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Manifold Perspectives

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May 24th – May 30th, 2009

Abstract. The study of the global properties of manifolds and their symmetries has had a great impact on both geometric and algebraic topology, and also on other branches of mathematics, such as algebra, differential geometry and analysis. The purpose of the meeting was to bring together active researchers from diverse areas to discuss these exciting current perspectives on the topology of manifolds.

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Introduction by the Organisers

The study of the global properties of manifolds and their symmetries has had a great impact on both geometric and algebraic topology, and also on other branches of mathematics, such as algebra, differential geometry and analysis. The purpose of the meeting was to bring together active researchers from diverse areas to discuss these exciting current perspectives on the topology of manifolds.

In keeping with the long-standing tradition at Oberwolfach, the program consisted of only 23 talks and allowed a lot of time for lively interactions and new connections to be made at the meeting. Among the 49 participants were mathematicians from Argentina, Belgium, Canada, China, Denmark, France, Germany, Holland, Spain, Switzerland, Turkey, UK and the US, including graduate students, recent Ph.Ds, and 8 women mathematicians. We appreciated (and used fully) the support for the participation of young scientists from the Oberwolfach Leibniz Graduate Students (OWLG) program and from the National Science Foundation (NSF).

In recent years there has been great progress in the exploration of the rigidity properties of manifolds with infinite fundamental groups focussed on the Farrell-Jones conjectures. An essential part of this progress has been the development of controlled topology. In addition, the deep ideas of Gromov in geometric group theory have profoundly influenced our view of the connection between the fundamental group and the topology of manifolds via algebraic K-theory and surgery theory.

The lectures of Goulnara Arzhantseva, Arthur Bartels, Wolfgang Lück, and Guoliang Yu gave us an overview of the state of the art on the Borel, Novikov and Farrell-Jones conjectures. In particular, Lück talked on his recent joint work with Bartels which proves the Borel Conjecture for hyperbolic groups and CAT(0) groups. Bartels showed in his talk how an easily stated topological condition on the Gromov boundary of a hyperbolic group leads to the existence of new families of closed aspherical manifolds (joint work with Lück and Weinberger).

These talks perhaps represented the most significant new results at the meeting, but we also heard a variety of excellent lectures on other important themes: moduli spaces of surfaces (Ramras) and their generalizations to high-dimensional manifolds and embeddings (Lambrechts, Randal-Williams), perspectives from index theory and analysis (Bunke, Ebert, Cortiñas), orbifolds (Holm, Henriques), symmetries of manifolds (Yalçın, Hanke), group theory (Leary, Nucinkis, Chatterji, Mineyev), 3-manifolds (Pitsch), as well as surgery theory (Crowley, Hughes). The opening talk by Kreck on codes and 3-manifolds, and the closing talk by Bartholdi on Julia sets and dynamics, nicely framed the program by discussing two very different topics that look outwards from manifolds to significant new applications for the future.

Timetable

Monday, May 25, 2009

9:30-10:15	Matthias Kreck, *Codes, manifolds and arithmetic*
10:30-11:15	Daniel Ramras, *Gauge theory and homotopical representation theory*
11:30-12:15	Pascal Lambrechts, *Rational homology of spaces of smooth embeddings*
16:00-16:45	Guoliang Yu, *Decomposition complexity and the bounded isomorphism conjectures*
17:00-17:45	Ergün Yalçın, *Constructing group actions via orbit categories*

Tuesday, May 26, 2009

9:15-10:00	Johannes Ebert, *Madsen-Tillmann spectra and index theory*
10:15-11:00	Igor Mineyev, *Topology and analysis of the Hanna Neumann conjecture*
11:30-12:15	Brita Nucinkis, *Finite classifying spaces for the family of virtually cyclic subgroups*
16:00-16:45	Wolfgang Pitsch, *Invariants of homology 3-spheres via trivial 2-cocycles*
17:00-17:45	Bruce Hughes, *Dihedral manifold approximate fibrations*

Wednesday, May 27, 2009

9:15-10:00	Wolfgang Lück, *Topological rigidity*
10:15-11:00	Indira Chatterji, *Bounded* $\mathbf{Z}$*-valued cocycles on connected Lie groups*
11:00-11:10	Conference Photograph
11:30-12:15	Oscar Randal-Williams, *Monoids of moduli spaces of manifolds*

Thursday, May 28, 2009

9:15-10:00	Guillermo Cortiñas, *A homotopy invariance theorem for commutative* C^**-algebras*
10:15-11:00	Diarmuid Crowley, *The smooth structure set of* $S^p \times S^q$
11:30-12:15	Arthur Bartels, *Aspherical manifolds and boundaries of hyperbolic groups*
16:00-16:45	Ian Leary, *An infinite Smith group*
17:00-17:45	André Henriques, *An Elmendorf theorem for orbifolds*

Friday, May 29, 2009

9:15-10:00	Goulnara Arzhantseva, *Uniform embeddings of groups into a Hilbert space*
10:15-11:00	Bernhard Hanke, *The stable free rank of symmetry of products of spheres*
11:30-12:15	Tara Holm, *Topological invariants of orbifolds*
16:00-16:45	Ulrich Bunke, *Adams operations in* K*-theory and secondary index theorems*
17:00-17:45	Laurent Bartholdi, *Insanely twisted rabbits*

Workshop: Manifold Perspectives

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Abstracts

Codes, manifolds and arithmetic

Matthias Kreck
(joint work with Volker Puppe)

After a brief introduction to error-correcting binary codes I defined for a code $C \subseteq (\mathbb{Z}/2)^r$ the lattice $L(C) \subseteq \mathbb{R}^r$ and noticed that if C is a self-dual code, then $L(C)$ is a unimodular lattice or equivalently a positive definite unimodular form over $\mathbb{Z}$. These are arithmetic object which are by no means understood. Thus the occurrence of self-dual codes is potentially leading to new lattices, in particular since the map from isomorphism classes of codes C to positive definite unimodular forms $C \mapsto L(C)$ is injective.

V. Puppe has associated to a $(2k+1)$-dimensional closed manifold M with involution τ with fixpoints $x_1, \dots, x_r$ a code $C(M,\tau) \subset (\mathbb{Z}/2)^r$ as the image of $H^k_{\mathbb{Z}/2}(M\mathbb{Z}/2) \to H^k(M^G;\mathbb{Z}/2) = (\mathbb{Z}/2)^r$ [2]. In joint work we answered the question which self-dual codes come from manifolds.

Theorem 1 (K-Puppe)**.** *All self-dual codes are of the form $C(M^3,\tau)$, i.e. they come from a 3-manifold.*

We define Spin-involutions and prove:

Theorem 2 (K-Puppe)**.** *All doubly even codes come from a* Spin-*involution.*

Finally I defined a product associating to self-dual codes $C_1, \dots, C_{2m+1}$ a self-dual code $C(C_1, \dots, C_{2m+1})$ which geometrically corresponds to the product of manifolds. This allows an extremely simple construction of codes C when $L(C)$ is $E_8, E_{16}, \dots$?

References

[1] Matthias Kreck and Volker Puppe. Involutions on 3-manifolds and self-dual, binary codes. *Homology, Homotopy Appl.*, 10(2):139–148, 2008.

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Gauge theory and homotopical representation theory

Daniel Ramras

Deformation K-theory serves as a homotopy theoretical analogue of the representation ring of a group. Computations suggest that for many infinite discrete groups G with compact classifying spaces, deformation K-theory agrees with topological K-theory of BG (but only above the rational cohomological dimension of G minus 2). The relationship with K-theory is reminiscent of the Atiyah-Segal theorem, while the failure in low dimensions is precisely analogous to the low dimensional failure in the Quillen-Lichtenbaum conjecture. When $BG = M$ is a manifold, this relationship can be interpreted in terms of gauge theory on principal bundles over

M. This perspective, together with work of Tyler Lawson, has been used to calculate deformation K-theory for products of surfaces, and to compute the homotopy type of the corresponding stable moduli spaces of representations. I'll discuss these results as well as joint work with Tom Baird which (conjecturally) explains the low-dimensional discrepancy between representation theory and K-theory.

Rational homology of spaces of smooth embeddings

Pascal Lambrechts

(joint work with Greg Arone, Victor Turchin, Ismar Volić)

Fix a smooth compact manifold M (maybe with boundary). We consider the spaces

$$\mathrm{Imm}(M, \mathbb{R}^d) \qquad \text{and} \qquad \mathrm{Emb}(M, \mathbb{R}^d)$$

of all immersions and embeddings of that manifold into some fixed euclidean space $\mathbb{R}^d$. These spaces are equipped with a suitable topology so that for example two embeddings $f_0, f_1 \colon M \hookrightarrow \mathbb{R}^d$ are isotopic if and only if there are in the same path-connected component of $\mathrm{Emb}(M, \mathbb{R}^d)$.

Thanks to a theory developed in the early 1960's by Smale and Hirsch, the homotopy type of the space of immersions $\mathrm{Imm}(M, \mathbb{R}^d)$ is very well understood. For example when M is parallelizable, then this space has the same homotopy type as the space

$$\mathrm{map}(M, \mathrm{Stiefel}_{\dim(M)}(\mathbb{R}^d))$$

of all maps from M into a fixed Stiefel manifold. Notice that the functor $M \mapsto \mathrm{Imm}(M, \mathbb{R}^d)$ is not a homotopy functor.

The techniques of Smale-Hirsch have been generalized by Goodwillie in the late 1980's to apply to the study of the space of embeddings, and some variations thereof, like the functor

$$\overline{\mathrm{Emb}}(M, \mathbb{R}^d) = \mathrm{hofibre}(\mathrm{Emb}(M, \mathbb{R}^d) \hookrightarrow \mathrm{Imm}(M, \mathbb{R}^d)).$$

It can be proved that when $d > 2\dim M$ and M is simply-connected then the homotopy type of $\overline{\mathrm{Emb}}(M, \mathbb{R}^d)$ depends only on the homotopy type of M. Our goal is to study the rational homotopy type of $\overline{\mathrm{Emb}}(M, \mathbb{R}^d)$ from the rational homotopy type of M.

Here is an example of the results we obtain

Theorem 1 ([3]). *Suppose that M and M' are two smooth compact manifolds of the same rational homotopy type. Then, for $d \geq 4\max(\dim M, \dim M')$ the spaces $\overline{\mathrm{Emb}}(M, \mathbb{R}^d)$ and $\overline{\mathrm{Emb}}(M', \mathbb{R}^d)$ have isomorphic rational homologies.*
Moreover these homologies are theoretically computable.

The key ingredients of the proof of this result are

- a description by Greg Arone [1] of the layers in the orthogonal Weiss tower of the functor

$$V \mapsto \Sigma^\infty \overline{\mathrm{Emb}}(M, V)$$

where V is a variable finite dimensional euclidean vector space; in particular these layers are rational homotopy invariants of M.
- Kontsevich formality of the little disk operad whose detailed proof is given in [5]. This implies that the spectral sequence associated to the orthogonal tower splits over the rationals. Notice that this spectral sequence is very much alike a Vassiliev spectral sequence for computing $\mathrm{H}_*(\overline{\mathrm{Emb}}(M, \mathbb{R}^d))$.

In the special case of (some variation of) the space of high-codimensional knots $\mathrm{Emb}(\hat{S}^1, \mathbb{R}^d)$ (where the hat over S^1 means that we consider *long* knots), our results shows that the Vassiliev spectral sequence computing these spaces of knots collapses and hence that the rational homology of that embedding space admits an explicit combinatorial description in terms of generalized chord diagrams [4]. Actually the proof of this special case is much easier and is a more direct consequence of manifold calculus and formality of the little disk operad.

We also have proved in [2] that a spectral sequence computing the rational *homotopy* groups of $\mathrm{Emb}(\hat{S}^1, \mathbb{R}^d)$ also collapses and hence is explicitly computable.

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Decomposition complexity and the bounded isomorphism conjectures

Guoliang Yu

(joint work with Erik Guentner, Romain Tessera)

Inspired by the property of finite asymptotic dimension of Gromov, we introduce the geometric concept of finite decomposition complexity to study the bounded Borel conjecture, the bounded Farrell-Jones L-theory isomorphism conjecture and stable Borel conjecture on topological rigidity of manifolds. Roughly speaking, a metric space has finite decomposition complexity when there is an algorithm to decompose the space into nice pieces in a certain asymptotic way. We prove the bounded Borel conjecture and the bounded Farrell-Jones L-theory isomorphism conjecture for spaces with finite decomposition complexity and the stable Borel conjecture for closed aspherical manifolds whose fundamental groups have finite decomposition complexity. We show that the class of groups with finite decomposition complexity includes all linear groups, subgroups of almost connected Lie groups, hyperbolic groups and elementary amenable groups, and is closed under various operations.

Theorem. *The bounded Borel conjecture and the bounded Farrell-Jones L-theory isomorphism conjecture hold for any metric space with finite decomposition complexity.*

Recall that a Riemannian manifold M is said to be uniformly contractible if for every $r > 0$, there exists $R \geq r$ such that every ball $B(x, r)$ in M with radius r can be contracted to a point in $B(x, R)$. If M is a uniformly contractible Riemannian manifold with bounded geometry and dimension greater than or equal to 5, then the bounded Borel conjecture for M implies that M is boundedly rigid in the sense that if another uniformly contractible Riemannian manifold M' with bounded geometry is boundedly homotopy equivalent to M, then M' is homeomorphic to M. In particular, if N is a closed aspherical manifold, then its universal cover is uniformly contractible. Our theorem implies that if a closed aspherical manifold N has dimension at least 5 and its fundamental group has finite decomposition complexity, then whenever another closed manifold N' is homotopy equivalent to N, their universal covers $\tilde{N}$ and $\tilde{N}'$ are homeomorphic.

The following result is a consequence of the above theorem.

Theorem. *The stable Borel conjecture holds for a closed aspherical manifold M whose fundamental group has finite decomposition complexity. Precisely, if another closed manifold M' is homotopy equivalent to M, then $M \times \mathbb{R}^n$ is homeomorphic to $M' \times \mathbb{R}^n$ for some n.*

We remark that any countable group can be given a proper length metric (if G is a finitely generated group, then the word length metric is such an example). The property of finite decomposition complexity for a countable group (viewed as a metric space with a proper length metric) is independent of the choices of the proper length metric.

The next theorem shows that the class of groups with finite decomposition complexity is very rich.

Theorem. *The collection of countable groups having finite decomposition complexity contains all countable linear groups* (*over a field of arbitrary characteristic*)*, all countable subgroups of an almost connected Lie group, all hyperbolic groups and all elementary amenable groups.*

Recall that a Lie group is said to be almost connected if it has finitely many connected components.

We point out that linear groups can have infinite asymptotic dimension (for example certain finitely generated subgroups of $GL(n, \mathbb{R})$).

The following theorem shows that the class of groups with finite decomposition complexity is very stable.

Theorem. *The collection of countable groups having finite decomposition complexity is closed under the formation of subgroups, products, extensions, free amalgamated products, HNN extensions and direct limits.*

Finally we mention that in joint work with Dan Ramras and Romain Tessera, we prove the bounded algebraic K-theory isomorphism conjecture for spaces with finite decomposition complexity.

References

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Constructing group actions via orbit categories

Ergün Yalçin

(joint work with Ian Hambleton, Semra Pamuk)

By classical Smith theory and by a theorem of Swan [6], it is known that a finite group G acts freely on a finite complex X homotopy equivalent to a sphere if and only if G does not include $\mathbf{Z}/p \times \mathbf{Z}/p$ as a subgroup for any prime p. One often expresses this condition simply by saying that G has rank equal to 1, where the rank of a finite group G is defined as the largest integer s such that $(\mathbf{Z}/p)^s \leq G$ for some prime p. One of the open questions in finite transformation group theory is whether every rank 2 finite group acts freely on a finite complex homotopy equivalent to a product of two spheres.

An important result in this direction is proved recently by Adem and Smith [1] which states that if a finite group G admits a finite G-CW-complex homotopy equivalent to a sphere with rank 1 isotropy groups, then G acts freely on a finite complex homotopy equivalent to a product of two spheres. Many rank 2 finite groups, including all p-groups, have linear representations V such that the induced action on the unit sphere $S(V)$ has rank 1 isotropy. But, some finite groups do not have linear actions with rank 1 isotropy. The smallest such group is the symmetric group $G = S_5$. We consider this group and give a construction of a finite G-CW-complex with the desired properties. In particular, we prove the following:

Theorem 1 ([4])**.** *The symmetric group $G = S_5$ admits a finite G-CW-complex X homotopy equivalent to a sphere such that if $X^H \neq \emptyset$ for some $H \leq G$, then H is a cyclic 2-group.*

In the construction we use orbit category methods developed by tom Dieck [2] and Lück [5]. Given a finite group G and a family $\mathcal{F}$ of subgroups of G (we always assume that $\mathcal{F}$ is closed under conjugation and taking subgroups), the orbit category $\Gamma = \mathrm{Or}_{\mathcal{F}} G$ is defined as the category whose objects are coset spaces G/H with $H \in \mathcal{F}$ and whose morphisms from G/H to G/K are given by G-maps $G/H \to G/K$.

