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Spectral asymmetry and Riemannian geometry. II

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1. INTRODUCTION

In Part I of this paper (6) we proved various index theorems for manifolds with boundary including an extension of the Hirzebruch signature theorem. We now propose to investigate the geometric and topological implications of these theorems in a variety of contexts.

In §2 we consider a generalization of the signature theorem involving unitary representations of the fundamental group. This leads to a new *differential* invariant of manifolds with given fundamental group G *independent of any Riemannian metric*, which generalizes previously known invariants for *finite* G studied in (4). In more detail the situation is as follows. We consider an oriented Riemannian manifold Y of odd dimension and a unitary representation $\alpha:\pi_1(Y)\to U(n)$. On the space of all exterior differential forms of even degree there is a natural self-adjoint operator B defined by

$$B\phi = i^l(-1)^{p+1}(*d-d*)\,\phi \quad (\deg\phi = 2p,\ \dim Y = 2l-1).$$

This can be naturally extended to give a self-adjoint operator B_α acting on even forms with coefficients in the flat vector bundle defined by α. We then consider the spectral function $\eta_\alpha(s)$ of this operator; that is,

$$\eta_\alpha(s) = \sum_{\lambda\neq 0}(\operatorname{sign}\lambda)\,|\lambda|^{-s},$$

where λ runs over the eigenvalues of B_α. We now compare $\eta_\alpha(s)$ with the corresponding function $\eta(s)$ for B, by putting

$$\tilde{\eta}_\alpha(s) = \eta_\alpha(s) - n\eta(s).$$

Our main result (Theorem (2·4)) is that $\tilde{\eta}_\alpha(0)$ is independent of the Riemannian metric and so is a differential invariant of (Y,α): we denote it by $\rho_\alpha(Y)$. It is a real number.

For $\pi_1(Y)$ finite, or more generally for representations α of $\pi_1(Y)$ which factor through a finite group G, our invariant $\rho_\alpha(Y)$ is a rational number which has the following alternative description. By cobordism theory one knows that some multiple NY of Y bounds an oriented manifold X so that the G-covering $N\tilde{Y}\to NY$ (associated to $\pi_1(Y)\to G$) extends to a G-covering $\tilde{X}\to X$. On $H^{2l}(X,\alpha)$, the cohomology of X with local coefficients, there is a natural hermitian form induced by the cup-product and the unitary structure of α. We denote by $\operatorname{sign}_\alpha(X)$ the signature of this hermitian form. Then $\rho_\alpha(Y)$ is given by

$$\rho_\alpha(Y) = \frac{n \operatorname{sign}(X) - \operatorname{sign}_\alpha(X)}{N}.$$

This description shows that $\rho_\alpha(Y)$ is essentially equivalent to the invariants $\sigma_g(\tilde{Y})$ defined in (4), §7. In fact $\sigma_g(\tilde{Y})$ as g varies gives a (class) function on $G - \{1\}$ and $\rho_\alpha(Y)$ is its 'Fourier transform', namely

$$\sigma_g(\tilde{Y}) = \sum_\alpha \rho_\alpha(Y) \bar{\chi}_\alpha(g) \quad \text{for} \quad g \neq 1,$$

where χ_α is the character of α and α runs over the isomorphism classes of irreducible representations of G.

For infinite $\pi_1(Y)$ and general α there does not appear to be at present any non-analytical definition of $\rho_\alpha(Y)$.

In section 3 we replace the operator B above by other operators such as the Dirac operator of a Spin (or Spin^c) manifold, and we get similar invariants independent of the metric but only after we have *reduced modulo Z*. This is because the 0-eigenvalue no longer represents cohomology (via harmonic forms) and so it produces integer jumps in $\eta(0)$. These R/Z invariants are cobordism invariants and for finite G, we can again identify them with the Q/Z-invariants introduced in (4), §7. In fact, for finite G there are three alternative topological definitions of these Q/Z-invariants with differing merits. The equivalence of two of these has been established by G. Wilson (11) and the equivalence with the third is dealt with in §5. For unitary cobordism we establish in §3 the equivalence of the analytic definition with one (and hence all) the topological definitions. This does not cover the case of Spin-cobordism or oriented-cobordism but these will eventually be included in a much more embracing general index theorem for flat bundles which will be the main topic of Part III.

In §4 we review some of the work of Chern–Simons (8) and its relation to our invariants. In a sense our invariants are more refined and contain some additional homotopy information. For instance, if Y is of dimension $4k-1$ and has a parallelism π we give (Theorem (4·14)) an explicit analytical formula for the Adams e-invariant of the element in the stable homotopy group π_{4k-1}^S defined by (Y, π). This formula may be used to investigate the behaviour of the e-invariant under finite coverings. We show (Proposition (4·16)) that its deviation from being multiplicative is precisely measured by the Q/Z-invariants for free finite group actions on Spin-manifolds discussed in §3. We also consider in some detail the question of conformal immersion or embedding of 3-manifolds in R^4. The Chern–Simons invariant provides an R/Z-obstruction to a conformal immersion. Our invariant $\eta(0)$ turns out to provide an R-obstruction to a conformal embedding (Proposition (4·20)).

As already mentioned, section 5 contains a proof of equivalence between two topological definitions of a cobordism invariant in Q/Z. In fact, this section is essentially a topological Appendix reviewing some elementary properties of K-theory with coefficients in Q/Z. Some of this material is in preparation for more extensive work in Part III.

The investigation of $\eta(s)$ for general self-adjoint elliptic operators, not connected with Riemannian geometry, will be undertaken in Part III.

2. The signature with local coefficients

In Part I we established a signature theorem for manifolds with boundary. In this section we shall consider a generalization involving cohomology with local coefficients. The main interest of this is that we end up with differential-topological invariants, *independent of any Riemannian metric.*

Let X be a $2l$-dimensional compact oriented manifold with boundary Y and let $\alpha:\pi_1(X)\to U(n)$ be a unitary representation of the fundamental group. This defines a *flat* vector bundle V_α over X, or equivalently a local coefficient system. Hence we have cohomology groups $H^*(X;V_\alpha)$ and $H^*(X,Y;V_\alpha)$, and these have a natural pairing into C given by the cup-product, the inner product on V_α and evaluation on the top cycle of X mod Y. This induces a non-degenerate form on $\hat{H}^*(X;V_\alpha)$, the image of the relative cohomology in the absolute cohomology (all coefficients in V_α). In the middle dimension, on $\hat{H}^l(X;V_\alpha)$, this form is hermitian for l even and skew-hermitian for l odd. In the odd case we convert the skew-hermitian form (z,ω) to a hermitian form† by putting

$$\langle z,\omega\rangle = (i^l z,\omega).$$

In both cases the signature of the hermitian form will be denoted by $\operatorname{sign}_\alpha(X)$. For the trivial one-dimensional representation $\operatorname{sign}_\alpha(X)$ reduces to $\operatorname{sign}(X)$ for l even and to 0 for l odd. More generally $\operatorname{sign}_\alpha(X) = 0$ for l odd and α an *orthogonal* representation. However, $\operatorname{sign}_\alpha(X)$ is not in general zero for l odd, which is why we have not restricted ourselves here to $4k$-dimensional manifolds.

Assume now that we have a Riemannian metric on Y, then we can define an elliptic operator on the space of all differential forms on Y of even degree with coefficients in V_α by

$$\phi \mapsto i^l(-1)^{p+1}(*d-d*)\phi \quad (\deg\phi = 2p). \tag{2·1}$$

Here $*$ is the usual $*$ operator on forms extended by the identity on V_α. It is easy to check that (2·1) is self-adjoint with respect to the natural inner product induced by the metric on Y and that on V_α. Thus we can define the η-function of the operator (2·1) which we shall denote by $\eta_\alpha(s)$. Note of course that $\eta_\alpha(s)$ exists for any representation $\alpha:\pi_1(Y)\to U(n)$: it is not necessary for α to extend to a representation of $\pi_1(X)$.

If we now assume our metric on Y is extended to a metric on X which is a product near Y, then we have the following generalization of Theorem (4·14) of Part I.

THEOREM (2·2).
$$\operatorname{sign}_\alpha(X) = n\int_X L(p)-\eta_\alpha(0),$$
where $L(p) = 0$ if l is odd and $L(p) = L_k(p_1,\ldots,p_k)$ is the Hirzebruch L_k-polynomial in the Pontrjagin forms of X when $l = 2k$.

† This sign convention (depending on l) is chosen so that our notation shall be consistent with that of (4), §6.

The proof of (2·2) is essentially identical with that of Theorem (4·14) of Part I. We shall not repeat the proof but will content ourselves with a few brief comments.

(1) The starting-point of the proof is to apply the general index formula (3·10) of Part I to the generalized signature operator A_ξ, where $\xi = V_\alpha$.

(2) The invariant theory of (1) is then used to identify the integrand in the index formula. Since V_α is flat its Chern forms vanish in positive dimensions so the integrand coincides with that for trivial α, namely $nL(p)$.

(3) The arguments of Part I, from (4·9) to (4·14), which lead to the elimination of the various integers $h, h\pm$, etc. from the final formula work just as well for cohomology with local coefficients. In addition to Poincaré duality and the de Rham theorems the only additional fact used was the Kodaira–de Rham L^2-decomposition theorem.

(4) For l odd we have to verify that (i) the tangential component of A_ξ is essentially two copies of the operator (2·1), (ii) the hermitian form on $\hat{H}^l(X; V_\alpha)$ (with signature $\operatorname{sign}_\alpha(X)$) is positive (negative) definite on the $+1$ (-1) eigenspaces of the involution τ defined by $\tau\phi = i^{p(p-1)+l} * \phi$ for ϕ a form of degree p with coefficients in V_α.

