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TOPOLOGICAL MANIFOLDS *

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0. Introduction.

Homeomorphisms – topological isomorphisms – have repeatedly turned up in theorems of a strikingly conceptual character. For example :

(1) (19^{th} century). There are continuously many *non-isomorphic* compact Riemann surfaces, but, up to homeomorphism, only one of each genus.

(2) (B. Mazur 1959). Every smoothly embedded $(n-1)$-sphere in euclidean n-space R^n bounds a topological n-ball.

(3) (R. Thom and J. Mather, recent work). Among smooth maps of one compact smooth manifold to another the topologically stable ones form a dense open set.

In these examples and many others, homeomorphisms serve to reveal basic relationships by conveniently erasing some finer distinctions.

In this important role, PL (= piecewise-linear)(**) homeomorphisms of simplicial complexes have until recently been favored because homeomorphisms in general seemed intractable. However, PL homeomorphisms have limitations, some of them obvious ; to illustrate, the smooth, non-singular self-homeomorphism $f : R \to R$ of the line given by $f(x) = x + \dfrac{1}{4} \exp(-1/x^2) \sin(1/x)$ can in no way be regarded as a PL self-homeomorphism since it has infinitely many isolated fixed points near the origin.

Developments that have intervened since 1966 fortunately have vastly increased our understanding of homeomorphisms and of their natural home, the category of (finite dimensional) topological manifolds(***). I will describe just a few of them below. One can expect that mathematicians will consequently come to use freely the notions of homeomorphism and topological manifold untroubled by the frustrating difficulties that worried their early history.

(*) This report is based on theorems concerning homeomorphisms and topological manifolds [44] [45] [46] [46 A] developed with R.C. Kirby as a sequel to [42]. I have reviewed some contiguous material and included a collection of examples related to my observation that $\pi_3(\mathrm{TOP/PL}) \neq 0$. My oral report was largely devoted to results now adequately described in [81], [82].

(**) A continuous map $f : X \to Y$ of (locally finite) simplicial complexes is called PL if there exists a simplicial complex X' and a homeomorphism $s : X' \to X$ such that s and fs each map each simplex of X' (affine) linearly into some simplex.

(***) In some situations one can comfortably go beyond manifolds [82]. Also, there has been dramatic progress with infinite dimensional topological manifolds (see [48]).

1. History.

A topological (= TOP) m-manifold M^m (with boundary) is a metrizable topological space in which each point has an open neighborhood U that admits an open embedding (called a *chart*) $f : U \to R_+^m = \{(x_1, \ldots, x_m) \in R^m | x_1 \geqslant 0\}$, giving a homeomorphism $U \approx f(U)$.

From Poincaré's day until the last decade, the lack of techniques for working with homeomorphisms in euclidean space R^m (m large) forced topologists to restrict attention to manifolds M^m equipped with atlases of charts $f_\alpha : U_\alpha \to R_+^m$, $\cup U_\alpha = M$, (α varing in some index set), in which the maps $f_\beta f_\alpha^{-1}$ (where defined) are especially tractable, for example all DIFF (infinitely differentiable), or all PL (piecewise linear). Maximal such atlases are called respectively DIFF or PL manifold structures. Poincaré, for one, was emphatic about the importance of the naked homeomorphism – when writing philosophically [68, §§ 1, 2] – yet his memoirs treat DIFF or PL manifolds only.

Until 1956 the study of TOP manifolds as such was restricted to sporadic attempts to prove existence of a PL atlas (= *triangulation conjecture*) and its essential uniqueness (= *Hauptvermutung*). For $m = 2$, Rado proved existence, 1924 [70] (Kerékjártó's classification 1923 [38] implied uniqueness up to isomorphism). For $m = 3$, Moise proved existence and uniqueness, 1952 [62], cf. amisproof of Furch 1924 [21].

A PL manifold is easily shown to be PL homeomorphic to a simplicial complex that is a so-called combinatorial manifold [37]. So the *triangulation conjecture* is that any TOP manifold M^m admits a homeomorphism $h : M \to N$ to a combinatorial manifold. The *Hauptvermutung* conjectures that if h and $h' : M \to N'$ are two such, then the homeomorphism $h'h^{-1} : N \to N'$ can be replaced by a PL homeomorphism $g : N \to N'$. One might reasonably demand that g be *topologically isotopic* to $h'h^{-1}$, or again *homotopic* to it. These variants of the Hauptvermutung will reappear in §5 and §15.

The Hauptvermutung was first formultated in print by Steinitz 1907 (see [85]). Around 1930, after homology groups had been proved to be topological invariants without it, H. Kneser and J.W. Alexander began to advertise the Hauptvermutung for its own sake, and the triangulation conjecture as well [47] [2]. Only a misproof of Nöebling [66] (for any m) ensued in the 1930's. Soberingly delicate proofs of triangulability of DIFF manifolds by Cairns and Whitehead appeared instead.

Milnor's proof (1956) that some 'well-known' S^3 bundles over S^4 are homeomorphic to S^7 but not DIFF isomorphic to S^7 strongly revived interest. It was very relevant ; indeed homotopy theory sees the failure of the Hauptvermutung (1969) as quite analogous. The latter gives the first nonzero homotopy group $\pi_3(\mathrm{TOP/O}) = Z_2$ of TOP/O ; Milnor's exotic 7-spheres form the second $\pi_7(\mathrm{TOP/O}) = Z_{28}$.

In the early 1960's, intense efforts by many mathematicians to unlock the geometric secrets of topological manifolds brought a few unqualified successes : for example the generalized Shoenflies theorem was proved by M. Brown [7] ; the tangent microbundle was developed by Milnor [60] ; the topological Poincaré conjecture in dimensions $\geqslant 5$ was proved by M.H.A. Newman [65].

Of fundamental importance to TOP manifolds were Černavskii's proof in 1968 that the homeomorphism group of a compact manifold is locally contractible [10] [11], and Kirby's proof in 1968 of the stable homeomorphism conjecture with the help of surgery [42]. Key geometric techniques were involved – a meshing idea in the former, a particularly artful torus furling and unfurling idea(*) in the latter. The disproof of the Hauptvermutung and the triangulation conjecture I sketch below uses neither, but was conceived using both. (See [44] [44 B] [46 A] for alternatives).

2. Failure of the Hauptvermutung and the triangulation conjecture.

This section presents the most elementary disproof I know. I constructed it for the Arbeitstagung, Bonn, 1969.

In this discussion $B^n = [-1, 1]^n \subset R^n$ is the standard PL ball ; and the sphere $S^{n-1} = \partial B^n$ is the boundary of B^n. $T^n = R^n/Z^n$ is the standard PL torus, the n-fold product of circles. The closed interval $[0, 1]$ is denoted I.

As starting material we take a certain PL automorphism α of $B^2 \times T^n, n \geqslant 3$, fixing boundary that is constructed to have two special properties (1) and (2) below. The existance of α was established by Wall, Hsiang and Shaneson, and Casson in 1968 using sophisticated surgical techniques of Wall (see [35] [95]). A rather naive construction is given in [80, § 5], which manages to avoid surgery obstruction groups entirely. To establish (1) and (2) it requires only the s-cobordism theorem and some unobstructed surgery with boundary, that works from the affine locus $Q^4 : z_1^5 + z_2^3 + z_3^2 = 1$ in C^3. This Q^4 coincides with Milnor's E_8 plumbing of dimension 4 ; it has signature 8 and a collar neighborhood of infinity $M^3 \times R$, where $M^3 = SO(3)/A_5$ is Poincaré's homology 3-sphere, cf. [61, § 9.8].

(1) *The automorphism β induced by α on the quotient T^{2+n} of $B^2 \times T^n$ (obtained by identifying opposite sides of the square B^2) has mapping torus*

$$T(\beta) = I \times T^{2+n}/\{(0, x) = (1, \beta(x))\}$$

not PL *isomorphic to T^{3+n} ; indeed there exists(**) a* PL *cobordism $(W ; T^{n+3}, T(\beta))$ and a homotopy equivalence of W to $\{I \times T^3 \# Q \cup \infty\} \times T^n$ extending the standard equivalences $T^{3+n} \simeq 0 \times T^3 \times T^n$ and $T(\beta) \simeq 1 \times T^3 \times T^n$. The symbol # indicates (interior) connected sum* [41].

(2) *For any standard covering map $p : B^2 \times T^n \to B^2 \times T^n$ the covering automorphism α_1 of α fixing boundary is* PL *pseudo-isotopic to α fixing boundary. (Covering means that $p\alpha_1 = \alpha p$). In other words, there exists a* PL *automorphism H of $(I ; 0, 1) \times B^2 \times T^n$ fixing $I \times \partial B^2 \times T^n$ such that $H | 0 \times B^2 \times T^n = 0 \times \alpha$ and $H | 1 \times B^2 \times T^n = 1 \times \alpha_1$.*

(*) Novikov first exploited a torus *furling* idea in 1965 to prove the topological invariance of rational Pontrjagin classes [67]. And this led to Sullivan's partial proof of the Hauptvermutung [88]. Kirby's *unfurling* of the torus was a fresh idea that proved revolutionary.

(**) This is the key property. It explains the exoticity of $T(\beta)$ – (see end of argument), and the property (2) – (almost, see [80, § 5]).

