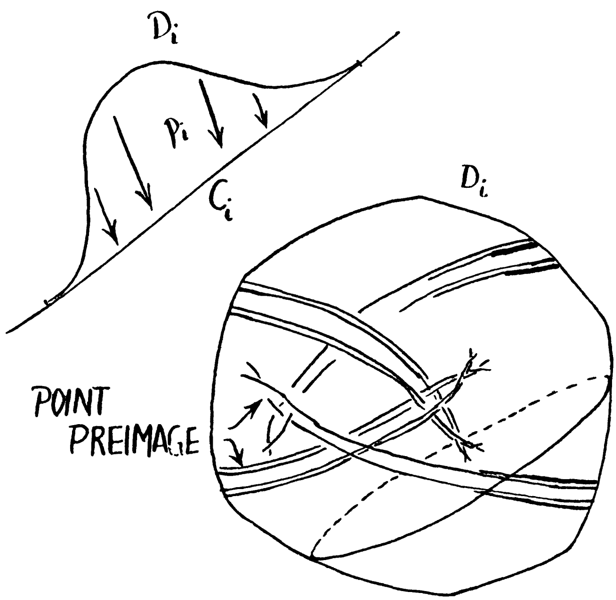


FIGURE 8

7. Outline of the proof of the double suspension theorem. Setting:
D^n, a homology sphere of dimension $n \geqslant 3$.
X, the circle.
$E = X * D^n = \Sigma^2 D^n$, the join of X and D^n = the double suspension of D^n.

THEOREM. *The space E and the $(n + 2)$-sphere S^{n+2} are homeomorphic.*

PROOF. In §4 we noted that one ought to think of the circle X as wildly embedded in the generalized manifold E. With this in mind, we analyze the wildness of X in E by exactly the same four steps employed in §§5 and 6 to analyze the wildness of Antoine's necklace.

7.1. Find a null sequence of crumpled $(n + 2)$-cubes $C_1, C_2, \ldots$ in E whose interiors miss X such that $(E - X) \cup (\bigcup C_i)$ is uniformly locally simply connected = 1-ULC:

For this step one must be able to detect certain small crumpled cubes homologically. This detection problem is discussed in detail in Supplement 13. The idea of that supplement is that a certain infinite 2-complex, called a grope, thickened and compactified appropriately, always yields a special kind of crumpled cube. We outline the application of that fact here.

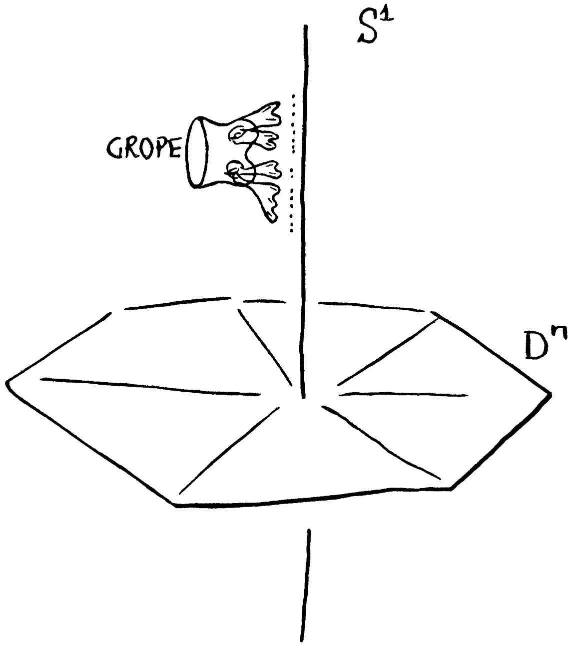


FIGURE 9

For simplicity we assume D^n equipped with a PL triangulation so that $E - X$ is a PL manifold. This assumption can be avoided. Then the idea is to take a small simple closed curve J in $E - X$ near X which does not bound a small disk missing X. Since D^n has trivial first homology, it is an easy matter to prove that J bounds a small disk-with-handles D_0 missing X. Furthermore the handles of D_0 may be taken to be arbitrarily small. The disk-with-handles D_0 serves as stage 0 of a grope. Similarly, handle curves in D_0 bound even smaller disks-with-handles in $E - X$ having tiny handles. These disks-with-handles, taken together form stage 1, D_1, of the grope. Disks-with-handles attached to the handle curves of D_1 form D_2, etc. By induction, one obtains a PL grope G in $E - X$, with handles converging in size to 0, PL embedded as a closed subset of $E - X$, the "fingers" converging to points of X. Using the join structure in E and general position in $E - X$, one easily spreads out the fingers of G in such a manner that the closure of G in E is a closed grope G^+ (Supplement 13), $G \subset E - X$, $G^+ - G \subset X$, $E - G^+$ $1 -$ ULC. If one then thickens G a bit in $E - X$ to a PL regular neighborhood N which pinches nearer and nearer to G near $G^+ \cap X = G^+ - G$, then $C = N \cup G^+$ is an $(n + 2)$-dimensional crumpled cube as described in Supplement 13, an exact high dimensional analogue of the Alexander crumpled cube encountered in the analysis of Antoine's necklace.

Just as it was possible in the case of Antoine's necklace to iterate the procedure and to find thereby a countable null sequence of crumpled cubes capturing all the wildness of X = Antoine's necklace in $E = S^3$, so it is possible in the case $X = S^1$ and $E = S^1 * D^n$. An exact argument appears in [23, §5]. This completes step 7.1.

7.2. Replace the crumpled cubes $C_1, C_2, \ldots$ of step 7.1 by real cubes $D_1, D_2, \ldots$ having the same boundary:

As in the analysis of Antoine's necklace, one assumes E embedded in a high dimensional Euclidean space, and one physically makes the replacement.

7.3. Prove that the resulting space $E' = (E - \bigcup C_i) \cup (\bigcup D_i)$ is homeomorphic with S^{n+2}:

The first matter of business is that of showing E' is a generalized $(n+2)$-manifold. Showing that it is an ENR consists essentially in constructing inverse homotopy equivalences $E' \to E \to E'$ which take each C_i to each D_i and fix $E' - \bigcup D_i$ and $E - \bigcup C_i$. From the existence of such homotopy equivalences, the local contractibility of E' follows easily; and closed locally contractible subspaces of Euclidean space are ENR's. One then has to establish the relevant homology condition $H_*(E', E' - \{x\}; Z) \cong H_*(E^{n+2}, E^{n+2} - \{0\}; Z)$ for each $x \in E'$. This condition follows from the corresponding condition for E essentially because C_i and D_i are indistinguishable homologically, both locally and globally. (See [23] for details.)

The next consideration is to note that $E' - X$ is 1-ULC. The C_i's were constructed precisely with that aim in mind. The actual argument is fussy but obviously bound to succeed (see [23] for details).

Finally, one is in a position to prove that E' is a manifold. To this end, one first shows that X sits nicely in the boundary of an $(n+2)$-cell B in E' as follows. The spaces E' and E have the same homotopy type. Therefore, since $\pi_1(E) = 1$, X contracts in E'. The contraction defines a singular cone cX over X in E'. The fact that $E' - X$ is 1-ULC allows one to adjust the cone so that the open cone lies entirely in the PL $(n+2)$-manifold $E' - X$. General position embeds the open cone in a PL fashion. A PL regular neighborhood N of $(cX) - X$ in $E' - X$ which pinches nearer and nearer to cX near X then forms with X the desired $(n+2)$-ball B which has 1-ULC complement in E'. We then have the setting of the following theorem of A. V. Černavskiĭ and C. L. Seebeck III from which we conclude immediately that E is a manifold ($S = \text{Bd } B$; $k = n + 2$):

THEOREM (ČERNAVSKIĬ [27]; SEEBECK [59]). *Suppose E' is a generalized k-manifold of dimension $k > 5$, that the nonmanifold set of E' is contained in a codimension-one submanifold S of E', and that $E' - S$ is 1-ULC. Then E' is a manifold and S is a locally flat submanifold.*

Unfortunately, neither the Černavskiĭ nor the Seebeck proofs have appeared in print. S. Ferry has proved a more general theorem which will be discussed a bit in Supplement 4 but which has also not as yet appeared. Fortunately the theorem can be proved in our case without too much difficulty by a radial engulfing argument which, acting only in $E' - B$, drags a local collar from $(\text{Bd } B) - X$ across X and thereby shows that $\text{Bd } B$ is locally collared at X from $E' - B$.