We shall now apply Theorem (2·2) to a manifold X of the form $Y \times I$ (with I the unit interval). Then $\partial X = Y \times \{1\} - Y \times \{0\}$ so that the term $\eta_\alpha(0)$ in (2·2) is now the difference

$$\eta_\alpha(\rho_1, 0) - \eta_\alpha(\rho_0, 0),$$

where $\eta_\alpha(\rho, s)$ is the η-function for Y with metric ρ (and for the representation α of $\pi_1(Y) = \pi_1(X)$). Since $H^*(X, \partial X; V_\alpha) \to H^*(X; V_\alpha)$ is now the zero homomorphism, $\operatorname{sign}_\alpha(X) = 0$ and (2·2) reduces to

$$\eta_\alpha(\rho_1, 0) - \eta_\alpha(\rho_0, 0) = n \int_{Y \times I} L(p), \tag{2·3}$$

where the right-hand side is computed from any metric on $Y \times I$ which connects ρ_1 to ρ_0 (and is a product near the ends). In particular the right-hand side depends on α only through its dimension n. Hence, if we define a *reduced* η-function by

$$\tilde{\eta}_\alpha(s) = \eta_\alpha(s) - n\eta(s),$$

where $\eta(s) = \eta_1(s)$, (2·3) implies the first part of:

THEOREM (2·4). *$\tilde{\eta}_\alpha(\rho, 0)$ is independent of the metric ρ. It is a diffeomorphism invariant of Y and α which we shall denote by $\rho_\alpha(Y)$. If $Y = \partial X$ with α extending to a unitary representation of $\pi_1(X)$ then*

$$\rho_\alpha(Y) = n \operatorname{sign}(X) - \operatorname{sign}_\alpha(X).$$

The last part of (2·4) comes by applying (2·2) again to α and the trivial representation and subtracting.

If $\pi_1(Y)$ is finite or, more generally, for representations α which factor through some finite quotient G of $\pi_1(Y)$, the invariant $\rho_\alpha(Y)$ coincides essentially with the invariant defined in (4), § 7, as we shall now explain.

Let $\tilde{X}$ be a compact oriented $2l$-dimensional manifold with boundary and let G be a finite group acting on $\tilde{X}$ and preserving orientation. Then G acts on $\hat{H}^l(\tilde{X}; R)$ preserving the bilinear form which is symmetric for l even and skew-symmetric for

l odd. We can treat both cases together by complexifying and considering, as before, the corresponding hermitian form. Using any auxiliary G-invariant inner product on $\hat{H}^l(\tilde{X};\mathbb{C})$ we get a G-invariant decomposition

$$\hat{H}^l = \hat{H}^l_+ \oplus \hat{H}^l_-,$$

where the hermitian form is positive definite on $\hat{H}^l_+$ and negative definite on $\hat{H}^l_-$. We define† a virtual representation $\operatorname{sign}(G,\tilde{X})$ by

$$\operatorname{sign}(G,\tilde{X}) = \hat{H}^l_+ - \hat{H}^l_-.$$

In terms of characters this means we have a function on G defined by

$$\operatorname{sign}(g,\tilde{X}) = \operatorname{trace}(g \mid \hat{H}^l_+) - \operatorname{trace}(g \mid \hat{H}^l_-).$$

For g the identity $\operatorname{sign}(g,\tilde{X})$ reduces to $\operatorname{sign}(\tilde{X})$ or 0 according as l is even or odd.

Suppose now that G acts freely on $\tilde{X}$ so that $X = \tilde{X}/G$ is a manifold and $\tilde{X} \to X$ a finite G-covering. Then for every representation $\alpha: G \to U(n)$ we have the corresponding representation of $\pi_1(X)$ (composing with $\pi_1(X) \to G$) which we still denote by α. We therefore have the integer invariants $\operatorname{sign}_\alpha(X)$ defined earlier. These are related to the function $\operatorname{sign}(g,\tilde{X})$ by the following lemma:

Lemma (2·5).

$$\operatorname{sign}(g,\tilde{X}) = \sum_\alpha \operatorname{sign}_\alpha(X)\,\bar{\chi}_\alpha(g),$$

where α runs over all irreducible representations of G and χ_α is the character of α.

Proof. The de Rham complex of $\tilde{X}$ has a G-module decomposition:

$$\begin{aligned}\Omega^*(\tilde{X}) &= \sum_\alpha \operatorname{Hom}_G([\alpha], \Omega^*(\tilde{X})) \otimes [\alpha]\\ &= \sum_\alpha (\Omega^*(\tilde{X}) \otimes [\alpha^*])^G \otimes [\alpha],\end{aligned}$$

where α^* is the dual of α and $(\ )^G$ denotes the G-invariant part. Taking cohomology we get

$$\begin{aligned}H^*(\tilde{X}) &= \sum_\alpha H^*(X, V_{\alpha^*}) \otimes [\alpha]\\ &= \sum_\alpha H^*(X, V_\alpha) \otimes [\alpha^*].\end{aligned}$$

Similar decompositions hold for $H^*(X,\partial X)$, $\hat{H}^*(X)$ and therefore $\hat{H}^l_+$ and $\hat{H}^l_-$. The lemma follows by taking characters, subtracting and recalling that $\chi_{\alpha^*} = \bar{\chi}_\alpha$.

Now let $\tilde{Y} \to Y$ be a finite G-covering, where Y is an oriented closed odd-dimensional manifold. Then for every representation α of G we have an invariant $\rho_\alpha(Y)$ as given in Theorem (2·4). Assume first that $Y = \partial X$ with $\tilde{Y} \to Y$ extending to a G-covering $\tilde{X} \to X$. Then by (2·4)

$$\rho_\alpha(Y) = \dim\alpha \,.\, \operatorname{sign}(X) - \operatorname{sign}_\alpha(X).$$

Multiply this by $\bar{\chi}_\alpha(g)$ and sum over all α. Since $\Sigma \dim\alpha\,.\,[\alpha^*]$ is the regular representation of G,

$$\Sigma \dim\alpha\,\bar{\chi}_\alpha(g) = 0 \quad \text{for} \quad g \neq 1.$$

† In (4), §6, the cases l odd and l even are treated differently. The present version is more uniform and leads to the same result.

Together with Lemma (2·5) this gives

$$-\operatorname{sign}(g, \tilde{X}) = \sum_{\alpha} \rho_\alpha(Y)\bar{\chi}_\alpha(g) \quad \text{for} \quad g \neq 1. \tag{2·6}$$

In (4), §7, an invariant $\sigma_g(Y)$ for free actions of G was defined by

$$\sigma_g(\tilde{Y}) = -\operatorname{sign}(g, \tilde{X}) \quad (g \neq 1),$$

provided $\tilde{Y} = \partial\tilde{X}$ with G acting freely on $\tilde{X}$. For such manifolds (2·6) shows that $\sigma_g(\tilde{Y})$ is determined by the $\rho_\alpha(Y)$;

$$\sigma_g(\tilde{Y}) = \sum_{\alpha} \rho_\alpha(Y)\bar{\chi}_\alpha(g) \quad \text{for} \quad g \neq 1. \tag{2·7}$$

Conversely, since $\rho_1(Y) = 0$, we can invert (2·7) and express $\rho_\alpha(Y)$ in terms of the $\sigma_g(\tilde{Y})$ by

$$\rho_\alpha(Y) = \frac{1}{|G|}\sum_{g\neq 1} \sigma_g(\tilde{Y})\{\chi_\alpha(g) - \dim\alpha\}. \tag{2·8}$$

In general it is not possible to find $\tilde{X}$ with $\tilde{X} = \partial\tilde{Y}$ as above but it is possible if we first replace $\tilde{Y}$ by some multiple $N\tilde{Y}$. Then $\sigma_g(\tilde{Y})$ is defined by

$$\sigma_g(\tilde{Y}) = \frac{1}{N}\sigma_g(N\tilde{Y}).$$

Since our invariant $\rho_\alpha(Y)$ is clearly additive in Y so that $\rho_\alpha(NY) = N\rho_\alpha(Y)$, equations (2·7) and (2·8) for Y follow from those for NY. Summarizing we have:

THEOREM (2·9). *For representations α of $\pi_1(Y)$ associated to a finite G-covering $\tilde{Y} \to Y$ the invariant $\rho_\alpha(Y)$ of Theorem (2·4) is the 'Fourier transform' of the invariant $\sigma_g(\tilde{Y})$ defined in (4), §7. The equations expressing one invariant in terms of the other are (2·7) and (2·8). In particular $\rho_\alpha(Y)$ is rational.*

Note. $\rho_\alpha(Y) = 0$ for $\alpha = 1$, but $\sigma_g(\tilde{Y})$ was not defined in (4) for $g = 1$. In fact it is convenient to extend the definition of σ_g to $g = 1$ by putting

$$\sigma_1 = -\sum_{g\neq 1} \sigma_g.$$

Then (2·7) holds also for $g = 1$. Moreover if $\tilde{Y} = \partial\tilde{X}$,

$$\sigma_1(\tilde{Y}) = -\operatorname{sign}\tilde{X} + |G|\operatorname{sign} X$$

measures the deviation from multiplicativity of the signature for manifolds with boundary.

We see now that our analytical methods using η-functions have produced in Theorem (2·4) a differential-topological invariant $\rho_\alpha(Y)$ which extends the previously known invariant $\sigma_g(\tilde{Y})$ from *finite* coverings to *infinite* coverings. It is by no means clear whether ρ_α could be defined by non-analytical means.

As a simple illustration let us check Theorem (2·9) explicitly when Y is the circle. So let α be the representation of $\pi_1(Y) \to U(1)$ taking the generator into $\exp(2\pi i a)$, where $0 < a < 1$. If $x \bmod 2\pi$ is a coordinate for Y then $v_\alpha(x) = \exp(iax)$ is a generating section of the bundle V_α and any other section is of the form $f(x)\, v_\alpha(x)$ with f a function on Y. The operator (2·1) becomes in this case

$$fv_\alpha \mapsto -i\frac{d}{dx}(fv_\alpha) = \left(-i\frac{df}{dx}+af\right)v_\alpha.$$

Its eigenvalues are therefore $\pm n + a$ and so its η-function is

$$\begin{aligned}\eta_\alpha(s) &= \sum_{n\geqslant 0}\frac{1}{(n+a)^s} - \sum_{n\geqslant 1}\frac{1}{(n-a)^s}\\ &= \frac{1}{a^s} - \sum_{n\geqslant 1}\frac{2as}{n^{s+1}} + s\Sigma_1,\end{aligned}$$

where Σ_1 is absolutely convergent near $s = 0$. Hence

$$\eta_\alpha(0) = 1-2a.$$

For the trivial representation the eigenvalues are just $\pm n$ so that $\eta_1(0) = 0$. Hence

$$\rho_\alpha(S^1) = 1-2a, \tag{2·10}$$

showing in particular that $\rho_\alpha(S^1)$ is rational if and only if a is rational – that is, if α factors through a finite group.