Let R be a commutative ring. A right $R\Gamma$-module M is a contravariant functor $M : \Gamma \to R$-Mod. For $M(G/H)$, we often write $M(H)$ and view an $R\Gamma$-module M as a direct sum of R-modules $M(H)$ together with conjugation and restriction maps between them. For a G-set X, the assignment $H \to R[X^H]$ defines an $R\Gamma$-module, denoted by $R[X^?]$. If all the isotropy subgroups of the G-action on X are in $\mathcal{F}$, then the module $R[X^?]$ is a projective $R\Gamma$-module. We call an $R\Gamma$-module M a free module if it is a direct sum of $R\Gamma$-modules of the form $R[G/K^?]$ with $K \in \mathcal{F}$.

Given a finite G-CW-complex X, associated to it there is a finite chain complex of $R\Gamma$-modules

$$\mathbf{C}(X^?; R): \ 0 \to R[X_n{}^?] \to \cdots \to R[X_1{}^?] \to R[X_0{}^?] \to 0$$

where X_i is the set of i-dimensional cells in X. If all the isotropy subgroups of a G-CW-complex X lie in $\mathcal{F}$, then $\mathbf{C}(X^?; R)$ is a chain complex of free $R\Gamma$-modules. Conversely, given a finite free chain complex of $R\Gamma$-modules, we can realize it as a chain complex of $R\Gamma$-modules coming from a G-CW-complex provided that the chain complex has the homology of a sphere and satisfies certain properties. In the orbit category setting, the concept of a homology sphere is defined in the following way:

Let $\underline{n} : \mathcal{F} \to \mathbf{Z}$ be a function constant on conjugacy classes. Throughout, we will assume that $\underline{n}$ is positive and monotone, i.e., it satisfies the condition that $0 \le \underline{n}(H) \le \underline{n}(K)$ for all $H, K \in \mathcal{F}$ with $K^g \le H$ for some $g \in G$. A chain complex

$$\mathbf{C}: \ 0 \to C_n \to \cdots \to C_1 \to C_0 \to 0$$

of $R\Gamma$-modules is called an R-homology $\underline{n}$-sphere if for every $H \in \mathcal{F}$, the complex $\mathbf{C}(H)$ has the R-homology of a sphere of dimension $\underline{n}(H)$. Notice that if $X = S(V)$ for some complex representation V of G, then $\mathbf{C}(X^?; R)$ is an R-homology $\underline{n}$-sphere with $\underline{n}(H) = 2\dim(V^H) - 1$ for $H \in \mathcal{F}$. We have the following realization theorem:

Theorem 2. *Let $\mathbf{C}$ be a finite free chain complex of $\mathbf{Z}\Gamma$-modules which is a $\mathbf{Z}$-homology $\underline{n}$-sphere. Suppose that $\underline{n}(H) \ge 3$ for all $H \in \mathcal{F}$, and that $C_i = 0$ for $i > \underline{n}(H)+1$ for all $H \in \mathcal{F}$. Then, there is a G-CW-complex X such that $\mathbf{C}(X^?; \mathbf{Z})$ is chain homotopy equivalent to $\mathbf{C}$.*

This theorem reduces the proof of Theorem 1 to an algebraic problem of constructing a finite free chain complex of $\mathbf{Z}\Gamma$-modules satisfying the above properties, where $G = S_5$ and the family $\mathcal{F}$ is the family of all cyclic 2-subgroups in G. We show that once one constructs a finite projective chain complex, then a finite free chain complex satisfying the above properties can also be obtained using a version of Wall's finiteness obstruction theory and then using some elimination methods for finite free chain complexes.

We construct a finite projective chain complex of $\mathbf{Z}\Gamma$-modules which is a $\mathbf{Z}$-homology $\underline{n}$-sphere for some dimension function $\underline{n}$. The construction is done one prime at a time for $p = 2, 3, 5$ and then the resulting complexes are glued together using an algebraic version of Postnikov section theory due to Dold [3]. The p-local chain complexes are constructed in such a way that they have the same dimension function.

The construction for the prime $p = 2$ is the most difficult one because the Sylow 2-subgroups of G have rank equal to 2. The starting point for this construction is the observation that $H = S_4$ controls 2-fusion in G and that H has an action on a 2-sphere with rank 1 isotropy (coming from the symmetries of a cube). This gives a finite projective chain complex of $R\Gamma_H$-modules where $R = \mathbf{Z}_{(2)}$ and Γ_H is the orbit category of H relative to the family of all cyclic 2-subgroups in H. To lift this chain complex to a finite projective chain complex of $R\Gamma_G$-modules, we again use the algebraic Postnikov section theory. The key ingredient for this lifting is the following theorem that we prove which can be considered as an orbit category version of Mislin's theorem in group cohomology.

Theorem 3. *Let G be a finite group, $R = \mathbf{Z}_{(p)}$, and let $\mathcal{F}$ be a family of p-subgroups in G. Suppose $H \leq G$ controls p-fusion in G. Then, for $R\Gamma_G$-modules M and N, the restriction map*

$$\operatorname{res}^G_H \colon \operatorname{Ext}^n_{R\Gamma_G}(M, N) \to \operatorname{Ext}^n_{R\Gamma_H}(\operatorname{res}^G_H M, \operatorname{res}^G_H N)$$

is an isomorphism for $n \geq 0$, provided that $C_G(Q)$ acts trivially on $M(Q)$ and $N(Q)$ for all $Q \in \mathcal{F}$.

For $p = 3$ and for $p = 5$, the constructions are easier since G has cyclic Sylow p-subgroups for these primes. This means, in particular, that when $p = 3$ or 5, there is a periodic projective resolution

$$0 \to R \to P_n \to \cdots \to P_0 \to R \to 0$$

over the group ring RG where $R = \mathbf{Z}_{(p)}$. We view this resolution as a finite projective chain complex $\mathbf{C}$ of $R\Gamma$-modules by including it as the chain complex sitting at the trivial subgroup $Q = 1$. We then add free $R\Gamma$-module summands to this complex to change its homology to match its dimension function with the dimension function that we obtained in the $p = 2$ case. The resulting complex has dimension function $\underline{n}$ which has values $\underline{n}(1) = n$ and $\underline{n}(Q) = k$ for every nontrivial subgroup $Q \in \mathcal{F}$ for some positive integers n and k such that $n + 1 = 3(k + 1)$ and $8 \mid n + 1$. So, the smallest complex with these conditions is a 23-dimensional complex.

It is an interesting question whether similar constructions can be repeated for other rank 2 finite groups. It is known that the finite group

$$\mathrm{Qd}(p) = (\mathbf{Z}/p \times \mathbf{Z}/p) \rtimes SL_2(p)$$

does not admit a finite G-CW-complex $X \simeq S^n$ with rank 1 isotropy (see Theorem 3.3 in [7]). So, a theorem in this direction should exclude groups which have $\mathrm{Qd}(p)$ as a subgroup. Another open question is the following:

Question 4. Does S_5 act smoothly on a sphere with rank 1 isotropy?

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Madsen-Tillmann spectra and index theory

Johannes Ebert

Let M be any oriented closed smooth n-manifold, let $B_M := B\operatorname{Diff}^+(M)$ and let $p : E_M \to B_M$ be the universal oriented M-bundle. Let $\mathrm{MTSO}(n) := \mathbb{T}\mathbf{h}(-L_n)$ be the Madsen-Tillmann spectrum, i.e. the Thom spectrum of the inverse $-L_n$ of the universal n-dimensional oriented vector bundle $L_n \to B\,\mathrm{SO}(n)$. There is a map $\alpha_M : B_M \to \Omega^\infty \mathrm{MTSO}(n)$, the *Madsen-Tillmann map*, see [4]. The importance of this construction is that many important cohomology classes of B_M (alias characteristic classes of smooth M-bundles) are induced from classes on $\Omega^\infty \mathrm{MTSO}(n)$ via the map α_M. For example, the rational cohomology of the unit component $\Omega^\infty_0 \mathrm{MTSO}(n)$ is given by the Thom isomorphism

$$\mathrm{th}\colon H^*(B\,\mathrm{SO}(n);\mathbb{Q}) \cong H^{*-n}(\mathrm{MTSO}(n);\mathbb{Q})$$

and the isomorphism

$$s\colon \Lambda H^*_{>0}(\mathrm{MTSO}(n);\mathbb{Q}) \cong H^*(\Omega^\infty_0 \mathrm{MTSO}(n);\mathbb{Q}),$$

where Λ is the functor which associates to a graded $\mathbb{Q}$-vector space the free-graded commutative algebra generated by it. Given any $c \in H^*(B\,\mathrm{SO}(n))$, then

$$p_!(c(T_vE_M)) = \alpha_M^* s\,\mathrm{th}(c), \tag{1}$$

where $T_vE_M \to E_M$ is the vertical tangent bundle.

Another source of characteristic classes of smooth fiber bundles, this time with values in the topological K-theory of the base space, is the index of natural differential elliptic operators. In this talk, we consider only self-adjoint operators. The case of general operators is parallel (and better known). Let D be a family of self-adjoint elliptic differential operators on the universal bundle $E_M \to B_M$. By [2], these data have an index $\mathrm{ind}(D) \in K^1(B_M)$. On the other hand, the Atiyah-Singer family index theorem holds and yields

$$\mathrm{ind}(D) = (p\circ\pi)_!(\mathrm{smb}_D)_{s.a.},$$

where $(\mathrm{smb}(D)_{s.a.} \in K^1(\mathbb{T}\mathbf{h}(T_vE_M))$ is the self-adjoint symbol class of D, see [1], and $(p\circ\pi)_! : K^1(\mathbb{T}\mathbf{h}(T_vE_M)) \to K^1(B_M)$ is the umkehr map in K-theory, which is defined as the composition of the Thom isomorphism $K^1(\mathbb{T}\mathbf{h}(T_vE_M)) \cong K^1(\mathbb{T}\mathbf{h}(-T_vE_M))$ and the map $K^1(\mathbb{T}\mathbf{h}(-T_vE_M)) \to K^1(B_M)$ induced by Pontrjagin-Thom collapse.

If the operator D is natural then there exists an element $\sigma_D \in K^1(\mathbb{T}\mathbf{h}(L_n))$ such that σ_D maps to $(\mathrm{smb})_{s.a}$ under the map $\mathbb{T}\mathbf{h}(T_vE_M) \to \mathbb{T}\mathbf{h}(L_n)$ which comes from the classifying map for the vertical tangent bundle. In this case

$$\operatorname{ind}(D) = \alpha_M^* \operatorname{th}^{-1} \sigma_D, \tag{2}$$

where $\operatorname{th} : K^1(\mathrm{MTSO}(n)) \to K^1(\mathbb{T}\mathbf{h}(L_n))$ is the Thom isomorphism.

Now let M be a $2m+1$-dimensional closed oriented manifold. The even signature operator [1] $D : \bigoplus_{p\geq 0} \mathcal{A}^{2p}(M) \to \bigoplus_{p\geq 0} \mathcal{A}^{2p}(M)$ is defined to be

$$D\phi = i^{m+1}(-1)^{p+1}(*d - d*)\phi$$

whenever $\phi \in \mathcal{A}^{2p}(M)$. It is a self-adjoint, elliptic differential operator, and it is natural. Furthermore, it is related to the Laplace-Beltrami operator on forms by $D^2 = \Delta$. Moreover

$$\ker(D) = \ker(\Delta) = \bigoplus_{p\geq 0} H^{2p}(M; \mathbb{C}) \tag{3}$$

by the Hodge theorem. Now choose a fiberwise smooth metric on the vertical tangent bundle of the universal M-bundle. The even signature operators on the fibers define a family of self-adjoint elliptic differential operators and hence we have an index $\operatorname{ind}(D) \in K^1(B_M)$. Here is our main result.

Theorem 4. [3] *For any odd-dimensional closed oriented manifold M, the family index of the even signature operator* $\operatorname{ind}(D) \in K^1(B_M)$ *is trivial.*

The proof is purely analytic (it uses spectral theory, Kuiper's theorem on the contractibility of the unitary group of a Hilbert space and, crucially, the constancy of the dimension of the kernel, which follows from (3)).

Because the proof of the vanishing of the index is purely analytical, the result (2) allows us to draw topological conclusions. Apply the Chern character to (2). A routine calculation of characteristic classes shows that $\operatorname{ch}(\alpha_M^* \operatorname{th}^{-1} \sigma_D) = \alpha_M^* s \operatorname{th} \mathcal{L}$, where $\sigma_D \in K^1(\mathbb{T}\mathbf{h}(L_n))$ is the universal symbol for the even signature operator and $\mathcal{L} \in H^{4*}(B\,\mathrm{SO}(2m+1); \mathbb{Q})$ is the Hirzebruch L-class. Therefore Theorem 4 implies

Theorem 5. [3] *For any closed oriented $2m+1$-manifold M, the Madsen-Tillmann map $\alpha_M : B_M \to \Omega_0^\infty \mathrm{MTSO}(2m+1)$ annihilates $s \operatorname{th} \mathcal{L} \in H^{4*-2m-1}(\Omega_0^\infty \mathrm{MTSO}(2m+1); \mathbb{Q})$.*

If $m = 1$, then kth component $\mathcal{L}_k$ generates $H^{4k}(BSO(3); \mathbb{Q})$. Therefore, the map $B_{M^3} \to \Omega_0^\infty \mathrm{MTSO}(3)$ induces the zero map in rational cohomology. This is in sharp contrast to the 2-dimensional case, where Madsen and Weiss [5] showed that $B_M \to \Omega_0^\infty \mathrm{MTSO}(2)$ induces an isomorphism in integral homology in degrees $* < g/2 - 1$, whenever M is a connected closed oriented surface of genus g.

I conjecture that the vanishing theorem 5 is the only vanishing theorem for these characteristic classes of smooth oriented bundles. More precisely, I state the following conjecture:

Conjecture 6. *Let $n \in \mathbb{N}$ and let $\mathcal{R}_n$ be a set of representatives for the isomorphism classes of smooth closed oriented n-manifolds. Let $\alpha : \coprod_{M\in\mathcal{R}} B_M \to \Omega^\infty \operatorname{MTSO}(n)$ be the universal Madsen-Tillmann map. Then*

$$\alpha_n^* : H^*(\Omega^\infty \operatorname{MTSO}(n);\mathbb{Q}) \to H^*(\textstyle\coprod_{M\in\mathcal{R}} B_M;\mathbb{Q})$$

is injective if n is even and if n is odd, then the kernel of α_n^* is the ideal that is generated by $s\,\mathrm{th}(\mathcal{L}_k)$, $4k \geq n$, where $\mathcal{L}_k$ is the component of the Hirzebruch class of degree $4k$.

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The topology and analysis of the Hanna Neumann conjecture.

Igor Mineyev

We present a topological and analytic approach to the Hanna Neumann conjecture. It was introduced by Hanna Neumann in 1956-1957 [8], [9], and is considered one of the outstanding conjectures of combinatorial and geometric group theory. The conjecture states that if A and B are nontrivial finitely generated subgroups of a free group Γ, then

$$rk\,(A \cap B) \leq (rk\,A - 1)(rk\,B - 1).$$

Hanna Neumann showed that the inequality

$$rk\,(A \cap B) \leq 2(rk\,A - 1)(rk\,B - 1)$$

always holds, improving the estimate of Howson [6] who was first to prove that the rank of $A \cap B$ is finite.

Walter Neumann [10] introduced the reduced rank of a free group, $\tilde{r}(A) := \max(0, rk\,A - 1)$, so that the conjecture rewrites as

$$\tilde{r}(A \cap B) \leq \tilde{r}(A) \cdot \tilde{r}(B),$$

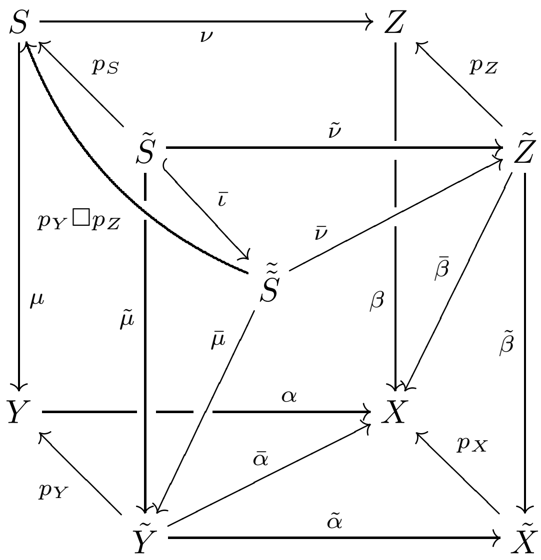
and suggested a stronger version of the conjecture, the *strengthened Hanna Neumann conjecture* in which the left hand side of the inequality is replaced with the sum of reduced ranks of the intersection of A with all the conjugates of B.

The conjecture has resisted numerous attempts. In the last 50 years or so there have been many partial results improving the estimate, reformulating the conjecture, and discussing related questions. See for example Burns [1], Stallings [11], Gersten [5], Dicks [2], Dicks and Formanek [3], Sergei Ivanov [7], Dicks and Ivanov [4].