Since E' is a manifold and has the homotopy type of E which has the homotopy type of S^{n+2}, $n + 2 > 5$, the high dimensional Poincaré conjecture implies that E' is a topological S^{n+2}.

7.4. Recover the original wild embedding of X in E from the tame embedding $X \subset E'$ via decomposition space techniques in such a way as to conclude that E' and E are homeomorphic:

The proof is exactly as indicated for the case of Antoine's necklace once one notes that each D_i is tame in E' since it has 1-ULC complement [27], [30]. Alternatively one can simply shrink the appropriate collar arcs via an engulfing argument which uses the 1-ULC condition but which makes no use of the 1-ULC taming theorem for codimension-one submanifolds.

Thus we conclude that $S^{n+2} \cong E' \cong E$ as desired.

Supplement 1. Recognizing manifolds of dimension $n \leq 2$. Illustrations. Manifolds of dimension ≤ 2 admit striking characterizations. For example, the circle or simple closed curve may be characterized as the only compact connected metric space containing a pair of points and separated by every pair of its points. This characterization allows one to prove beautiful theorems such as the following from plane topology:

S1.1. THEOREM. *Suppose the locally connected, connected, compact metric subspace X of the plane E^2 separates points x and y in E^2. Then some simple closed curve J in X separates x and y in E^2.*

PROOF. We describe J as follows. Let U be the complementary domainof X in E^2 containing x. Let I be its boundary. Let V be the complementary domain of I in E^2 containing y. Let J be its boundary. Note that J separates x from y in E^2. (See Figure 10.) In order to prove that J is a simple closed curve requires two lemmas in addition to the characterization cited above:

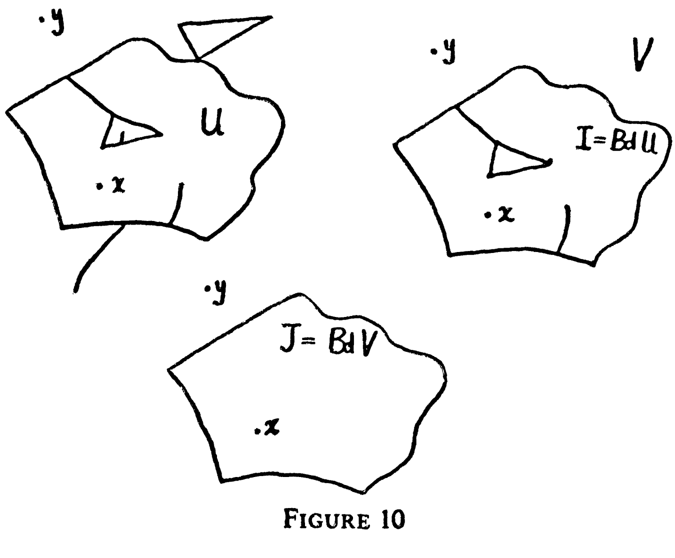


FIGURE 10

S1.2. LEMMA. *If U is a complementary domain of a planar locally connected continuum, then* Bd U *is a locally connected cöntinuum.* [53].

S1.3. LEMMA. *If U is a complementary domain of a planar locally connected continuum, then each point of* Bd U *is arcwise accessible from U.* [53].

Assuming the two lemmas, we note first that J is connected by two applications of Lemma 2. We then take any two points p and q of J. We must show that p and q separate J. Both p and q are arcwise accessible from

$U \subset E^2 - J$ by Lemma 3. Both p and q are arcwise accessible from $V \subset E^2 - J$ by Lemmas 2 and 3 taken together. Thus there is a simple closed curve K in E^2 intersecting J precisely at p and q, one arc of $K - J$ in U, the other in V (Figure 11). But J must be separated by K, for otherwise the Jordan curve theorem implies that the complementary domain of K missing J hits both U and V, and that would imply that J cannot separate x from y, a contradiction.

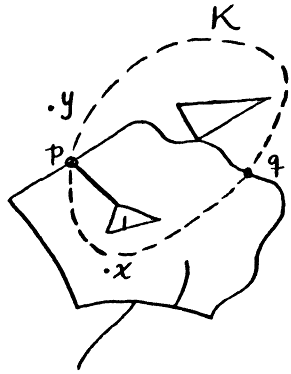


FIGURE 11

The 2-dimensional sphere S^2 also admits a striking characterization called the Kline sphere characterization proved by R. H. Bing **[10]**:

S1.4. THEOREM. *The 2-sphere is the only nondegenerate locally connected, connected, compact metric space which is separated by no pair of its points but is separated by each of its simple closed curves.*

As an application of this theorem we mention a beautiful theorem of R. L. Moore originally proved by other means (see **[51]**, **[52]**).

S1.5. THEOREM (MOORE **[52]**). *Suppose f: $S^2 \to X$ is a surjective map from the 2-sphere S^2 onto a Hausdorff space X such that, for each $x \in X$, $S^2 - f^{-1}(x)$ is nonempty and connected. Then X is homeomorphic with S^2.*

PROOF. Since X is Hausdorff, f is a closed map, and X is thus locally connected, connected, compact, and metric.

Let x and y be points of X. By hypothesis, neither $f^{-1}(x)$ nor $f^{-1}(y)$ separates S^2. Thus $f^{-1}(x) \cup f^{-1}(y)$ does not separate S^2 (apply, say, the Mayer-Vietoris sequence for homology). It follows that $X - \{x, y\}$ is connected.

Let J be a simple closed curve in X made up of two arcs A and B intersecting precisely at their endpoints. Then $f^{-1}(A)$ and $f^{-1}(B)$ are connected since each $f^{-1}(x)$ is connected. But $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(\text{Bd}\, A) = f^{-1}(\text{Bd}\, B)$ is not connected. A Mayer-Vietoris argument suffices to prove therefore that $f^{-1}(J)$ separates S^2. Consequently J separates X. By the Kline sphere characterization X is a 2-sphere.

Supplement 2. Characterizations of n-manifolds via partitionings. R. H. Bing **[12]** has given a characterization of S^3 in terms of partitionings. O. G. Harrold **[41]**, **[42]** has given similar characterizations of high dimensional manifolds. The characterizations are difficult to apply since they assume so much explicit geometric structure.

Supplement 3. Characterizations of Q-manifolds. Torunczyk's beautiful theorem [65] states the following.

THEOREM. *Let E denote a locally compact separable metric ANR. Then E is a Q-manifold if and only if the following general position criterion is satisfied:*

For each n, each map $f: B^n \to E$, and each $\varepsilon > 0$, f can be ε approximated by a map onto a Z-set in E.

Supplement 4. Open questions concerning the characterization of generalized manifolds. The disjoint disk property of §1 has been used in various forms to distinguish certain nonmanifolds from manifolds ever since R. H. Bing's example in [13]. In its present form and as a hypothesis sufficient to pick out manifolds among certain generalized manifolds it first appeared in our first proof of the double suspension theorem.

In the arbitrary generalized manifold, the disjoint disk property can fail miserably. J. W. Cannon and R. J. Daverman [25] have described a generalized n-manifold E for every $n \geqslant 3$ and a simple closed curve J in E such that every shrinking of J in E contains an open subset of E in its image. The space E also has the property that $E \times E^1 = E^{n+1}$ although every product line $\{x\} \times E^1$ is wild.

R. J. Daverman has proved that if E is a generalized manifold of dimension $\geqslant 3$, then $E \times E^2$ has the disjoint disk property. Still unresolved is the following question.