Now consider only those representations which factor through

$$\pi_1(Y) \cong Z \to Z/mZ = G,$$

then $a = k/m$ with $k = 1, 2, \ldots, m-1$. Now using (2·7) to compute σ_l from (2·10) we get

$$\sigma_l = \sum_{k=1}^{m-1}\left(1-\frac{2k}{m}\right)\exp\left(-\frac{2\pi i l k}{m}\right) \quad (l \in Z/mZ).$$

But for any mth root of unity $u \neq 1$ we have

$$\sum_{k=1}^{m-1} u^k = -1 \quad \text{and} \quad \sum_{k=1}^{m-1} k u^k = \frac{m}{u-1},$$

and so taking $u = \exp 2\pi i l/m$ for $l \neq 0$ we get

$$\sigma_l = -1-\frac{2}{u-1} = \frac{1+u}{1-u} = i\cot\frac{\pi l}{m}. \tag{2·11}$$

Now we recall that the invariants $\sigma_g(\tilde{Y})$ of (4), §7, can be computed from any G-manifold $\tilde{X}$ with boundary $\tilde{Y}$ (the action of G on $\tilde{X}$ being not necessarily free) by the formula

$$\sigma_g(\tilde{Y}) = L(g, \tilde{X}) - \operatorname{sign}(g, \tilde{X}),$$

where $L(g, \tilde{Y})$ is an explicitly defined number depending on the fixed-point set of g in $\tilde{X}$. In particular, when $\tilde{Y}$ is the circle, $\tilde{X}$ the disc and g rotation through an angle θ,

$$L(g, \tilde{X}) = -i\cot\theta/2.$$

Since $H^2(\tilde{X}) = 0$ in this case we have $\operatorname{sign}(g, \tilde{X}) = 0$ and hence

$$\sigma_g(\tilde{Y}) = -i\cot\theta/2,$$

which checks with (2·11) for $\theta = 2\pi l/m$.

Similar explicit calculations can be made when Y is a lens space of any odd dimension. We shall not carry these out here but will content ourselves with the following Proposition which identifies $\eta(0)$ for lens spaces:

PROPOSITION (2·12). *Let the cyclic group G of order m act on $R^{2n} = R^2 \oplus R^2 \oplus \ldots \oplus R^2$ so that its generator rotates the j-th copy of R^2 through an angle θ_j. Assume G acts freely on $S^{2n-1} \subset R^{2n}$ and form the lens space $Y = S^{2n-1}/G$. Then for the standard metric on Y let $\eta(s)$ denote the η-function of the operator* (2·1). *The value at $s = 0$ is given by the generalized Dedekind sum:*

$$\eta(0) = \frac{i^{-n}}{m} \sum_{k=1}^{m-1} \prod_{j=1}^{n} \cot \tfrac{1}{2}k\theta_j.$$

Proof. Let us write $\eta(Y, s)$ and $\eta(\tilde{Y}, s)$ to distinguish between the η-functions on the lens space Y and on the sphere $\tilde{Y}$. Clearly we have

$$\eta(\tilde{Y}, s) = \sum_\alpha \eta_\alpha(Y, s),$$

where α runs over the m irreducible characters of G. Now by definition

$$\rho_\alpha(Y) = \eta_\alpha(Y, 0) - \eta(Y, 0),$$

hence

$$\eta(\tilde{Y}, 0) = \sum_\alpha \rho_\alpha(Y) + m\eta(Y, 0).$$

But the sphere $\tilde{Y}$ admits an orientation reversing isometry which therefore takes our basic operator (2·1) for $\tilde{Y}$ into its negative. This shows $\eta(\tilde{Y}, s) \equiv 0$ and so

$$\begin{aligned} \eta(Y, 0) &= -\frac{1}{m} \sum_\alpha \rho_\alpha(Y) \\ &= \frac{1}{m} \sum_{g \neq 1} \sigma_g(\tilde{Y}) \quad \text{by (2·7) for} \quad g = 1. \end{aligned}$$

On the other hand $\sigma_g(\tilde{Y})$ can be computed by taking $S^{2n-1} = \tilde{Y}$ to bound the unit ball $\tilde{X}$ in R^{2n} and then we get

$$\begin{aligned} \sigma_g(\tilde{Y}) &= L(g, \tilde{X}) - \operatorname{sign}(g, \tilde{X}) \\ &= i^{-n} \prod_{j=1}^{n} \cot \tfrac{1}{2}k\theta_j, \end{aligned}$$

using the definition of $L(g, \tilde{X})$ as in (4), §7 (here we have written $g = \xi^k$, where ξ is the generator of G referred to in the Proposition). This completes the proof.

The general relation between the invariants σ_g and ρ_α suggests that we should generalize our η-invariants in yet another direction. Suppose quite generally that A is an elliptic self-adjoint operator commuting with the action of some compact group G, then the λ-eigenspace E_λ of A will be a finite-dimensional G-module and so we can define a function

$$\eta(g, s) = \sum_{\lambda \neq 0} \operatorname{sign} \lambda \operatorname{Tr}(g \mid E_\lambda) |\lambda|^{-s} \tag{2·13}$$

for any $g \in G$. In particular we can consider such η-functions when A is our basic operator (2·1) and G is a group of orientation preserving isometries. For example, when Y is the circle and g is rotation through an angle α,

$$\eta(g,s) = 2i \sum_{n=1}^{\infty} \frac{\sin n\alpha}{n^s}.$$

Suppose first that G is finite and acts freely on our manifold $\tilde{Y}$ with $Y = \tilde{Y}/G$. Then we have

$$\eta(g,s) = \sum_{\alpha} \eta_\alpha(s)\bar{\chi}_\alpha(g), \tag{2·14}$$

which shows that $\eta(g,s)$ is meromorphic in s and finite for $s = 0$. Comparing with (2·7) we see that

$$\eta(g,0) = \sigma_g(\tilde{Y}) - R(g)\,\eta_1(0), \tag{2·15}$$

where R is the regular representation of g. Hence

$$\eta(g,0) = \sigma_g(\tilde{Y}) \quad \text{for} \quad g \neq 1, \tag{2·16}$$

which shows that $\eta(g,0)$ is independent of the metric provided $g \neq 1$. In fact one can give a direct treatment of the function $\eta(g,s)$ not assuming that G acts freely and (2·16) will still hold provided g has no fixed-points on $\tilde{Y}$. For this one needs first to extend the results of (1) to group actions and this will be done in (2). After that it is straightforward to extend Part I of this paper also to include group actions.

If G acts on X with free action on ∂X, then Theorem (4·14) of Part I gets replaced by

$$\operatorname{sign}(g,X) = \int_{X^g} \omega - \eta(g,0),$$

where X^g is the fixed-point set of the element $g \neq 1$ of G, and the integrand ω is the characteristic class expression occurring in the G-signature theorem of (4), §6.

3. G-Cobordism invariants

In the preceding section we studied the η-function of the operator related to signature in connexion with coverings. We now proceed to do the same for the other classical operators. In general terms the situation is quite similar but with one all-important difference which we now explain.

For any self-adjoint elliptic operator A the η-function

$$\eta_A(s) = \sum_{\lambda \neq 0} \operatorname{sign}\lambda\, |\lambda|^{-s}$$

discards the zero-eigenvalue. In general, when we vary A continuously the zero-eigenvalue moves so that $\eta_A(s)$, and in particular $\eta_A(0)$, is *not* a continuous function of A. However the jumps in $\eta_A(0)$ are simply due to eigenvalues changing sign as they cross zero and this means $\eta_A(0)$ has only *integer jumps*. This will be proved more formally in a later section but for the classical first order systems we are investigating now it is a consequence of the explicit index formula for manifolds with boundary established in Theorem (3·10) of Part I:

$$\operatorname{index} D = \int_X \alpha_0(x)\,dx - \frac{h+\eta(0)}{2}.$$

Here $\eta(0)$ refers to an operator on $Y = \partial X$. Since the integrand is continuous in the coefficients of the operators and index D is an integer we see that, if we put

$$\xi(s) = \frac{h+\eta(s)}{2} \tag{3·1}$$

then $\xi(0)$ has only integer jumps; this is a slightly more refined statement than saying $\eta(0)$ has integer jumps.

In our geometrically defined classical operators the data which is varied consists of the Riemannian metric on Y and possibly also a metric and connexion on an auxiliary vector bundle. Now in the particular case in which the null space of our operator consists of *harmonic forms* this has a dimension h equal to the appropriate Betti number and is *independent of the Riemannian metric*. Thus we get no jump for $\eta(0)$ in this case and the same holds for harmonic forms in a local coefficient system. It is for this reason that in section 2 we were able to define real-valued invariants $\rho_\alpha(Y)$. For other classical operators we have no such control over the integer h and we can only define invariants in R/Z. In return these invariants will be stronger than those of section 2: they will be cobordism invariants not just diffeomorphism invariants.

Suppose now that Y is a closed odd-dimensional Spin-manifold (or more generally a Spin^c-manifold). For any choice of metric on Y we then have the Dirac operator D acting on the space of Spinors: this is self-adjoint. Moreover we can twist D by any unitary representation α of $\pi_1(Y)$ to obtain a new self-adjoint operator D_α, analogously to the case of forms. Put

$$\xi_\alpha(s) = \xi_{D_\alpha}(s),$$

where $\xi(s)$ is related to the usual function $\eta(s)$ by (3·1). Finally put

$$\tilde{\xi}_\alpha(0) = \xi_\alpha(0) - \xi_n(0) \mod Z, \tag{3·2}$$

where $n = \dim \alpha$ and $\xi_n(s)$ refers to the trivial n-dimensional representation so that $\xi_n(0) = n\xi_D(0)$. Thus $\tilde{\xi}_\alpha(0)$ takes values in R/Z.

The analogue of Theorem (2·4) is

THEOREM (3·3). *$\tilde{\xi}_\alpha(0)$ is independent of the metric on Y. It is a cobordism invariant of (Y, α) in the sense that $\tilde{\xi}_\alpha(0) = 0$ if $Y = \partial X$ (as Spin^c-manifolds) with α extending to a unitary representation of $\pi_1(X)$.*

The only differences between this and (2·4) are that we use Theorem (4·2) of Part I (instead of the signature formula) and that we work always *modulo integers*.

Remark. More cobordism invariants can be obtained if we replace the Dirac operator D by the Dirac operator with coefficients in an auxiliary vector bundle V associated to the Spin-structure.

If we consider only those α which factor through a finite quotient G of $\pi_1(Y)$ then $\tilde{\xi}_\alpha(0) \in Q/Z$. This follows from the fact that the corresponding cobordism group is *finite* so that some multiple of Y bounds.

Just as in section 2 we can in this situation replace $\tilde{\xi}_\alpha(0)$ by a function on G – its Fourier transform, defined by

$$f(g) = \sum_\alpha \tilde{\xi}_\alpha(0)\,\overline{\chi}_\alpha(g), \tag{3·4}$$

where α runs over the irreducible representations of G. Clearly f must be viewed as a complex-valued function *modulo characters*. Now G-cobordism invariants of this form were defined in (4), § 7, by use of the fixed-point formula in the G-index theorem. We propose to show that our present invariants (for finite G), given by (3·4), coincide with those of (4), § 7. In fact there are *four* different definitions of these invariants and we shall see that they are all equivalent. The definitions may be summarized as follows.