In (2) choose p to be the 2^n-fold covering derived from scalar multiplication by 2 in R^n. (Any integer > 1 would do as well as 2.) Let $\alpha_0 (= \alpha), \alpha_1, \alpha_2, \ldots$ be the sequence of automorphisms of $B^2 \times T^n$ fixing boundary such that α_{k+1} covers α_k, i.e. $p\alpha_{k+1} = \alpha_k p$. Similarly define $H_0 (= H), H_1, H_2, \ldots$ and note that H_k is a PL concordance fixing boundary from α_k to α_{k+1}. Next define a PL automorphism H' of $[0, 1) \times B^2 \times T^n$ by making $H' | [a_k, a_{k+1}] \times B^2 \times T^n$, where $a_k = 1 - \frac{1}{2^k}$, correspond to H_k under the (oriented) linear map of $[a_k, a_{k+1}]$ onto $[0, 1] = I$. We extend H' by the identity to $[0, 1) \times R^2 \times T^n$. Define another self-homeomorphism H'' of $[0, 1) \times B^2 \times T^n$ by $H'' = \varphi H' \varphi^{-1}$ where

$$\varphi(t, x, y) = (t, (1-t)x, y) \quad .$$

Finally extend H'' by the identity to a bijection

$$H'' : I \times B^2 \times T^n \to I \times B^2 \times T^n \quad .$$

It is also continuous, hence a homeomorphism. To prove this, consider a sequence $q_1, q_2, \ldots$ of points converging to $q = (t_0, x_0, y_0)$ in $I \times B^2 \times T^n$. Convergence $H''(q_j) \to H''(q)$ is evident except when $t_0 = 1, x_0 = 0$. In the latter case it is easy to check that $p_1 H''(q_j) \to p_1 H''(q) = 1$ and $p_2 H''(q_j) \to p_2 H''(q) = 0$ as $j \to \infty$, where p_i, $i = 1, 2, 3$ is projection to the i-th factor of $I \times B^2 \times T^n$. It is not as obvious that $p_3 H''(q_j) \to p_3 H''(q) = y_0$. To see this, let

$$\widetilde{H}_k : I \times B^2 \times R^n \to I \times B^2 \times R^n$$

be the universal covering of H_k fixing $I \times \partial B^2 \times R^n$. Now

$$\sup \{ | p_3 z - p_3 \widetilde{H}_k z | \quad ; \quad z \in [0, 1] \times B^2 \times R^n \} \equiv D_k$$

is finite, being realized on the compactum $I \times B^2 \times I^n$. And, as $\widetilde{H}_k$ is clearly $\theta_k^{-1} \widetilde{H}_0 \theta_k$, where $\theta_n(t, x, y) = (t, x, 2^n y)$, we have $D_k = \frac{1}{2^k} D_0$. Now D_k is $\geqslant$ the maximum distance of $p_3 H_k$ from p_3, for the quotient metric on $T^n = R^n / Z^n$; so $D_k \to 0$ implies $p_3 H''(q_j) \to p_3 H''(q) = y_0$, as $j \to \infty$.

As the homeomorphism H'' is the identity on $I \times \partial B^2 \times T^n$ it yields a self-homeomorphism g of the quotient $I \times T^2 \times T^n = I \times T^{2+n}$. And as

$$g | 0 \times T^{2+n} = 0 \times \beta \quad ,$$

and $g | 1 \times T^{2+n} =$ identity, g gives a homeomorphism h of $T(\beta)$ onto

$$T(\text{id}) = T^1 \times T^{2+n} = T^{3+n}$$

by the rule sending points (t, z) to $g^{-1}(t, z)$ – hence $(0, z)$ to $(0, \beta^{-1}(z))$ and $(1, z)$ to $(1, z)$

The homeomorphism $h : T^{3+n} \approx T(\beta)$ *belies the Hauptvermutung*. Further, (1) offers a certain PL cobordism $(W; T^{3+n}, T(\beta))$. Identifying T^{3+n} in W to $T(\beta)$ under h we get a closed topological manifold

$$X^{4+n} \simeq \{T^1 \times T^3 \# Q \cup \infty\} \times T^n$$

($\simeq$ indicating homotopy equivalence).

If it had a PL manifold structure the fibering theorem of Farrell [19] (or the author's thesis) would produce a PL 4-manifold X^4 with $w_1(X^4) = w_2(X^4) = 0$ and signature $\sigma(X^4) \equiv \sigma(S^1 \times T^3 \# Q \cup \infty) \equiv \sigma(Q \cup \infty) \equiv 8$ mod. 16, cf. [80, § 5]. Rohlin's theorem [71] [40] cf. § 13 shows this X^4 doesn't exist. *Hence X^{4+n} has no* PL *manifold structure.*

Let us reflect a little on the generation of the homeomorphism $h : T(\beta) \approx T^{3+n}$. The behaviour of H'' is described in figure 2-a (which is accurate for B^1 in place of B^2 and for $n = 1$) by partitioning the fundamental domain $I \times B^2 \times I^n$ according to the behavior of H''. The letter α indicates codimension 1 cubes on which H'' is a conjugate of α.

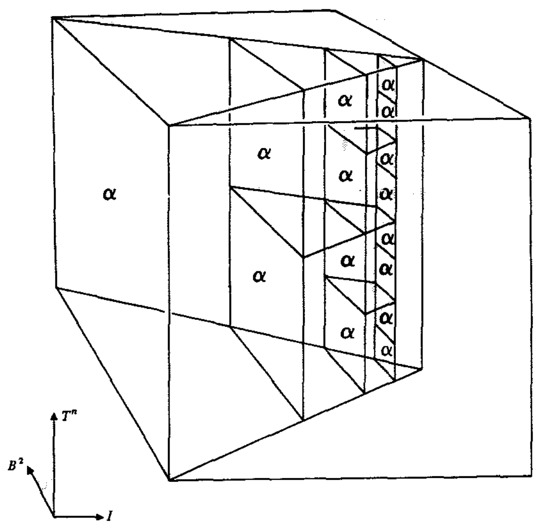

Figure 2a

Observe the infinite ramification (2^n-fold) into smaller and smaller domains converging to all of $1 \times 0 \times T^n$. In the terminology of Thom [92, figure 7] this reveals the failure of the Hauptvermutung to be a *generalized catastrophe* !

Remark 2.1. — Inspection shows that $h : T(\beta) \approx T^{3+n}$ is a Lipschitz homeomorphism and hence X^{4+n} is a Lipschitz manifold as defined by Whitehead [98] for the pseudogroup of Lipschitz homeomorphisms — see §4. A proof that $T(\beta) \approx T^{3+n}$ (as given in [44]) using local contractibility of a homeomorphism group would not reveal this as no such theorem is known for Lipschitz homeomorphisms. Recall that a theorem of Rademacher [69] says that every Lipschitz homeomorphism of one open subset of R^m to another is almost everywhere differentiable.

3. The unrestricted triangulation conjecture.

When a topological manifold admits no PL manifold structure we know it is not homeomorphic to a simplicial complex which is a combinatorial manifold [37]. But it may be homeomorphic to *some* (less regular) simplicial complex — i.e. triangulable in an unrestricted sense, cf. [79]. For example $Q \cup \infty$ (from §2) is triangulable and Milnor (Seattle 1963) asked if $(Q \cup \infty) \times S^1$ is a topological manifold even though $Q \cup \infty$ obviously is not one. If so, the manifold X^{4+n} of § 2 is easily triangulated.

If all TOP manifolds be triangulable, why not conjecture that that every locally triangulable metric space is triangulable ?

Here is a construction for a compactum X that is *locally triangulable* but is *non-triangulable.* Let L_1, L_2 be closed PL manifolds and

$$(W \,; L_1 \times R \,, L_2 \times R)$$

an invertible(*) PL cobordism that is not a product cobordism. Such a W exists for instance if $\pi_1 L_i = Z_{257}$ and $L_1 \not\cong L_2$, compare [78]. It can cover an invertible cobordism $(W', L_1 \times S^1, L_2 \times S^1)$ [77, § 4]. To the Alexandroff compactification $W \cup \infty$ of W adjoin $\{(L_1 \times R) \cup \infty\} \times [0, 1]$ identifying each point $(x, 1)$ in the latter to the point x in $W \cup \infty$. The resulting space is X. See Figure 3-a. The properties of X and of related examples will be demonstrated in [83]. They complement Milnor's examples [57] of homeomorphic complexes that are PL (combinatorially) distinct, which disproved an *unrestricted Hauptvermutung.*

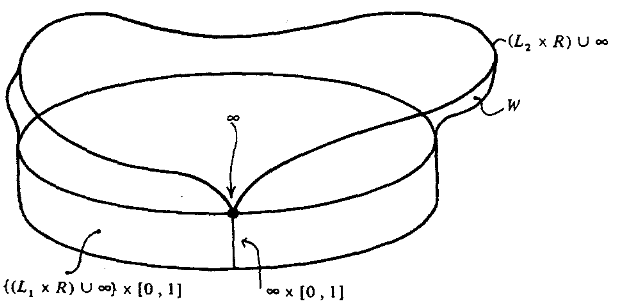

Figure 3a

(*) This means that W can be expressed as a union $W = C_1 \cup C_2$, where C_i is a closed collar neighborhood of $L_i \times R$ in W.

4. Structures on topological manifolds.

Given a TOP manifold M^m (without boundary) and a pseudo-group G of homeomorphisms(*) of one open subset of R^m to another, the problem is to find and classify G-structures on M^m. These are maximal "G-compatible" atlases $\{U_\alpha, f_\alpha\}$ of charts (= open embeddings) $f_\alpha : U_\alpha \to R^m$ so that each $f_\beta f_\alpha^{-1}$ is in G. (Cf. [29] or [48].)

One reduction of this problem to homotopy theoretic form has been given recently by Haefliger [28] [29]. Let $G(M^m)$ be the (polyhedral quasi-) space (**) of G-structures on M. A map of a compact polyhedron P to $G(M)$ is by definition a G-foliation $\mathfrak{F}$ on $P \times M$ transverse to the projection $p_1 : P \times M \to P$ (i.e. its defining submersions are transverse to p_1)(***). Thus, for each $t \in P$, $\mathfrak{F}$ restricts to a G-structure on $t \times M$ and, on each leaf of $\mathfrak{F}$, p_1 is an open embedding. Also note that $\mathfrak{F}$ gives a G_P-structure on $P \times M$ where G_P is the pseudo-group of homeomorphisms of open subsets of $P \times R^m$ locally of the form $(t, x) \to (t, g(x))$ with $g \in G$. If G consists of PL or DIFF homeomorphisms and $P = [0, 1]$, then $\mathfrak{F}$ gives (a fortiori) what is called a *sliced concordance* of PL or DIFF structures on M (see [45] [46]).

We would like to analyse $G(M^m)$ using Milnor's tangent R^m-microbundle $\tau(M)$ of M, which consists of total space $E(\tau M) = M \times M$, projection $p_1 : M \times M \to M$, and (diagonal) section $\delta : M \to M \times M$, $\delta(x) = (x, x)$. Now if ξ^m is *any* R^m microbundle over a space X we can consider $G^\perp(\xi)$ the space of G-foliations of $E(\xi)$ transverse to the fibers. A map $P \to G^\perp(\xi)$ is a G-foliation $\mathfrak{F}$ defined on an open neighborhood of the section $P \times X$ in the total space $E(P \times \xi) = P \times E(\xi)$ that is transverse to the projection to $P \times X$. Notice that there is a natural map

$$d : G(M^m) \to G^\perp(\tau M^m) \quad ,$$

which we call the *differential.* To a G-foliation $\mathfrak{F}$ of $P \times M$ transverse to p_1, it assigns the G-foliation $d\mathfrak{F}$ on $P \times M \times M = E(P \times \tau(M))$ obtained from $\mathfrak{F} \times M$

(*) e.g. the PL isomorphisms, or Lipschitz or DIFF or analytic isomorphisms. Do not confuse G with the stable monoid $G = \cup G_n$ of § 5.5.