S4.1. *Question.* Is there a generalized manifold E of dimension $\geqslant 4$ such that $E \times E^1$ does not have the disjoint disk property?

R. D. Edwards has just recently announced [36] the following beautiful theorem.

S4.2. THEOREM. *Suppose $f: M \to E$ is a surjective proper map from an n-manifold M onto a generalized n-manifold E such that, for each $x \in E$, $f^{-1}(x)$ is cell-like, $n \geqslant 5$. Then f can be approximated by homeomorphisms if and only if E satisfies the disjoint disk property.*

The best theorems previously known were those of F. Tinsley [64a] and J. W. Cannon [22].

Edwards' argument essentially depends only on radial engulfing. Taken together with some of his earlier work, it supplies a new proof of the double suspension theorem. In this regard, it is interesting to note that the proof that we have described can also be presented in such a way that it depends only on radial engulfing as well. When Edwards' theorem is taken together with Daverman's theorem about $E \times E^2$, we can deduce that each generalized manifold which is a cell-like image of a manifold M is a Cartesian factor of the manifold $E \times E^2 \cong M \times E^2$.

S4.3. *Question.* If E is a generalized manifold which is a cell-like image of a manifold, is $E \times E^1$ a manifold?

S4.4. *Question.* Is every generalized n-manifold a cell-like image of an n-manifold?

The best known result regarding Question S4.4 is the following.

S4.5. THEOREM (CANNON, BRYANT, AND LACHER [23]). *A generalized n-manifold E of dimension $n \geqslant 5$ whose nonmanifold set has dimension k, $2k + 2 \leqslant n$, is a cell-like image of a manifold.*

The low dimensional ($n = 3, 4$) versions of Question S4.4 and Theorem S4.5 are also unresolved. To some extent they are tied up with the low dimensional Poincaré conjectures. In this regard see for example the thesis of M. Brin [17].

The method we have exposited in examining Antoine's necklace and the double suspension of a homology sphere yields the following theorem.

S4.6. THEOREM. *Suppose that E is a generalized n-manifold of dimension $n > 5$ and that the nonmanifold set M of E has dimension $k \leqslant n - 3$. Then there is a generalized n-manifold E' and a cell-like map $p: E' \to E$ such that the nonmanifold set M' in E' has dimension $\leqslant n - 3$ and has 1-ULC complement in E'.*

The basic unresolved question regarding the application of Theorem S4.6 is the following.

S4.7. *Question.* Is E' a manifold? Is E' a manifold if M' lies in a topological polyhedron in E' of dimension $\leqslant n - 3$?

The resolution of question S4.4 seems to require some extension of the Černavskiĭ-Seebeck theorem quoted and used in §7. Two nice generalizations have just recently been proved by S. Ferry and T. Chapman.

S4.8. THEOREM (S. FERRY). *Suppose E is a generalized n-manifold, $n \geqslant 5$, that S is a generalized $(n - 1)$-manifold lying in E as a closed subset with $E - S$ 1-ULC, and suppose the nonmanifold set of E lies in S. If $S \times E^1$ is a manifold, then E is a manifold.*

We do not have the details of the Chapman result. But it is reportedly similar to the Ferry theorem and involves an $(n - 2)$-dimensional generalized submanifold S of E, where $E - S$ is a true manifold satisfying appropriate homotopy properties.

Perhaps a codimension-three version of the Ferry-Chapman results would resolve Question S4.7.

S4.9. *Question.* Is every generalized n-manifold a Cartesian factor of an n-manifold?

A tantalizing unresolved question is the following.

S4.10. *Question.* Suppose $f: M \to Y$ is a cell-like mapping from a compact n-manifold M onto a compact metric space Y. Must Y be a generalized n-manifold? Equivalently, must Y be finite dimensional? An ANR?

Question S4.10 is closely related to the old question:

S4.11. *Question.* Is a compact metric space of finite homological dimension also of finite covering dimension? (Added in proof: R. D. Edwards has announced that S.4.10 and S.4.11 are equivalent questions.)

Supplement 5. Homogeneity. A space X is homogeneous if, for each $x, y \in X$, there is a homeomorphism $h: X \to X$ taking x to y. Every connected manifold (without boundary) is homogeneous, and homogeneity has often been suggested as an appropriate condition to add to generalized manifold to get a possible characterization of manifold. We suspect however, from our experience constructing generalized manifolds as decomposition spaces, that there are nonmanifolds that are nevertheless homogeneous generalized manifolds.

S5.1. *Question*. Does there exist a generalized manifold which is not a manifold but is nevertheless homogeneous?

A related, possibly easier, question is the following.

S5.2. *Question*. Does there exist a wild embedding $f: S^2 \to E^3$ that is homogeneous in the following sense: for each $x, y \in f(S^2)$, there is a homeomorphism $h: E^3 \to E^3$ taking $f(S^2)$ to $f(S^2)$ and x to y?

Supplement 6. Locally flat approximations of codimension-one submanifolds of manifolds. Recently F. D. Ancel and J. W. Cannon [2] proved a theorem of which the following is perhaps the most important special case.

S6.1. THEOREM. *Suppose $f: X \to E$ is a closed embedding of an $(n-1)$-manifold X in an n-manifold E, $n \geqslant 5$. Then f can be approximated by locally flat embeddings.*

The techniques used in analyzing the double suspension theorem supply another proof of the codimension-one approximation theorem stated above:

PROOF. One simply follows the same four steps. Proofs are exactly the same except for step 3 (compare 6.3 and 7.3). For that step one can apply the Černavskiĭ-Seebeck-Ferry theorem directly with $S = f(X)$ to deduce that E' is a manifold and $f(X)$ is a locally flat submanifold. The map $p: E' \to E$ can be approximated by homeomorphisms (L. C. Siebenmann [61], J. W. Cannon [21], or R. D. Edwards [36]). A homeomorphic approximation h to p takes the locally flat $f(X) \subset E'$ to a locally flat approximation $hf(X)$ to the wild $f(X) \subset E$.

S6.2. GENERALIZATION. Using the Ferry theorem in Supplement 4 one can allow X to be a generalized $(n-1)$-manifold with $X \times E^1$ a manifold.

Supplement 7. The high dimensional Hosay-Lininger-Daverman theorem and its generalizations. N. Hosay [43], L. L. Lininger [47], and R. J. Daverman [29] in dimension $n = 3$ and R. J. Daverman [32] in dimensions $n \geqslant 5$ have proved the following beautiful theorem.

S7.1. THEOREM. *Suppose S is a topological $(n-1)$-sphere in S^n $(n \neq 4)$ and C is the closure of one of the complementary domains of S in S^n. Then C can be reembedded in S^n so that $S^n -$ Im(Int C) is an n-ball.*

PROOF FOR $n \geqslant 5$. Let B^n denote an n-ball and let $E = C \cup_{S=\mathrm{Bd}\,B^n} B^n$ be the generalized n-manifold which results when $X = \mathrm{Bd}\, C$ and $\mathrm{Bd}\, B^n$ are identified. It is not difficult to check that E has the disjoint disk property. The technique indicated in Supplement 6 shows that E is a cell-like image of an n-manifold. R. D. Edwards' theorem from Supplement 4 implies that E is an n-manifold. The Poincaré conjecture implies that E is S^n.

S7.2. GENERALIZATION. Using Ferry's generalization of the Černavskiĭ-Seebeck theorem (Supplement 4), one can generalize the Hosay-Lininger-

Daverman theorem to allow S to be replaced by a generalized $(n-1)$-manifold whose product with E^1 is a manifold. Using the Ancel-Cannon theorem [1], [2], one can get a weak version even with generalized $(n-1)$-manifolds which need not be E^1 cofactors of an n-manifold.