I. *The analytical definition.* This is our invariant $\tilde{\xi}_\alpha(0)$, constructed from the Dirac operator on Y, or equivalently the function f given by (3·4).

II. *The K-theory definition.* As will be explained later in section 5, the representation α defines an element $[\alpha] \in K^{-1}(Y; Q/Z)$. Since Y is a Spin^c-manifold we have a direct image homomorphism

$$K^{-1}(Y; Q/Z) \to Q/Z.$$

Applying this to $-[\alpha]$ gives an element of Q/Z.

III. *The fixed-point definition.* If $\tilde{Y} \to Y$ is the given G-covering we assume that $\tilde{Y} = \partial\tilde{X}$ (as Spin^c-manifolds) with the action of G extending to $\tilde{X}$ (but not necessarily freely). We then define $f(g)$ to be the contribution of the fixed-point set $\tilde{X}^g$ of g in the Lefschetz formula for the Dirac operator (4), § 5. Because of the formula on closed manifolds, f modulo characters of G is independent of the choice of $\tilde{X}$.

IV. *The differential-geometric definition.* Let V_α be the complex vector bundle over Y defined by α: note that it is *flat*. Assume now that $Y = \partial X$ (as Spin^c-manifolds) *and* that there is a complex vector bundle W on X (not necessarily flat) which extends V_α. Choose any connexion on W which extends the flat connexion on V_α and choose any metric on X. Then our invariant (depending on α) is defined to be

$$\int_X (\operatorname{ch} W - n)\,\mathscr{T}(X), \tag{3·5}$$

where $\mathscr{T} = e^{\frac{1}{2}c_1}\,\hat{\alpha}$ is the total Todd polynomial.

To see that definition IV is independent of all choices we first note that the differential form $(\operatorname{ch} W - n)$ is zero on Y (because $W_Y = V_\alpha$ is flat) and hence represents a relative cohomology class in $H^*(X, Y; R)$. It is easy to show (see Lemma (5·8)) that this class is independent of the connexion on W. Moreover (3·5) now has a cohomological interpretation and the class of $\mathscr{T}(X)$ in $H^*(X; R)$ is independent of the metric on X. Finally, if we make another choice (X', W'), then by glueing W to W' over $M = X \cup -X'$ we can apply the integrality of (3·5) for closed Spin^c-manifolds to deduce that our invariant, taken modulo Z, is independent of the choice of (X, W).

Definitions I and II should be viewed as the fundamental ones since they are intrinsically defined on Y. Definitions III and IV involve an extraneous manifold $\tilde{X}$ (or X) which may not always exist. On the other hand III and IV, when they apply, have the merit of being more readily computable.

Definitions II, III, IV are purely topological, so their identification is a problem in K-theory. In fact the equivalence of II and III has been proved by G. Wilson in (11) while the proof that II $\sim$ IV proceeds on somewhat similar lines and will be given in §5.

The main problem is therefore to bridge the gap between the analysis and the topology by proving that definition I is equivalent to one of the other three. Now the equivalence I $\sim$ IV is an immediate consequence of the general index formula of Part I applied to the Dirac operator on X with coefficients in W (see formula (4·3) of Part I). An alternative direct approach proving the equivalence of I $\sim$ III will be given in (2) using the extension of Part I to group actions.

The best result in this direction would of course be to prove the equivalence of the two basic definitions I and II. This does not follow from I $\sim$ III and III $\sim$ II because III is only defined under a more restrictive hypothesis. In fact if we consider only manifolds Y which are *stably almost complex* (a much stronger condition than being Spin^c) then III is always defined, that is we can always find X with $Y = \partial X$ and a bundle W on X extending V on Y. This follows from complex cobordism theory and is equivalent to saying that $\Omega^U_q(BU(n)) = 0$ for odd q, a consequence of the fact that $H*(BU(n), Z)$ and $\Omega^U_* = \Omega^U$ (point) both vanish in odd dimensions (see (10), p. 144). Thus we can for the present state

THEOREM (3·6). *If Y is stably almost complex then definitions I and II coincide.*

This theorem should be viewed as a *Riemann–Roch Theorem for flat bundles associated to finite groups*. As such it is a special case of a more general index theorem for flat bundles which we shall prove in Part III. In particular our proof of the general theorem will make no use of cobordism theory and it will include the identification of definitions I and II without the restriction imposed in Theorem (3·6). It will also include the case of oriented manifolds which are not Spin^c – on which we still have interesting operators.

As we explained at the beginning of this section the main difference between the signature operator and the general Dirac operator is that in the latter case we do not have control of the zero eigenvalue and hence we only get invariants in R/Z. It is, however, possible to get invariants in R if we impose suitable restrictions on our metric so as to eliminate the zero eigenvalue. For the Dirac operator D of a Spin manifold† this is particularly simple in view of the result of Lichnerowicz (9). We recall from (9) that

$$D^2\psi = \nabla^*\nabla\psi + \tfrac{1}{4}R\psi, \tag{3·7}$$

where ∇ is the total covariant derivative, ∇^* its adjoint and R is the *scalar curvature*. On a closed manifold (3·7) implies

$$\langle D\psi, D\psi\rangle = \langle\nabla\psi, \nabla\psi\rangle + \tfrac{1}{4}\langle R\psi, \psi\rangle \tag{3·8}$$

† Here we work strictly with Spin and not the more general Spin^c.

and hence $R > 0$ implies that there are no (non-zero) harmonic Spinors (solutions of $D\psi = 0$ or $D^2\psi = 0$). Essentially the same reasoning applies to the non-compact 'elongated' manifold $\hat{X}$ introduced in Part I (we recall that for a compact manifold X with boundary Y, we put $\hat{X} = X \cup (Y \times R^-)$). More precisely let ψ be an L^2-harmonic Spinor on $\hat{X}$ then†, as shown in Part I, §3, $D\psi = 0$ and ψ has an expansion of the form

$$\psi(y,u) = \sum_{\lambda \neq 0} e^{|\lambda| u} \psi_\lambda(0)\, \phi_\lambda(y)$$

in the cylinder $u \leqslant 0$. This shows that ψ decays exponentially as $u \to -\infty$. Moreover for the covariant derivative $\nabla\psi$ we have

$$\nabla = \nabla_u + \nabla_y, \quad \|\nabla_y \phi_\lambda(y)\|^2_Y < C(1+\lambda^2),$$

where $\| \; \|_Y$ denotes the L^2-norm on Y. Hence $\nabla\psi$ is also exponentially decaying and so (using Green's formula for X_U and then letting $U \to -\infty$) we deduce as in Part I, §3, that

$$\langle \nabla^*\nabla\psi, \psi\rangle = \langle \nabla\psi, \nabla\psi\rangle.$$

Lichnerowicz's argument now applies and shows that, if $R > 0$, there are no harmonic L^2-Spinors on $\hat{X}$. Since $R(X) > 0$ implies $R(Y) > 0$ we also have no harmonic Spinors on the closed manifold Y. Hence Theorem (4·2) of Part I reduces now to

THEOREM (3·9). *If the scalar curvature of the Spin-manifold X is positive then*

$$\int_X \hat{A}(p) = \tfrac{1}{2}\eta(0),$$

where $\eta(s)$ is the η-function of the Dirac operator on $Y = \partial X$.

A similar result holds for D_α, the Dirac operator twisted by a unitary representation $\alpha : \pi_1(Y) \to U(n)$. The formula becomes

$$n\int_X \hat{A}(p) = \tfrac{1}{2}\eta_\alpha(0). \tag{3·10}$$

Suppose now we put

$$f(Y,\alpha,\rho) = \tfrac{1}{2}(\eta_\alpha(0) - n\eta(0)),$$

where we have explicitly indicated the metric ρ on Y. Then (3·9) and (3·10) assert that $f(Y,\alpha,\rho) = 0$ if $Y = \partial X$ with α extending to $\pi_1(X)$ and ρ extending so as to have positive scalar curvature. Theorem (4·2) of Part I applied to $Y \times I$ shows that the difference $f(Y,\alpha,\rho_1) - f(Y,\alpha,\rho_0)$ is always an integer, and is in fact the index of our basic boundary value problem. As long as ρ has positive scalar curvature, so that there are no zero-eigenvalues, this boundary value problem varies continuously with ρ and hence its index is constant. Hence $f(Y,\alpha,\rho)$ depends only on (Y,α) and on the component of ρ in $\mathscr{R}^+(Y)$ – the space of those metrics on Y which have positive scalar curvature. In this differential-geometric sense we have a refined invariant in R which reduces modulo Z to the cobordism invariant studied earlier.

† We deal with + and − Spinors together so D here is the total Dirac operator ($D \oplus D^*$ in the notation of Part I, §3).

We conclude this section with a comment on the way our invariants $\tilde{\xi}_\alpha(0)$ depend on the representation α. For a finite group G the (classes of) representations are finite in number and, under continuous variation, they remain constant. However, for an infinite group Γ we can certainly have a continuous family α_t of unitary representations in which the isomorphism class is not constant. This is trivially the case for $\Gamma = Z$ and α_t the family of representations $Z \to U(1)$ given by $1 \mapsto \exp(2\pi i t)$. For this family we explicitly computed the Signature invariant $\eta_{\alpha_t}(0)$ in section 2 and found the value $1-2t$ (for $0 < t < 1$), which therefore depends non-trivially on t. This example is typical of the abelian case, but we shall now show that, for representations $\alpha: \Gamma \to SU(n)$, the invariants $\tilde{\xi}_\alpha(0)$ are constant under continuous variation of α.

Let α_t be a 1-parameter family of unitary representations $\pi_1(Y) \to SU(n)$. Then using the differential-geometric definition† IV we see that

$$\tilde{\xi}_{\alpha_1}(0) - \tilde{\xi}_{\alpha_0}(0) = \int_{I\times Y} (\operatorname{ch} W - n)\mathscr{T}(I \times Y) \mod Z, \tag{3·11}$$

where W denotes the bundle over $I \times Y$ whose restriction to $t \times Y$ is just V_{α_t}. More precisely, if $\tilde{Y}$ is the universal covering of Y, then W is the quotient of $I \times \tilde{Y} \times C^n$ by the action of $\Gamma = \pi_1(Y)$ given by

$$(t, \tilde{y}, u) \mapsto (t, \gamma(\tilde{y}), \alpha_t(\gamma)\, u) \quad \text{for} \quad \gamma \in \Gamma.$$

Thus W has a natural flat $SU(n)$-connexion on each $t \times Y$ and this can be extended to a full $SU(n)$-connexion on $I \times Y$ (this extension amounts to giving an action of $\partial/\partial t$ or equivalently of identifying W with a bundle pulled up from Y). With such a choice of connexion it follows that the curvature of W is a multiple of dt and so the only non-zero component of $\operatorname{ch} W - n$ is the first Chern form in dimension 2. Since the structure group of W is assumed to be $SU(n)$ the first Chern form (trace of the curvature) vanishes identically and so (3·11) gives zero as asserted.