(**) Formally such a space X is a contravariant functor $X : P \to [P, X]$ from the category of PL maps of compact polyhedra (denoted P, Q etc.) to the category of sets, which carries union to fiber product. Intuitively X is a space of which we need (or want or can) only know the maps of polyhedra to it.

(***) A *G-foliation* on a space X is a maximal G-compatible atlas $\{V_\alpha, g_\alpha\}$ of topological submersions $g_\alpha : V_\alpha \to R^m$. (See articles of Bott and Wall in these proceedings.) A map $g : V \to W$ is a *topological submersion* if it is locally a projection in the sense that for each x in V there exists an open neighborhood W_x of $g(x)$ in W a space F_x and an open embedding onto a neighborhood of x, called a *product chart* about x, $\varphi : F_x \times W_x \to V$ such that $g\varphi$ is projection $p_2 : F_x \times W_x \to W_x \subset W$. One says that g is *transverse* to another submersion $g' : V \to W'$ if for each x, φ can be chosen so that $F_x = W'_x \times F'_x$ and $g'\varphi$ is projection to W'_x an open subset of W'. This says roughly that the leaves (= fibers) of f and g intersect in general position. Above they intersect in points.

by *interchanging* the factors M. If P is a point, the leaves of $d\mathcal{F}$ are simply

$$\{P \times M \times x \mid x \in M\} \quad .$$

Clearly $d\mathcal{F}$ is transverse to the projection $P \times p_1$ to $P \times M$.

THEOREM 4.1. CLASSIFICATION BY FOLIATED MICROBUNDLES. — *The differential*

$$d : G(M^m) \to G^{\perp}(\tau M^m)$$

is a weak homotopy equivalence for each open (metrizable) m-manifold M^m with no compact components.

Haefliger deduces this result (or at least the bijection of components) from the topological version of the Phillips-Gromov transversality theorem classifying maps of M transverse to a TOP foliation. (See [29] and J.C. Hausmann's appendix).

As formulated here, 4.1 invites a direct proof using Gromov's distillation of immersion theory [25] [26]. This does not seem to have been pointed out before, and it seems a worthwhile observation, for I believe the transversality result adequate for 4.1 requires noticeably more geometric technicalities. In order to apply Gromov's distillation, there are two key points to check. For any $C \subset M^m$, let $G_M(C) = \text{inj lim}\,\{G(U) \mid C \subset U \text{ open in } M\}$.

(1) *For any pair $A \subset B$ of compacta in M, the restriction map $\pi : G_M(B) \to G_M(A)$ is micro-gibki* – i.e., given a homotopy $f : P \times I \to G_M(A)$ and $F_0 : P \times 0 \to G_M(B)$ with $\pi F_0 = f \mid P \times 0$ there exists $\epsilon > 0$ and $F : P \times [0, \epsilon] \to G_M(B)$ so that $\pi F = f \mid P \times [0, \epsilon]$. Chasing definitions one finds that this follows quickly from the TOP isotopy extension theorem (many-parameter version) or the relative local contractibility theorem of [10] [17].

(2) *d is a weak homotopy equivalence for $M^m = R^m$.* Indeed, one has a commutative square of weak homotopy equivalences

$$\begin{array}{ccc} G(R^m) & \xrightarrow{d} & G^{\perp}(\tau R^m) \\ \simeq\downarrow & & \downarrow\simeq \\ G_{R^m}(0) & \xleftarrow{\simeq} & G^{\perp}(\tau R^m \mid 0) \end{array}$$

in which the verticals are restrictions and the bottom comes from identifying the fiber of $\tau R^m \mid 0$ to R^m, cf. [27].

Gromov's analysis applies (1) and (2) and more obvious properties of G, $G^{\perp}$ to establish 4.1. Unfortunately, M doesn't always have a handle decomposition over which to induct ; one has to proceed more painfully chart by chart.

We can now pass quickly from a bundle theoretic to a homotopy classification of G-structures. Notice that if $f : X' \to X$ is any map and ξ^m is a R^m microbundle over X equipped with a G-folitation $\mathcal{F}$, transverse to fibers, defined on an open neighborhood of the zero section X, then $f^*\xi$ over X' is similarly equipped with a pulled-back foliation $f^*\mathcal{F}$. This means that equipped bundles behave much like bundles. One can use Haefliger's notion of "gamma structure" as in [29] to deduce for numerable equipped bundles the existence of a universal one $(\gamma_G^m, \mathcal{F}_G)$ over a base space $B_{\Gamma(G)}$(*). There is a map $B_{\Gamma(G)} \to B_{\mathrm{TOP}(m)}$ classifing γ_G^m as an R^m-microbundle ; we make it a fibration. Call the fiber $\mathrm{TOP}(m)/\Gamma(G)$. One finds that there is a weak homotopy equivalence $G^{\perp}(\xi) \simeq \mathrm{Lift}(f \text{ to } B_{\Gamma(G)})$, to the space of liftings to $B_{\Gamma(G)}$ of a fixed classifying map $f : X \to B_{\mathrm{TOP}(m)}$ for ξ^m. Hence one gets

THEOREM 4.2. — *For any open topological m-manifold M^m, there is a weak homotopy equivalence $G(M) \simeq \mathrm{Lift}(\tau \text{ to } B_{\Gamma(G)})$ from the space of G-structures $G(M)$ on M to the space of liftings to $B_{\Gamma(G)}$ of a fixed classifying map map $\tau : M \to B_{\mathrm{TOP}(m)}$ for $\tau(M)$.*

Heafliger and Milnor observe that for $G = \mathrm{CAT}^m$ the pseudo-group of CAT isomorphisms of open subsets of R^m — CAT meaning DIFF (= smooth C^∞), or PL (= piecewise linear) or TOP (= topological) — one has

$$\pi_i\,(\mathrm{CAT}(m)/\Gamma\,(\mathrm{CAT}^m)) = 0 \quad , \quad i < m \quad . \tag{4.3}$$

Indeed for CAT = TOP, 4.2 shows this amounts to the obvious fact that $\pi_0(G(S^i \times R^{m-i})) = 0$. Analogues of 4.2 with DIFF or PL in place of TOP can be proved analogously(**) and give the other cases of (4.3). Hence one has

THEOREM 4.4. — *For any open topological manifold M^m, there is a natural bijection $\pi_0\mathrm{CAT}^m(M^m) \simeq \pi_0\mathrm{Lift}(\tau \text{ to } B_{\mathrm{CAT}(m)})$.*

This result comes from [44] for $m \geqslant 5$. Lashof [50] gave the first proof that was valid for $m = 4$. A stronger and technically more difficult result is sketched in [63] [45]. It asserts a weak homotopy equivalence of a "sliced concordance" variant of $\mathrm{CAT}^m(M^m)$ with $\mathrm{Lift}(\tau \text{ to } B_{\mathrm{CAT}(m)})$. This is valid without the openness restriction if $m \neq 4$. For open M^m (any m), it too can be given a proof involving a micro-gibki property and Gromov's procedure.

5. The product structure theorem.

THEOREM 5.1 (*Product structure theorem*). — *Let M^m be a* TOP *manifold, C a closed subset of M and σ_0 a* CAT (= DIFF *or* PL) *structure on a neighborhood of C in M. Let Σ be a* CAT *structure on $M \times R^s$ equal $\sigma_0 \times R^s$ near $C \times R^s$. Provide that $m \geqslant 5$ and $\partial M \subset C$.*

Then M has a CAT *structure σ equal σ_0 near C. And there exists a* TOP *isotopy (as small as we please) $h_t : M_\sigma \times R^s \to (M \times R^s)_\Sigma$, $0 \leqslant t \leqslant 1$, of h_0 = identity, fixing a neighborhood of $C \times R^s$, to a* CAT *isomorphism h_1.*

It will appear presently that this result is the key to TOP handlebody theory and transversality. The idea behind such applications is to reduce TOP lemmas to their DIFF analogues.

(*) Alternatively, for our purpose, $B_{\Gamma(G)}$ can be the ordered simplicial complex having one d-simplex for each equipped bundle over the standard d-simplex that has total space in some $R^n \subset R^\infty$.

(**) The forgetful map $\varphi : B_{\Gamma(\mathrm{PL}^m)} \to B_{\mathrm{PL}(m)}$ is more delicate to define. One can make $B_{\Gamma(\mathrm{PL}^m)}$ a simplicial complex, then define φ simplex by simplex.

It seems highly desirable, therefore, to prove 5.1 as much as possible by pure geometry, without passing through a haze of formalism like that in § 4. This is done in [46]. Here is a quick sketch of proof intended to advertise [46].

First, one uses the CAT s-cobordism theorem (no surgery !) and the handle-straightening method of [44] to prove – without meeting obstructions –

THEOREM 5.2 (*Concordance implies isotopy*). — *Given M and C as in* 5.1, *consider a* CAT *structure Γ on $M \times I$ equal $\sigma_0 \times I$ near $C \times I$, and let $\Gamma | M \times 0$ be called $\sigma \times 0$.* (Γ is called a concordance of σ rel C).

There exists a TOP *isotopy (as small as we please) $h_t : M_\sigma \times I \to (M \times I)_\Sigma$, $0 \leq t \leq 1$, of h_0 = identity, fixing $M \times 0$ and a neighborhood of $C \times I$, to a* CAT *isomorphism h_1.*

Granting this result, the Product Structure Theorem is deduced as follows.

In view of the relative form of 5.2 we can assume $M = R^m$. Also we can assume $s = 1$ (induct on s !). Thirdly, it suffices to build a concordance Γ (= structure on $M \times R^s \times I$) from $\sigma \times R^s$ to Σ rel $C \times R^s$. For, applying 5.2 to the concordance Γ we get the wanted isotopy. What remains to be proved can be accomplished quite elegantly. Consider Figure 5-a.