Supplement 8. Blankinship's generalizations of Antoine's necklace. Antoine's necklace was constructed in a solid torus $B^2 \times S^1$. Blankinship [16] described analogous wild Cantor sets in higher dimensions. He considered high dimensional thickened tori $B^2 \times S^1 \times \cdots \times S^1$ ($n-2$ S^1 factors) and the $n-2$ projection maps p_i: $B^2 \times S^1 \times \cdots \times S^1 \to B^2 \times S^1$ onto B^2 and the ith S^1 factor. If one then performs a stage of the Antoine construction, putting four linked tori in the image $B^2 \times S^1$, then the inverses under p_i of those four tori are again of the form $B^2 \times S^1 \times \cdots \times S^1$ but are small only in the ith factor. The next stage of the construction calls for the projection of the new thickened tori onto a new factor; this stage produces in each of the new thickened tori four even smaller thickened tori that are small in two S^1 factors. A sequence of $n-2$ stages finally produces 4^{n-2} thickened tori that are smaller in all $n-2$ of the S^1 factors. An infinite iteration leads finally to a wild Cantor set in the original $B^2 \times S^1 \times \cdots \times S^1$.

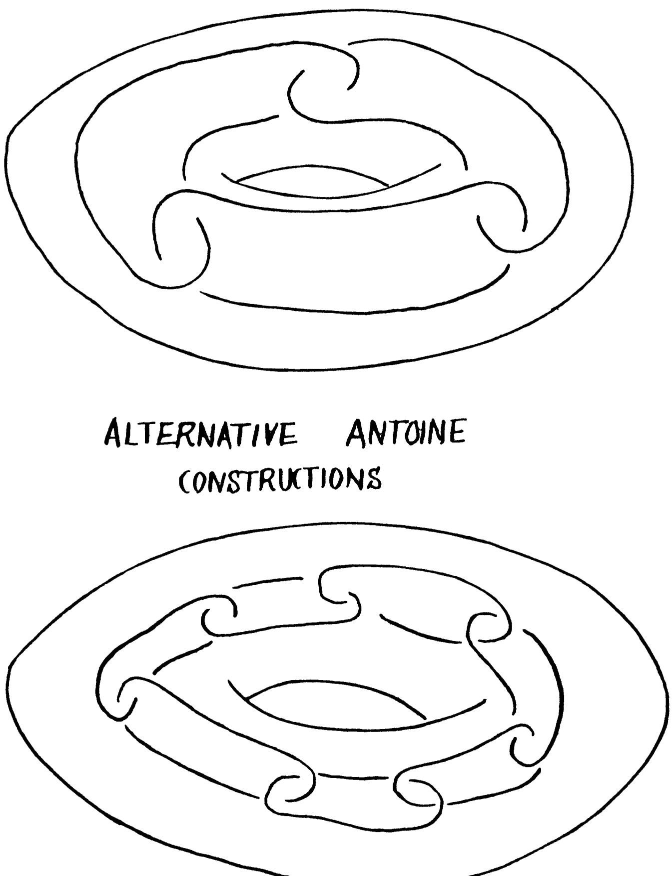
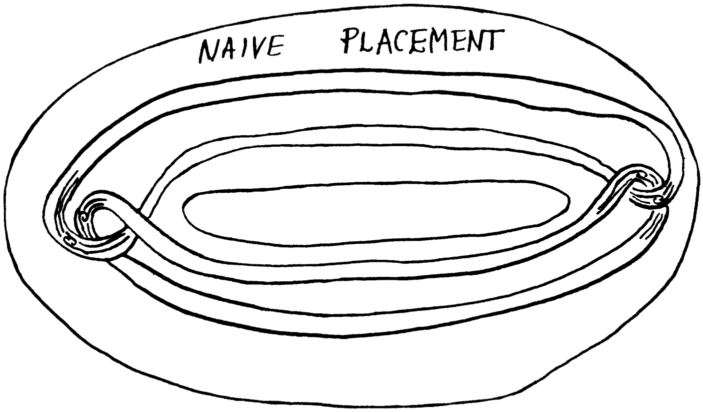


FIGURE 12



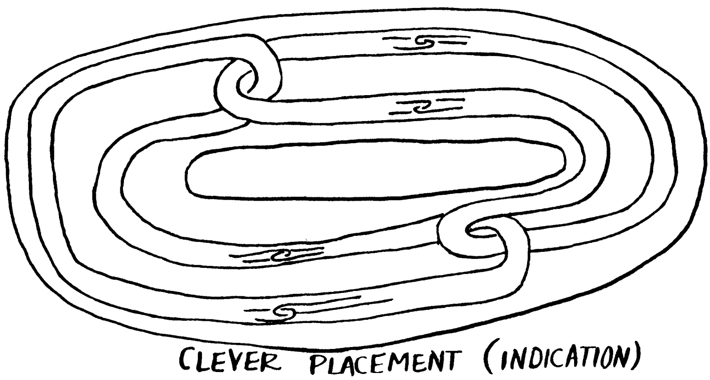


FIGURE 13 FIGURE 14

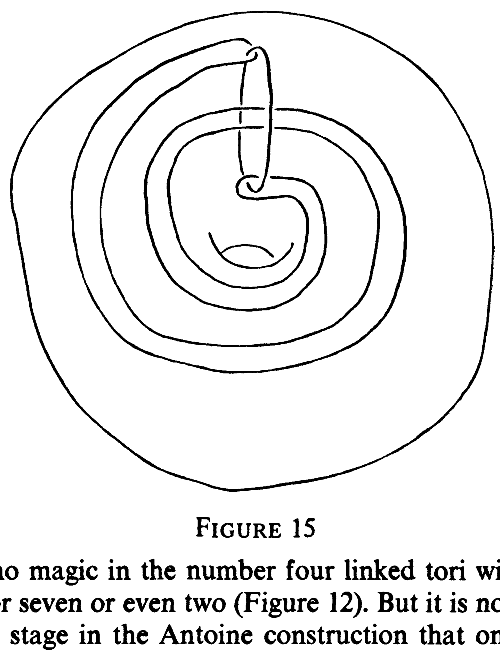


FIGURE 15

Now there is no magic in the number four linked tori within a torus. One might use three or seven or even two (Figure 12). But it is not clear that if one uses two at each stage in the Antoine construction that one can so cleverly arrange the tori that, in the limit, their diameters go to zero so that a Cantor set results (see Figure 13). R. H. Bing's very clever proof that the sewing of two solid Alexander horned spheres yielded S^3 consisted precisely in showing that such a placement is possible for two linked tori (see Figure 14). Bing's example [13] of a countable null nonshrinkable cellular decomposition of S^3 depends precisely on the possibility of showing that such a clever placement is impossible for two linked tori provided that they are required to circle the previous stage more than once (see Figure 15). However, we leave it as an exercise for the reader to show that in higher dimensions the Antoine-Blankinship construction using Figure 15 can be so cleverly arranged that it describes a Cantor set. In dimensions $\geqslant 5$ one can use the disjoint disk property and apply general theorems. This exercise disposes of the most obvious attempts to construct an example of a countable null nonshrinkable cellular decomposition of S^n, $n \geqslant 4$. Thus we have the following problem.

S8.1. PROBLEM. Construct a countable null nonshrinkable cellular decomposition of S^n, $n \geqslant 4$. (Added in proof: R. J. Daverman has constructed an example.)

Supplement 9. The status of cell-like decomposition space theory of manifolds. Edwards' theorem, Theorem S4.2, brings the theory of cell-like upper semicontinuous decompositions to a certain degree of completion. However, the study of the local topology of 3-manifolds, in which [14], [15] Bing's 1-ULC taming theorems play roles analogous to Edwards' theorem, really matured only as the nonobvious implications of Bing's theorems began to be worked out (see [19]). Enough theory now exists to allow a rather deep and explicit study of the various forms of wildness in spaces and subspaces. R. J. Daverman's work has created the most interesting examples to date, but much remains to be done.

Many problems remain unresolved in dimensions 3 and 4. M. Starbird, following work of E. Woodruff, D. Everett, and M. Starbird [63], [67], [38], seems to have proved for E^3 the appropriate analogue of Edwards' theorem. However, the consequences of Starbird's theorem have also not been explored in any depth.