The approach mostly has been combinatorial and asymptotically all the new estimates stayed the same as the original estimate by Hanna Neumann, with the coefficient 2 in the right hand side.

First observe that the reduced rank of (the fundamental group of) a graph X can be expressed as the first ℓ^2-Betti number of a Γ-action on a forest with quotient X.

We define a *system* of graphs, or more generally, complexes, which is a commutative diagram



arising from the data (Γ, A, B). Then the strengthened conjecture is restated in terms of the Murray-von Neumann dimension of the ℓ^2-homology of entries of the system. We provide several sufficient conditions for the conjecture: in terms of existence of Γ-orthonormal bases on the ℓ^2-homology Hilbert Γ-modules (the diagonal approach), and in terms of existence of certain injective maps between Hilbert Γ-modules (the square approach). We further prove statements that look very close to those sufficient conditions. This gives conceptually new approaches to the conjecture.

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The type of the classifying space for the family of virtually cyclic subgroups

Brita E.A. Nucinkis
(joint work with Dessislava H. Kochloukova and Conchita Martínez-Pérez)

Throughout this note let G denote a discrete group and let $\mathfrak{F}$ be a family of subgroups of G, closed under conjugation and taking subgroups. A G-CW-complex X is called a model for $E_{\mathfrak{F}}G$ if X^H is contractible for all $H \in \mathfrak{F}$ and is empty otherwise. By generalising either Milnor's or Segal's construction for EG [11, 13], one can show that models for $E_{\mathfrak{F}}G$ exist. These constructions, however, do not give finite dimensional or finite type models in general. The three main examples of families $\mathfrak{F}$ considered are:

- The trivial family $\mathfrak{F} = \{e\}$. Here a model for $E_{\mathfrak{F}}G$ is a model for EG, the universal cover of a $K(G,1)$.
- The family $\mathfrak{F} = \mathfrak{F}in$ of all finite subgroups. Here $E_{\mathfrak{F}}G$ is a model for the classifying space for proper action, denoted $\underline{E}G$.
- The family $\mathfrak{F} = \mathfrak{VC}$, the family of all virtually cyclic subgroups. We'll be dealing with this case here. In this case $E_{\mathfrak{F}}G$ is denoted by $\underline{\underline{E}}G$.

We say X is a model of finite type if there are finitely many G-orbits in each dimension. For $\mathfrak{F} = \mathfrak{F}in$, there are many examples of groups admitting finite type or even cocompact models for $\underline{E}G$. For G polycyclic, we can take $\mathbb{R}^n$ as a cocompact model for $\underline{E}G$. It was also shown [5] that elementary amenable groups of type FP_∞ admit a cocompact model for $\underline{E}G$. For word-hyperbolic groups, the Rips-complex is a cocompact model for $\underline{E}G$ [10]. See [9] for more examples.
On the other hand, for $\underline{\underline{E}}G$, so far we only know of one cocompact example: the point as a model for $\underline{\underline{E}}G$ for G virtually cyclic. Under certain circumstances one can build models for $\underline{\underline{E}}G$ from models for $\underline{E}G$ by attaching orbits of cells in small dimensions:

Theorem 1. [3, Juan-Pineda and Leary] *Let G be a discrete group such that every infinite virtually cyclic subgroup is contained in a unique maximal virtually cyclic subgroup, which is equal to its own normalizer. Let $\mathcal{C}_0$ (resp $\mathcal{C}_n$) be a set of conjugacy class representatives of orientable (resp. non-orientable) maximal* *infinite virtually cyclic subgroups of* G. *Let* $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_n$. *Then for any model of* $\underline{E}G$ *a model for* $\underline{\underline{E}}G$ *can be made by adding:*

(1) *orbits of* 0*-cells indexed by* $\mathcal{C}$,
(2) *orbits of* 1*-cells indexed by* $\mathcal{C}_0 \cup \{1,2\} \times \mathcal{C}_n$,
(3) *orbits of* 2*-cells indexed by* $\mathcal{C}$.

Example 2. Let G be a word-hyperbolic group, which is not virtually cyclic. Then G satisfies the condition of Theorem 1 [3], but has infinitely many conjugacy classes of maximal virtually cyclic subgroups [1, Cor. 5.1.B]. Hence G has a finite dimensional model for $\underline{\underline{E}}G$ with finitely many orbits of cells in dimension 3 and above. Since G has infinitely many conjugacies of maximal virtually cyclic subgroups, it cannot admit a finite type model for $\underline{\underline{E}}G$.

Furthermore, it was shown [3] that $\mathbb{Z} \times \mathbb{Z}$ does not admit a finite type model for $\underline{\underline{E}}G$. These examples led to the following conjecture:

Conjecture 3. [3, Juan-Pineda and Leary] *Let* G *be a group, for which there is a finite model for* $\underline{\underline{E}}G$. *Then* G *is virtually cyclic.*

We will now outline a purely algebraic proof that this conjecture holds for elementary amenable groups, see [6]. For this we consider finiteness conditions in Bredon cohomology, which can be viewed as the algebraic mirror for classifying spaces with respect to a family $\mathfrak{F}$. Let $\mathcal{O}_{\mathcal{F}}G$ denote the orbit category, which is the category with objects the transitive G-sets G/H with $H \in \mathfrak{F}$ and with morphisms being G-maps. We denote by $[G/H, G/K] = mor_{\mathcal{O}_{\mathcal{F}}G}(G/H, G/K)$ the morphism set, which is non-empty if and only if $H^g \leq K$ for some $g \in G$. A Bredon-module is a contravariant functor

$$M : \mathcal{O}_{\mathcal{F}}G \to \mathcal{A}b.$$

After evaluation, $M(G/H)$ is in a natural way a $N_G(H)/H$-module. We have a notion of exactness, direct products and coproducts. Furthermore, this category has enough projectives and we can therefore define Bredon cohomology. The building blocks for free Bredon-modules are the free abelian groups on the morphism sets, $\mathbb{Z}[-, G/K]$ and we can view finitely generated free modules as $\mathbb{Z}[-, \Omega]$, where $\Omega = \sqcup_{i=1}^{n} G/K_i$ is the disjoint union of finitely many transitive G-sets with stabilizers in $\mathfrak{F}$. We say a group is of type Bredon-FP_∞ if the constant functor $\mathbb{Z}_{\mathfrak{F}}$ has a resolution with finitely generated projectives in each dimension. Let X be a model for $E_{\mathfrak{F}}G$. Then the augmented cellular chain complex gives a free Bredon resolution of $\mathbb{Z}_{\mathfrak{F}}$ by taking fixed points:

$$C_*(X^{(-)}) \twoheadrightarrow \mathbb{Z}_{\mathfrak{F}}.$$

Furthermore:

Theorem 4. [8, Lück-Meintrup] *A group admits a model for* $E_{\mathfrak{F}}G$ *of finite type if and only if* G *is of type Bredon-*FP_∞ *and there is a model for* $E_{\mathfrak{F}}G$ *with a finite* 2*-skeleton.*

To consider finiteness conditions for the family of virtually cyclic subgroups we consider the following two conditions on a group G:

(1) G has $max - vc$, the maximal condition on virtually cyclic subgroups, and has finitely many conjugacy classes of maximal virtually cyclic subgroups.
(2) There are finitely many virtually cyclic subgroups $H_1, ..., H_n$ and each virtually cyclic subgroup is subconjugated to one of the H_i $(i = 1, ..., n)$.

Condition (1) obviously implies condition (2), yet the converse does not hold. The famous construction by Higman-Neumann-Neumann [12, 6.4.6] of a group, in which all non-trivial elements are conjugate, satisfies (2) but not (1).

We say a group is of type $\underline{\underline{\mathrm{FP}}}_\infty$ if it is of type Bredon-FP_∞ for the family of virtually cyclic subgroups. A group is of type $\underline{\underline{\mathrm{FP}}}_n$ if there is a Bredon-projective resolution of $\mathbb{Z}_{\mathfrak{VC}}$which is finitely generated in dimension less or equal n.

Theorem 5. [6, Kochloukova, Martínez-Pérez and Nucinkis] *Let G be an elementary amenable group of type $\underline{\underline{\mathrm{FP}}}_\infty$. Then G is virtually cyclic.*

To prove this theorem, we first reduce the problem to a completely group theoretic one by showing that a group is of type $\underline{\underline{\mathrm{FP}}}_0$ if and only if G satisfies condition (2) above. Furthermore we show that groups of type $\underline{\underline{\mathrm{FP}}}_n$ satisfy (2) and have the property that centralizers of virtually cyclic subgroups are of type FP_n. We then reduce to the case of nilpotent-by-abelian groups. This involves showing that $\underline{\underline{\mathrm{FP}}}_n$ is preserved by restricting to a subgroup of finite index and by using the fact that elementary amenable groups of type FP_∞ are nilpotent-by-abelian-by-finite [2, 4]. The theorem now follows from the following two propositions, which can proved purely group-theoretically:

Proposition 6. [6] *Let G be a nilpotent-by-abelian group of type* FP_∞ *such that the centralizer $C_G(g)$ is of type* FP_∞ *for all $g \in G$. Then G is polycyclic.*

Proposition 7. [6] *Let G be a nilpotent-by-abelian group satisfying condition (1). Then G is virtually cyclic.*

Proposition 6 ties in with an old question of J.C. Lennox [7], who asked whether abelian-by-polycyclic groups, in which centralisers of elements are finitely generated, are polycyclic. We also have some partial answers to this question. Let us conclude that together with Proposition 7 Lennox' question can be rephrased as:

Question 8. *Are elementary amenable groups of type $\underline{\underline{\mathrm{FP}}}_1$ virtually cyclic?*

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Invariants of homology 3-spheres via trivial cocycles

Wolfgang Pitsch

Let Σ_g be an oriented surface of genus g with a little disc D^2 on it, and $\mathcal{M}_g$ its mapping class group. i.e. $\mathcal{M}_g = \pi_0(\mathrm{Diff}(\Sigma_g;\ \text{rel. } D^2))$. Fix a Heegaard splitting of the oriented sphere S^3, that is a decomposition of S^3 into two handlebodies of genus g, $S^3 = \mathcal{H}_g \cup_{\iota_g} -\mathcal{H}_g$, where ι_g is a diffeomorphism of Σ_g along which the handlebodies are glued together. Then we get a map from $\mathcal{M}_g$ to the set $\mathcal{V}(3)$ of diffeomorphism classes of closed oriented 3-manifolds, by twisting the Heegaard splitting:

$$\begin{array}{ccc} \mathcal{M}_g: & \longrightarrow & \mathcal{V}(3) \\ \phi & \longmapsto & S^3_\phi = \mathcal{H}_g \cup_{\iota_g \phi} -\mathcal{H}_g. \end{array}$$

The little disc allows to define a stabilization map $\mathcal{M}_g \hookrightarrow \mathcal{M}_{g+1}$ which is compatible with the above construction. Furthermore, restricting the diffeomorphisms of either handlebodies to their boundary we get two subgroups $\mathcal{A}_g, \mathcal{B}_g \subset \mathcal{M}_g$, namely those classes which contain diffeomorphisms that extend to the exterior handlebody $-\mathcal{H}_g$ and to the interior handlebody $\mathcal{H}_g$ respectively.

The fundamental theorem on Heegaard splittings, due to Singer [2], now says:

Theorem 1. *The map*

$$\begin{array}{ccc} \lim_g \mathcal{B}_g \setminus \mathcal{M}_g / \mathcal{A}_g & \longrightarrow & \mathcal{V}(3) \\ \phi & \longmapsto & S^3_\phi = \mathcal{H}_g \cup_{\iota_g \phi} -\mathcal{H}_g. \end{array}$$

is well defined and is a bijection.

Let $\mathcal{T}_g$ be the Torelli group, that is the kernel of the canonical map $\mathcal{M}_g \to Aut(H_1(\Sigma_g, \mathbb{Z}))$. Twisting the Heegaard splitting by maps in the Torelli group produces homology 3-spheres, and all are obtained in this way. So, if $\mathcal{S}(3)$ denotes the subset of $\mathcal{V}(3)$ of homology spheres, we get an induced bijection

$$\begin{array}{ccc} \lim_g \mathcal{T}_g / \sim & \longrightarrow & \mathcal{S}(3) \\ \phi & \longmapsto & S^3_\phi = \mathcal{H}_g \cup_{\iota_g \phi} -\mathcal{H}_g, \end{array}$$

where $\sim$ is the equivalence relation on $\mathcal{T}_g$ by the double cosets in $\mathcal{M}_g$. Any invariant of homology spheres with values in the integers, say F, can be viewed via this bijection as a sequence of compatible maps F_g on the Torelli groups T_g that are constant on equivalence classes. There is now an associated family of trivial 2-cocycles defined by

$$C_{F_g}(\phi, \psi) = F_g(\phi\psi) - F_g(\phi) - F_g(\psi)$$

In this talk we characterize those trivial cocycles that come from invariants and we explain how to recover the invariant out of the 2-cocycles. For instance we show that only the trivial map $F = 0$ is associated to the 0-cocycle, or equivalently that no non-trivial invariant is a group homomorphism. The characterization turns out to be purely cohomological. In particular we show how to construct the Casson invariant in a purely algebraic and elementary way, and characterize it as being the unique invariant whose cocycle C_g is a pull-back of a cocycle defined on an abelian quotient of the Torelli group. The results in this talk have appeared in [1].

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Dihedral manifold approximate fibrations

Bruce Hughes

(joint work with Qayum Khan)

We begin with a classical theorem, attributed to T.A. Chapman [1], on sucking and wrapping up manifolds over the real line. Recall that the infinite cyclic group C_∞ acts on the real line $\mathbb{R}$ by integer translations.

Theorem (Chapman)**.** *Let W be a connected topological manifold of dimension > 4. The following statements are equivalent:*

(1) *The space W is finitely dominated, and there exists a cocompact, free, discontinuous C_∞-action on W.*
(2) *There exists a proper bounded fibration $W \to \mathbb{R}$.*
(3) *There exists a manifold approximate fibration $W \to \mathbb{R}$.*
(4) *There exist a C_∞-action on W and C_∞-manifold approximate fibration $W \to \mathbb{R}$.*
(5) *There exists a manifold approximate fibration $M \to S^1$ such that $\overline{M}$ is homeomorphic to W.*

This theorem can be viewed as an answer to the question:

> *When does a finitely dominated manifold W admit a cocompact, free, discontinuous action of the infinite cyclic group C_∞?*

There are two essentially different answers. The first, spelled out by conditions (2) and (3), is that W admits a proper map to $\mathbb{R}$ with bounded or controlled versions of the homotopy lifting property (called a bounded or approximate fibration). The equivalence of the bounded and controlled versions is the main advance of Chapman's paper [1], and it is part of the phenomenon called *sucking*.

The second answer, formulated in conditions (4) and (5), is that the approximate fibration $W \to \mathbb{R}$ can be made equivariant with respect to some C_∞-action on W, or that $W \to \mathbb{R}$ can be *wrapped-up* to an approximate fibration $M \to S^1$. Chapman's wrapping-up construction is a variation of Siebenmann's twist-gluing construction (where the twist is the identity) used by him [5] in his formulation of Farrell's fibering theorem [2]. From this point of view, the question that is being raised is:

> *Given a discrete subgroup Γ of isometries on $\mathbb{R}$ and a manifold approximate fibration $W \to \mathbb{R}$, can the Γ-action on $\mathbb{R}$ be "approximately lifted" to a free, discontinuous Γ-action on W, so that there is a Γ-manifold approximate fibration $W \to \mathbb{R}$?*

Chapman proves that this can always be done when $\Gamma = C_\infty$ and $\dim W > 4$.

Our formulation of Chapman's theorem does not appear explicitly in [1]. Most aspects of this theorem were discovered independently by Ferry [3]. An analysis of many of the details appears in [4].

The theme of this talk is to extend Chapman's results from the smallest infinite discrete group of isometries on $\mathbb{R}$, namely the infinite cyclic group C_∞, to the largest discrete group of isometries on $\mathbb{R}$, namely the infinite dihedral group D_∞. In particular, our main result can be viewed as an answer to the question:

> *When does a finitely dominated manifold W admit a cocompact, free, discontinuous action of the infinite dihedral group D_∞?*

The main technical issues that arise involve the fact that D_∞ has torsion; it has a non-trivial finite subgroup, namely the cyclic group C_2 of order two, which acts by reflection on $\mathbb{R}$ fixing the origin. The lifted D_∞-action on W contains a C_2-action on W with a nonempty invariant subset. For notation, consider the dihedral groups $D_\infty := C_\infty \rtimes_{-1} C_2$ and $D_N := C_N \rtimes_{-1} C_2$ with $N \geq 1$. In particular, $D_1 = C_2$. There are short exact sequences

$$0 \longrightarrow C_\infty \longrightarrow D_\infty \longrightarrow C_2 \longrightarrow 0, \qquad 0 \longrightarrow N \cdot C_\infty \longrightarrow D_\infty \longrightarrow D_N \longrightarrow 0.$$

Fix presentations: $C_\infty = \langle T \mid \, \rangle$, $C_2 = \langle R \mid R^2 = 1 \rangle$, and

$$D_\infty = \langle R, T \mid R^2 = 1, RT = T^{-1}R \rangle.$$

The D_∞-action on $\mathbb{R}$ by isometries is given by $R(x) = -x$ and $T(x) = x + 1$.