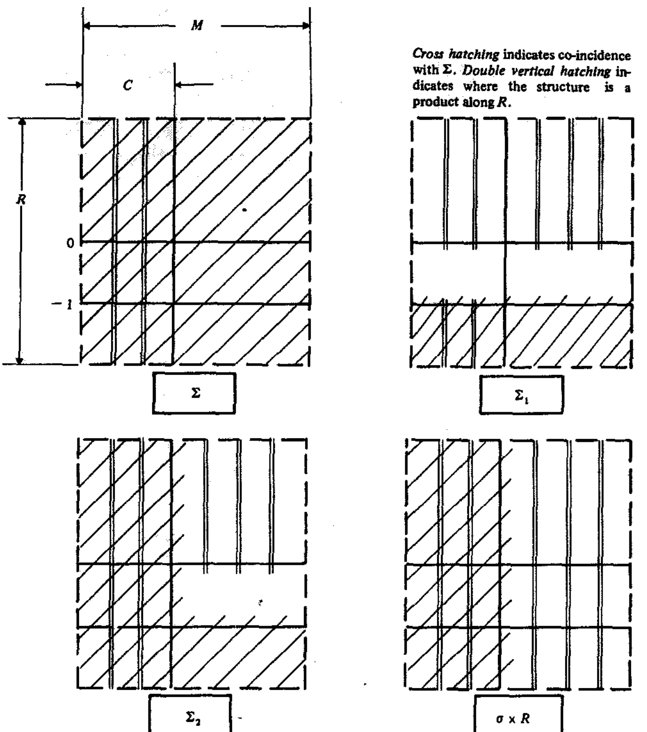

Figure 5a

We want a concordance rel $C \times R$ from Σ to $\sigma \times R$. First note it suffices to build Σ_2 with the properties indicated. Indeed Σ_2 admits standard (sliced) concordances rel $C \times R$ to $\sigma \times R$ and to Σ. The one to $\sigma \times R$ comes from sliding R over itself onto $(0, \infty)$. The region of coincidence with $\sigma \times R$ becomes total by a sort of window-blind effect. The concordance to Σ comes from sliding R over itself onto $(-\infty, -1)$. (Hint : The structure picked up from Σ_2 at the end of the slide is the same as that picked up from Σ).

It remains to construct Σ_2. Since $M \times R = R^{m+1}$, we can find a concordance (not rel $C \times R$) from Σ to the standard structure, using the STABLE homeomorphism theorem(*) [42]. Now 5.2 applied to the concordance gives Σ_1, which is still standard near $M \times [0, \infty)$. Finally an application of 5.2 to $\Sigma_1 | N \times [-1, 0]$, where N is a small neighborhood of C, yields Σ_2. The change in $\Sigma_1 | M \times 0$ (which is standard) on $N \times 0$ offered by 5.2 is extended productwise over $M \times [0, \infty)$. This completes the sketch.

It is convenient to recall here for later use one of the central results of [44]. Recall that $\mathrm{TOP}_m/\mathrm{PL}_m$ is the fiber of the forgetful map $B_{\mathrm{PL}(m)} \to B_{\mathrm{TOP}(m)}$. And TOP/PL is the fiber the similar map of stable classifying spaces $B_{\mathrm{PL}} \to B_{\mathrm{TOP}}$. Similarly one defines $\mathrm{TOP}_m/\mathrm{DIFF}_m \equiv \mathrm{TOP}_m/\mathrm{O}_m$ and $\mathrm{TOP/DIFF} \equiv \mathrm{TOP/O}$.

THEOREM 5.3(**) (Structure theorem).— $\mathrm{TOP/PL} \simeq K(Z_2, 3)$ *and*

$$\pi_k(\mathrm{TOP}_m/\mathrm{CAT}_m) = \pi_k(\mathrm{TOP/CAT})$$

for $k < m$ *and* $m \geqslant 5$. *Here* CAT = PL *or* DIFF.

Since $\pi_k(\mathrm{O}_m) = \pi_k(\mathrm{O})$ for $k < m$, we deduce that $\pi_k(\mathrm{TOP}, \mathrm{TOP}_m) = 0$ for $k < m > 5$, a weak stability for TOP_m.

Consider the second statement of 5.3 first. Theorem 4.4 says that

$$\pi_k(\mathrm{TOP}_m/\mathrm{CAT}_m) = \pi_0(\mathrm{CAT}^m(S^k \times R^{m-k})) \equiv S_k^m$$

for $k < m \geqslant 5$. Secondly, 5.1 implies $S_k^m = S_k^{m+1} = S_k^{m+2} = \cdots, m \geqslant k$. Hence $\pi_k(\mathrm{TOP}_m/\mathrm{CAT}_m) = \pi_k(\mathrm{TOP/CAT})$.

We now know that $\pi_k(\mathrm{TOP/PL})$ is the set of isotopy classes of PL structures on S^k if $k \geqslant 5$. The latter is zero by the PL Poincaré theorem of Smale [84], combined with the stable homeomorphism theorem [42] and the Alexander isotopy. Similarly one gets $\pi_k(\mathrm{TOP/DIFF}) = \Theta_k$ for $k \geqslant 5$. Recall $\Theta_5 = \Theta_6 = 0$ [41].

The equality $\pi_k(\mathrm{TOP/PL}) = \pi_k(\mathrm{TOP/DIFF}) = \pi_k(K(Z_2, 3))$ for $k \leqslant 5$ can be deduced with ease from local contractibility of homeomorphism groups and the surgical classification [35] [95], by $H^3(T^5 ; Z_2)$, of homotopy 5-tori. See [43] [46 A] for details.

Combining the above with 4.4 one has a result of [44].

(*) Without this we get only a theorem about compatible CAT structures on STABLE manifolds (of Brown and Gluck [8]).

(**) For a sharper result see [63] [45], and references therein.

CLASSIFICATION THEOREM 5.4. — *For $m \geqslant 5$ a* TOP *manifold M^m (without boundary) admits a* PL *manifold structure iff an obstruction $\Delta(M)$ in $H^4(M;Z_2)$ vanishes. When a* PL *structure Σ on M is given, others are classified (up to concordance or isotopy) by elements of $H^3(M\,;Z_2)$.*

Complement. — Since $\pi_k(\mathrm{TOP/DIFF}) = \pi_k(\mathrm{TOP/PL})$ for $k < 7$ (see above calculation), the same holds for DIFF in low dimensions.

Finally we have a look at low dimensional homotopy groups involving

$$G = \lim\{G_n \,|\, n \geqslant 0\}$$

where G_n is the space of degree± 1 maps $S^{n-1} \to S^{n-1}$. Recall that $\pi_n G = \pi_{n+k} S^k$, k large. G/CAT is the fiber of a forgetful map $B_{\mathrm{CAT}} \to B_G$, where B_G is a stable classifying space for spherical fibrations (see [15], [29]).

$$\begin{tikzcd}[column sep=small]
 & & \pi_4 G \arrow[dl] \arrow[dr] & & \\
 & \pi_4 G/0 \arrow[rr] \arrow[dd] & & \pi_4 G/\mathrm{TOP} \arrow[dr] \arrow[dd, "\partial"'] & \\
\pi_4 \mathrm{TOP/O} \arrow[ur] \arrow[dr] & & & & \pi_3 \mathrm{TOP/O} \\
 & \pi_3 \mathrm{O} \arrow[rr] \arrow[dr] & & \pi_3 \mathrm{TOP} \arrow[ur] \arrow[dl] & \\
 & & \pi_3 G & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dl] \arrow[dr] & & \\
 & Z \arrow[rr] \arrow[dd] & & \boxed{Z} \arrow[dr, "\text{(onto)}"] \arrow[dd, "\partial"'] & \\
0 \arrow[ur] \arrow[dr] & & & & Z_2 \\
 & Z \arrow[rr] \arrow[dr, "\text{(onto)}"'] & & \boxed{Z \oplus Z_2} \arrow[ur] \arrow[dl] & \\
 & & Z_{24} & &
\end{tikzcd}$$

The left hand commutative diagram of natural maps is determined on the right. Only $\pi_3\mathrm{TOP}$ is unknown(*). So the exactness properties evident on the left leave no choice. Also ∂ must map a generator of $\pi_4 G/\mathrm{TOP} = Z$ to $(12,1)$ in $Z \oplus Z/2Z = \pi_3\mathrm{TOP}$.

The calculation with PL in place of O is the same (and follows since $\pi_i(\mathrm{PL/O}) = \Gamma_i = 0$ for $i \leqslant 6$).

6. Simple homotopy theory [44] [46 A].

The main point is that every compact TOP manifold M (with boundary ∂M) has a preferred simple homotopy type and that two plausible ways to define it are equivalent. Specifically, a handle decomposition of M or a combinatorial triangulation of a normal disc-bundle to M give the same simple type.

The second definition is always available. Simply embed M in R^n, n large, with normal closed disc-bundle E [31]. Theorem 5.1 then provides a small homeomorphism of R^n so that $h(\partial E)$, and hence $h(E)$, is a PL submanifold.

(*) That $\pi_4 G/\mathrm{TOP}$ is Z (not $Z \oplus Z_2$) is best proved by keeping track of some normal invariants in disproving the Hauptvermutung, see [46A]. Alternatively, see 13.4 below.

Working with either of these definitions, one can see that the preferred simple type of M and that of the boundary ∂M make of $(M, \partial M)$ a finite Poincaré duality space in the sense of Wall [95], a fact vital for TOP surgery.

The Product structure theorem 5.1 makes quite unnecessary the bundle theoretic nonsense used in [44] (cf. [63]) to establish preferred simple types.

7. **Handlebody theory** (statements in [44 C] [45], proofs in [46 A]).

7.1. The main result is that handle decompositions exist in dimension $\geqslant 6$. Here is the idea of proof for a closed manifold M^m, $m \geqslant 6$. Cover M^m by finitely many compacta $A_1, \ldots, A_k$, each A_i contained in a co-ordinate chart $U_i \approx R^m$. Suppose for an inductive construction that we have built a handlebody $H \subset M$ containing $A_1 \cup \ldots \cup A_{i+1}$, $i \geqslant 0$. The Product Structure Theorem shows that $H \cap U_i$ can be a PL (or DIFF) m-submanifold of U_i after we adjust the PL (or DIFF) structure on U_i. Then we can successively add finitely many handles onto H in U_i to get a handlebody H' containing $A_1 \cup \ldots \cup A_i$. After k steps we have a handle decomposition of M.

A TOP Morse function on M^m implies a TOP handle decomposition (the converse is trivial) ; to see this one uses the TOP isotopy extension theorem to prove that a TOP Morse function without critical points is a bundle projection. (See [12] [82, 6.14] for proof in detail).

Topological handlebody theory as conceived of by Smale now works on the model of the PL or DIFF theory (either). For the sake of those familiar with either, I describe simple ways of obtaining transversality and separation (by Whitney's method) of attaching spheres and dual spheres in a level surface.

LEMMA 7.2. (Transversality). — *Let $g : R^m \to R^m, m \geqslant 5$, be a* STABLE *homeomorphism. In R^m, consider $R^p \times 0$ and $0 \times R^q, p + q = m$, with 'ideal' transverse intersection at the origin. There exists an ϵ-isotopy of g to $h : R^m \to R^m$ such that $h(R^p \times 0)$ is transverse to $0 \times R^q$ is the following strong sense. Near each point $x \in h^{-1}(0 \times R^q) \cap R^p \times 0$, h differs from a translation by at most a homeomorphism of R^m respecting both $R^p \times 0$ and $0 \times R^q$.*

Furthermore, if C is a given closed subset of R^m and g satisfies the strong transversality condition on h above for points x of R^m near C, then h can equal g near C.