4. RELATION WITH CHERN–SIMONS INVARIANTS

In this section we shall explain the relation between our work and the results of Chern–Simons (8). We begin with some rather general constructions involving characteristic classes of connexions.

Let Y be a compact oriented manifold of dimension l and let $\mathscr{C} = \mathscr{C}(Y)$ denote the space of all C^∞ connexions on the tangent bundle of Y. Then $\mathscr{C}$ is an infinite-dimensional (affine) linear space. It will be convenient formally to consider exterior differential forms on $\mathscr{C}$ (and on $\mathscr{C} \times Y$). This presents no serious problems but the reader may, if he prefers, always replace $\mathscr{C}$ in what follows by a finite-dimensional submanifold. Now let us lift the tangent bundle of Y to $\mathscr{C} \times Y$. Here it acquires a natural connexion θ, namely the connexion which is trivial in the $\mathscr{C}$-direction and, on $c \times Y$, coincides with the connexion parametrized by c. Let K denote the curvature of θ, and let f be any invariant polynomial on the Lie algebra of $GL(l, R)$, then we can form the corresponding characteristic differential form $f(K)$ on $\mathscr{C} \times Y$. By partial integration over

† Note that, when IV is defined, the equivalence I ~ IV (as a consequence of Part I) does not require finiteness of Γ.

Y we then obtain a differential form ω_f on $\mathscr{C}$ of degree $= 2\deg f - l$. Since $f(K)$ is closed and since integration over Y commutes with d, it follows that ω_f is also closed.

We shall be primarily concerned with the special case when $l = 4k-1$ and $f(K)$ is a form of degree $4k$ (some polynomial in the Pontrjagin forms). Then ω_f is a closed 1-form and since $\mathscr{C}$ is simply connected (being a linear space) this implies that

$$\omega_f = dF \tag{4·1}$$

for some smooth function F on $\mathscr{C}$, determined up to an additive constant. Because of (4·1), for any path γ on $\mathscr{C}$ joining two connexions c_0, c_1 we have

$$F(c_1) - F(c_0) = \int_\gamma \omega_f. \tag{4·2}$$

The integral in (4·2) can be expressed more explicitly as follows. The path $\gamma: I \to \mathscr{C}$ defines a connexion θ_γ for the tangent bundle of Y pulled up to $I \times Y$. If K_γ denotes the curvature of θ_γ then, essentially by definition of ω_f,

$$\int_\gamma \omega_f = \int_{I\times Y} f(K_\gamma). \tag{4·3}$$

Since the tangent bundle of $I \times Y$ is naturally the direct sum of the pull-back of the tangent bundles of Y and of I, we can take on $I \times Y$ the direct sum $\tilde{\theta}_\gamma$ of the connexion θ_γ with the trivial connexion on I. The curvature is then $\tilde{K}_\gamma = 0 \oplus K_\gamma$ and so if we extend† f to an invariant polynomial of $GL(4k, R)$ we have $f(\tilde{K}_\gamma) = f(K_\gamma)$. Now let ϕ be *any* connexion on $I \times Y$ which coincides with $\tilde{\theta}_\gamma$ near the boundary then

$$\int_{I\times Y} f(\tilde{K}_\gamma) = \int_{I\times Y} f(K(\phi)). \tag{4·4}$$

This is proved in the usual way by considering the family $s\phi + (1-s)\tilde{\theta}_\gamma$ as a connexion over $J \times I \times Y$, where J is the interval $0 \leqslant s \leqslant 1$: $f(K)$ for this connexion is a closed form so its integral over $\partial(J \times I) \times Y$ vanishes, but the form is zero on $J \times \partial I \times Y$ (because the family is constant near $\partial I \times Y$) and the integral over $\partial J \times I$ gives (4·4).

Combining (4·2), (4·3) and (4·4) we see that

$$F(c_1) - F(c_0) = \int_{I\times Y} f(K(\phi)) \tag{4·5}$$

for any connexion ϕ on $I \times Y$ which coincides near the boundary with a connexion of the form $\tilde{\theta}_\gamma$. In particular (4·5) holds if, near the boundary, ϕ is a *product connexion* (so that $\phi = \tilde{\theta}_\gamma$ with γ constant near 0 and 1).

We now restrict ourselves to Riemannian connexions, so let $\mathscr{R}$ denote the space of all Riemannian metrics on Y and let $\mathscr{R} \to \mathscr{C}$ be the map which associates to a metric its Levi–Civita connexion. Then the 1-form ω_f and the function F can be pulled back to $\mathscr{R}$ and will be denoted by the same symbols. If ρ_0, ρ_1 are two metrics on Y, and if

† This extension exists and is unique because f has dimension $4k$ and so involves only the Pontrjagin forms $p_1, \ldots, p_k$ which occur already for $GL(2k, R)$, and $2k \leqslant 4k-1$.

ρ is any metric on $I \times Y$ which coincides near the boundary with the product metric (for ρ_0 on $\{0\} \times Y$ and ρ_1 on $\{1\} \times Y$), we can apply (4·5) and deduce

$$F(\rho_1) - F(\rho_0) = \int_{I \times Y} f(K(\rho)), \tag{4·6}$$

where $K(\rho)$ is the Riemannian curvature of ρ. In the notation of previous sections we have

$$f(K(\rho)) = f(p_1, \ldots, p_k),$$

p_i denoting the Pontrjagin forms of ρ and f being regarded as a polynomial in the basic invariants of $GL(4k, R)$, namely the Pontrjagin forms. In particular we shall take $f = L_k$, the Hirzebruch L-polynomial, and we compare (4·6) with the signature theorem for $I \times Y$ (Theorem (4·14) of Part I):

$$\operatorname{sign}(I \times Y) = \int_{I \times Y} L(p) - \{\eta(\rho_1, 0) - \eta(\rho_0, 0)\}.$$

As observed in section 2 we have $\operatorname{sign}(I \times Y) = 0$ and so

$$\eta(\rho_1, 0) - \eta(\rho_0, 0) = \int_{I \times Y} L(p). \tag{4·7}$$

Comparing with (4·6) we see that

$$\eta(\rho, 0) = F(\rho, 0) + C$$

for some constant C. Recall now that F was only determined up to an additive constant by the equation (4·1). Thus *our spectral invariant* $\eta(\rho, 0)$ *provides a natural choice for* F, the indefinite integral of ω_f.

Remark. Theorem (4·14) of Part I could be easily differentiated with respect to ρ, to yield a formula for $d\eta(\rho, 0)$, were it not for the restriction that our metric has to be a product near the boundary. The somewhat involved discussion concerning families of connexions was designed to get round this difficulty without explicit computation.

For the other classical operators, including the Dirac operator, we have similar results except that we must work modulo integers. As observed in section 3, if we put

$$\xi(\rho, s) = \tfrac{1}{2}\{h(\rho) + \eta(\rho, s)\},$$

where $h(\rho)$ and $\eta(\rho, s)$ are the dimension of the space of harmonic spinors and the η-function of the Dirac operator (for the metric ρ), then $\xi(\rho, 0)$ gives a smooth function

$$\bar{\xi}: \mathscr{R} \to \mathrm{R}/\mathrm{Z}, \tag{4·8}$$

and as before, using Theorem (4·2) of Part I, its differential $d\bar{\xi}$ can be identified on R with the 1-form ω_f, where $f = \hat{A}_k$ is the Hirzebruch $\hat{A}$-polynomial. If we restrict ourselves to the subspace $\mathscr{R}^+ \subset \mathscr{R}$ representing metrics with positive scalar curvature then the theorem of Lichnerowicz (9) implies that $\xi(\rho, 0)$ is a continuous function $\mathscr{R}^+ \to \mathrm{R}$ and so provides a natural indefinite integral for ω_f.

The functions $\bar{\xi}: \mathscr{R} \to \mathrm{R}/\mathrm{Z}$ given by the Dirac operator and its generalizations are in fact *conformal invariants*, that is to say $\bar{\xi}(\phi\rho) = \bar{\xi}(\rho)$, where ϕ is any positive smooth function on Y. To see this we take a metric on $I \times Y$ of the form $\sigma = \psi\tilde{\rho}$ where $\tilde{\rho}$ is the product metric defined by ρ and ψ is a positive function such that

$$\psi(t,y) = 1 \quad \text{for} \quad t \leqslant \tfrac{1}{4},$$
$$\psi(t,y) = \phi(y) \quad \text{for} \quad t \geqslant \tfrac{3}{4}.$$

Now apply Theorem 4·2 of Part I and we get

$$\xi(\phi\rho) - \xi(\rho) = \int_{I\times Y} \hat{A}_k(p(\sigma)) \mod \mathrm{Z}.$$

Now we use the fact that the *Pontrjagin forms are conformally invariant*,† so that

$$\hat{A}_k(p(\sigma)) = \hat{A}_k(p(\tilde{\rho})) = 0$$

since $\tilde{\rho}$ is the product metric. Hence $\xi(\phi\rho) = \xi(\rho)$ as required. Of course since we have an explicit formula for the differential $d\xi$ the conformal invariance can also be verified directly without using the invariance of the Pontrjagin forms.