Our main theorem, which we now state, contains analogues of all parts of Chapman's theorem.

Theorem (Main Theorem)**.** *Let W be a connected topological manifold of dimension > 4. The following statements are equivalent:*

(1) *The space W is finitely dominated, and there exists a cocompact, free, discontinuous D_∞-action on W.*

(2) *There exist a free* C_2*-action on* W *and proper* C_2*-bounded fibration* $W \to \mathbb{R}$.
(3) *There exist a free* C_2*-action on* W *and* C_2*-manifold approximate fibration* $W \to \mathbb{R}$.
(4) *There exist a free* D_∞*-action on* W *and* D_∞*-manifold approximate fibration* $W \to \mathbb{R}$.
(5) *For every* $N \geq 1$*, there exist a free* D_N*-action on a manifold* M *and* D_N*-manifold approximate fibration* $M \to S^1$ *whose induced infinite cyclic cover* $\overline{M}$ *is homeomorphic to* W.

The prototypical example of a manifold W satisfying the conditions of the Main Theorem is the product $W = S^n \times \mathbb{R}$, where the D_∞-action on W is given by $T(x,t) = (x, t+1)$ and $R(x,t) = (-x, -t)$.

Most of the work in this talk applies to settings more general than the real line in the main theorem. We now discuss some of the highlights.

The first is a variation on the well-known property that close maps into an ANR B are closely homotopic. In the compact case, closeness is measured uniformly by using a metric on B. In the non-compact case, open covers of B are used. In this talk we are confronted with metrically close maps into a non-compact ANR (B,d) which we want to conclude are metrically closely homotopic. We isolate a condition, called *finite isometry type* (which essentially means that there exists only finitely many simplicial isometry types of cones of vertices), that allows us to do this. Here is the result.

Theorem (Metrically Close Maps)**.** *Suppose* (B,d) *is a triangulated metric space with finite isometry type. For every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that: if* X *is any space and* $f, g\colon X \to B$ *are two* δ*-close maps, then* f *and* g *are* ϵ*-homotopic rel*

$$X_{(f=g)} := \{x \in X \mid f(x) = g(x)\}.$$

Likewise, we need a metric version of Chapman's manifold approximate fibration sucking theorem, which has an open cover formulation in Chapman's work. We again find the finite isometry type condition suitable for our purposes.

Theorem (Metric MAF Sucking)**.** *Suppose* (B,d) *is a triangulated metric space with finite isometry type, and let* $m > 4$*. For every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that: if* M *is an* m*-dimensional manifold and* $p\colon M \to B$ *is a proper* δ*-fibration, then* p *is* ϵ*-homotopic to a manifold approximate fibration* $p'\colon M \to B$.

The following result is an equivariant version of Metric MAF Sucking Theorem, valid for many finite subgroups of $O(n)$. We will need it only for the simplest non-trivial case of $C_2 \leq O(1)$ when we perform dihedral wrapping up. However, we expect future applications in the theory of locally linear actions of finite groups on manifolds.

Theorem (Orthogonal Sucking)**.** *Suppose* G *is a finite subgroup of* $O(n)$ *acting freely on* S^{n-1} *such that* $\mathbb{R}^n/G$ *has finite isometry type,*[1] *and let* $m > 4$*. For every*

[1] Immediately after the talk, André Henriques and Ian Leary showed us that the finite isometry condition is always satisfied in this situation.

$\epsilon > 0$ there exists $\delta > 0$ such that: if M is a free G-manifold of dimension m and $p\colon M \to \mathbb{R}^n$ is a proper G-δ-fibration, then p is G-ϵ-homotopic to a G-manifold approximate fibration $p'\colon M \to \mathbb{R}^n$.

The following result is the dihedral version of wrapping-up.

Theorem (Dihedral Wrapping-up)**.** *Let W is a topological manifold of dimension > 4 and equipped with a free C_2-action generated by $R\colon W \to W$. Suppose $p\colon W \to \mathbb{R}$ is a C_2-MAF.*

(1) *There exists a cocompact, free, discontinuous D_∞-action on W extending the C_2-action.*
(2) *The C_2-MAF $p : W \to \mathbb{R}$ is properly C_2-homotopic to a D_∞-MAF $\tilde{p}\colon W \to \mathbb{R}$.*

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Topological rigidity

Wolfgang Lück
(joint work with Arthur Bartels)

Let G be a discrete group and let R be an associative ring with unit. We explain and state the following conjectures and discuss their relevance.

Kaplansky Conjecture. If G is torsionfree and R is an integral domain, then 0 and 1 are the only idempotents in RG.

Conjecture. Suppose that G is torsionfree. Then $K_n(\mathbb{Z}G)$ for $n \leq -1$, $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish.

Novikov Conjecture. Higher signatures are homotopy invariants.

Borel Conjecture. An aspherical closed manifold is topologically rigid.

Serre's Conjecture. A group of type FP is of type FF.

Conjecture. If G is a finitely presented Poincaré duality group of dimension $n \geq 5$, then it is the fundamental group of an aspherical homology ANR-manifold.

Farrell-Jones Conjecture. Let G be torsionfree and let R be regular. Then the assembly maps for algebraic K- and L-theory

$$\begin{array}{rcl} H_n\big(BG;\mathbb{K}_R\big) & \to & K_n(RG); \\ H_n\big(BG;\mathbb{L}_R^{\langle -\infty\rangle}\big) & \to & L_n^{\langle -\infty\rangle}(RG), \end{array}$$

are bijective for all $n \in \mathbb{Z}$.

There is a more complicate version of the Farrell-Jones Conjectures which makes sense for all groups and rings and allows twistings of the group ring. We explain that it implies all the other conjectures mentioned above provided that in the Kaplansky Conjecture R is a field of characteristic zero and in the Borel Conjecture the dimension is greater or equal to five. We present the following result:

Theorem [Bartels-Lück]. Let $\mathcal{FJ}$ be the class of groups for which the Farrell-Jones Conjecture is true in its general form. Then:

(1) Hyperbolic groups belong to $\mathcal{FJ}$;
(2) CAT(0) groups belong to $\mathcal{FJ}$;
(3) Cocompact lattices in almost connected Lie groups belong to $\mathcal{FJ}$;
(4) Fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary) belong to $\mathcal{FJ}$;
(5) If G_0 and G_1 belong to $\mathcal{FJ}$, then also $G_0 * G_1$ and $G_0 \times G_1$;
(6) If G belongs to $\mathcal{FJ}$, then any subgroup of G belongs to $\mathcal{FJ}$;
(7) Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). If each G_i belongs to $\mathcal{FJ}$, then also the direct limit of $\{G_i \mid i \in I\}$.
(8) Let $1\colon H \to G \xrightarrow{p} Q \to 1$ be an extension of groups. If Q and for all virtually cyclic subgroups $V \subseteq Q$ the preimage $p^{-1}(V)$ belongs to $\mathcal{FJ}$, then G belongs to $\mathcal{FJ}$;

Since certain prominent constructions of groups yield colimits of hyperbolic groups, the class $\mathcal{FJ}$ contains many interesting groups, e.g. limit groups, Tarski monsters, groups with expanders and so on. Some of these groups were regarded as possible counterexamples to the conjectures above but are now ruled out by the theorem above.

There are also prominent constructions of closed aspherical manifolds with exotic properties, e.g. whose universal covering is not homeomorphic to Euclidean space, whose fundamental group is not residually finite or which admit no triangulation. All these constructions yield fundamental groups which are CAT(0) and hence yield topologically rigid manifolds.

However, the Farrell-Jones Conjecture is open for instance for solvable groups, $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, mapping class groups or automorphism groups of finitely generated free groups.

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Bounded Z-valued cocycles on connected Lie groups

Indira Chatterji

(joint work with Guido Mislin, Christophe Pittet, Laurent Saloff-Coste)

Given a locally compact and compactly generated group G and a $\mathbf{Z}$-subgroup of G, generated by an element $g \in G$ one can compare the length of $g^n \in G$ (for the length coming from a compact generating set of G, or a Riemannian metric on G if G happens to be a connected Lie group) with n, which is the length of g^n as an element of the $\mathbf{Z}$-subgroup. If those two lengths have the same asymptotic behavior, the $\mathbf{Z}$-subgroup generated by g is quasi-isometrically embedded. In this case we say that it is *undistorted* and we call it *distorted* otherwise. Distortion never occurs for semi-simple elements when there is enough non-positive curvature but does for parabolic ones and for Heisenberg groups. Given a locally compact group G and a central $\mathbf{Z}$-extension

$$0 \to \mathbf{Z} \to E \to G \to \{e\}$$

it is a classical result in group theory that the law on the group E is given by a 2-cocycle $\gamma : G \times G \to \mathbf{Z}$ expressing the failure of a section $s : G \to E$ to be a homomorphism. If that 2-cocycle γ is bounded, it is a straightforward computation that the inclusion $\mathbf{Z} \to E$ is undistorted. Hence a natural question is the following.

Question. *If a central $\mathbf{Z}$-extension of a group G is undistorted, what can we say about the class of the 2-cocycle defining it?*

In the case of topological extensions of connected Lie groups, we provide a complete answer to this question and more precisely the following.

Main Theorem (Chatterji, Mislin, Pittet, Saloff-Coste). *Let G be a connected Lie group. The following conditions are equivalent.*

(1) *The radical $\sqrt{G}$ of G is linear.* (Topologists might prefer to think of the equivalent condition that the closure of the commutator subgroup of $\sqrt{G}$ is simply-connected.)
(2) *The map $H^n_{Bb}(G, \mathbf{Z}) \to H^n_B(G, \mathbf{Z})$ is surjective for all $n \geq 2$.* (In words, each Borel cohomology class of G with $\mathbf{Z}$-coefficients can be represented by a Borel bounded cocycle.)
(3) *The map $H^2_{Bb}(G, \mathbf{Z}) \to H^2_B(G, \mathbf{Z})$ is surjective.* (In words, each Borel cohomology class of G of degree two with $\mathbf{Z}$-coefficients can be represented by a Borel bounded cocycle.)
(4) *The class in $H^2_B(G, \pi_1(G))$ defined by the universal cover of G can be represented by a Borel bounded cocycle.*
(5) *The natural inclusion $\pi_1(G) \to \tilde{G}$ of the fundamental group of G into the universal cover of G is undistorted.*

Here the radical $\sqrt{G}$ of G is the maximal connected solvable subgroup of G. The groups $H^n_B(G, \mathbf{Z})$ are Borel cohomology classes of G with $\mathbf{Z}$-coefficients. Those are, as in the classical group cohomology, classes of cocycles $\gamma : G^n \to \mathbf{Z}$ up to co-boundaries, except that they are assumed to be Borel measurable. Moore [9] showed that $H^2_B(G, \mathbf{Z})$ classify topological central $\mathbf{Z}$-extensions of G (i.e covers with $\mathbf{Z}$-fibers) and Wigner [11] that $H^n_B(G, \mathbf{Z}) \simeq H^n(BG, \mathbf{Z})$, where BG is a classifying space for G.

Example. The "toy example" one should keep in mind is the following: Take $\mathbf{H}$ be the 3-dimensional Heisenberg group and consider $\mathbf{H} \times S^1$, its center is $\mathbf{R} \times S^1$. Take the discrete central subgroup $\mathbf{Z}$ generated by $(1, t)$ with 1 generating $\mathbf{Z}$ in $\mathbf{R}$, and t of infinite order in S^1; this central subgroup of $\mathbf{H} \times S^1$ is discrete. Define $G := (\mathbf{H} \times S^1)/\mathbf{Z}$. It is a nilpotent connected Lie group with $[G, G]$ homeomorphic to $\mathbf{R}$, embedded in the maximal torus $S^1 \times S^1$ of G in a dense way (hence not closed). It follows that $\pi_1([G, G])$ is trivial but $\pi_1\left(\overline{[G, G]}\right) = \mathbf{Z}^2$.

Remarks.

- The proof given in [5] shows that one can relax the boundedness hypothesis in Conditions (2), (3) and (4), of the above theorem: assuming that the representative cocycle has sub-linear growth, leads in each case to an equivalent condition. Similarly, assuming that $\pi_1(G) \to \tilde{G}$ has sub-linear distortion is equivalent to Condition (5).
- The Main Theorem assumes *integer* coefficients. Let us emphasise that this is in contrast with the case of *real* coefficients (as studied in [8], [3], [1]),

where the map $H^*_{Bb}(G, \mathbf{R}) \to H^*_B(G, \mathbf{R})$ conjectured to be onto for semi-simple Lie groups with finite center [4] but is never onto for G a connected solvable Lie group for which the right hand side does not vanish for all positive degrees as the left hand side is always 0. This map is not onto in degree 3 for G the universal cover of $SL(2, \mathbf{R})$ [8], thus, in general it is not onto for semi-simple Lie groups either. However it is onto for semi-simple Lie groups with finite center in the top dimension as this is equivalent to the simplicial volume conjecture solved by Lafont-Schmidt [7].

As a corollary to our Main Theorem, we obtain the following generalization of Gromov's [6] and Bucher-Karlsson's [2] theorems.

Corollary. Let G be a virtually connected Lie group with linear radical. Each class in the image of the natural map $H^*(BG, \mathbf{R}) \to H^*(BG^\delta, \mathbf{R})$ can be represented by a cocycle whose set of values on all singular simplices of BG^δ is finite.

Steps in the proof of Main Theorem. That (4) implies (5) is a generalisation to topological groups of an easy exercise in case of discrete groups. The linearity of the radical $\sqrt{G}$ of G amounts to a trivial fundamental group of the closure of the commutator subgroup of $\sqrt{G}$. So, non-linearity of that radical gives us an element in that fundamental group, which creates a distorted copy of $\mathbf{Z}$ in the universal cover of G (this uses a crucial result proved in [10] in the nilpotent case), and that in turn amounts to the existence of a unbounded class in degree 2, hence that gives us (5) implies (1). To see that (1) implies (3) (which in turn easily implies (4)), if we assume the radical to be linear, then standard Lie group structure theory allows us to reduce to the case of semi-simple Lie groups, which we then can compute. Finally, the equivalence with Condition (2), the boundedness of $\mathbf{Z}$-valued classes in all degrees, is a consequence of the observation that the $\mathbf{R}$-cohomology of a connected Lie group is generated by 2-dimensional classes and classes which are in the image via an inflation map of primary characteristic classes of a semi-simple Lie group, which according to Bucher-Karlsson's work in [2] and [1] have continuous, bounded representatives.

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Monoids of moduli spaces of manifolds

Oscar Randal-Williams
(joint work with Søren Galatius)

The work of Tillmann, Madsen, Weiss and Galatius [5, 3, 4, 1] has firmly linked the study of diffeomorphism groups (or mapping class groups) of oriented surfaces to the spectrum known as $\mathbf{MTSO}(2)$, which is the Thom spectrum of the complement to the canonical vector bundle over $BSO(2)$, and to *cobordism categories.*

Let $\theta : \mathbf{X} \to BO(d)$ be a fibration, and define a θ-structure on a d-dimensional vector bundle $V \to B$ to be a fibrewise linear isomorphism $V \to \theta^*\gamma_d$. Let $\mathrm{Bun}(V, \theta^*\gamma_d)$ denote the space of such maps. Recall from [1] that the cobordism category $\mathcal{C}_\theta$ is a topological category roughly defined as follows. It has objects subsets $M \subset \mathbb{R}^\infty$ which are compact $(d-1)$-dimensional submanifolds, along with a θ-structure on $\epsilon^1 \oplus TM$. It has as morphisms subsets $W \subset [0, a] \times \mathbb{R}^\infty$ which are compact, collared d-manifolds, having boundary in $\{0, a\} \times \mathbb{R}^\infty$, along with a θ-structure on TW. Composition is via union of subsets of $\mathbb{R} \times \mathbb{R}^\infty$. Galatius, Madsen, Tillmann and Weiss [1] proved that there is a homotopy equivalence

$$B\mathcal{C}_\theta \simeq \Omega^{\infty-1}\mathbf{MT}\theta$$

where $\mathbf{MT}\theta$ is the Thom spectrum of the complement to the bundle classified by the map θ.

We define another category $\mathcal{C}_\theta^\bullet$, the *pointed cobordism category,* which is roughly the subcategory of $\mathcal{C}_\theta$ whose objects are connected and contain the origin, and whose morphisms are connected and contain the interval $[0, a] \times \{0\}$. Our first theorem is as follows.

Theorem A. *Let $d \geq 2$ and $\theta : \mathbf{X} \to BO(d)$ be a tangential structure such that $\mathbf{X}$ is path connected and S^d admits a θ-structure. Then the map*

$$B\mathcal{C}_\theta^\bullet \to B\mathcal{C}_\theta \simeq \Omega^{\infty-1}\mathbf{MT}\theta$$

is a weak homotopy equivalence.