Proof of 7.2 — For the first statement $\epsilon/2$ isotop g to diffeomorphism g' using Ed Connell's theorem [14] (or the Concordance-implies-epsilon-isotopy theorem 5.2), then $\epsilon/2$ isotop g' using standard DIFF techniques to a homeomorphism h' which will serve as h if $C = \emptyset$.

The further statement is deduced from the first using the flexibility of homeomorphisms. Find a closed neighborhood C' of C near which g is still transverse such that the frontier $\dot{C}'$ misses $g^{-1}(0 \times R^q) \cap (R^p \times 0)$ — which near C is a discrete collection of points. Next, find a closed neighborhood D of $\dot{C}'$ also missing $g^{-1}(0 \times R^q) \cap (R^p \times 0)$, and $\delta : R^m \to (0, \infty)$ so that $d(gx, 0 \times R^q) < \delta(x)$ for x in $D \cap (R^p \times 0)$. If $\epsilon : R^m \to (0, \infty)$ is sufficiently small, and h' in the first paragraph is built for ϵ, Cernavskii's local contractibility theorem [11] (also [17] and [82, 6.3]) says that there exists a homeomorphism h equal g on C' and equal h' outside $C' \cup D$ so that $d(h', g) < \delta$. This is the wanted h.

7.3. THE WHITNEY LEMMA.

The TOP case of the Whitney process for eliminating pairs of isolated transverse intersection points (say of M^p and N^q) can be reduced to the PL case [99] [37]. The Whitney 2-disc is easily embedded and a neighborhood of it is a copy of R^m, $m = p + q$. We can arrange that either manifold, say M^p, is PL in R^m, and(*) N^q is PL near M^q in R^m. Since $5 \leqslant m = p + q$, we can assume $q \leqslant m/2$; so N^q can now be pushed to be PL in R^m by a method of T. Homma, or by one of R.T. Miller [54 A], or again by the method of [44], applied pairwise [44 A] (details in [73]). Now apply the PL Whitney lemma [37]. On can similarly reduce to the original DIFF Whitnev lemma [99].

7.4. CONCLUSION.

The s-cobordism theorem [37] [39], the boundary theorem of [76], and the splitting principle of Farrell and Hsiang [20] can now be proved in TOP with the usual dimension restrictions.

8. Transversality (statements in [44 C] [45], proofs in [46 A]).

If $f : M^m \to R^n$ is a continuous map of a TOP manifold without boundary to R^n and $m - n > 5$, we can homotop f to be transverse to the origin $0 \in R^n$. Here is the idea. One works from chart to chart in M to spread the transversality, much as in building handlebodies. In each chart one uses the product structure theorem 5.1 to prepare for an application of the *relative* DIFF transversality theorem of Thom.

Looking more closely one gets a *relative* transversality theorem for maps $f : M^m \to E(\xi^n)$ with target *any* TOP R^n-microbundle ξ^n over *any* space. It is parallel to Williamson's PL theorem [100], but is proved only for $m \neq 4 \neq m - n$. It is indispensible for surgery and cobordism theory.

9. Surgery.

Surgery of compact manifolds of dimension $\geqslant 5$ as formulated by Wall [95] can be carried out for TOP manifolds using the tools of TOP handlebody theory. The chief technical problem is to make the self-intersections of a framed TOP immersion $f : S^k \times R^k \to M^{2k}$ of S^k, $k \geqslant 3$, transverse (use Lemma 7.2 repeatedly), and then apply the Whitney lemma to find a regular homotopy of f to an embedding when Wall's self-intersection coefficient is zero.

In the simply connected case one can adapt ideas of Browder and Hirsch [4].

Of course TOP surgery constantly makes use of TOP transversality, TOP simple homotopy type and the TOP s-cobordism theorem.

(*) Use of the strong transversality of 7.2 makes this trivial in practice.

10. **Cobordism theory : generalities.**

Let Ω_n^{TOP} [respectively Ω_n^{STOP}] be the group of [oriented] cobordism classes of [oriented] closed n-dimensional TOP manifolds. Thom's analysis yields a homomorphism

$$\theta_n : \Omega_n^{\mathrm{TOP}} \to \pi_n(\mathrm{MTOP}) = \varinjlim_k \pi_{n+k}(\mathrm{MTOP}(k)) \quad .$$

Here $\mathrm{MTOP}(k)$ is the Thom space of the universal TOP R^k-bundle γ_{TOP}^k over $B_{\mathrm{TOP}(k)}$ – obtained, for example, by compactifying each fiber with a point (cf. [49]) and crushing these points to one. The Pontrjagin Thom definition of θ_n uses a stable relative existence theorem for normal bundles in euclidean space – say as provided by Hirsch [30] and the Kister-Mazur Theorem [49].

Similarly one gets Thom maps

$$\theta_n : \Omega_n^{\mathrm{STOP}} \to \pi_n(\mathrm{MSTOP}), \qquad \text{and} \qquad \theta_n : \Omega_n^{\mathrm{SPINTOP}} \to \pi_n(\mathrm{MSPINTOP})\,,$$

and more produced by the usual recipe for cobordism of manifolds with a given, special, stable structure on the normal bundle [86, Chap. II].

THEOREM 10.1. — *In each case above the Thom map $\theta_n : \Omega_n \to \pi_n(M)$ is surjective for $n \neq 4$, and injective for $n \neq 3$.*

This follows immediately from the transversality theorem.

PROPOSITION 10.2. — *$B_{\mathrm{SO}} \otimes Q \simeq B_{\mathrm{STOP}} \otimes Q$, where Q denotes the rational numbers.*

Proof. — $\pi_i(\mathrm{STOP/SO}) = \pi_i(\mathrm{TOP/O})$ is finite for all i by [40] [44] cf. § 5, STOP/SO being fiber of $B_{\mathrm{SO}} \to B_{\mathrm{STOP}}$. (See § 15 or [90] for definition of $\otimes Q$).

PROPOSITION 10.3. — $\pi_* \mathrm{MSO} \otimes Q \cong \pi_* \mathrm{MSTOP} \otimes Q$.

Proof. — From 10.2 and the Thom isomorphism we have

$$H_*(\mathrm{MSO}\,; Q) \cong H_*(\mathrm{MSTOP}\,; Q) \quad .$$

Now use the Hurewicz isomorphism (Serre's from [75]).

PROPOSITION 10.4. — *$\Omega_*^{\mathrm{SO}} \otimes Q \cong \Omega_*^{\mathrm{STOP}} \otimes Q$ each being therefore the polynomial algebra freely generated by CP_{2n}, $n \geqslant 1$.*

Proof of 10.4. — The uncertainty about dimensions 3 and 4 in 10.1 cannot prevent this following from 10.2. Indeed, $\Omega_3^{\mathrm{STOP}} \to \pi_3 \mathrm{MSTOP}$ is injective because every TOP 3-manifold is smoothable (by Moise et al., cf. [80, § 5]). And

$$\Omega_4^{\mathrm{STOP}} \to \pi_4 \mathrm{MSTOP}$$

is rationally onto because $\Omega_4^{\mathrm{SO}} \to \pi_4 \mathrm{MSTOP}$ is rationally onto.

Since $\pi_i(\mathrm{STOP/SPL}) = \pi_i(\mathrm{TOP/PL})$ is Z_2 for $i = 3$ and zero for $i \neq 3$ the above three propositions can be repeated with SPL in place of SO and dyadic rationals $Z[\frac{1}{2}]$ in place of Q. The third becomes :

PROPOSITION 10.5. — $\Omega_*^{SPL} \otimes Z[\frac{1}{2}] \cong \Omega_*^{STOP} \otimes Z[\frac{1}{2}]$.

Next we recall

PROPOSITION 10.6. — (S.P. Novikov). $\Omega_*^{SO} \to \Omega_*^{STOP}$ *is injective.*

This is so because every element of Ω_*^{SO} is detected by its Stiefel-Whitney numbers (homotopy invariants) and its Pontrjagin numbers (which are topological invariants by 10.2).

In view of 10.2 we have canonical Pontrjagin characteristic classes p_k in

$$H^{4k}(B_{STOP}\ ; Q) = H^{4k}(B_{SO}\ ; Q)$$

and the related Hirzebruch classes $L_k = L_k(p_1, \ldots, p_k) \in H^{4k}$. Hirzebruch showed that $L_k : \Omega_{4k}^{SO} \otimes Q \to Q$ sending a $4k$-manifold M^{4k} to its characteristic number $L_k(M^{4k}) = L_k(\tau(M^{4k}))\,[M^{4k}] \in Q$ is the signature (index) homomorphism. From 10.2 and 10.4, it follows that the same holds for STOP in place of SO. Hence we have

PROPOSITION 10.7. — *For any closed oriented* TOP *$4k$-manifold M^{4k} the signature $\sigma(M^{4k})$ of the rational cohomology cup product pairing $H^{2k} \otimes H^{2k} \to H^{4k} = Q$ is given by $\sigma(M^{4k}) = L_k(\tau M^{4k})\,[M^{4k}] \in Z$.*

11. Oriented cobordism.

The first few cobordism groups are fun to compute geometrically — by elementary surgical methods, and the next few pages are devoted to this.

THEOREM 11.1. — $\Omega_n^{STOP} \simeq \Omega_n^{SO} \oplus R_n$ *for $n \leqslant 7$, and we have $R_n = 0$ for $n \leqslant 3$, $R_4 \leqslant Z_2, R_5 = 0$, $R_6 = Z_2$, $R_7 \leqslant Z_2$.*

Proof of 11.1. — For $n = 1, 2, 3$, $\Omega_n^{STOP} = \Omega_n^{SO} = 0$ is seen by smoothing.

For $n = 4$, first observe that $Z = \Omega_4^{SO} \to \Omega_4^{STOP}$ maps Z to a summand because the signature of a generator CP_2 is 1 which is indivisible. Next consider the Z_2 characteristic number of the first stable obstruction $\Delta \in H^4(B_{STOP}\ ; Z_2)$ to smoothing. It gives a homomorphism $\Omega_4^{STOP} \to Z_4$ killing Ω_4^{SO}. If

$$\Delta(M^4) \equiv \Delta(\tau(M^4))\,[M^4] = 0 \quad ,$$

then, by 5.3, $M^4 \times R$ has a DIFF structure Σ. Push the projection $(M^4 \times R)_\Sigma \to R$ to be transversal over $0 \in R$ at a DIFF submanifold M' and behold a TOP oriented cobordism M to M'. Thus $R_4 \leqslant 0$.