One question which should really be capable of resolution at the present time is the following:

S9.1. *Question.* If G is a cell-like upper semicontinuous decomposition of E^3, is $(E^3/G) \times E^1$ homeomorphic with E^4?

Of course many special cases are known.

In dimension four one needs some cellularity criterion; one needs analogues to the 1-ULC taming theorems (possibly using a more geometric version of the 1-ULC property); etc.

Supplement 10. Poincaré's homology sphere. This supplement together with the three which follow it presents a rather complete analysis of the relationship of homology spheres, perfect groups and wildness. In order to have one concrete example in mind, we describe Poincaré's original nonsimply connected homology 3-sphere here.

Poincaré gave the first example of a nonsimply connected homology 3-sphere D^3 in 1904 [54, pp. 493–498] in terms of the following Heegaard diagram [54, p. 494] (Figure 16). For an explanation of the figure see

Poincaré's paper or, for example, Rolfsen's book [56, p. 245].

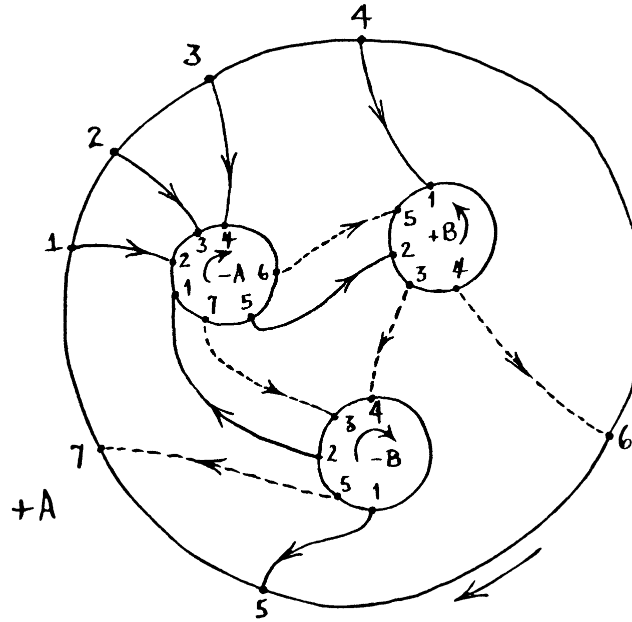


FIGURE 16

Weber and Seifert in 1933 [66] showed that Poincaré's homology sphere is homeomorphic with the so-called (spherical) dodecahedral space. The dodecahedral space is the orbit space of a fixed point free action of the binary icosahedral group of order 120 on the 3-sphere S^3. It follows immediately from covering space theory that $\pi_1(D^3)$ is the binary icosahedral group. This discontinuous group has as fundamental region in S^3 a regular spherical

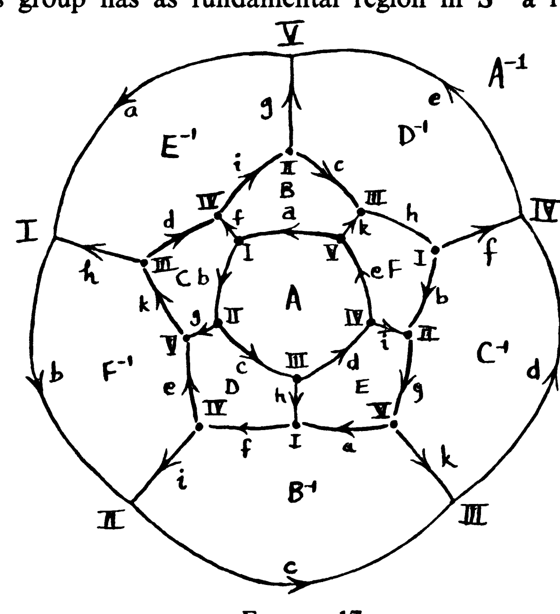


FIGURE 17

dodecahedron of edge angle $2\pi/3$. Exactly 120 of these regular dodecahedra form a tessellation of S^3. One obtains D^3 by identifying opposite faces of a fundamental region with a twist of $\pi/5$ as indicated in Figure 17 which is taken from Weber and Seifert [66, p. 243].

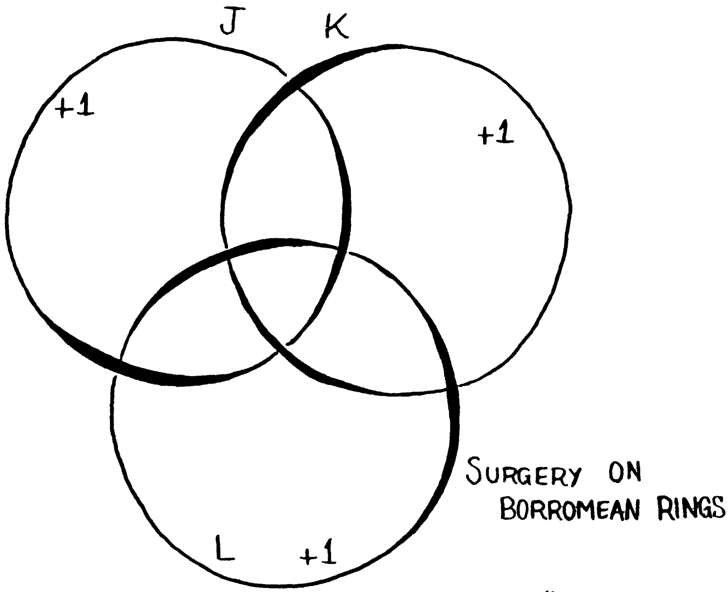


FIGURE 18

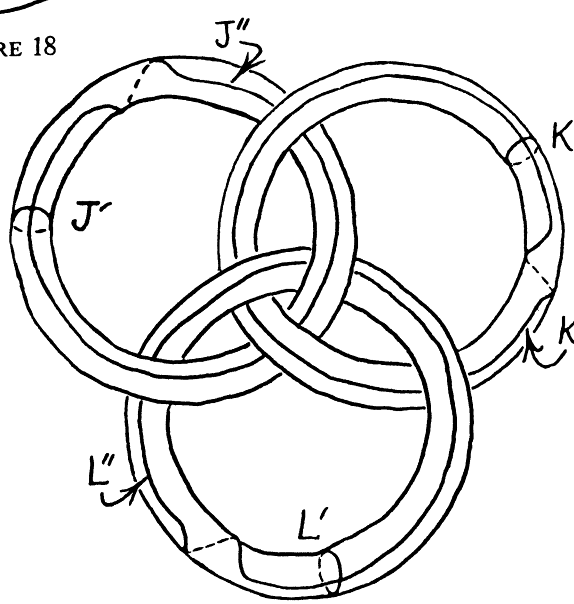


FIGURE 19

We shall use the following description of D^3 in Supplement 11 to conclude geometrically that $\pi_1(D^3)$ is a perfect group. The description is a so-called surgery diagram; we first saw this description in Rolfsen's book [56, p. 269]; we give it here as Figure 18. The meaning of the diagram of Figure 18 is as follows (see Figure 19). Let $N(J)$, $N(K)$, and $N(L)$ denote disjoint tubular neighborhoods of J, K, and L, respectively, in S^3. Let J', K', and L' denote meridional curves on Bd $N(J)$, Bd $N(K)$, and Bd $N(L)$ as pictured in Figure 19. Let J'' K'', and L'' denote curves on Bd $N(J)$, Bd $N(K)$, and Bd $N(L)$ as pictured in Figure 19 that circle the respective boundaries once longitudinally and once meridionally. Cut S^3 apart along Bd $N(J) \cup$ Bd $N(K) \cup$ Bd $N(L)$ to obtain four compact components, S_0, T_1, T_2, and T_3, corresponding respectively to $S^3 - \text{Int}(N(J) \cup N(K) \cup N(L))$, $N(J)$, $N(K)$, and $N(L)$. Identify the copy of Bd $N(J)$ in S_0 with the copy of Bd $N(J)$ in T_1 in a new way so that J' and J'' are interchanged by the identification. Sew T_2 and T_3 to S_0 similarly, interchanging K' and K'', L' and L''. One obtains thereby a new compact three manifold $S_0 \cup T_1 \cup T_2 \cup T_3$ homeomorphic with D^3.