In dimension two, this can be improved.

Theorem B. *Let $\theta : \mathbf{X} \to BO(2)$ be a tangential structure such that $\mathbf{X}$ is path connected and that S^2 admits a θ-structure. Let $\mathcal{D} \subseteq \mathcal{C}_\theta^\bullet$ be a full subcategory.*

Then the inclusion

$$B\mathcal{D} \to B\mathcal{C}_\theta^\bullet$$

is a weak homotopy equivalence of each component of $B\mathcal{D}$ onto a component of $B\mathcal{C}_\theta^\bullet$. Additionally, the monoid of endomorphisms of any object $c \in \mathcal{C}_\theta^\bullet$ is homotopy commutative.

The implication of these theorems are as follows. Suppose θ is a tangential structure satisfying the hypotheses of Theorem B. Pick an object $c \in \mathcal{C}_\theta^\bullet$. Then

$$\mathrm{End}_{\mathcal{C}_\theta^\bullet}(c) \simeq \coprod_{[F]} \mathrm{Bun}_c(TF, \theta^*\gamma_2)/\!\!/\mathrm{Diff}(F)$$

is a full subcategory of $\mathcal{C}_\theta^\bullet$, where the disjoint union is over diffeomorphism types of θ-surfaces with one boundary circle and Bun_c denotes the space of those bundle maps which agree with that of c on the boundary of F. By Theorems A and B the group completion of this topological monoid is homotopy equivalent to a collection of path components of $\Omega^\infty\mathbf{MT}\theta$.

In cohomology (certainly over a field $\mathbb{F}$, and possibly more generally), the group completion theorem of McDuff–Segal [6] asserts that for a homotopy commutative monoid $\mathcal{M}$, the natural map $\mathcal{M} \to \Omega B\mathcal{M}$ induces a map on cohomology which is an isomorphism after taking $\pi_0(\mathcal{M})$-invariants:

$$H^*(\mathcal{M};\mathbb{F})^{\pi_0(\mathcal{M})} \xleftarrow{\simeq} H^*(\Omega_0 B\mathcal{M};\mathbb{F}).$$

In our setting we obtain that

$$H^*(\Omega_0^\infty\mathbf{MT}\theta;\mathbb{F}) \longrightarrow H^*\left(\coprod_{[F]} \mathrm{Bun}_c(TF, \theta^*\gamma_2)/\!\!/\mathrm{Diff}(F);\mathbb{F}\right)$$

is an isomorphism onto the $\pi_0(\mathrm{End}_{\mathcal{C}_\theta^\bullet}(c))$-invariants; that is, that this map is injective and that its image is precisely those characteristic classes of bundles of θ-surfaces with boundary which are invariant under gluing on trivial bundles. This statement can perhaps be generalised to higher-dimensional manifolds.

Question. Is there a homotopy commutative monoid $\mathcal{M}$ constructed from classifying spaces of diffeomorphism groups of oriented d-manifolds (with boundary) and composition given by gluing manifolds along their boundary, such that the natural map

$$H^*(\Omega_0^\infty\mathbf{MTSO}(d)) \longrightarrow H^*(\mathcal{M})^{\pi_0(\mathcal{M})}$$

to the $\pi_0(\mathcal{M})$-invariants is surjective, with kernel as conjectured by J. Ebert in this volume?

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A homotopy invariance theorem for commutative C^*-algebras

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(joint work with Andreas Thom)

Let X be a compact Hausdorff space, and consider the (C^*-) algebra $C(X)$ of continuous functions $X \to \mathbb{C}$. The assignment $X \mapsto C(X)$ gives a functor from the category Comp of compact Hausdorff spaces to the category $\mathfrak{Comm}/\mathbb{C}$ of commutative $\mathbb{C}$-algebras. The talk was about the homotopy invariance theorem below, which shows that if a functor satisfies three algebraic conditions, then the composite $X \mapsto F(C(X))$ turns out to be homotopy invariant. As a corollary we obtained a positive answer to a conjecture of Jonathan Rosenberg [3] that the negative algebraic K-theory groups of $C(X)$ are homotopy invariant functors of X.

Theorem 1. *Let F be a functor from commutative $\mathbb{C}$-algebras to abelian groups. Assume*

(1) *If $X \in$ Comp is the union of closed subsets X_1 and X_2 and X_1 retracts onto $X_1 \cap X_2$, then the following sequence is exact:*
$$0 \to F(X) \to F(X_1) \oplus F(X_2) \to F(X_1 \cap X_2) \to 0$$
(2) *F commutes with filtering colimits.*
(3) *If $Y \subset \mathbb{C}^n$ is a smooth affine algebraic variety and $\mathcal{O}(Y)$ is the algebra of regular algebraic maps $Y \to \mathbb{C}$, then $F(\mathcal{O}(Y)) = 0$.*

Then the functor Comp $\to \mathfrak{Ab}$, *$X \mapsto F(C(X))$ is homotopy invariant.*

Remark 2. The proof of the theorem above was built upon an argument of Eric Friedlander and Mark E. Walker, in their proof that $K_{<0}(\mathbb{C}(\Delta^n)) = 0$ ([1]).

Example 3. The algebraic K-theory functors K_n ($n \in \mathbb{Z}$) satisfy the first two conditions of the theorem above (of these, the first is a difficult theorem of Suslin and Wodzicki [4]). If moreover $n < 0$, also the third condition is satisfied; this is because rings of the form $\mathcal{O}(Y)$ with Y smooth affine are noetherian regular, and $K_{<0}$ is known to vanish for noetherian regular rings [2]. Similarly, if R is noetherian regular then it is K-regular, i.e., $i_n : K_n(R) \to K_n(R[t_1, \ldots, t_p])$ is an isomorphism for all $n \in \mathbb{Z}$ and $p \geq 1$, and thus the functor $F_n(R) = \operatorname{coker}(i_n)$ satisfies the conditions of the theorem.

The following corollary is immediate from the theorem and the example above.

Corollary 4. (Rosenberg's conjecture, [3, 3.7]) *For $n \leq 0$, $X \mapsto K_n(C(X))$ is homotopy invariant.*

Rosenberg showed (using excision and other standard arguments) that his conjecture implied the following result.

Corollary 5. (Rosenberg, [3, 3.8]) $K_n(C(X)) = [\Sigma^{-n}, bu]$ $(n < 0)$.

Using the theorem and the example above, and applying a similar argument to Rosenberg's, we also get:

Corollary 6. ([3, 3.5]) *Commutative C^*-algebras are K-regular.*

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The smooth structure set of $S^p \times S^q$

Diarmuid J. Crowley

Let M be a closed, oriented manifolds of dimension at least 5, either smooth or topological and let $\mathcal{S}^{Diff}(M)$ and $\mathcal{S}^{Top}(M)$ be the smooth, respectively topological structure sets of M. In [S], Siebenmann proved that the topological surgery exact sequence for M is a long exact sequence of abelian groups. In [N] Nicas asked if the same might be true of the smooth surgery exact sequence. Later Weinberger [W] showed that the forgetful map $F : \mathcal{S}^{Diff}(M) \to \mathcal{S}^{Top}(M)$ is finite-to-one with image containing a subgroup of finite index and in a remark left open whether the image of F is in general a subgroup of $\mathcal{S}^{Top}(M)$.

Responding to a question of Wolfgang Lück to Andrew Ranicki in Bonn in 2007, I was lead to a calculate $\mathcal{S}^{Diff}(S^p \times S^q)$ for $p, q \geq 2$ and $p + q \geq 5$. This gives explicit, elementary examples showing that the smooth surgery exact sequence cannot be an exact sequence of groups and even that the image of the smooth structure set in the topological structure set is not a subgroup.

The topological structure set of $S^p \times S^q$ for $p, q \geq 2$ and $p+q \geq 5$ was calculated in [R1] and also in [K-L].

Proposition 1. [R1, K-L] *There is an isomorphism of abelian groups*

$$\mathcal{S}^{Top}(S^p \times S^q) \cong \pi_p(G/Top) \times \pi_q(G/Top).$$

When $M = S^n$, $\mathcal{S}^{Diff}(S^n) = \Theta_n$ is the group of diffeomorphism classes of homotopy n-spheres. This is a finite abelian group by [K-M] and the smooth surgery exact sequence for S^n is a long exact sequence of abelian groups

$$\ldots \longrightarrow \pi_{n+1}(G/O) \xrightarrow{\theta^{Diff}} L_{n+1}(e) \xrightarrow{\omega^{Diff}} \Theta_n \xrightarrow{\eta^{Diff}} \pi_n(G/O) \xrightarrow{\theta^{Diff}} L_n(e).$$

The image of ω^{Diff} is $bP_{n+1} \subset \Theta_n$, the group of homotopy spheres bounding parallelisable manifolds.

The following examples best illustrate the main theorem and its applications. We use the isomorphism $\pi_4(G/O) \cong \mathbb{Z}$.

Theorem 2. *For $S^3 \times S^4$ there is a short exact sequence of pointed sets*

$$0 \longrightarrow \mathbb{Z}_{28} \xrightarrow{\omega^{bP}} \mathcal{S}^{Diff}(S^3 \times S^4) \xrightarrow{\eta} \mathbb{Z} \longrightarrow 0$$

where, $\mathbb{Z}_{28} \cong bP_8 = \Theta_7$ acts transitively on the fibres of η, $7 \cdot bP_8 \cong \mathbb{Z}_4$ acts freely on all of $\mathcal{S}^{Diff}(S^3 \times S^4)$ but $4 \cdot bP_8 \cong \mathbb{Z}_7$ acts freely on $\eta^{-1}(v)$ if and only if $v \in \mathbb{Z}$ is divisible by 7.

Corollary 3. *The sets $\mathcal{S}^{Diff}(S^3 \times S^4)$ and $\mathcal{N}^{Diff}(S^3 \times S^4) \equiv \mathbb{Z}$ cannot be given group structures such that η^{Diff} is a homomorphism. The same holds for $\mathcal{S}^{Diff}(S^3 \times S^4)$ and both $\omega^{bP} : bP_8 \to \mathcal{S}^{Diff}(S^3 \times S^4)$ and $\omega : L_8(e) \to \mathcal{S}^{Diff}(S^3 \times S^4)$.*

Theorem 4. *For $S^4 \times S^4$ there is a short exact sequence of pointed sets*

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\omega^{\Theta}} \mathcal{S}^{Diff}(S^4 \times S^4) \xrightarrow{i^* \circ \eta} \mathbb{Z} \times \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}_7 \longrightarrow 0$$

where $\mathbb{Z}_2 \cong \Theta_8$ acts freely and transitively on the fibres of $i^* \circ \eta$ and $\partial(u, v) = uv$ mod 7.

Corollary 5. *The image of the forgetful map $\mathcal{S}^{Diff}(S^4 \times S^4) \to \mathcal{S}^{Top}(S^4 \times S^4)$ is not a subgroup.*

Theorem 6. *Define the integer t_i by $t_{4k} = |\mathrm{Cok}(\pi_{4k}(G/O) \xrightarrow{F} \pi_{4k}(G/Top))|$ so that $t_4 = 2$ and $t_{4k} = |bP_{4k}|$ if $k > 1$ and by $t_i = 0$ if $i \neq 0$ mod 4. For $p, q \geq 2$ and $p + q \geq 5$, there is an exact sequence of pointed sets*

$$0 \longrightarrow \Theta_{p+q} \xrightarrow{\omega^{\Theta}} \mathcal{S}^{Diff}(S^p \times S^q) \xrightarrow{i^* \circ \eta} \prod_{i=1}^{2} \pi_{p_i}(G/O) \xrightarrow{\partial} 8 t_p t_q \cdot bP_{p+q} \longrightarrow 0$$

where Θ_{p+q} acts transitively on the fibres of $i^* \circ \eta^{Diff}$ and the map ∂ is the composite

$$\prod_{i=1}^{2} \pi_{p_i}(G/O) \longrightarrow \prod_{i=1}^{2} \pi_{p_i}(G/Top) \cong L_p(e) \times L_q(e) \xrightarrow{\alpha(p,q)} L_{p+q}(e) \xrightarrow{\omega^{Diff}} bP_{p+q}$$

with $\alpha(p, q)$ is the canonical product on L-groups.

We next describe the action of Θ_{p+q} on $\mathcal{S}^{Diff}(S^p \times S^q)$. We adopt the convention that if $p + q$ is odd then q is assumed even. If $(p, q) = (4j - 1, 4k)$ let $i_{4k} : S^{4k} \hookrightarrow$

$S^{4j-1} \times S^{4k}$ be the standard inclusion, let $\phi_{4k} : \pi_{4k}(G/O) \cong \mathbb{Z}$ be the projection onto the free part of $\pi_{4k}(G/O)$ and define the surjection

$$d : \mathcal{S}^{Diff}(S^{4j-1} \times S^{4k}) \to \mathbb{Z},\ [N, f] \mapsto d([N, f]) := \phi_{4k}(i^*_{4k}(\eta^{Diff}([N, f]))).$$

Theorem 7. *For $p, q \geq 2$ and $p + q \geq 5$, the action of Θ_{p+q} on $\mathcal{S}^{Diff}(S^p \times S^q)$ is free unless $p = 4j - 1$ and $q = 4k$ in which case the stabilisers of the action are all subgroups of $bP_{4(j+k)}$ of odd order. The stabilisers of the action of $bP_{4(j+k)}$ are given by*

$$(bP_{4(j+k)})_{[N,f]} = 8\, d([N, f])\, t_{4j}\, t_{4k} \cdot bP_{4(j+k)}.$$

For example: $8\, t_4\, t_4 \cdot bP_8 \cong \mathbb{Z}_7$, $8\, t_4\, t_8 \cdot bP_{12} \cong \mathbb{Z}_{31}$, $8\, t_4\, t_{12} \cdot bP_{16} \cong \mathbb{Z}_{127}$ and $8\, t_8\, t_8 \cdot bP_{16} \cong \mathbb{Z}_{127}$.

The proofs of the above theorems given in [C1] are based on three key points: the smooth surgery obstruction map factors through the topological surgery obstruction map, the product formulae for the surgery obstruction of a product of degree one normal maps and the composition formulae for normal invariants of compositions. The later are worked out in great detail in [R2] where $\mathcal{S}^{Top}(S^p \times S^q)$ is discussed in illuminating detail.

An interesting next step is the classification of smooth manifolds homotopy equivalent to $S^p \times S^q$ via the computation of the action of the group of self-equivalences of $S^p \times S^q$ on $\mathcal{S}^{Diff}(S^p \times S^q)$. We achieve this goal in [C1] by other methods for $S^3 \times S^4$ and $S^4 \times S^4$.

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Aspherical manifolds and boundaries of hyperbolic groups

Arthur Bartels
(joint work with Wolfgang Lück, Shmuel Weinberger)

The *Borel Conjecture* asserts that if a group is the fundamental group of a closed aspherical manifold, then this manifold is unique up to homeomorphism. So in this case the group should determine not only a homotopy type, but also a homeomorphism type.
There is the natural corresponding existence question:

> For which groups G is there a closed aspherical manifold M such that its fundamental group is G?

There are two obvious obstructions to the existence of such a manifold:

- If $G = \pi_1(M)$ is the fundamental group of a closed aspherical manifold, then $M \simeq BG$. Thus there is a finite CW-model for BG. In particular, G is finitely generated, even finitely presented.
- If $G = \pi_1(M)$ is the fundamental group of a closed aspherical manifold, then the homology of G satisfies Poincaré duality. This means that there should be n such that

$$H^k(G; \mathbf{Z}[\mathbf{G}]) \cong \begin{cases} \mathbf{Z} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}.$$

Let now G be a hyperbolic group. Then the Rips complex is a cocompact model for the classifying space for proper actions. In particular, if G is in addition torsion free, then the quotient of the Rips complex by the action of G is a finite CW-model for BG. For the second condition, the boundary ∂G of G is important. There is the following result, due to Bestvina and Mess [1]:

Theorem. *Let G be a torsion-free hyperbolic group. Then G is a Poincaré duality group if and only if the boundary of G has the Cech cohomology of a sphere.*

The following was the main result of the talk:

Theorem. *Let G be a torsion-free hyperbolic group. If $\partial G \cong S^{n-1}$, $n \geq 6$, then there is an n-dimensional closed aspherical manifold M such that $\pi_1(M) = G$.*

The proof relies on the Farrell-Jones conjecture for hyperbolic groups, the total surgery obstruction, the Quinn resolution obstruction, the surgery theory of *ANR*-homology manifolds and the above mentioned work of Bestvina and Mess on the boundaries of hyperbolic groups.

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An infinite Smith group

Ian J. Leary
(joint work with G. Arzhantseva, M. R. Bridson, T. Januszkiewicz, P. H. Kropholler, A. Minasyan, J. Świątkowski)

We say that a discrete group G is a *Smith group* if for every G-CW-complex X such that $\dim(X) < \infty$ and X is contractible, we have that the fixed point set X^G is non-empty.