For $n \geqslant 5$ note that any oriented TOP manifold M^n is oriented cobordant to a simply connected one M' by a finite sequence of 0 and 1-dimensional surgeries. But, for $n = 5, H^4(M'\ ; Z_2) \cong H_1(M'\ ; Z_2) = 0$ so M' is smoothable. Hence $R_5 = 0$.

For $n = 6$ we prove

PROPOSITION 11.2. — *The characteristic number $\Delta w_2 : \Omega_6^{STOP} \to Z_2$ is an isomorphism.*

Proof. — It is clearly non-zero on any non-smoothable manifold $M^6 \simeq CP_3$, since $w_2(M^6) = w_2(CP_3) \neq 0$, and we will show that such a M^6 exists in 15.7 below.

Since $\Omega_6^{SO} = 0$ it remains to prove that Δw_2 is injective. Suppose $\Delta w_2(M^6) = 0$ for oriented M^6. As we have observed, we can assume M is simply connected. Consider the Poincaré dual $D\Delta$ of $\Delta = \Delta(\tau M)$ in

$$H_2(M^6\,;Z_2) = H_2(M^6\,;Z) \otimes Z_2 = \pi_2(M^6) \otimes Z_2$$

and observe that it can be represented by a locally flatly embedded 2-sphere $S \subset M^6$. (Hints : Use [24], or find an immersion of $S^2 \times R^4$ [52] and use the idea of Lemma 7.1).

Note that $\Delta|(M-S) = \Delta(M-S)$ is zero because $\Delta[x] = x \cdot D\Delta$ (the Z_2 intersection number) for all $x \in H_2(M\,;Z_2)$. Thus $M-S$ is smoothable.

A neighborhood of S is smoothed, there being no obstruction to this ; and S is made a DIFF submanifold of it. Let N be an open DIFF tubular neighborhood of S. Now $0 = \Delta w_2[M] = w_2[D\Delta] = w_2[S]$ means that $w_2(\tau M)|S$ is zero. Hence $N = S^2 \times R^4$. Killing S by surgery we produce M', oriented cobordant to M, so that, writing $M_0 = M - S^2 \times \mathring{B}^4$, we have $M' = M_0 + B^3 \times S^3$ (union with boundaries identified). Now M' is smoothable since M_0 is and there is no further obstruction. As $\Omega_6^{SO} = 0$, Proposition 11.2, is established.

PROPOSITION 11.3. — *The characteristic number* $(\beta\Delta)w_2 : \Omega_7^{STOP} \to Z_2$ *is injective, where* $\beta = Sq^1$.

Proof of 11.3. — We show the $(\beta\Delta)w_2[M] = 0$ implies M^7 is a boundary. Just as for 11.2, we can assume M is simply connected. Then $\pi_2 M = H_2(M\,;Z)$ and we can kill any element of the kernel of $w_2 : H_2(M\,;Z) \to Z_2$, by surgery on 2-spheres in M. Killing the entire kernel we arrange that w_2 is injective.

We have $0 = (\beta\Delta)w_2[M] = w_2[D\beta\Delta]$. So the Poincaré dual $D\beta\Delta$ of $\beta\Delta$ is zero as w_2 is injective.

Now $\beta\Delta = 0$ means Δ is reduced integral ; indeed β is the Bockstein

$$\delta : H^4(M\,;Z_2) \to H^5(M\,,Z)$$

followed by reduction mod 2. But

$$H^5(M\,;Z) \cong H^5(M\,;Z_2), \text{ since } H_2(M\,;Z) \cong H_2(M\,;Z_2)$$

(both isomorphisms by reduction). Thus $\beta\Delta = 0$ implies $\delta\Delta = 0$, which means Δ is reduced integral. Hence $D\Delta$ is reduced integral. Since the Hurewicz map $\pi_3 M \to H_3(M\,;Z)$ is onto, $D\Delta$ is represented by an embedded 3-sphere S. Following the argument for dimension 6 and recalling $\pi_2 O = 0$, we can do surgery on S to obtain a smoothable manifold.

12. Unoriented cobordism (*).

Recalling calculations of Ω_i^{O} and Ω_i^{SO} from Thom [91] we get the following table

i	4	5	6	7
Ω_i^{SO}	Z	Z_2	0	0
$\Omega_i^{STOP}/\Omega_i^{SO} = R_i$	$\leqslant Z_2$	0	Z_2	$R_7 \leqslant Z_2$
Ω_i^{O}	$Z_2 \oplus Z_2$	Z_2	$Z_2 \oplus Z_2 \oplus Z_2$	Z_2
$\Omega_i^{TOP}/\Omega_i^{O}$	$\leqslant R_4 \leqslant Z_2$	Z_2	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2 \oplus R_7$

The only non zero-entry for $0 < i < 4$ would be $\Omega_2^{O} = Z_2$.

To deduce the last row from the first three, use the related long exact sequences (from Dold [16])

(12.1)
$$\begin{array}{ccccccccc}
\ldots \to \Omega_i^{SO} & \to & \Omega_i^{O} & \xrightarrow{(\partial,d)} & \Omega_{i-1}^{SO} \oplus \Omega_{i-2}^{O} & \xrightarrow{j} & \Omega_{i-1}^{SO} & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots \to \Omega_i^{STOP} & \to & \Omega_i^{TOP} & \xrightarrow{(\partial,d)} & \Omega_{i-1}^{STOP} \oplus \Omega_{i-2}^{TOP} & \xrightarrow{j} & \Omega_{i-1}^{STOP} & \to & \ldots
\end{array}$$

If transversality fails $\pi_4(M\,?)$ should replace $\Omega_4^{?}$ in the TOP sequence. (See [93, §6], [3] for explanation).

All the maps are forgetful maps except those marked j and (∂, d). The map j kills the second summand, and is multiplication by 2 on the first summand (which is also the target of j).

At the level of representatives, ∂ maps M^i to a submanifold M^{i-1} dual to $w_1(M^i)$, and d maps M^i to $M^{i-2} \subset M^{i-1}$ dual to $w_1(M^i)|M^{i-1}$

The map d is onto with left inverse φ defined by associating to M^{i-2} the RP^2 bundle associated to $\lambda \oplus \epsilon^2$ over M^{i-2}, where λ is the line bundle with

$$w_1(\lambda) = w_1(M^{i-2})$$

and ϵ^2 is trivial.

The diagram (12.1) gives us the following generators for $S_i \equiv \Omega_i^{TOP}/\Omega_i^{O}$.

$S_4 \leqslant Z_2$: Any M^4 with $\Delta(M) \neq 0$ --- if it exists.

$S_5 = Z_2$: Any M^5 detected by Δw_1.

$S_6 = Z_2 \oplus Z_2$: Non-smoothable $M_1^6 \simeq CP_3$, detected by Δw_2 ;

$$M_2^6 \simeq RP_2 \times (Q \cup \infty)$$

(*) Added in proof : A complete calculation of Ω_*^{TOP} has just been announced by Brumfiel, Madsen and Milgram (Bull. AMS to appear).

detected by Δw_1^2. $M_2^6 \times R$ can be $\mathrm{RP}_2 \times \widetilde{X}^5$, where $\widetilde{X}^5$ is the universal covering of a manifold constructed in [80, § 5].

$S_7 = 2Z_2 \oplus R_7 : M_1^7 = \varphi M_1^5$ detected by Δw_1^3; M_2^7 is detected by $\Delta w_2 w_1$, $M_2^7 = T(\rho)$, the mapping torus of an orientation reversing homeomorphism of $M_1^6 \simeq \mathrm{CP}_3$ homotopic to complex conjugation in CP_3. Such a ρ exists because conjugation doesn't shift the normal invariant for $M_1^6 \simeq \mathrm{CP}_3$ in $[\mathrm{CP}_3, \mathrm{G/TOP}]$. Finally M_3^7 a generator of R_7 detected by $(\beta\Delta) w_2$ (if it exists).

13. Spin cobordism.

The stable classifying space B_{SPINTOP} is the fiber of $w_2 : B_{\mathrm{STOP}} \to K(Z_2, 2)$. So $\pi_i B_{\mathrm{SPINTOP}}$ is 0 for $i \leqslant 3$ and equals $\pi_i B_{\mathrm{TOP}}$ for $i \geqslant 3$. Topological spin cobordism is defined like smooth spin cobordism Ω_*^{SPIN} but using TOP manifolds. Thus $\Omega_*^{\mathrm{SPINTOP}}$ is the cobordism ring for compact TOP manifolds M equipped with a spin structure – i.e. a lifting to B_{SPINTOP} of a classifying map $M \to B_{\mathrm{TOP}}$ for $\tau(M)$ – or equivalently for the normal bundle $\nu(M)$.

THEOREM 13.1. — *For* $n \leqslant 7$, $\Omega_n^{\mathrm{SPINTOP}}$ *is isomorphic to* Ω_n^{SPIN}, *which for* $n = 0, 1, \ldots, 8$ *has the values* $Z, Z_2, Z_2, 0, Z, 0, 0, 0, Z \oplus Z$ [59] [86]. *The image of the forgetful map* $Z = \Omega_4^{\mathrm{SPIN}} \to \Omega_4^{\mathrm{SPINTOP}} = Z$ *is the kernel of the stable triangulation obstruction* $\Delta : \Omega_4^{\mathrm{SPINTOP}} \to Z_2$.

The question whether Δ is zero or not is the question whether or not Rohlin's congruence for signature $\sigma(M^4) \equiv 0 \bmod 16$ holds for all topological spin manifolds M^4. Indeed $\sigma(M^4) \equiv 8\Delta(M^4) \bmod 16$, $\Delta(M^4)$ being 0 or 1.

Proof of 13.1. — The isomorphism $\Omega_n^{\mathrm{SPINTOP}} \cong \Omega_n^{\mathrm{SPIN}}$ for $n \leqslant 3$ comes from smoothing.

Postponing dimension 4 to the last, we next show $\Omega_n^{\mathrm{SPINTOP}}/\Omega_n^{\mathrm{SPIN}} = 0$ for $n = 5, 6, 7$. Note first that a smoothing and a topological spin structure determine a unique smooth spin structure. The argument of § 11 shows that the only obstruction to performing oriented surgery on M^n to obtain a smooth manifold is a characteristic number, viz. 0, Δw_2, $(\beta\Delta) w_2$ for $n = 5, 6, 7$ respectively. But $w_2(M^n) = 0$ for any spin topological manifold. It remains to show that the surgeries can be performed so that each one, say from M to M', thought of as an elementary cobordism $(W^{n+1}; M^n, M'^n)$, can be given a topological spin structure extending that of M. The only obstruction to this occurs in $H^2(W, M; Z_2)$, which is zero except if the surgery is on a 1-sphere. And in that case we can obviously find a possibly different surgery on it (by spinning the normal bundle !) for which the obstruction is zero.