Supplement 11. Perfect groups. A group π is perfect if it is its own commutator subgroup: $\pi = [\pi, \pi]$. The characteristic property of the fundamental group $\pi = \pi_1(D^n)$ of a homology n-sphere D^n, $n \geqslant 2$, is that it is a perfect group: $\pi/[\pi, \pi] \cong H_1(D^n;Z) \cong H_1(S^n;Z) = 0$.

Let D^3 denote Poincaré's homology sphere as described in Supplement 10. We illustrate the fact that $\pi = \pi_1(D^3)$ is perfect by giving a geometric proof that $H_1(D^3;Z) \cong 0$.

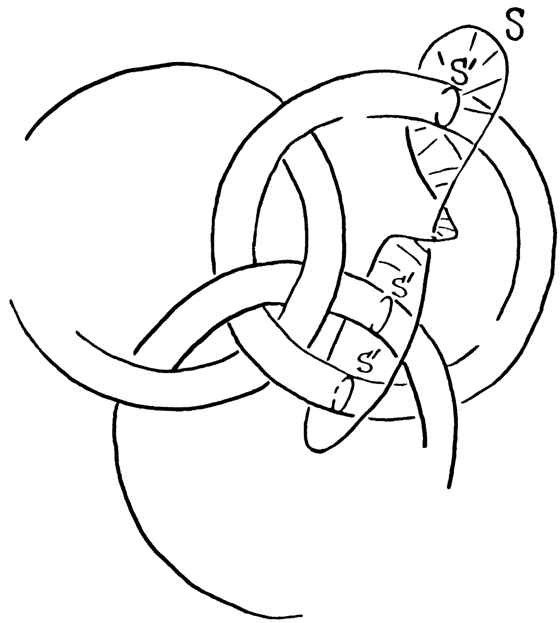
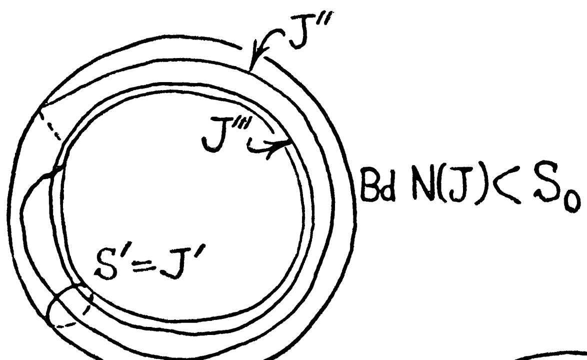


FIGURE 20

Our surgery description of D^3 (Figures 18 and 19) realized D^3 as a union $S_0 \cup T_1 \cup T_2 \cup T_3$ with boundaries sewn together in a special way. Let S be any simple closed curve in D^3, representing an element of $H_1(D^3)$. Since T_1, T_2, and T_3 are regular neighborhoods in D^3 of 1-dimensional subpolyhedra of D^3, S may be adjusted so that it lies in Int S_0. Thinking of S_0 for a moment as a submanifold of S^3, and using the fact that S contracts to a point in S^3 along a singular disk transverse ('orthogonal') to the three original curves J, K, and L, we see easily that S is one boundary curve of a (singular) disk-with-holds D_0 in S_0 such that each of the other boundary curves S' of D_0 is parallel on $\mathrm{Bd}(N(J) \cup N(K) \cup N(L))$ to one of the original meridional curves J', K', or L' (see Figure 20). Thinking next of S' as being in the boundary of $T_1 \cup T_2 \cup T_3$ in D^3, we see that S' is isotopic in $T_1 \cup T_2 \cup T_3$ to one of the curves J''', K''', or L''' (see Figure 21). Finally, J''' evidently bounds a disk-with-a-single-handle in S_0 (Figure 22), as do also K''' and L'''. That is, S' is null-homologous (bounds a singular disk-with-handles) in D^3. The disk-with-holes D_0 and the disks-with-handles bounded by the S' curves together form a singular disk-with-handles bounded by S. Thus S is null-homologous, and D^3 has trivial first homology as claimed.



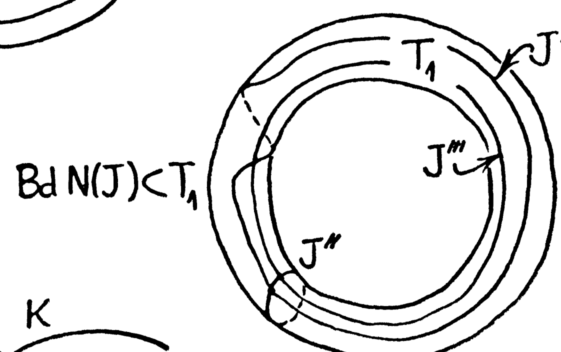


FIGURE 21

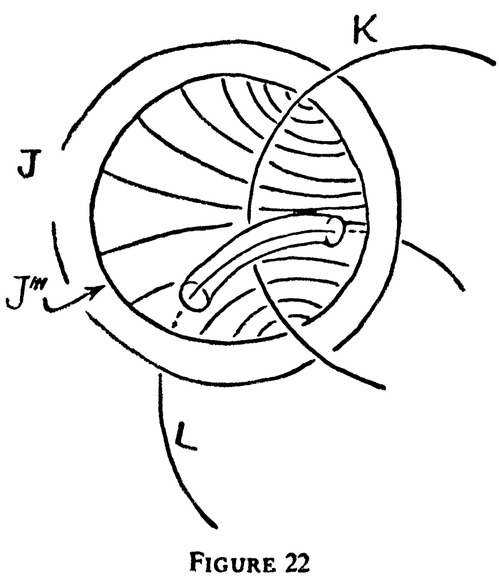


FIGURE 22

Supplement 12. Perfect groups and wildness. Every group π has a maximum perfect subgroup $\omega(\pi)$. The group $\omega(\pi)$ may be easily characterized in any of the following three equivalent ways:

(1) $\omega(\pi)$ is the subgroup of π generated by the union of all perfect subgroups of π.

(2) $\omega(\pi)$ is the intersection of all the groups in the transfinite derived series of π.

(3) $\omega(\pi)$ consists of those elements of π that are products of commutators of elements of π that are products of commutators of elements of π that are products . . . (and so on for countably many steps).

For reasons that we shall presently explain, we choose to call $\omega(\pi)$ the wild group of π.

We now give a first justification for the term 'wild group.' Even more compelling justifications will appear in Supplement 13. In solving the locally spherical sphere problem [20], we were led to consider the following situation (see Figure 23).