It is easy to see that any quotient of a Smith group is a Smith group. If G is a strictly ascending countable union of subgroups G_i, then there is a G-tree (with vertex set $\coprod_i G/G_i$, and one edge from gG_i to gG_{i+1} for each $g \in G$ and i) which shows that G cannot be a Smith group. In particular, any countable Smith group must be finitely generated.

A theorem of P. A. Smith implies that every finite group of prime power order is a Smith group [8]. On the other hand, the Conner-Floyd examples [2] and generalizations (which the author learned from R. Oliver, see for example [7]) show that there are no other finite Smith groups. In the early 1990's, P. H. Kropholler asked if there is an infinite Smith group. He was interested in this question because of implications for the class $\mathbf{H}\mathfrak{F}$ of groups [6]. A homological argument shows that the Thompson group F cannot be in $\mathbf{H}\mathfrak{F}$, and hence neither can any group containing F, but this was the only method known to show that groups do not lie in $\mathbf{H}\mathfrak{F}$. It follows easily from the definition that an infinite Smith group cannot lie in $\mathbf{H}\mathfrak{F}$.

The following is the main result from [1]. In the statement 'mod-p acyclic' means 'having the same Cech cohomology with mod-p coefficients as a point'.

Theorem 1. *There is an infinite group Q such that if X is any Hausdorff space of finite covering dimension, then for any action of Q on X and for any prime p such that X is mod-p acyclic, the fixed point set X^Q is also mod-p acyclic.*

The group Q can be chosen to be a directed colimit of word hyperbolic groups. Alternatively, the group Q can be chosen to be simple, to have property T, and to contain any specified countable group C as a subgroup.

If p and q are distinct primes, then one can construct an action of a cyclic group of order p on a mod-q acyclic 2-dimensional simplicial complex without a global fixed point. Hence the group Q in the statement has a stronger fixed point property than any (non-trivial) finite group.

On the other hand, finite p-groups have a fixed point property for arbitrary actions on compact Hausdorff mod-p acyclic spaces, whereas the group Q does not. The group Q can certainly be chosen to be non-amenable. The space of all finitely-additive probability measures on Q is a compact Hausdorff contractible space with a Q-action that (by definition of amenability) has no fixed point.

The theorem is proved by constructing, for each $n \geq 0$ and for each prime p, a group $G_{n,p}$ which is non-elementary word hyperbolic and has the following (n,p)-generation property: there exists a set $S = \{s_1, \ldots, s_{n+2}\}$ of elements such that

S generates $G_{n,p}$, but any proper subset of S generates a finite p-subgroup. Any group with this (n,p)-generation property has a fixed point theorem for actions on mod-p acyclic spaces of covering dimension at most n. The construction of the groups $G_{n,p}$ uses the theory of systolic groups [5]. The group Q is constructed as a common infinite quotient of the groups $G_{n,p}$, which implies that Q has the (n,p)-generation property for all n and p.

The same techniques can also be used to prove the following result, concerning Kropholler's hierarchy of groups. See [4] or [6] for a definition of the stages of Kropholler's hierarchy. In the statement ω_1 denotes the first uncountable ordinal.

Theorem 2. *For each ordinal $\alpha \leq \omega_1$, the class $\mathbf{H}_\alpha\mathfrak{F}$ is strictly contained in the class $\mathbf{H}_{\alpha+1}\mathfrak{F}$.*

Previously only the cases $\alpha = 0$, 1, and 2 of this Theorem were known. The proof relies on the following Lemma.

Lemma 3. *For any countable group H, any $n \geq 0$ and any prime p, there is a group $H_{n,p}$ which is a quotient of the group $G_{n,p}$ and which contains the group H as a subgroup. If H is in the class $\mathbf{H}\mathfrak{F}$, then so is $H_{n,p}$.*

The proof that each $H_{n,p}$ is in $\mathbf{H}\mathfrak{F}$ uses Dahmani's relative Rips complex [3].

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An Elmendorf theorem for orbifolds

André Henriques

(joint work with David Gepner)

1. Elmendorf's theorem

By way of motivation, let us briefly recall the basics of the homotopy theory of G-spaces. We first fix a family[1] $\mathcal{F}$ of closed subgroups of G, closed under

[1] We do not require that $\mathcal{F}$ be closed under taking subgroups.

conjugation. Although everything we do is relative to $\mathcal{F}$, we generally omit it from the notation.

A G-space is said to be cellular if it can be constructed by successively attaching cells of the form $D^n \times G/H$ for $H \in \mathcal{F}$. And a map $f : M \to N$ is called a weak equivalence is for every $H \in \mathcal{F}$, the induced map between fixed point sets $M^H = \mathrm{map}_G(G/H, M) \to N^H = \mathrm{map}_G(G/H, N)$ is a weak equivalence of spaces.

Definition The *orbit category* Orb_G of the topological group G (we suppress $\mathcal{F}$ from the notation) is the full subcategory of G-spaces on the G-orbits $O = G/H$ for $H \in \mathcal{F}$. An Orb_G-space is a continuous contravariant functor from Orb_G to spaces.

Cellular objects and weak equivalences also exist in the category of Orb_G-spaces. The latter are the maps that induce weak equivalences upon evaluating at any orbit $O \in \mathrm{Orb}_G$.

Elmendorf's Theorem The functor

$$\Phi : \{G\text{-spaces}\} \longrightarrow \{\mathrm{Orb}_G\text{-spaces}\}$$

sending a G-space M to the Orb_G-space $\Phi(M) : O \mapsto \mathrm{map}_G(O, M)$ admits a homotopy inverse

$$\Psi : \{\mathrm{Orb}_G\text{-spaces}\} \longrightarrow \{G\text{-spaces}\}$$

and natural transformations from $\Psi\Phi$ and $\Phi\Psi$ to the respective identity functors that provide weak equivalences

$$\Psi\Phi(M) \xrightarrow{\sim} M, \qquad \Phi\Psi(X) \xrightarrow{\sim} X \tag{1}$$

for all G-spaces M and Orb_G-spaces X. Moreover, if M or X are cellular, then the maps (1) are homotopy equivalences.

The above theorem can be interpreted as giving an explicit equivalence of homotopy theories between the categories of G-spaces and Orb_G-spaces. The original proof [1] restricts to the case in which G is a compact Lie group and $\mathcal{F}$ is the family of all closed subgroups of G. A proof that works in full generality can be found in [2, Section VI.6].

Remark A modern way of rephrasing Elmendorf's theorem is in terms of a Quillen equivalence

$$L : \{\mathrm{Orb}_G\text{-spaces}\} \rightleftarrows \{G\text{-spaces}\} : R.$$

The translation from the previous formulation is given by $\Phi = R$ and $\Psi = L \circ \mathrm{cof}$, where cof is the cofibrant replacement functor in Orb_G-spaces. The existence of a natural transformation $\Phi\Psi \to 1$ is then due to the exceptional fact that the unit map $X \to RL(X)$ is an isomorphism whenever X is cofibrant.

2. Statement of results

Fix a family $\mathcal{F}$ of allowed isotropy groups, by which we mean an essentially small class of topological groups closed under isomorphisms, and subject to a mild paracompactness condition. Once again, although everything is done relative to $\mathcal{F}$, we shall generally omit it from the notation. Given a group G, let $\mathbb{B}G := (G \rightrightarrows *)$ denote the topological groupoid with trivial object space $*$, arrow space G, and composition given by multiplication in G.

Definition A map of stacks $f : \mathscr{X} \to \mathscr{Y}$ is called *faithful* if for every point p of $\mathscr{X}$, the map $\mathrm{aut}(p) \to \mathrm{aut}(f(p))$ is a closed inclusion.

Let $\mathbb{M}\mathrm{ap}^{\mathrm{rep}}(\mathbb{B}H, \mathbb{B}G)$ be the topological groupoid whose objects are the maps $\mathbb{B}H \to \mathbb{B}G$ that induce closed inclusions $H \hookrightarrow G$. We then let $\mathrm{Orb}^{\mathrm{rep}}$ denote the topologically enriched category whose objects are the groups in $\mathcal{F}$, and whose hom-spaces are the fat geometric realizations of the above groupoids:

$$\mathrm{Orb}^{\mathrm{rep}}(H, G) := \| \mathbb{M}\mathrm{ap}^{\mathrm{rep}}(\mathbb{B}H, \mathbb{B}G) \|. \tag{2}$$

The category $\mathrm{Orb}^{\mathrm{rep}}$ is a topological version of the category of orbit stacks $\mathscr{M}_G$ and faithful maps[2] between them. Indeed, the map $\mathrm{Orb}^{\mathrm{rep}}(H, G) \to \mathrm{map}^{\mathrm{rep}}(\mathscr{M}_H, \mathscr{M}_G)$ is a weak equivalence. We also introduce a category $\mathrm{Orb}^{\mathrm{full}}$ with morphisms

$$\mathrm{Orb}^{\mathrm{full}}(H, G) := \big\| \mathbb{M}\mathrm{ap}^{\mathrm{full}}(\mathbb{B}H, \mathbb{B}G) \big\|,$$

where $\mathbb{M}\mathrm{ap}^{\mathrm{full}}(\mathbb{B}H, \mathbb{B}G)$ is the groupoid of all maps. That category corresponds to the *full* subcategory of stacks on the orbit stacks $\mathscr{M}_G$.

Definition An $\mathrm{Orb}^{\mathrm{rep}}$-space is a continuous contravariant functor from $\mathrm{Orb}^{\mathrm{rep}}$ to spaces. Similarly, an $\mathrm{Orb}^{\mathrm{full}}$-space is a continuous contravariant functor from $\mathrm{Orb}^{\mathrm{full}}$ to spaces.

Our main theorem provides a description of the homotopy theory of cellular stacks (with isotropy groups in $\mathcal{F}$).

Main Theorem *There are equivalences of homotopy theories*

$$\left\{ \begin{matrix} \text{Cellular local quotient stacks,} \\ \text{faithful maps} \end{matrix} \right\} \approx \left\{ \mathrm{Orb}^{\mathrm{rep}}\text{-spaces} \right\}, \tag{3}$$

and

$$\left\{ \begin{matrix} \text{Cellular stacks (not necessarily} \\ \text{local quotients), all maps} \end{matrix} \right\} \approx \left\{ \mathrm{Orb}^{\mathrm{full}}\text{-spaces} \right\}. \tag{4}$$

Making sense of the above statement requires a considerable amount of extra structure on our categories. Namely, we need notions of *weak equivalence*, *fibrant object* and *cofibrant object*. Now let f be a functor between such categories, which

[2]We use the superscript "rep" because *faithful* maps are essentially the same thing as *representable* maps. For example, they agree on cellular local quotient stacks with Lie group stabilizers.

respects the extra structure in the sense that it sends weak equivalences to weak equivalences, fibrant objects to fibrant objects, and cofibrant objects to cofibrant objects. We say that f is an *equivalence of homotopy theories* if it is:

• *Homotopically fully faithful:* For all pairs of fibrant and cofibrant objects X and Y in the source category, the induced map on derived mapping spaces $\mathrm{map}(Y,X) \to \mathrm{map}\big(f(Y), f(X)\big)$ is a weak homotopy equivalence.

• *Homotopically essentially surjective:* For any fibrant and cofibrant object X of the target category there is a fibrant and cofibrant object Y of the source category and a weak equivalence between $f(Y)$ and X.

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Uniform embeddings of groups into a Hilbert space

Goulnara N. Arzhantseva

Let (G,d) be a metric space. Typically, G is a countable group endowed with a proper left invariant metric. For example, G is finitely generated and d is the word length metric associated to a finite generating set of G. Let $\mathcal{H}$ be a separable Hilbert space.

Definition 1 (Gromov' 1995)**.** A map $f\colon G \to \mathcal{H}$ is said to be a *uniform embedding* (also called a *coarse embedding*) if for $g_n, h_n \in G$, $n \in \mathbb{N}$,

$$d(g_n, h_n) \to \infty \ \text{ if and only if } \ \|f(g_n) - f(h_n)\| \to \infty.$$

Examples. We write, for brevity, UE for a uniform embedding or a uniformly embeddable group.

Free groups are UE. Define a map f as follows:

$$\mathbb{F}_k \ni g \xmapsto{f} \delta_{g[1]} + \cdots + \delta_{g[t]} \in \mathcal{H} = \ell^2(\text{edges}),$$

where $g[1], \ldots, g[t]$ are the edges of the unique geodesic path from g to 1 in the Cayley graph of the free group $\mathbb{F}_k$ of rank k and $\delta_{g[i]}$ is the characteristic function of the i-th edge. Obviously, $\|f(g) - f(h)\| = \sqrt{d(g,h)}$ for any $g, h \in G$. Hence, f is a uniform embedding.

Equivariant UE. If G acts properly by isometries on $\mathcal{H}$ then the map $f(g) := g \cdot 0$ is a uniform embedding. Thus, all a-T-menable groups (also known as groups with the Haagerup property) are uniformly embeddable. In particular, all amenable groups are uniformly embeddable.

Weakly amenable groups are UE. All groups with Guoliang Yu's property A are uniformly embeddable [6].

Non-UE groups. There are finitely generated groups with no uniform embeddings into a Hilbert space. The existence of such groups is shown by means of the probabilistic method through an appropriate notion of random groups, see [4] and [3] for details.

The study of uniformly embeddable groups is highly motivated by a spectacular result of Yu [6]: a finitely generated group G uniformly embeddable into a Hilbert space satisfies the Novikov conjecture on homotopy invariance of higher signatures.

The following new invariant describes how close any uniform embedding of the group into a Hilbert space can be to a quasi-isometry.

Definition 2. The *Hilbert space compression value* of G is defined to be the number

$$R(G) = \sup_f \{\alpha \geqslant 0 \mid d(g,h)^\alpha \leqslant \|f(g) - f(h)\|\},$$

where the supremum is taken over all 1-Lipschitz maps $f\colon G \to \mathcal{H}$.

In case G acts by isometries on $\mathcal{H}$, we can restrict to equivariant 1-Lipschitz maps $f\colon G \to \mathcal{H}$ in the previous definition. This leads to the notion of the *equivariant* Hilbert space compression value, denoted by $R_G(G)$.

Remarks.

- Obviously, $0 \leqslant R_G(G) \leqslant R(G) \leqslant 1$;
- By definition, $R(G) > 0$ implies that G is uniformly embeddable;
- If $R_G(G) > 0$ then G acts properly by isometries on $\mathcal{H}$, i.e. G is a-T-menable;
- If G is amenable then $R_G(G) = R(G)$ (Gromov' 2005);
- If $R(G) > 1/2$ then G is weakly amenable (Guentner & Kaminker' 2004);
- If $R_G(G) > 1/2$ then G is amenable (Guentner & Kaminker' 2004).

Many groups are known to have Hilbert space compression value 1. This holds for free abelian groups, free groups, word hyperbolic groups, for any group acting properly and co-compactly on a finite dimensional CAT(0)-cubical complex, for co-compact lattices in arbitrary Lie groups, and all lattices in semi-simple Lie groups, see, for example, [2] for bibliography. The already mentioned groups that are not uniformly embeddable into a Hilbert space have compression 0.

In [1], we give first examples of finitely generated groups having an intermediate, with values in $(0,1)$, Hilbert space compression values. In particular, we show

Theorem 3 (Arzhantseva, Guba, Sapir' 2006)**.**

(1) *For Richard Thompson's group F we have $R(F) = R_F(F) = 1/2$.*

(2) *Let H be a finitely generated group with fixed generating set. Suppose that its growth function satisfies the condition $\gamma(n) \succeq n^k$, for some $k > 0$. Then*

$$R(\mathbb{Z} \wr H) \leqslant \frac{1 + k/2}{1 + k}.$$

Kasparov and Yu proved the Novikov conjecture for groups uniformly embeddable into a uniformly convex Banach space, see [5]. This motivates our study of uniformly convex Banach space compression values and functions, see [2].

Theorem 4 (Arzhantseva, Druţu, Sapir' 2007). *For every $\alpha \in [0,1]$ there exists a finitely generated group G_α of asymptotic dimension at most 2, and such that $R(G_\alpha) = \alpha =$ "uniformly convex Banach space compression value".*

In particular, we provide the first example of a finitely generated group G such that

- G is uniformly embeddable into a Hilbert space but with compression 0.
- G is weakly amenable and the Hilbert compression value is 0.
- G has the uniformly convex Banach space compression 0.

Open problems.
(i) What is the compression of Grigorchuk's group of intermediate growth?
(ii) Is there an amenable group of Hilbert space compression value $< 1/2$?
(iii) Is any uniformly embeddable group weakly amenable?
(iv) Does there exist a finitely generated group with no uniform embedding into any uniformly convex Banach space?

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The stable free rank of symmetry of products of spheres

Bernhard Hanke

Conjecture 1 (Adem-Browder, 1988). *Let p be a prime. If $(\mathbb{Z}/p)^r$ acts freely on $S^{n_1} \times S^{n_2} \times \ldots \times S^{n_k}$, then r is less than or equal to k.*

Classical Smith theory implies this statement for $k = 1$. The case $k = 2$ was solved by Heller in 1959. Furthermore, it is known that this conjecture holds for $n_1 = n_2 = \ldots = n_k$ (Carlsson, 1982; Adem-Browder, 1988).