We are now ready to explain the relation between our invariants and those of Chern–Simons (8). In the first place we should make it clear that we are concerned only with the special case of the Chern–Simons theory involving the tangent bundle and cohomology in the highest dimension. Moreover, for simplicity, we shall consider only the case in which Y is an oriented boundary: $Y = \partial X$. Since $\dim Y$ is odd, $2Y$ is always an oriented boundary, and so we are at most ignoring a factor of 2. For every polynomial $f(p_1, \ldots, p_k)$ of dimension $4k$ and having *integer coefficients* Chern and Simons define a function $F: \mathscr{C} \to R/Z$. In the simplest case when Y is parallelizable F is defined by

$$F(c) = \int_{\pi}^{c} \omega_f, \tag{4·9}$$

where the integration is taken along any path joining c to the *flat connexion* defined by the parallelism π. Note that, by the definition of ω_f, $F(c)$ is actually given by an integral over $I \times Y$:

$$F(c) = \int_{I\times Y} f(K(\phi)),$$

where $K(\phi)$ is the curvature of the connexion ϕ on $I \times Y$ defined by the path, from π to c, of connexions on Y. If π' is another parallelism the difference

$$\int_{\pi'}^{c} \omega_f - \int_{\pi}^{c} \omega_f = \int_{\pi'}^{\pi} \omega_f$$

is an *integer* because it can be interpreted cohomologically as the relative characteristic number of $I \times Y$ defined by $f(p_1, \ldots, p_k)$ using the framings π and π' on the two boundary components. In general when Y is not parallelizable the definition of F is more sophisticated, but if, as we are assuming, $Y = \partial X$ it can be expressed as an integral over X, namely

$$F(c) = \int_X f(p(\tilde{c})), \tag{4·10}$$

† In fact the Weyl curvature tensor (kernel of the contraction giving the Ricci tensor) is conformally invariant and the Pontrjagin forms involve only the Weyl tensor.

where $\tilde{c}$ is any connexion on X which extends the product connexion given by c near the boundary. If X' is another choice for X we can glue X and $-X'$ together to form a closed oriented manifold M. The connexions $\tilde{c}$ and $\tilde{c}'$ on X, X' fit to give a smooth connexion γ on M and

$$\int_X f(p(\tilde{c})) - \int_{X'} f(p(\tilde{c}')) = \int_M f(p(\gamma)),$$

which is an integer, namely the Pontrjagin number of M defined by f. Thus (4·10) gives a unique value in R/Z independent of the choice of X. If Y has a parallelism π then taking $c = \pi$ in (4·10) we again get an integer value for the integral (the relative characteristic number) so that $F(\pi) = 0$ in R/Z. This shows that (4·10) is compatible with the previous definition (4·9).

We see therefore that our invariants are essentially the same as the Chern–Simons invariants except that we allow ourselves certain polynomials with *rational* coefficients, e.g. $f = L_k, \hat{A}_k$. Of course these polynomials give *integer* characteristic numbers on closed manifolds precisely because they are indices of elliptic operators. Thus the basic difference between our invariants and those of (8) can be summarized as follows. The theory of (8) is analogous to (and derived from) the theory of characteristic classes: it gives *odd-dimensional* 'characteristic classes' for *vector bundles with connexion*. On the other hand our invariants are analogous to index invariants of manifolds: they are defined for *odd-dimensional manifolds with connexion*.

If N_k denotes the L.C.M. of the denominators of the coefficients of the polynomial $\hat{A}_k$, so that $f = N_k \hat{A}_k$ has integer coefficients, then $N_k \bar{\xi}$ is a Chern–Simons invariant where $\bar{\xi}$ is the invariant of (4·8) and Y is assumed to be a Spin-manifold. Since these invariants take values in R/Z it is clear that $\bar{\xi}$ is a more refined invariant than $N_k \bar{\xi}$. We shall now exploit this extra refinement to derive an analytical expression for the Adams e-invariant.

We recall first that, by the Pontrjagin–Thom construction, the stable homotopy groups of spheres can be identified with the cobordism groups of framed manifolds. In particular a parallelism π on a manifold Y induces a framing† on Y and hence defines an element

$$[Y, \pi] \in \pi_n^S \quad \text{where} \quad n = \dim Y.$$

The Adams e-invariant for $n = 4k-1$ is a homomorphism

$$e: \pi_{4k-1}^S \to Q/Z,$$

which can be defined as follows (10). According to (10) the Spin-cobordism group in dimension $4k-1$ is zero and so $Y = \partial X$ for some Spin-manifold X, the Spin-structure induced on Y coinciding with the trivial Spin-structure defined by π. We now put

$$\left.\begin{aligned} e[Y,\pi] &= \hat{A}(X) && \text{if } k \text{ is even} \\ &= \tfrac{1}{2}\hat{A}(X) && \text{if } k \text{ is odd,} \end{aligned}\right\} \tag{4·11}$$

† A framing means a trivialization of the stable normal bundle and up to homotopy this is equivalent to a trivialization of the stable tangent bundle. A parallelism is a trivialization of the tangent bundle.

where $\hat{A}(X)$ is the relative characteristic number defined by $\hat{A}$ using the trivialization π on the boundary. Because $\hat{A}(M)$ is integral for a closed Spin-manifold (and divisible by 2 if $\dim M \equiv 4 \bmod 8$) it follows that $e[Y,\pi]$ is well-defined as an element of Q/Z by (4·11), independent of the choice of X.

It is natural to ask whether there is any analytical way of computing $e[Y,\pi]$ using Y alone. We shall solve this problem by using our spectral invariants. Consider first the case when k is even and define $F:\mathscr{C}\to R/Z$ by

$$F(c) = \int_X \hat{A}(p(\tilde{c})) \mod Z, \tag{4·12}$$

where $\tilde{c}$ is any connexion on X extending the product connexion defined by c near Y. The discussion at the beginning of this section shows that

$$dF = \omega_{\hat{A}}, \tag{4·13}$$

while the restriction of F to $\mathscr{R}$ coincides (by our main theorem) with the spectral invariant $\xi(\rho,0)$ associated to the Dirac operator. On the other hand, when $c=\pi$ we have $F(c) = e[Y,\pi]$. Hence

$$\begin{aligned} e[Y,\pi] &= F(c) = F(\rho) + \{F(c)-F(\rho)\} && \mod Z \\ &= \xi(\rho,0) + \int_\rho^\pi \omega_{\hat{A}} && \mod Z. \end{aligned}$$

In particular we can take for ρ the metric ρ_π defined by π, in which the vector-fields of the parallelism are defined to be orthonormal. Then the above formula for $e[Y,\pi]$ involves only analysis on Y. For k odd we must divide by 2 and for this to work we have to know that $\xi(\rho,0)$ has *even integer* jumps. Returning to Theorem (4·2) of Part I this means knowing that index D is even. But in dimensions $8q+4$ the Spin bundles are quaternionic so that the spaces of harmonic spinors (satisfying the various boundary conditions) are all of even complex dimension. Thus index D is even (just as in the case of closed manifolds) which gives the integrality of $\frac{1}{2}\hat{A}$.

Summarizing therefore we have established

THEOREM (4·14). *Let Y be a smooth manifold of dimension $4k-1$ and π a parallelism on Y. Denote by ρ_π the Riemannian metric defined by π and put*

$$\xi(\rho_\pi, s) = \tfrac{1}{2}\{h(\rho_\pi) + \eta(\rho_\pi, s)\},$$

where η is the η-function of the Dirac operator and h is the dimension of the space of harmonic spinors. Then the Adams e-invariant of $[Y,\pi]$ is given by

$$e[Y,\pi] = \epsilon(k)\left\{\xi(\rho_\pi,0) - \int_\pi^{\rho_\pi} \omega_{\hat{A}}\right\} \mod Z,$$

where $\epsilon(k) = 1$ if k is even and $\epsilon(k) = \frac{1}{2}$ if k is odd.

Remarks. (1) This is Theorem 5 of (5) although an error of sign has been corrected here.

(2) The integral in (4·14) is in effect an integral over $I\times Y$ (see the explanation following (4·9)) but this is quickly reduced to an explicit integral over Y. When multiplied by N_k it gives the Chern–Simons invariant for $N_k\hat{A}$ and the metric ρ_π.

Only in very special cases such as homogeneous spaces can eigenvalues be explicitly computed so that Theorem (4·14) is of more theoretical than practical interest. However, it can be applied to the general question raised in (7) concerning the behaviour of the e-invariant under finite coverings. Suppose therefore that $\tilde{Y} \to Y$ is a finite covering and that we pull back the parallelism π on Y to a parallelism $\tilde{\pi}$ on $\tilde{Y}$. Applying (4·14) we see that

$$e[\tilde{Y}, \tilde{\pi}] - n e[Y, \pi] = \epsilon(k)\{\xi(\tilde{\rho}_{\tilde{\pi}}, 0) - n\xi(\rho_\pi, 0)\} \mod Z, \tag{4·15}$$

where n is the degree of the covering (the integral contributions $\int \omega_{\hat{A}}$ cancel out). If $\alpha: \pi_1(Y) \to O(n)$ is the representation† associated to the covering then

$$\xi(\tilde{\rho}_{\tilde{\pi}}, 0) - n\xi(\rho_\pi, 0) = \tilde{\xi}_\alpha(0)$$

and is independent of ρ when viewed as an element of Q/Z. In fact, when k is odd, we can divide by 2 because the Spin bundles are quaternionic and so all dimensions are even. Thus $\epsilon(k)\,\tilde{\xi}_\alpha(0)$ is well defined in Q/Z and (4·15) gives

PROPOSITION (4·16). *For a finite covering $\tilde{Y} \to Y$ of parallelized manifolds the e-invariant satisfies*

$$e[\tilde{Y}, \tilde{\pi}] - n e[Y, \pi] = \epsilon(k)\,\tilde{\xi}_\alpha(0) \mod Z,$$

where n is the degree of the covering, dim $Y = 4k-1$, $\epsilon(k) = 1$ *(for k even) and* $\frac{1}{2}$ *(for k odd). Finally α denotes the representation $\pi_1(Y) \to O(n)$ associated to the covering and $\tilde{\xi}_\alpha(0)$ is the spectral invariant of the Dirac operator defined by* (3·2). *In particular* $\epsilon(k)\,\tilde{\xi}_\alpha(0) \bmod Z$ *depends only on the covering and on the Spin structure given by the parallelism.*

If we drop the factor $\epsilon(k)$ from the definition of the e-invariant we obtain a slightly weaker invariant (for k odd) denoted by e_C (whereas $e = e_R$). Clearly for e_C (4·16) becomes

$$e_C[\tilde{Y}, \tilde{\pi}] - n e_C[Y, \pi] = \tilde{\xi}_\alpha(0) \mod Z. \tag{4·17}$$

We can now replace the invariant $\tilde{\xi}_\alpha(0)$ by any one of the other three equivalent definitions (note that our manifolds are framed hence certainly stably almost complex). In particular the *e-invariant difference in* (4·17) *can be expressed as a K-theory characteristic number or alternatively in terms of the fixed-point invariants of* (4).

To improve these results to cover the e-invariant (and not e_C) we have to work with real K-theory. This will be covered by Part III and to a certain extent by (2).