Finally we deal with dimension 4. If $\Delta(M^4) = 0$ for any spin 4-manifold, then M^4 is spin cobordant to a smooth spin manifold by the proof of 11.1. Next suppose M^4 is a topological spin manifold such that the characteristic number $\Delta(M^4)$ is not zero. If we can show that $\sigma(M^4) \equiv 8\Delta(M^4) \bmod 16$ the rest of 13.1 will follow, including the fact that $\Omega_4^{\mathrm{SPINTOP}} \cong Z$ rather than $Z \oplus Z_2$. We can assume M^4 connected (by surgery).

LEMMA 13.2. — *For any closed connected topological spin 4-manifold M^4, there exists a (stable)* TOP *bundle ξ over S^4 and a degree 1 map $M \to S^4$ covered by a* TOP *bundle map $\nu(M) \to \xi$. This ξ is necessarily fiber homotopically trivial. A similar result (similarly proved) holds for smooth spin manifolds.*

Proof of 13.2. — Since any map $M_0 \equiv M - (\text{point}) \to B_{\text{SPINTOP}}$ is contractible, $\nu(M)|M_0$ is trivial, and so $\nu(M) \to \xi$ exists as claimed. Now ξ is fiber homotopically trivial since it is – like $\nu(M)$ – reducible, hence a Spivak normal bundle for S^4. (Cf. proof in [40].)

LEMMA 13.3. — *A fiber homotopically trivialized* TOP *bundle ξ over the 4-sphere is (stably) a vector bundle iff $\frac{1}{3} p_1(\xi)[S^4] \equiv 0$ mod. 16.*

Proof of 13.3. — Consider the homomorphism $\frac{1}{3} p_1 : \pi_4 G/\text{TOP} \to Z$ given by associating the integer $\frac{1}{3} p_1(\xi)[S^4]$ to a such a bundle ξ over S^4. The composed map $\frac{1}{3} p_1 : \pi_4 G/0 \to Z$ sends a generator η to $\pm 16 \in Z$. Indeed, by Lemma 13.2 , DIFF transversality, and the Hirzebruch index theorem, $\frac{1}{3} p_1(\eta)$ is the least index of a closed smooth spin 4-manifold, which is ± 16 by Rohlin's theorem [40]. The lemma follows if we grant that $\pi_4 G/\text{TOP} = Z$ (not $Z \oplus Z_2$).

Now we complete 13.1. In $Z/16\,Z$ we have

$$\sigma(M^4) = \frac{1}{3} p_1(\tau M)[M] = \frac{1}{3} p_1(\xi)[S^4] = 8\Delta(\xi)[S^4] = 8\Delta(\tau(M))[M^4] = 8\Delta(M^4)$$

the third equality coming from the last lemma.

π_4G/TOP $= Z$ is used in 13.3 and in all following sections. So we prove it as

PROPOSITION 13.4. — *The forgetful map π_4 G/O $\to \pi_4$ G/TOP is $Z \xrightarrow{\times 2} Z$.*

Proof of 13.4. — (cf. naïve proof in [46A]). Since the cokernel is π_3TOP/O $=$ π_3TOP/PL $= Z_2$, it suffices to show that $\frac{1}{3} p_1 : \pi_4$ G/TOP $\to Z$ in the proof of 13.3 sends some element ζ to $\pm 8 \in Z$.

Such a ζ is constructed as follows. In §2, we constructed a closed TOP manifold X^{4+n} with $w_1(X) = w_2(X) = 0$ and a homotopy equivalence $f : N^4 \times T^n \simeq X^{4+n}$, where N^4 is a certain homology manifold (with one singularity) having $\sigma(N^4) = \pm 8$. Imitating the proof of 13.2 with N^4 and $\nu' = f^*\nu(X^{4+n})|N^4$ in place of M^4 and $\nu(M^4)$ we construct ξ over S^4 and $\nu' \to \xi$ over the degree 1 map $N^4 \to S^4$. This ξ is fiber homotopically trivial because $\nu(X), f^*\nu(X)$ and ν' are Spivak normal bundles. Let ξ represent ζ in π_4G/TOP.

It remains to show $\frac{1}{3} p_1(\zeta) = \pm 8$. First we reduce n to 1 in $f : N^4 \times T^n \simeq X^{4+n}$ by using repeatedly a splitting principle valid in dimension ≥ 6. (eg. use the TOP version of [76], or just the PL or DIFF version as in the latter part of 5.4 (a) in [80]). Consider the infinite cyclic covering

$$\overline{f} : N^4 \times R \simeq \overline{X}^5 \quad \text{of} \quad f : N^4 \times T^1 \simeq X^5 .$$

Splitting as above, we find that $CP_2 \times \overline{X}^5 \approx Y^8 \times R$ for some 8-manifold Y^8. Thus using the index theorem 10.7, and the multiplicativity of index and L-classes we have

$$\pm 8 = \sigma(N^4) = \sigma(CP_2 \times N) = \sigma(Y^8) = L_2(Y^8) = L_1(CP_2)\, L_1(\overline{X}^5)\, [CP_2 \times N] =$$

$$L_1(\overline{X}^5)\, [N^4] = - L_1(\nu')\, [N^4] = -\frac{1}{3} p_1(\nu')\, [N^4] = -\frac{1}{3} p_1(\xi)\, [S^4] = -\frac{1}{3} p_1(\xi).$$

(We have suppressed some natural (co)homology isomorphisms).

14. The periodicity of Casson and Sullivan.

A geometric construction of a "periodicity" map

$$\pi : G/PL \to \Omega^4 G/PL$$

was discovered by A.J. Casson in early 1967 (unpublished)(*).

He showed that the fiber of π is $K(Z_2, 3)$, and used this fact with the ideas of Novikov's proof of topological invariance of the rational Pontrjagin classes to establish the Hauptvermutung for closed simply connected PL manifolds $M^m, m \geqslant 5$, with $H^3(M^m, Z_2) = 0$. (Sullivan had a slightly stronger result [88]).

Now precisely the same construction produces a periodicity map π' in a homotopy commutative square

$$\text{(14.1)} \qquad \begin{array}{ccc} G/PL & \xrightarrow{\pi} & \Omega^4 G/PL \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \Omega^4\varphi} \\ G/TOP & \xrightarrow{\pi'} & \Omega^4 G/TOP \end{array}$$

The construction uses TOP versions of simply connected surgery and transversality. Recalling that the fiber of φ is $K(Z_2, 3)$ we see that $\Omega^4\varphi$ is a homotopy equivalence. Hence π' must be a homotopy equivalence. Thus $(\pi')^{-1} \circ (\Omega^4 \varphi)$ gives a homotopy identification of π to φ ; and an identification of the fiber of π to the fiber TOP/PL of φ. Thus TOP/PL had been found (but not identified) in 1967 !

The perfect periodicity $\pi' : G/TOP \simeq \Omega^4(G/TOP)$ is surely an attractive feature of TOP. It suggests that topological manifolds bear the simplest possible relation to their underlying homotopy types. This is a broad statement worth testing.

(*) Essentially the same construction was developed by Sullivan and Rourke later in 1967-68, see [72]. The "periodicity" π is implicit in Sullivan's analysis of G/PL as a fiber produc of $(G/PL)_{(2)}$ and $B_O \otimes Z[\frac{1}{2}]$ over $B_O \otimes Q$, [88] [89].

15. Hauptvermutung and triangulation for normal invariants; Sullivan's thesis(*).

Since $\mathrm{TOP/PL} \xrightarrow{j} \mathrm{G/PL} \xrightarrow{\varphi} \mathrm{G/TOP} \xrightarrow{\Delta} K(Z_2, 4)$ is a fibration sequence of H-spaces see 15.5 we have an exact sequence for any complex X

$$H^3(X; Z_2) = [X, \mathrm{TOP/PL}] \to [X, \mathrm{G/PL}] \xrightarrow{\varphi_*} [X, \mathrm{G/TOP}] \xrightarrow{\Delta_*} H^4(X; Z_2)$$

Examining the kernel and cokernel of φ using Sullivan's analysis of $\mathrm{G/PL})_{(2)}$(**) we will obtain

THEOREM 15.1. — *For any countable finite dimensional complex X there is an exact sequence of abelian groups* :

$$H^3(X; Z_2)/\mathrm{Image}\, H^3(X; Z) \xrightarrow{j_*} [X, \mathrm{G/PL}] \xrightarrow{\varphi_*} [X, \mathrm{G/TOP}] \xrightarrow{\Delta_*} \{\mathrm{Image}\,(H^4(X; Z) + \mathrm{Sq}^2 H^2(X; Z_2))\}$$

The right hand member is a subgroup of $H^4(X; Z_2)$, and j^* comes from

$$K(Z_2, 3) \simeq \mathrm{TOP/PL} \xrightarrow{j} \mathrm{G/PL}$$

In 1966-67, Sullivan showed that φ_* is injective provided that the left hand group vanishes. Geometrically interpreted, this implies that a homeomorphism $h: M' \to M$ of closed simply connected PL manifolds of dimension $\geqslant 5$ is homotopic to a PL homeomorphism if $H^3(M; Z_2)/\mathrm{Image}\, H^3(M; Z) = 0$, or equivalently if $H^4(M; Z)$ has no 2-torsion [88]. Here $[M, \mathrm{G/PL}]$ is geometrically interpreted as a group of *normal invariants*, represented by suitably equipped degree 1 maps $f: M' \to M$ of PL manifolds to M, cf. [95]. The relevant theorem of Sullivan is :

(15.2) *The Postnikov K-invariants of* G/PL, *except for the first, are all odd ; hence*

$$(\mathrm{G/PL})_{(2)} = \{K(Z_2; 2) \times_{\delta\mathrm{Sq}^2} K(Z_{(2)}, 4)\} \times K(Z_2, 6) \times K(Z_{(2)}, 8) \times K(Z_2, 10) \times K(Z_{(2)}, 12) \times \dots,$$

where $Z_{(2)} = Z\left[\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right]$ *is* Z *with* $\frac{1}{p}$ *for all odd primes* p *adjoined.* This is one of the chief results of Sullivan's thesis 1966 [87]. For expositions of it see [72] [13] [74] [89].

(*) Section 15 (indeed §§10-16) discusses corollaries of $\pi_3(\mathrm{TOP/PL}) = Z_2$ collected in spring 1969. For further information along these lines, the reader should see work of Hollingsworth and Morgan (1970) and S. Morita (1971) (added in proof).