In the following we abbreviate the product of spheres by X.

Our main result verifies a stable version of Conjecture 1.

Theorem 2 (H., 2009)**.** *If p is a prime, $p > 3\dim X$ and $(\mathbb{Z}/p)^r$ acts freely on X, then r is less than or equal to the number of odd spheres in X.*

This result implies that the maximal rank of an elementary abelian p-group acting freely on X is equal to the number of odd spheres provided that $p > 3\dim X$.

Theorem 2 implies the following well known estimate of the *free toral rank* of X.

Theorem 3 (Halperin, 1977)**.** *If $(S^1)^r$ acts freely on X, then r is less than or equal to the number of odd spheres in X.*

However, even for very large primes Theorem 2 cannot be deduced from this result by some sort of limiting process: In 1999 Browder constructed free actions of $(\mathbb{Z}/p)^r$ on $(S^m)^k$ for each odd $m \geq 3$ and $k \geq r$, $p > km/2$, which are exotic in the sense that they cannot be extended to $(S^1)^r$-actions.

Our proof of Theorem 2 is based on the observation that in spite of Browder's result the cohomology theory of $(\mathbb{Z}/p)^r$-actions for large p bears some similarities to the cohomology theory of $(S^1)^r$-actions.

Let us first sketch a proof of Theorem 3.

Assume $G = (S^1)^r$ acts freely on X. Then the Borel space $X_G = EG \times_G X$ is homotopy equivalent to the orbit space X/G. This is a closed manifold and consequently $H^*(X_G;\mathbb{Q})$ is finite dimensional over $\mathbb{Q}$.

Rational homotopy theory in its description via differential forms due to Sullivan (1977) yields a commutative cochain algebra (E^*, d) over $\mathbb{Q}$ whose cohomology is isomorphic to $H^*(X_G;\mathbb{Q})$. We have an algebra isomorphism

$$E^* \cong \mathbb{Q}[t_1, \ldots, t_r] \otimes M^*$$

where each t_i is of degree 2 and M^* is a cochain algebra freely generated by one odd generator for each odd sphere and one even and one odd generator for each even sphere in X. There is a short exact sequence of cochain algebras

$$\mathbb{Q}[t_1, \ldots, t_r] \to E^* \to M^*$$

with trivial differential on $\mathbb{Q}[t_1, \ldots, t_r]$, which in some sense models the Borel fibration $X \hookrightarrow X_G \to BG$. In particular, M^* is a minimal model of X.

One now argues that $H^*(X_G;\mathbb{Q})$ can only be finite dimensional, if the number of even generators in E^* is not larger then the number of odd generators. For this objective one deforms the differential d so that all even generators become cocycles and the cohomology of E^* is still finite dimensional. The estimate in Theorem 3 now follows from Krull's principal ideal theorem applied to the ideal generated by the differentials of odd generators of E^*.

Rational cochain complexes cannot be used for a proof of Theorem 2, because the rational homotopy type of BG is trivial for $G = (\mathbb{Z}/p)^r$.

Instead of rational homotopy theory *tame homotopy* is a more promising approach. This theory was invented by Dwyer (1979) and is modelled on Quillen's rational homotopy theory (1969), but without immediately losing p-torsion information for all primes p.

For our purpose we need a description of tame homotopy theory via differential forms. Our aim is to replace the rational cochain algebra used in the proof of Theorem 3 by a small commutative cochain algebra E^* over $\mathbb{F}_p$ which calculates $H^*(X_G; \mathbb{F}_p)$ when $G = (\mathbb{Z}/p)^r$ is acting on X. For odd p one might hope that E^* is of the form

$$E^* \cong \mathbb{F}_p[t_1, \ldots, t_r] \otimes \Lambda(s_1, \ldots, s_r) \otimes M^*$$

where M^* is a free graded algebra over $\mathbb{F}_p$ with generators as before and $s_1, \ldots, s_r$ are generators in degree 1 (corresponding to generators of $H^1(BG; \mathbb{F}_p)$).

One can show that this is impossible in general. However, we have the following approximative result.

Theorem 4 (H., 2009)**.** *Let $p > 3 \dim X$ and let $G = (\mathbb{Z}/p)^r$ act freely on X. Then there is an $\mathbb{F}_p$-cochain algebra E^* of the described form which calculates the $\mathbb{F}_p$-cohomology of X_G up to degree* $\dim X + 1$*. Moreover and more importantly, the cohomology of $H^*(E)$ vanishes in degrees* $\dim X + 1$ *and* $\dim X + 2$*.*

One can now prove that the cohomology of $E^*/(s_1, \ldots, s_r)$ is again finite dimensional. The argument starts with the observation that all t_i represent nilpotent cohomology classes in $H^*(E)$ due to the vanishing of $H^*(E)$ in two consecutive degrees. Once the finite dimensionality of $H^*(E/(s_1, \ldots, s_r))$ has been established, Theorem 2 can be discussed in a similar way as Theorem 3.

The proof of Theorem 4 uses a tame analogue of Sullivan's construction of minimal models. This is based on work of Cenkl-Porter (1984) and of Sörensen (1992).

We believe that our methods are not sufficient to establish the general form of Conjecture 1 for small primes. This problem remains open.

More details are available in my paper *The stable free rank of symmetry of products of spheres*, Invent. Math. (published online 5 May 2009).

Topological invariants of orbifolds

Tara Holm

(joint work with Rebecca Goldin, Megumi Harada)

The aim of this talk was to present work in a nearly complete manuscript [5]. During the course of the workshop, my coauthor Megumi Harada and I made progress improving the results in this manuscript, and this abstract reflects those changes. We are grateful to the Oberwolfach Mathematics Institute for the opportunity to make progress on this project, and to discuss it with the other topologists at the workshop. The new and improved version of the manuscript will appear on the arXiv shortly.

The notion of an orbifold has been present in topology since the 1950s [12, 13]. More recently, orbifolds have played an important role in differential and algebraic geometry, and in mathematical physics. A fundamental theme is to compute topological invariants associated to an orbifold. Initially these invariants were defined in a similar manner to manifold invariants, taking into account the orders of the local isotropy groups. More recently, there has been a flurry of work defining and computing stringy versions of these invariants, first introduced in the seminal work of Dixon, Harvey, Vafa, and Witten [3]. These authors discovered that in order to capture the contributions of the orbifold singularities when defining an associated conformal field theory, it was necessary to add extra factors, corresponding to the so-called twisted sectors. Examples of orbifold invariants that take into account these twisted sectors include the orbifold cohomology defined by Chen and Ruan [2] and the full orbifold K-theory introduced by Jarvis, Kaufmann, and Kimura [9]. The monograph [1] is a useful reference for these stringy invariants.

In the symplectic category, orbifolds arise as symplectic quotients. The symmetries of a symplectic manifold M may be encoded as a Hamiltonian group action by a compact Lie group G. The moment map is a G-equivariant map $\Phi : M \to \mathfrak{g}^*$. When α is a central regular value, the level set $\Phi^{-1}(\alpha)$ is a G-invariant submanifold of M. Moreover, the action of G on a regular level set is **locally free**: it has only finite stabilizers. Thus, at a regular value the symplectic reduction $M/\!/_\alpha G = \Phi^{-1}(\alpha)/G$ is an orbifold. In fact, Marsden and Weinstein proved that the symplectic form descends to a symplectic structure on the quotient $M/\!/_\alpha G$.

Standard techniques from equivariant symplectic geometry may be used to compute topological invariants of smooth symplectic quotients. These are described in Kirwan's seminal work [10] for cohomological invariants, and in [8] for K-theory. When the Lie group is a compact abelian Lie group T that acts with finite stabilizers, we may adapt these techniques to stringy invariants [7, 4, 6]. We employ techniques coming from algebraic topology, most notably equivariant cohomology and equivariant K-theory. This allows us to compute effectively the orbifold cohomology and the full orbifold K-theory of abelian symplectic quotients. Our most recent work along these lines uses the moment map to study the topology of the level set $\Phi^{-1}(\alpha)$ itself.

We now set our notation. Let $H \subseteq T$ be a closed Lie subgroup (i.e. a subtorus). Let $\Phi_H : M \to \mathfrak{h}^*$ be the induced H-moment map obtained as the composition of $\Phi : M \to \mathfrak{t}^*$ with the linear projection $\mathfrak{t}^* \to \mathfrak{h}^*$. Suppose $\alpha \in \mathfrak{h}^*$ is a regular value of Φ_H, and let $Z := \Phi_H^{-1}(\alpha) \subseteq M$ denote the corresponding level set. Since α is a regular value, Z is a smooth submanifold of M. Finally, let ξ be a **generic** element in $\mathfrak{t}$: one whose exponential generates the entire group T. Recall that $\Phi^\xi := \langle \Phi, \xi \rangle$ denotes the corresponding component of the moment map. The technical lemma that we need is the following.

Lemma 1. *Let $Z := \Phi_H^{-1}(\alpha)$ be a level set of the moment map for the H-action at a regular value. The map*

$$f := \Phi^\xi|_Z : Z \to \mathbb{R}$$

is a Morse-Bott function on Z.

The key to proving this is a local normal form result that describes a neighborhood and the moment map at any point $p \in M$. This lemma allows us to prove that there is no additive torsion in the integral full orbifold K-theory, when we have control over the critical set of the function f. Let

$$\mathfrak{X} := [M//_\alpha H] = [Z/H] \tag{2}$$

denote the quotient stack associated to the locally free H-action on Z. This is an orbifold, i.e. a Deligne-Mumford stack in the differentiable category.

Theorem 3. *Let $\mathfrak{X}$ be an orbifold constructed as in (2). Suppose that there exists $\xi \in \mathfrak{t}$ such that $f = \Phi^\xi|_Z$ is proper and bounded below. If the H-equivariant K-theory of $Crit(f)$ is concentrated in K^0, and is free of additive torsion in K^0, then the integral full orbifold K-theory $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ of $\mathfrak{X}$ contains no additive torsion.*

The power of this theorem is the class of examples to which it applies. We initially proved this theorem only for toric orbifolds arising via the Delzant construction [11] in equivariant symplectic geometry. This class includes weighted projective spaces.

While at Oberwolfach, we showed that this result also applies to a large collection of symplectic reductions of homogeneous spaces. Two examples are shown in the figure below. Figure 1 shows a circle reduction of $\mathcal{F}\ell ags(\mathbb{C}^3)$ that has $\mathbb{Z}/2\mathbb{Z}$

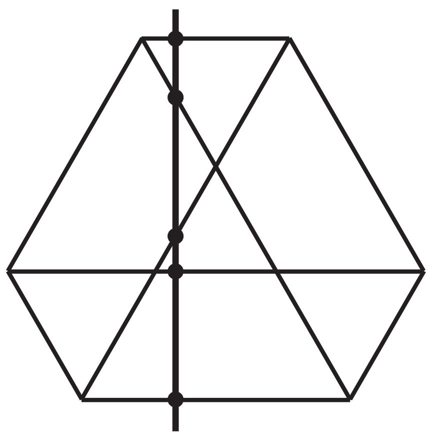

Figure 1. $\mathcal{F}\ell ags(\mathbb{C}^3)//S^1$

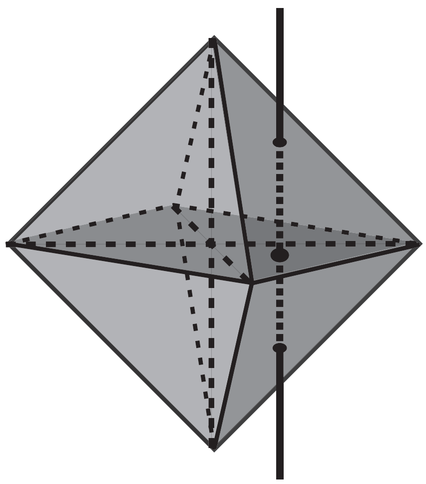

Figure 2. $\mathcal{G}r_2^+(\mathbb{R}^7)//T^2$

singularities. The intersection of the vertical line with the moment polytope represents the moment polytope for the symplectic reduction. It is easy to see in this example that the Morse-Bott function has isolated S^1-orbits as its critical sets. Those critical sets correspond to the points where the vertical line intersects the edges of the full polytope, and are indicated by the dots. Those dots that occur at the intersection of the vertical line with a horizontal edge in the full polytope have a $\mathbb{Z}/2\mathbb{Z}$ stabilizer.

Figure 2 indicates a T^2 reduction of $\mathcal{G}r_2^+(\mathbb{R}^7)$ that also has $\mathbb{Z}/2\mathbb{Z}$ singularities. In this example, careful analysis of the critical sets shows that there are two isolated T^2 orbits, where the vertical line intersects an exterior wall of the full polytope, and one critical set whose quotient is a weighted projective space $\mathbb{C}P^1_{1,2}$, where the vertical line intersects the horizontal plane inside the polytope. Since our main theorem applies to weighted projective spaces, the full orbifold K-theory of $\mathcal{G}r_2^+(\mathbb{R}^7)//T^2$ is still torsion free.

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Adams operations in K-theory and secondary index theorems

Ulrich Bunke

(joint work with Thomas Schick)

We consider a generalized cohomology theory and define the $\mathbb{Z}$-graded $\mathbb{R}$-vector space $\mathtt{V} := E \otimes_{\mathbb{Z}} \mathbb{R}$. For a smooth manifold M we define $\Omega^*(M, \mathtt{V}) := C^\infty(M, \Lambda^* T^*M \otimes_{\mathbb{R}} \mathtt{V})$ with the $\mathbb{Z}$-grading by the total degree. We let $d : \Omega^*(M, \mathtt{V}) \to \Omega^{*+1}(M, \mathtt{V})$ be the de Rham differential, and we write $\Omega^*_{cl}(M, \mathtt{V}) := \ker(d : \Omega^*(M, \mathtt{V}) \to \Omega^{*+1}(M, \mathtt{V}))$ for the subspace of closed forms. We identify $H^*(M; \mathtt{V})$ with the singular cohomology of M with coefficients in $\mathtt{V}$. Integration over simplices induces a natural transformation

$$\mathtt{Rham} : \Omega^*_{cl}(M, \mathtt{V}) \to H^*(M; \mathtt{V}) \ .$$

There is a canonical natural transformation of cohomology theories

$$\mathbf{ch} : E^*(X) \to H^*(X; \mathtt{V}) \ .$$

Definition 1. *A **smooth extension** of the generalized cohomology theory E is a quadruple $(\hat{E}, R, I, a)$, where*

A.1 *$\hat{E}$ is a contravariant functor from the category of smooth manifolds to $\mathbb{Z}$-graded abelian groups.*

A.2 *R is a natural transformation of $\mathbb{Z}$-graded abelian group-valued functors*

$$R : \hat{E}^*(M) \to \Omega^*_{cl}(M, \mathtt{V}) \ .$$

A.3 *I is a natural transformation of $\mathbb{Z}$-graded abelian group-valued functors*

$$I : \hat{E}^*(M) \to E^*(M) \ .$$

A.4 *a is a natural transformation of $\mathbb{Z}$-graded abelian group-valued functors*

$$a : \Omega^{*-1}(M, \mathtt{V})/\mathtt{im}(d) \to \hat{E}^*(M) \ .$$

These objects have to satisfy the following relations:

R.1 $R \circ a = d$

R.2 *For all manifolds M the diagram*

$$\begin{tikzcd} \hat{E}^*(M) \arrow[r, "R"] \arrow[d, "I"] & \Omega^*_{cl}(M, \mathtt{V}) \arrow[d, "\mathtt{Rham}"] \\ E^*(M) \arrow[r, "\mathbf{ch}"] & H^*(M, \mathtt{V}) \end{tikzcd}$$

commutes.

R.3 *For all manifolds M the sequence*

$$E^{*-1}(M) \stackrel{\mathbf{ch}}{\to} \Omega^{*-1}(M)/\mathtt{im}(d) \stackrel{a}{\to} \hat{E}^*(M) \stackrel{I}{\to} E^*(M) \to 0 \qquad (2)$$

is exact.

We introduce the notation $SF(M) := F(S^1 \times M)$ for a functor F defined on manifolds. There are integration maps $\int : SE^{*+1} \to E^*$ and $\int : S\Omega^{*+1}(\ldots, \mathtt{V}) \to \Omega^*(\ldots, \mathtt{V})$.

Definition 3. *A smooth extension **with integration** of E is a quintuple $(\hat{E}, R, I, a, \int)$, where $(\hat{E}, R, I, a)$ is a smooth extension of E and $\int$ is a natural transformation*

$$\int : S\hat{E}^{*+1} \to \hat{E}^*$$

such that

(1) *$\int \circ (t^* \times \mathtt{id})^* = -\int$, where $t : S^1 \to S^1$ is given by $t(z) := \bar{z}$.*

(2) *$\int \circ p^* = 0$, where $p : S^1 \times M \to M$ is the projection, and*

(3) *the diagram*

$$\begin{tikzcd}
S\Omega^*(M,\mathtt{V}) \arrow[r, "a"] \arrow[d, "\int"] & S\hat{E}^{*+1}(M) \arrow[r, "I"] \arrow[d, "\int"] \arrow[rr, bend left, "R"] & SE^{*+1}(M) \arrow[d, "\int"] & S\Omega^{*+1}_{cl}(M,E) \arrow[d, "\int"] \\
\Omega^{*-1}(M,\mathtt{V}) \arrow[r, "a"] & \hat{E}^*(M) \arrow[r, "I"] \arrow[rr, bend right, "R"'] & E^*(M) & \Omega^*_{cl}(M,E)
\end{tikzcd}$$

commutes for all manifolds M.