Finally we should point out that Proposition (4·16) also holds for framed manifolds. The only place where we used a strict parallelism π rather than a stable parallelism was in the explicit choice of metric ρ_π in (4·14). However, any metric would have done equally well and, in any case, as far as (4·16) is concerned the metric finally disappears from the formula.

We turn now to questions concerning conformal immersions and for simplicity we shall deal only with the case of immersing 3-manifolds in R^4. Let Y be an orientable 3-manifold, then Y is parallelizable and by the Hirsch–Smale theory it can be immersed differentiably in R^4. In (8) Chern and Simons showed how their invariants can be used to provide obstructions to conformal immersions. We shall review this from our present point of view.

† For a Galois covering with group G this would be the regular representation.

Let Y be an oriented 3-manifold immersed in R^4 and let $g: Y \to S^3$ be the Gauss map. Then $TY \simeq g^*(TS^3)$, where T denotes the tangent bundle, so that Y inherits a parallelism π from that of S^3 (given by right translation in the group of unit quaternions). Moreover g^* takes the Riemannian connexion on S^3 into the Riemannian connexion on Y (for the metric ρ induced by the immersion). Letting ω_p denote as before the closed 1-form on the space $\mathscr{C}$ of connexions on Y associated to the first Pontrjagin form p_1 we have

LEMMA (4·18). $\int_\pi^\rho \omega_p = 2d$, *where d is the degree of the Gauss map.*

Proof. Recall that, essentially by definition,

$$\int_\pi^\rho \omega_p = \int_{I\times Y} p_1(c),$$

where c is the connexion on TY (pulled up to $I \times Y$) corresponding to any path in $\mathscr{C}$ joining the flat connexion π to the Riemannian connexion of ρ. Since both these connexions are induced by g we can choose c to come from a path of connexions c' on S^3 and then

$$\int_{I\times Y} p_1(c) = d\int_{I\times S^3} p_1(c') = d\int_{\pi_0}^{\rho_0} \omega_p,$$

where π_0, ρ_0 denote the standard parallelism and metric on S^3. To compute this last expression we now view S^3 as the boundary of the ball B^4 which we endow with a metric $\tilde{\rho}_0$ having the properties

(i) near the boundary $\tilde{\rho}_0$ is the product of ρ_0 and the standard metric on I,

(ii) $\tilde{\rho}_0$ is invariant under the reflexion

$$\tau: (x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4).$$

Then
$$\int_{\pi_0}^{\rho_0} \omega_p = \int_{B^4} p_1(\tilde{\rho}_0) - p_1(B^4, \pi_0),$$

where $p_1(B^4, \pi_0)$ denotes the relative Pontrjagin number given by the parallelism π_0 on $S^3 = \partial B^4$: this is the same as the Pontrjagin number of the standard 4-dimensional bundle over S^4 (underlying the quarternionic Hopf bundle) and so is equal† to -2. Since $p_1(\tilde{\rho}_0)$ is invariant under τ, which is orientation reversing, it follows that the integral over B^4 vanishes. Thus

$$\int_{\pi_0}^{\rho_0} \omega_p = 2,$$

which completes the proof of the Lemma.

As we have seen $\int_\pi^\rho \omega_p$ is a conformal invariant of ρ and modulo Z it is independent of π. In fact, as noted by Chern and Simons, this statement can be improved by a factor 2. More precisely the canonical bundle over the suspension of $SO(3)$ has

† The sign depends on various orientation conventions but is not of great importance here.

Pontrjagin number† 2 and so $\int_{\pi}^{\pi'} \omega_p$ is an even integer. Thus $\alpha(\rho) = \frac{1}{2}\int_{\pi}^{\rho} \omega_p$ is a well-defined conformal invariant of ρ with values in R/Z (independent of π). Lemma 4·18 shows that *this invariant vanishes for metrics which admit a conformal immersion in* R^4. This is the result of Chern–Simons.

We now return to our spectral invariant $\eta(\rho, 0)$ for the operator on even forms of Theorem (4·14) of Part I. As we have seen this defines a function

$$F: \mathscr{R} \to \mathrm{R}$$

on the space $\mathscr{R}$ of Riemannian metrics on Y which satisfies $df = \frac{1}{3}\omega_p$. Its precise relation to the Chern–Simons invariant $\alpha(\rho)$ is given by

PROPOSITION (4·19). *$2\alpha(\rho)$ is the* $\bmod Z$ *reduction of $3F(\rho)$. Moreover*

$$S(Y) = \alpha(\rho) - \tfrac{3}{2}F(\rho) \bmod Z$$

is an invariant of the oriented manifold Y (independent of ρ): it takes the values 0 or $\frac{1}{2}$. If X is a Spin manifold with boundary Y then $S(Y) = \frac{1}{2}\operatorname{sign}(X) \bmod Z$.

Proof. Let $Y = \partial X$ with X a Spin-manifold compatible with a given parallelism π of Y. Then

$$\alpha(\rho) = \frac{1}{2}\int_{\pi}^{\rho} \omega_p = \frac{1}{2}\left\{\int_X p_1(\tilde{\rho}) - p_1(X, \pi)\right\} \quad \bmod \mathrm{Z},$$

where as usual $\tilde{\rho}$ is a metric on X extending ρ. The relative characteristic number $p_1(X, \pi)$ is *even* because p_1 is divisible‡ by 2 in the cohomology of B Spin (4). Thus

$$\alpha(\rho) = \frac{1}{2}\int_X p_1(\tilde{\rho}) \quad \bmod \mathrm{Z}.$$

But by our main theorem, since $L_1 = \frac{1}{3}p_1$,

$$F(\rho) = \eta(\rho, 0) = \frac{1}{3}\int_X p_1(\tilde{\rho}) - \operatorname{Sign}(X).$$

Hence $$\alpha(\rho) = \tfrac{3}{2}F(\rho) + \tfrac{3}{2}\operatorname{Sign} X \quad \bmod \mathrm{Z},$$

showing that $$2\alpha(\rho) = 3F(\rho) \quad \bmod \mathrm{Z}.$$

Moreover in the congruence

$$\alpha(\rho) - \frac{3F(\rho)}{2} = \frac{\operatorname{Sign}(X)}{2} \quad \bmod Z$$

the left side is independent of X while the right side is independent of ρ. Thus each defines an invariant $S(Y)$ depending only on Y and the Proposition is proved.

Remarks. (1) It has been pointed out to us by N. Hitchin that $S(Y) = \frac{1}{2}\sigma(Y) \bmod Z$, where $\sigma(Y)$ is the number of 2-primary summands in $H^2(Y; Z)$. This is a special case of a more general result of Brumfiel and Morgan (Topology 12 (1973), 105–122).

† Equivalently over the suspension of S^3 the Pontrjagin number is 4. The weights of the adjoint representation $S^3 \to SO(3)$ are $\pm 2x$ so $p_1 = 4x^2$ and x^2 represents the generator.

‡ In general, $p_1 = \omega_2^2 \bmod 2$.

(2) If (Y, ρ) admits an orientation reversing isometry then

$$F(\rho) = -F(\rho) \quad \text{and} \quad \alpha(\rho) = -\alpha(\rho) \quad \operatorname{mod} Z.$$

Since F is real-valued this implies $F(\rho) = 0$, but α taking values in R/Z can have order 2. In fact the Proposition shows that $\alpha(\rho) = S(Y)$ will now be independent of ρ. For $Y = S^3$ we have $S(Y) = 0$ so $\alpha(\rho) = 0$ for the standard metric ρ. On the other hand for $Y = SO(3)$ we have $S(Y) = \frac{1}{2}$ and so $\alpha(\rho) = \frac{1}{2}$ for the standard metric. As noted by Chern and Simons this shows that $SO(3)$ cannot be immersed conformally in R^4.

We shall now show that our function $F(\rho)$ gives an obstruction to conformal *embeddings* in R^4.

PROPOSITION (4·20). *Assume (Y, ρ) can be conformally embedded in R^4 then* $F(\rho) = 0$.

Proof. Since F is a conformal invariant we may assume ρ is the metric induced by an embedding of Y in R^4. Let X be the interior of Y then (4·18) can be rewritten

$$\int_X p_1(\tilde{\rho}) - p_1(X, \pi) = 2d,$$

where $\tilde{\rho}$ is a suitable extension of ρ (coinciding with a product near Y). Hence

$$F(\rho) = \frac{1}{3}\int_X p_1(\tilde{\rho}) - \operatorname{Sign} X = \frac{1}{3}(2d + p_1(X, \pi)) - \operatorname{Sign} X.$$

To show that $F(\rho) = 0$ we shall prove that

(i) $\operatorname{Sign} X = 0$,

(ii) $p_1(X, \pi) = -2d$.

For (i) we note that the quadratic form on $H^2(X, Y) = H^2_{\text{comp}}(X - Y) \cong H_2(X - Y)$, given by the intersection of cycles, can be computed via the open inclusion

$$j: X - Y \to R^4$$

(i.e. $a.b = j_*(a).j_*(b)$) and hence is identically zero (since $H_2(R^4) = 0$). For (ii) we compare π with the standard parallelism π' of R^4 and deduce that

$$p_1(X, \pi) = p_1(I \times Y; \pi, \pi') = dp_1(I \times S^3; \pi_0, \pi') = -2d$$

as computed in the proof of (4·18). Here $p_1(I \times Y, \pi, \pi')$ denotes the relative Pontrjagin number of $I \times Y$ with the trivializations π, π' on $1 \times Y, 0 \times Y$ respectively.

5. K-THEORY WITH COEFFICIENTS IN Q/Z.

This section is in the nature of a technical appendix in which we give the definition of K-theory with coefficients in Q/Z and derive a number of elementary properties. In particular we shall explain how a bundle associated to a representation α of a finite group G defines an element $[\alpha]$ of $K^{-1}(Y, Q/Z)$, where $\pi_1(Y) = G$, and we shall prove the statement made in section 3 concerning the equivalence of definitions II and IV.

The material of this section will also serve as a basis for a more systematic approach in Part III in connexion with the general index theorem for flat bundles.

We first define mod n K-theory. For this we take the standard map

$$f_n : S^1 \to S^1$$

of degree n given by $z \mapsto z^n$ and we let M_n denote its 'cofibre' – that is, the mapping cylinder C_n of f_n with the initial S^1 collapsed to a point. Homotopically M_n is a circle with a 2-cell attached by a map of degree n so that $H^2(M_n, Z) \cong Z/nZ$ is the only non-trivial cohomology group. We put

$$K(X; Z/nZ) = K(X \times M_n; X \times \text{point}). \tag{5·1}$$

Equivalently, and more briefly, we can write

$$K(X; Z/nZ) = K(1_X \times f_n), \tag{5·2}$$

where for any map g we denote by $K(g)$ the reduced K-group of the cofibre of g.