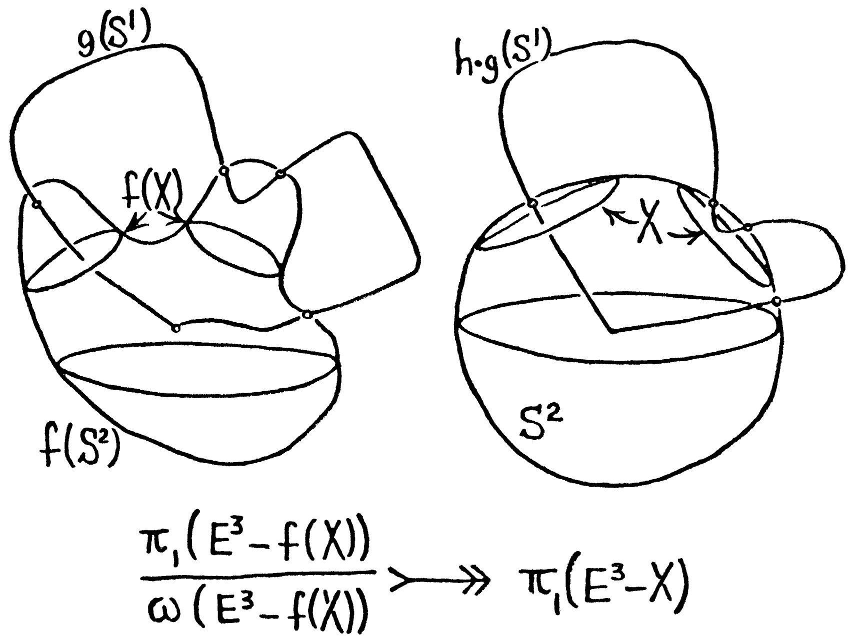


FIGURE 23

Suppose S^2 is the standard unit 2-sphere in E^3 and f: $S^2 \to E^3$ is an arbitrary, possibly wild, embedding. Let X be an arbitrary compact subset of S^2. In general, X will be tame in E^3 while $f(X)$ will be wild; as a consequence, the fundamental groups $\pi_1(E^3 - X)$ and $\pi_1(E^3 - f(X))$ need not be isomorphic. Nevertheless, there is a clear visual relationship between the holes in X, as seen in the components of $S^2 - X$, and the holes in $f(X)$, as seen in the components of $f(S^2 - X)$. Using this visual relationship, it is not difficult to define a surjective homomorphism φ: $\pi_1(E^3 - f(X)) \to \pi_1(E^3 - X)$ as follows: choose base points in the interiors in E^3 of the two 2-spheres $f(S^2)$ and S^2; take a loop g: $S^1 \to E^3 - f(X)$ representing an element of $\pi_1(E^3 - f(X))$; extend the map f^{-1}: $f(S^2) \to S^2$ to a map h: $f(S^2) \cup g(S^1) \to E^3$ in such a manner that $h[g(S^1) \cap \operatorname{Int} f(S^2)] \subset \operatorname{Int} S^2$, $h[g(S^1) \cap \operatorname{Ext} f(S^2)] \subset \operatorname{Ext} S^2$, and h(base point) = (base point); then $\varphi[g] = [h \circ g]$ defines the homomorphism φ. In general, φ has kernel. The kernel of φ is precisely the wild group of $\pi_1(E^3 - f(X))$. That is, the wild group captures at least globally the geometric wildness in the set $f(X)$. By working locally one can show that the wild groups capture algebraically precisely the geometric wildness of $f(X)$.

Supplement 13. Perfect groups, the grope, Alexander's horned crumpled cube, *BPL*, and Bing's sewing of two Alexander horned cubes. We seek a geometric interpretation of the wild group of a group π. For this purpose we use the third characterization of the wild group of a group described in Supplement 12: the wild group of π consists of those elements of π which are products of commutators of products of commutators of products of commutators, etc. We realize π as the fundamental group of some topological space Y. We let f: $S^1 \to Y$ be a loop representing an element $[f]$ of $\omega(\pi)$. A loop represents an element of π that is a product of commutators if and only if it bounds a disk-with-handles (singular) in Y [60, p. 173] (see Figure 24). That $[f]$ is a product of commutators of elements that are products of commutators of elements that are products of commutators, . . . says precisely, therefore, that f bounds a certain infinite (singular) 2-complex in Y made up of infinitely many disks-with-handles sewn together as indicated in Figure 25. The figure indicates that f bounds a disk-with-handles D_0, whose handle curves, a', b', c', d', bound disks-with-handles, whose handle curves bound disks-with-handles, etc. An infinite (nonsingular) 2-complex G formed by sewing together disks-with-handles as indicated in the figure, boundary curves of stage i being identified with handle curves of the preceding stage $i - 1$, is called an (open) grope because of its multitudinous fingers. D. R. McMillan, Jr., suggested this terminology.

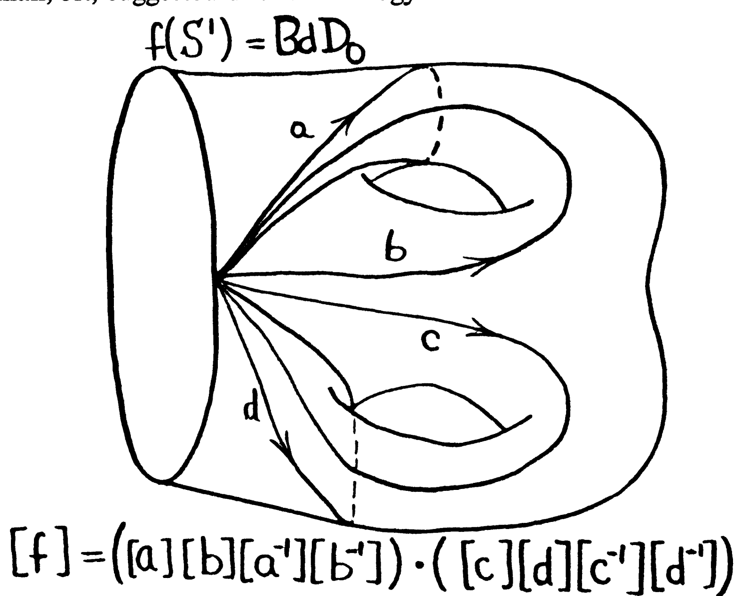


FIGURE 24

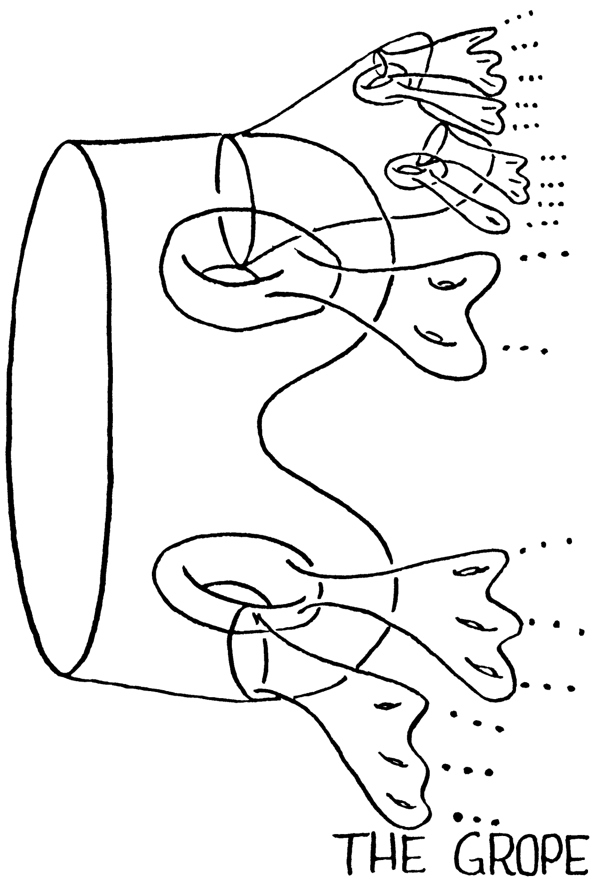


FIGURE 25

This terminology suggests any number of bad puns. For example, the simplest nontrivial grope consists at each stage entirely of disks-with-a-single-handle. This grope should clearly be called the fundamental grope. We indicate in Figure 26 how the fundamental (open) grope G and, more generally, all open gropes can be (PL) embedded in E^3. The idea is to proceed stage by stage, embedding stage i in pillboxes (2-handles) attached along the unknotted (inductive assumption) handle curves of stage $i-1$. If stage i is embedded with unknotted handles, the induction can continue. If the pillboxes are taken to have diameters convergent to 0 with i, then the closure of G in E^3 is what we call a closed grope G^+. The closed grope may also be defined abstractly as the Freudenthal compactification of G. The set $G^+ - G$ is a closed 0-dimensional set (a Cantor set in the case of the fundamental grope).