Assume now that E is a multiplicative cohomology theory. Then the functor E^* has values in graded commutative rings. In particular, $E^* := E^*(*)$ is a $\mathbb{Z}$-graded ring, and $H^*(M;\mathtt{V})$ and $\Omega^*(M,\mathtt{V})$ are $\mathbb{Z}$-graded rings as well. In this case we can make the following definition.

Definition 4. *A* ***multiplicative smooth extension*** *is a smooth extension $(\hat{E},R,I,a)$ such that $\hat{E}^*$ takes values in $\mathbb{Z}$-graded commutative rings, the transformations R and I are multiplicative, and the identity*

$$x \cup a(\alpha) = a(R(x)\wedge\alpha) \quad , \quad x \in \hat{E}^*(M)\ , \alpha \in \Omega^*(M,\mathtt{V})/\mathtt{im}(d)$$

holds true.

There is an obvious notion of a natural transformation between two smooth extensions with integration $(\hat{E},R,I,a,\int)$ and $(\hat{E}',R',I',a',\int')$ of E.

Theorem 5 (**Uniqueness**)**.** *Let E be a generalized cohomology theory. We assume that E^* is countably generated and rationally even. If $(\hat{E},R,I,a,\int)$ and $(\hat{E}',R',I',a',\int')$ are two smooth extensions with integration, then they are isomorphic by a unique natural isomorphism. If E and the extensions are in addition multiplicative, then the isomorphism preserves the ring structures.*

The flat theory

$$\hat{E}^*_{flat}(M) := \ker\Big(R : \hat{E}^*(M) \to \Omega^*_{cl}(M,\mathtt{V})\Big)$$

is a homotopy invariant functor on smooth manifolds with values $\mathbb{Z}$-graded abelian groups.

Theorem 6 (**the flat theory**)**.** *Assume that E is as in Theorem 5. If $(\hat{E},R,I,a,\int)$ is a smooth extension of E with integration, then there exists a canonical isomorphism*

$$\hat{E}^*_{flat} \xrightarrow{\sim} E\mathbb{R}/\mathbb{Z}^{*-1} \ .$$

More details can be found in [BS09].

By these theorems there is a unique multiplicative smooth extension of complex K-theory, see [BS07] for existence and further structures. We now fix $k \in$

$\{-1, 2, 3, \dots\}$ and consider the Adams operation $\Psi^k : K[\frac{1}{k}] \to K[\frac{1}{k}]$ on complex K-theory.

Theorem 7 (**Lift of Adams operations**, [Bun])**.** *There exists a unique lift of the Adams operation to a natural transformation $\hat{\Psi}^k : \hat{K}(\dots)[\frac{1}{k}] \to \hat{K}(\dots)[\frac{1}{k}]$ of functors on the category of compact manifolds.*

Theorem 8 (**Cannibalistic class**, [Bun])**.** *If $\pi : E \to B$ is a submersion with compact E which is smoothly K-oriented by o_π, then there exists a natural smooth refinement $\hat{\rho}^k(o_\pi) \in \hat{K}^0(E)[\frac{1}{k}]$ of the cannibalistic class $\rho^k(T^v\pi)^{-1} \in K^0(E)[\frac{1}{k}]^*$.*

Theorem 9 (**Riemann-Roch**, [Bun])**.** *If $\pi : E \to B$ is a smoothly K-oriented proper submersion over a compact base, then the following diagram commutes:*

$$\begin{tikzcd} \hat{K}^*(E)[\frac{1}{k}] \arrow[r, "\hat{\Psi}^k"] \arrow[d, "\hat{\pi}_!^K"] & \hat{K}^*(E)[\frac{1}{k}] \arrow[d, "\hat{\pi}_!^K(\hat{\rho}^k(o_\pi)\cup\dots)"] \\ \hat{K}^{*-n}(B)[\frac{1}{k}] \arrow[r, "\hat{\Psi}^k"] & \hat{K}^{*-n}(B)[\frac{1}{k}] \end{tikzcd} \quad . \tag{10}$$

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Insanely twisted rabbits

Laurent Bartholdi

Dynamical systems are concerned with the study of a map $f : X \to X$ under iteration; and, in particular, in its asymptotic behaviour. To specialize a little, I will concentrate on X a manifold, and f a branched covering; i.e. a map that is locally a covering, except at a *branch locus* where its behaviour is locally the quotient map under a finite group.

Even more specifically, consider X a 2-dimensional sphere, with isolated branch points where the local behaviour, in a complex chart, is $z \mapsto z^n$. Fatou already recognized the relevance to the asymptotic behaviour of f of the *forward orbit* of critical points [3]; set then

$$P_f = \bigcup_{n\geq 1} f^n(\text{critical points}).$$

If P_f is finite, then f is called a *Thurston map*. For example, identify S^2 with $\mathbb{P}^1(\mathbb{C})$; then $f(z) = z^2 - 1$ is a Thurston map, with $P_f = \{0, -1, \infty\}$.

One wishes to understand "isotopy+conjugacy" classes of Thurston maps, or more precisely classes under *combinatorial equivalence*: two maps f, g are equivalent if there exist $\phi_0, \phi_1 \in \mathrm{Homeo}^+(S^2)$, isotopic rel P_f, such that

$$\begin{tikzcd} (S^2, P_f) \arrow[r, "\phi_0"] \arrow[d, "f"'] & (S^2, P_g) \arrow[d, "g"] \\ (S^2, P_f) \arrow[r, "\phi_1"'] & (S^2, P_g). \end{tikzcd}$$

Thurston gives a combinatorial criterion for combinatorial equivalence of a Thurston map to a rational map:

Theorem 1 (Thurston, [2]). *Let f be a Thurston map, and assume that $S^2 \setminus P_f$ is hyperbolic as an orbispace*[1]*. Then f is equivalent to a rational map if and only if f admits no "Thurston obstruction".*

In that case, furthermore, the rational map is unique up to conjugation by $PSL_2(\mathbb{C})$.

A Thurston obstruction is a collection $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of simple curves in $S^2 \setminus P_f$, non-peripheral (i.e. surrounding at least two post-critical points) and mutually non-homotopic, such that the operator $f^* : \mathbb{Q}\Gamma \to \mathbb{Q}\Gamma$,

$$f^*(\gamma_i) = \sum_{\delta \text{ component of } f^{-1}(\gamma_i) \in \Gamma} \frac{1}{\deg(f : \delta \to \gamma_i)} \delta$$

has spectral radius ≥ 1.

1. Insanely twisted rabbits

It is, *a priori*, difficult to apply this criterion in concrete situations. Douady and Hubbard ask for instance the following question. Consider the set $\mathscr{R}$ of Thurston maps with one fixed critical point, and one critical cycle of length 3. It is easy to see that no such map is obstructed, and therefore all combinatorial equivalence classes admit a unique complex representative. There are then three possibilities, the "rabbit" polynomial $f_R(z) \approx 1 + (-0.1226 + 0.7449i)z^2$, the "corabbit" $f_C \approx 1 + (-0.1226 - 0.7449i)z^2$ and the "airplane" $f_A \approx 1 - 1.7549z^2$.

Consider now a Dehn twist T of $\mathbb{C}$ around the two non-critical values of the f_R-orbit of 0. The map $T^m f_R$ is again a branched covering, and belongs to $\mathscr{R}$; therefore it is equivalent to precisely one of f_R, f_C, f_A. Which one?

This question was asked by J. Hubbard, see [6] and remained open for long. We prove:

Theorem 2 (☺ & Nekrashevych, [1]). *Write m in base 4, as $m = \sum_{i=0}^{\infty} m_i 4^i$ with $m_i \in \{0,1,2,3\}$ and almost all $m_i = 0$ if m is non-negative, and almost all $m_i = 3$ if m is negative.*

[1]e.g. if $\#P_f \geq 5$. More generally, for $x \in S^2$ let n_x denote the supremum of $\{\gcd_{f^n(y)=x} \deg(f^n : y \to x)\}$; then the orbispace is hyperbolic if and only if $2 - \sum_x (1 - \frac{1}{n_x}) < 0$.

If at least one of the m_i is 1 or 2, then $T^m f_R$ is equivalent to f_A. Otherwise, it is equivalent to f_R if m is non-negative, and to f_C if m is negative.

2. Bisets

The idea of the proof is the following: one constructs an algebraic encoding of the branched covering. Consider more generally a Thurston map f, with post-critical set P_f. Choose a basepoint $* \in S^2 \setminus P_f$. Set $G = \pi_1(S^2 \setminus P_f, *)$, and let $\mathscr{F}$ denote homotopy classes of paths in $S^2 \setminus P_f$ starting at $*$ and ending at some element of $f^{-1}(*)$.

The set $\mathscr{F}$ naturally admits two commuting actions of G: the right action is obtained by prepending to $e \in \mathscr{F}$ a loop based at $*$; while the left action is obtained by appending to $e \in \mathscr{F}$, ending at $z \in f^{-1}(*)$, the unique f-lift starting at z of a loop based at $*$.

Say furthermore that two bisets ${}_F\mathscr{F}_F$ and ${}_G\mathscr{G}_G$ are isomorphic if there exists an isomorphism $\phi : F \to G$ and a bijection $\psi : \mathscr{F} \to \mathscr{G}$ such that

$$\phi(f)\psi(e)\phi(f') = \psi(fef') \text{ for all } f, f' \in F, e \in \mathscr{F}.$$

By a result of V. Nekrashevych [5], the biset $\mathscr{F}$ is a complete invariant for the map f up to combinatorial equivalence.

Furthermore, the biset of a map f is explicitly computable (it is free of rank $\deg(f)$ qua right G-set), and functorial; in particular, if m is a mapping class of $S^2 \setminus P_f$, then the invariant of mf is $m \otimes_G \mathscr{F}$, where ${}_G m_G$ is the biset G with actions $f \cdot e \cdot f' = m(f)ef'$.

It is then a simple matter to compute explicitly the invariant of $T^n f_R$ for all $n \in \mathbb{Z}$, and to construct isomorphisms between them.

3. Moduli space

Yet another biset gives us a more conceptual understanding of the "insanely twisted rabbit" problem: fix a 4-element set $P = \{a, b, c, d\} \subset S^2$, and let M denote the mapping class group of $S^2 \setminus P$. Changing our notation slightly, let $\mathscr{R}$ denote the set of isotopy classes rel P of degree-2 maps $S^2 \to S^2$, critical at a and b, fixing a and mapping $b \to c \to d \to b$.

Then $\mathscr{R}$ is an M-biset for pre- and post-composition. "Hurwitz classes" of maps are $(M \times M)$-orbits of $\mathscr{R}$; while combinatorial equivalence classes of maps are diagonal orbits of M on $\mathscr{R}$.

In our case, this biset may be identified with the biset of a rational map as follows. Consider

$$\mathcal{T} = \{\tau : S^2 \to \mathbb{P}^1(\mathbb{C}) \text{ homeomorphism}\}/\text{isotopy rel } P \text{ and Möbius maps}$$

the Teichmüller space of (S^2, P). By Riemann's Uniformization theorem, there exists a unique homeomorphism $\tau' : S^2 \to \mathbb{P}^1(\mathbb{C})$ such that τf and τ' induce the same complex structure on S^2; i.e. such that $\tau f(\tau')^{-1}$ is a rational map on $\mathbb{P}^1(\mathbb{C})$.

We denote by $f^* : \mathcal{T} \to \mathcal{T}$ the transformation $\tau \mapsto \tau'$:

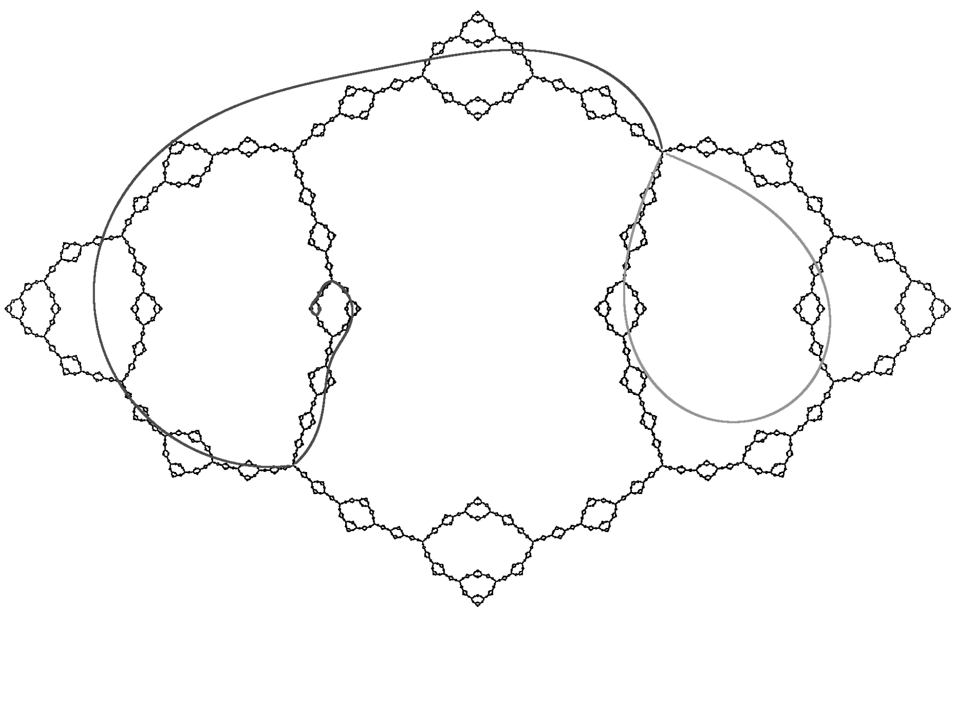
$$\begin{tikzcd}
(S^2, P_f) \arrow[r, "\tau'"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow f^*"] & (\mathbb{P}^1(\mathbb{C}), \tau'(P_f)) \arrow[d, "\text{rational}"] \\
(S^2, P_f) \arrow[r, "\tau"'] & (\mathbb{P}^1(\mathbb{C}), \tau(P_f))
\end{tikzcd}$$

We also denote by $\mathcal{V}$ the "moduli" space of embeddings of P in $\mathbb{P}^1(\mathbb{C})$ up to Möbius transformations. There is a map $\pi : \mathcal{T} \to \mathcal{V}$, which restricts $\tau : S^2 \to \mathbb{P}^1(\mathbb{C})$ to P; it is a regular covering map, with covering group M.

Thurston's theorem is proven as follows: one shows that f^* is weakly contracting for the Teichmüller metric, and therefore admits either a fixed point on $\mathcal{T}$ (which gives a complex metric preserved by f) or (after further analysis) an obstruction.

Let us return to our rabbits. We note, by direct computations, that f_R^* projects, on $\mathcal{V}$, to the inverse of a rational map; i.e. $\pi = g\pi f_R^*$ for a rational map g. Explicitly, one may set $a = \infty$ and $b = 0, c = 1, d = w$ using Möbius transformations; this normalization identifies $\mathcal{V}$ with $\mathbb{C} \setminus \{0,1\}$. Then $g(w) = 1 - w^{-2}$, and $\mathscr{R}$ is the biset associated with g.

The map g admits three fixed points, corresponding respectively to the rabbit, corabbit and airplane. Now elements of M are represented by loops in $\mathcal{V}$. Finding the combinatorial class of $T^n f_R$ amounts to finding the fixed point of $T^n f_R^*$ on $\mathcal{T}$. Therefore, the following recipe, taking place in $\mathcal{V}$, finds the combinatorial equivalence class of mf_R for any mapping class m: represent m by a path in $\mathcal{V}$; lift that path through g; append to it its own lift starting at its endpoint; and continue. Lifts under g get shorter and shorter, so the process converges exponentially to a fixed point of g. For $m = T$, the image in $\mathcal{V}$ is as follows, "proving" that Tf_R is equivalent to an airplane since the path converges to -1.7549:



A movie also illustrates this with $m = T$; it actually represents the semidirect product dynamical system

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 1 - w^{-2} \\ 1 - z^2 w^{-2} \end{pmatrix},$$

tracing the w-coordinate in the upper corner and the Julia set above that coordinate in the main view. In particular, this Julia set at w-fixed points represents a "rabbit", "corabbit" or "airplane" in its z-coordinate. This 2-dimensional, hyperbolic, post-critically finite dynamical system had already been considered in [4].

The movie may be downloaded from the website

`http://www.uni-math.gwdg.de/laurent/pub/rabbit/`

as "Following the lifts for generator S of the mapping class group" (or as the file named `follow_S.avi`).

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