The various properties of K-theory, applied to $X \times M_n$, immediately yield corresponding properties of mod n K-theory. In particular this extends to a periodic cohomology theory of period 2 and the exact sequence associated to the map $1_X \times f_n$ gives the expected coefficient exact sequence

$$\begin{array}{ccc} K^*(X) & \xrightarrow{n} & K^*(X) \\ \nwarrow \delta & & \swarrow \\ & K^*(X; Z/nZ) & \end{array}$$

For a point P we have

$$K^0(P; Z/nZ) \cong Z/nZ, \quad K^1(P; Z/nZ) = 0.$$

If X is a Spin^c manifold the direct image map in K-theory (3), § 3,

$$K^*(X \times M_n) \to K^*(M_n)$$

induces a direct image map

$$K^*(X; Z/nZ) \to K^*(\text{point}; Z/nZ).$$

For Q/Z coefficients we now put

$$K^*(X; Q/Z) = \varinjlim K^*(X; Z/nZ), \tag{5·3}$$

the map being induced by the diagrams

$$\begin{array}{ccc} S^1 & \xrightarrow{f_n} & S^1 \\ \downarrow 1 & & \downarrow f_m \\ S^1 & \xrightarrow{f_{nm}} & S^1 \end{array}$$

In view of the definition (5·2) of mod n K-theory as a relative group we can consider $K(X; Z/nZ)$ as constructed from triples (E, F, α), where E, F are vector bundles on $X \times S^1$ and α is an isomorphism

$$(1_X \times f_n)^* E \cong (1_X \times f_n)^* F.$$

Replacing X by its suspension it follows that $K^{-1}(X; Z/nZ)$ can be constructed from triples (E, F, α), where now E, F are vector bundles on $X \times S^2$ and α is an isomorphism

$$(1_X \times \Sigma f_n)^* E \cong (1_X \times \Sigma f_n)^* F,$$

$\Sigma f_n : S^2 \to S^2$ denoting the suspension of f_n.

To obtain a somewhat more convenient representation for $K^{-1}(X; Z/nZ)$ we now observe that in $K(S^2)$ we have

$$\xi^n - 1 = n(\xi - 1),$$

where ξ is the class of the Hopf bundle H. Let us fix a definite isomorphism (for each $n \geq 2$)

$$nH \cong H^n \oplus (n-1)\,1, \tag{5·4}$$

where 1 is the trivial line-bundle: the homotopy class of (5·4) is certainly unique because $\pi_2(U(n)) = 0$. Now suppose that E, F are two vector bundles on X and that α is an isomorphism $nE \cong nF$. In view of (5·4) E and F become naturally isomorphic after tensoring with $H^n \oplus (n-1)\,1$. Since H^n may be identified with $(\Sigma f_n)^* H$ we see that $V = E \otimes (H \oplus (n-1)\,1)$ and $W = F \otimes (H \oplus (n-1)\,1)$ are bundles on $X \times S^2$ with a definite isomorphism $(1_X \times \Sigma f_n)^* V \cong (1_X \times \Sigma f_n)^* W$. Thus we have established

PROPOSITION (5·5). *A triple (E, F, α), where F, E are vector bundles on X and α is an isomorphism $nE \cong nF$ defines a natural element $[E, F, \alpha] \in K^{-1}(X; Z/nZ)$.*

Remark. An alternative approach would have been to *define* $K^{-1}(X; Z/nZ)$ using such triples and then to have shown it had the required exact sequence properties. On the whole the definition using the spaces M_n is probably quicker to set up.

Suppose now that E, F are bundles on X and that $\alpha : nE_Y \cong nF_Y$ is an isomorphism over a closed subspace Y of X. On the one hand, just restricting to Y, we have by (5·5) an element $[E_Y, F_Y, \alpha] \in K^{-1}(Y; Z/nZ)$. On the other hand we also have an element $[nE, nF, \alpha] \in K(X, Y)$. The following lemma asserts that these are compatible:

LEMMA (5·6). *The elements $[E_Y, F_Y, \alpha]$ and $-[nE, nF, \alpha]$ have the same image in $K(X, Y; Z/nZ)$, namely*

$$\delta[E_Y, F_Y, \alpha] = -p[nE, nF, \alpha],$$

where $\delta : K^{-1}(Y; Z/nZ) \to K(X, Y; Z/nZ)$ is the coboundary and

$$p : K(X, Y) \to K(X, Y; Z/nZ)$$

is 'reduction modulo n'.

Proof. Consider the diagram

$$\begin{tikzcd} Y \times S^2 \arrow[r, "\theta"] \arrow[d, "\phi"'] \arrow[dr, "\psi"] & Y \times S^2 \arrow[d] \\ X \times S^2 \arrow[r] & X \times S^2 \end{tikzcd}$$

where the vertical maps are induced by the inclusion $Y \to X$ and the horizontal maps by $\Sigma f_n : S^2 \to S^2$. From E, F, α we construct the two bundles

$$V = E \otimes (H \oplus (n-1)\,1), \quad W = F \otimes (H \oplus (n-1)\,1)$$

on $X \times S^2$ together with an isomorphism $\psi^* V \cong \psi^* W$. This gives an element Ψ of $K(\psi)$ which restricts to $[E_Y, F_Y, \alpha]$ in $K(\theta)$ and to $[nE, nF, \alpha]$ in $K(\phi)$. The Lemma now follows formally. To see this we replace our diagram by the homotopy equivalent diagram of inclusions

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

using the mapping cylinders so that $A = B \cap C$. Now consider the diagram of K-groups

$$\begin{tikzcd} & K(B, A) \arrow[d] \arrow[dr] & \\ K(D, A) \arrow[ur] \arrow[r] \arrow[dr] & K(B \cup C, A) \arrow[r] & K^1(D, B \cup C) \\ & K(C, A) \arrow[u] \arrow[ur] & \end{tikzcd}$$

The middle row is exact while the vertical arrows express $K(B \cup C, A)$ as a direct sum (by excision). Hence the top and bottom maps from $K(D, A) \to K^1(D, B \cup C)$ must cancel, that is they must differ by -1. In our case the top map applied to our element Ψ gives $\delta[E_Y, F_Y, \alpha]$ while the bottom map applied to the same element gives

$$p[nE, nF, \alpha].$$

COROLLARY (5·7). *In the situation of* (5·6) *assume further that X is a compact even dimensional Spin^c manifold with boundary Y. Applying the direct image homomorphisms*

$$\lambda: K^{-1}(Y; Z/nZ) \to Z/nZ,$$
$$\mu: K(X, Y) \to Z,$$

we have
$$\lambda[E, F, \alpha] = -\mu[nE, nF, \alpha] \quad \mathrm{mod}\, n.$$

Passing to cohomology this gives

$$\lambda[E, F, \alpha] = -\{\mathrm{ch}\,[nE, nF, \alpha] . \mathscr{T}(X)\}[X],$$

where $\mathscr{T}$ is the Todd class of X.

Proof. λ factors as $\upsilon\delta$ where $\upsilon: K(X, Y; Z/nZ) \to Z/nZ$ is also a direct image homomorphism. Since μ reduced mod n agrees with υ the result follows at once from (5·6).

We come now to bundles associated to representations of a finite group G. Let B_G be the classifying space of G, B_G^N its N-skeleton (a finite complex), then we shall define

$$K^*(B_G; Q/Z) = \varprojlim_N K^*(B_G^N; Q/Z).$$

Since the rational cohomology of B_G is trivial it follows that

$$K^{-1}(B_G^{2N}) \otimes Q = 0, \quad \tilde{K}^0(B_G^{2N+1}) \otimes Q = 0.$$

Applying the coefficient sequence for $Z \to Q \to Q/Z$ to B_G^N and letting $N \to \infty$ we deduce that

$$K^{-1}(B_G; Q/Z) \overset{\delta}{\cong} \tilde{K}^0(B_G).$$

Now we have the standard homomorphism

$$R(G) \to K^0(B_G),$$

which assigns to each representation α of G the corresponding bundle on B_G. Restricting to the augmentation ideal maps $I(G) \to \tilde{K}^0(B_G)$ and hence composing with δ^{-1} we get a map $\gamma: I(G) \to K^{-1}(B_G; Q/Z)$.

Thus to any representation $\alpha: G \to U(k)$ we get an element

$$\gamma(\alpha - k) \in K^{-1}(B_G; Q/Z).$$

If now we have a representation

$$\alpha: \pi_1(Y) \to U(k),$$

which factors through G we can pull pack this element by the map

$$K^{-1}(B_G; Q/Z) \to K^{-1}(Y; Q/Z)$$

to get finally an element $[\alpha] \in K^{-1}(Y; Q/Z)$. This is the construction referred to in Definition II of section 3.

In more concrete terms this means that if V is the bundle over Y associated to α then we can find a trivialization ϕ of nV_α (pulling it back from B_G) and any two such become homotopic after passing to some larger multiple mnV_α. The triple (V_α, I_k, ϕ), where I_k is the trivial bundle of dimension k, thus defines an element in $K^{-1}(Y; Z/nZ)$ whose image in $K^{-1}(Y; Q/Z)$ is independent of the choice of n and ϕ.

The equivalence of definitions II and IV of section 3 is then an immediate consequence of Corollary (5·7) and the following Lemma:

LEMMA (5·8). *Let X be a manifold with boundary Y and assume that the bundle V_α associated to the representation $\alpha: \pi_1(Y) \to G \to U(k)$ extends to a vector bundle W on X. Take any connexion on W which extends the flat connexion on V_α, and let* ch W *denote the differential form on X representing the total Chern character and constructed from the connexion. Then* (ch $W - k$) *vanishes on Y and n* (ch $W - k$) *represents in $H^*(X, Y)$ the Chern character of $[nW, I_{nk}, \phi] \in K(X, Y)$.*

Proof. The Chern character of the relative element $[nW, I_{nk}, \phi]$ can be constructed from a connexion on nW which extends the trivial connexion on Y whereas

$$n(\mathrm{ch}\, W - k)$$

uses the flat connexion of nV_α on Y. The difference between these connexions on Y leads to a differential form on $Y \times I$ (which we can view as a collar in X) vanishing at both ends, and it is enough to show that this represents the zero class in

$$H^{\mathrm{ev}}(Y \times I, Y \times \partial I; R) \cong H^{\mathrm{odd}}(Y; R).$$

But since our trivialization ϕ on nV_α is induced from B_G, so is this cohomology class and hence it must be zero because B_G has trivial cohomology over R.

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