The reader should take occasion to note that the grope G is precisely the geometric realization of the arbitrary element of a perfect group. Take the closed grope G^+ in E^3 as described in the preceding paragraph. Delete $G^+ - G$ from E^3, and let N denote a (PL) regular neighborhood of G in $E^3 - (G^+ - G)$. Examine the set $C = N \cup G^+$. Then C is a compact subset of E^3 which has G^+ as a strong deformation retract. The set Bd C is always a 2-sphere, and, in the case where G is the fundamental grope, C is precisely the wild crumpled cube described in §5 as J. W. Alexander's wild crumpled cube [22]. In other words, the Alexander crumpled cube is the fundamental geometric realization of the nontrivial perfect group.

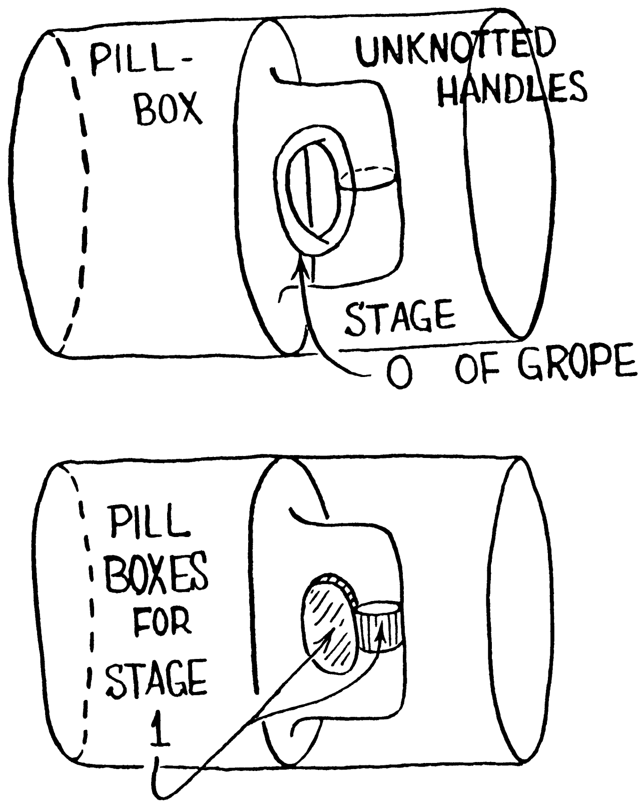


FIGURE 26

The fact that C is a compact absolute retract and that C deformation retracts to G^+ gives geometric content to some algebraic facts long utilized. Namely, the quotients of a group by the terms of its derived series (recall Supplement 12, characterization 2) may be used as a good approximation to the group itself. The fact that G^+ is contractible even though G need not be shows that quotients by terms of the derived series come 'within Freudenthal compactification' of capturing the group exactly. Of course, in order that the boundary $[f]$ of stage 0 of G contract, one need not use all of G^+. It suffices to choose 'half' of the handle curves of stage $i-1$ (one from each transverse pair) and to sew disks-with-handles to these to form stage i. One obtains what might be called a semigrope G. It corresponds geometrically to the algebraic lower central series of a group. As hinted above, f contracts in the Freudenthal compactification G^+ of the (open) semigrope. This observation lends geometric content to the idea of approximating a group through its quotients by terms of the lower central series.

The application of the preceding discussion to the double suspension of homology spheres requires that we examine the embedding of a grope not only in E^3 but also in arbitrary manifolds of dimension $n \geqslant 5$. For this we need the following beautiful theorem of W. B. R. Lickorish and L. C. Siebenmann [46]:

THEOREM. *Fix integers $n \geqslant 5$ and k, $2k + 1 \leqslant n$. Let G be a locally finite simplicial complex of dimension k. Consider all closed PL embeddings $f: G \to M$ of G into arbitrary PL n-manifolds M. Consider regular neighborhoods $N(f(G), M)$ of $f(G)$ in M. Then the PL homeomorphism type of the pairs $(N(f(G), M), f(G))$ are classified by the tangent PL microbundle of the open PL manifolds* Int $N(f(G), M)$. *Equivalently, the homeomorphism types are in one-one correspondence with the homotopy set $[G, BPL]$, where BPL is the classifying space for stable tangent PL microbundles.*

COROLLARY. *Let G be a grope and let $f: G \to M$ be any closed PL embedding of G into an n-manifold, $n \geqslant 5$. Let N be a regular neighborhood of $f(G)$ in M. Then the PL homeomorphism type of the pair $(N, f(G))$ is uniquely determined by G and n.*

PROOF OF THE COROLLARY. The homotopy set $[G, BPL]$ consists of a single element [23]. Thus the Lickorish-Siebenmann theorem applies.

In order to understand regular neighborhoods of an open PL grope in a PL manifold of dimension $\geqslant 5$, it therefore suffices to consider in detail some single carefully chosen regular neighborhood of some carefully chosen embedding. We take for our embedding the PL embedding of G in $E^3 = E^3 \times \{0\} \subset E^3 \times E^{n-3} = E^n$ described earlier in this section.

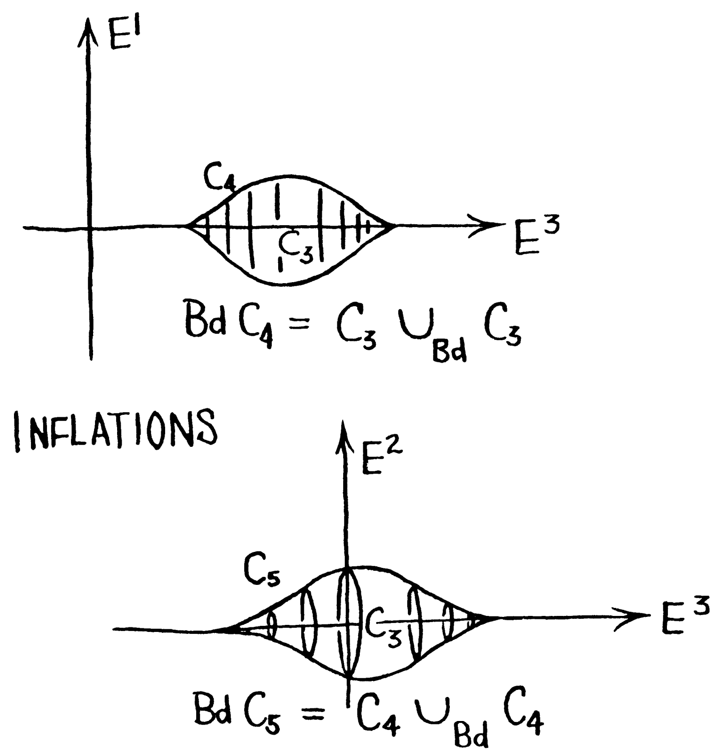


FIGURE 27

THEOREM. *Let* $G \subset G^+ \subset E^3 \times \{0\} \subset E^3 \times E^{n-3} = E^n$, $n \geqslant 3$, *be as described above. Let* N_n *be a regular neighborhood of* G *in* $E^n - (G^+ - G)$. *Then* $C_n = N_n \cup G^+$ *is a compact subset of* E^n *which strong deformation retracts to* G^+ *and is bounded by an* $(n-1)$*-sphere.*

INDICATION OF PROOF. The case $n = 3$ is proved by inspection. One then proceeds by induction. The set C_4 is what R. J. Daverman [31] has called the simple inflation of C_3 (see Figure 27). One easily sees that Bd C_4 is the union of two copies of C_3 sewn together by the identity map of their boundaries. In the case of the fundamental grope G, Bd C_4, is thus the identity sewing of two Alexander crumpled cubes. R. H. Bing proved in [11] that the identity sewing of two Alexander crumpled cubes is S^3. Thus Bd $C_4 = S^3$. A modification of Bing's proof establishes the same result for the arbitrary grope G [26]. Similarly, C_n is the inflation of C_{n-1} for all $n \geqslant 5$ so that Bd C_n is the union of two copies of C_{n-1} sewn together by the identity map of their boundaries. The inductive generalization of Bing's theorem (see [48] or [24]) proves that Bd $C_n = S^{n-1}$ for all n.

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