(**) The localisation at 2, $A_{(2)} = A \otimes Z_{(2)}$ of a space A will occur below, only for countable H-spaces A such that, for countable finite dimensional complexes X, $[X, A]$ is an abelian group (usually a group of some sort of stable bundles under Whitney sum). Thus E.H. Brown's representation theorem offers a space $A_{(2)}$ and map $A \to A_{(2)}$ so that $[X, A] \otimes Z_{(2)} = [X, A_{(2)}]$. For a more comprehensive treatment of localisation see [89]. The space $A \otimes Q$ is defined similarly.

Sullivan's argument adapts to prove

(15.3) *The Postnikov K-invariants of* G/TOP *are all odd ; Hence*

$$(\mathrm{G/TOP})_{(2)} = K(Z_2, 2) \times K(Z_{(2)}, 4) \times K(Z_2, 6) \times K(Z_{(2)}, 8) \times \dots$$

Indeed his argument needs only the facts that (1) TOP surgery works, (2) the signature map $Z = \pi_{4k}(\mathrm{G/TOP}) \to Z$ is $\times\, 8$ (even for $k = 1$, by 13.4), and (3) the Arf invariant map $Z_2 = \pi_{4k+2}(\mathrm{G/TOP}) \to Z_2$ is an isomorphism.

Alternatively (15.2) $\Rightarrow$ (15.3) if we use $\Omega^4(\mathrm{G/PL}) \simeq \mathrm{G/TOP}$ from §14.

Remark 15.4.

It is easy to see directly that the 4-stage of G/TOP must be $K(Z_2, 2) \times K(Z, 4)$. For the only other possibility is the 4 stage of G/PL with K-invariant $\delta \mathrm{Sq}^2$ in $H^5(K(Z_2, 2), Z) = Z_2$. Then the fibration $K(Z_2, 3) = \mathrm{TOP/PL} \to \mathrm{GL/PL} \to \mathrm{G/TOP}$ would be impossible. (Hint : Look at the induced map of 4 stages and consider the transgression onto $\delta\mathrm{Sq}^2$). This remark suffices for many calculations in dimension $\leqslant 6$. On the other hand it is not clear to me that (15.2) $\Rightarrow$ (15.3) without geometry in TOP.

Proof that Kernel $\varphi \cong H^3(X ; Z_2)/\text{image } H^3(X ; Z)$.

This amounts to showing that for the natural fibration

$$\Omega\mathrm{G/TOP} \to \mathrm{TOP/PL} \to \mathrm{G/PL}$$

the image of $[X, \Omega\mathrm{G/TOP}]$ in $[X, \mathrm{TOP/PL}] = H^3(X ; Z_2)$, consists of the reduced integral cohomology classes. Clearly this is the image of $[X, \Omega(\mathrm{G/TOP})_{(2)}]$ under $\Omega(\mathrm{G/TOP})_{(2)} \xrightarrow{j_{(2)}} (\mathrm{TOP/PL})_{(2)} = \mathrm{TOP/PL}$. Now $j_{(2)}$ is integral reduction on the factor $K(Z_{(2)}, 3)$ of $\Omega(\mathrm{G/TOP})_{(2)}$ because $\pi_4(\mathrm{G/TOP}) \to \pi_3(\mathrm{TOP/PL})$ is onto, and it is clearly zero on other factors. The result follows. The argument comes from [13] [72].

Proof that $\mathrm{Coker}(\varphi) = \{\text{Image } H^4(X ; Z) + \mathrm{Sq}^2 H^2(X ; Z_2)\}$.

The following lemma is needed. Its proof is postponed to the end.

LEMMA 15.5. — *The triangulation obstruction* $\Delta : B_{\mathrm{TOP}} \to K(Z_2, 4)$ *is an H-map.*

Write $\varphi : A \to B$ for $\varphi : \mathrm{G/PL} \to \mathrm{G/TOP}$ and let $\varphi_4 : A_4 \to B_4$ be the induced map of Postnikov 4-stages, which have inherited H-space structure. Consider the fibration $A_4 \xrightarrow{\varphi_4} B_4 \xrightarrow{\Delta_4} K(Z_2, 4)$.

Assertion (1). — $(\Delta_4)_*[X, B_4] = \{\text{Image } H^4(X ; Z) + \mathrm{Sq}^2 H^2(X ; Z_2)\}$.

Proof of (1). — Since $B_4 = K(Z_2, 2) \times K(Z, 4)$ and

$$[X, B_4] = H^2(X ; Z_2) \oplus H^4(X ; Z)$$

what we have to show is that the class of Δ_4 in

$$[B_4, K(Z_2, 4)] = H^4(K(Z_2, 2) \times K(Z, 4) ; Z_2) =$$

$$= H^4(K(Z_2, 2), Z_2) \oplus H^4(K(Z, 4) ; Z_2)$$

is (Sq^2, ρ) where ρ is reduction mod 2.

The second component of Δ_4 is $\Delta_4 | K(Z, 4)$ which is indeed ρ since

$$Z = \pi_4 \mathrm{G/TOP} \twoheadrightarrow Z_2 = \pi_4 K(Z_2, 4) \quad .$$

The first component $\Delta_4 \mid K(Z_2, 2)$ can be Sq^2 or 0 a priori, but it cannot be 0 as that would imply $A_4 \simeq K(Z_2, 2) \times K(Z, 4)$. This establishes Assertion (1).

Assertion (2). – $(\Delta_4)_*[X, B_4] = \Delta_*[X, B]$ by the projection $B \to B_4$.

Proof of (2). – In view of 15.5, localising B_4 and B at 2 does not change the left and right hand sides. But after localization, we have equality since $B_{(2)}$ is the product (15.3).

The theorem follows quickly

$$[X, A]/\varphi_*[X, A] = \Delta_*[X, B] = (\Delta_4)_*[X, B_4] = \{\mathrm{Image}\, H^4(X, Z) + \mathrm{Sq}^2 H^2(X; Z_2)\}$$

The three equalities come from Iemma 15.5 and (1) and (2) respectively.

It remains now to give

Proof of Lemma 15.5 (S. Morita's, replacing something more geometrical).

We must establish homotopy commutativity of the square

$$\begin{array}{ccc} B_{\mathrm{TOP}} \times B_{\mathrm{TOP}} & \xrightarrow{\sigma} & B_{\mathrm{TOP}} \\ {\scriptstyle \Delta \times \Delta}\downarrow & & {\scriptstyle \Delta}\downarrow \\ K(Z_2, 4) \times K(Z_2, 4) & \xrightarrow{\alpha} & K(Z_2, 4) \end{array}$$

where σ represents Whitney sum and α represents addition in cohomology.

Now $\alpha \circ (\Delta \times \Delta)$ represents $\Delta \times 1 + 1 \times \Delta$ in $H^4(B_{\mathrm{TOP}} \times B_{\mathrm{TOP}} \; ; \; Z_2)$. Also $\Delta \circ \sigma$ certainly represents something of the form $\Delta \times 1 + 1 \times \Delta + \Sigma$, where Σ is a sum of products $x \times y$ with x, y each in one of $H^i(B_{\mathrm{TOP}}; Z_2) = H^i(B_{\mathrm{PL}} \; ; Z_2)$, for $i = 1,2$ or 3. Since $\Delta \circ \sigma$ restricted to $B_{\mathrm{PL}} \times B_{\mathrm{PL}}$ is zero, Σ must be zero.

Theorem 15.1 is very convenient for calculations. Let M be a closed PL manifold, m-manifold $m \geqslant 5$, and write $\mathcal{S}_{\mathrm{CAT}}(M)$, CAT = PL or TOP, for the set of h-cobordism classes of closed CAT m-manifolds M' equipped with a homotopy equivalence $f: M' \to M$. (See [95] for details).

There is an exact sequence of pointed sets (extending to the left) :

$$\ldots \to [\Sigma M, \mathrm{G/CAT}] \to L_{m+1}(\pi, w_1) \to \mathcal{S}_{\mathrm{CAT}}(M) \xrightarrow{\nu} [M, \mathrm{G/CAT}] \to L_m(\pi, w_1) \quad .$$

It is due to Sullivan and Wall [95]. The map ν equips each $f: M' \to M$(above) as a CAT normal invariant. Exactness at $\mathcal{S}_{\mathrm{CAT}}(M)$ is relative to an action of $L_{m+1}(\pi, w_1)$ on it. Here $L_k(\pi, w_1)$ is the surgery group of Wall in dimension k for fundamental group $\pi = \pi_1 M$ and for orientation map $w_1 = w_1(M)\, ; \pi \to Z_2$. There is a generalisation for manifolds with boundary. Since the PL sequence maps naturally to the TOP sequence, our knowledge of the kernel and cokernel of

$$[M, \mathrm{G/PL}] \to [M, \mathrm{G/TOP}]$$

will give a lot of information about $\mathcal{S}_{\mathrm{PL}}(M) \to \mathcal{S}_{\mathrm{TOP}}(M)$. Roughly speaking failure of triangulability in $\mathcal{S}_{\mathrm{TOP}}(M)$ is detected by non triangulability of the TOP normal invariant ; and failure of Hauptvermutung in $\mathcal{S}_{\mathrm{PL}}(M)$ cannot be less than its failure for the corresponding PL normal invariants.

In case $\pi_1 M = 0$, one has $\mathcal{S}_{\mathrm{CAT}}(M) \cong \mathcal{S}_{\mathrm{CAT}}(M_0) \cong [M_0, \mathrm{G/CAT}]$ where M_0 is M with an open m-simplex deleted, and so Theorem 15.1 here gives complete information.

Example 15.6. — The exotic PL structure Σ on $S^3 \times S^n$, $n \geqslant 2$, from

$$1 \in H^3(S^3; Z_2) = Z_2$$

admits a PL isomorphism $(S^3 \times S^n)_\Sigma \cong S^3 \times S^n$ homotopic (not TOP isotopic) to the identity.

Example 15.7. — For $M = \mathrm{CP}_n$ (= complex projective space), $n \geqslant 3$, the map $[M_0, \mathrm{G/PL}] \to [M_0, \mathrm{G/TOP}]$ is injective with cokernel $Z_2 = H^4(M_0 . Z_2)$. This means that 'half' of all manifolds $M' \simeq \mathrm{CP}_n$, $n \geqslant 3$, have PL structure. Such a PL structure is unique up to isotopy, since $H^3(\mathrm{CP}_n, Z_2) = 0$.

16. **Manifolds homotopy equivalent to real projective space P^n.**

After sketching the general situation, we will have a look at an explicit example of failure of the Hauptvermutung in dimension 5.

From [54] [94] we recall that, for $n \geqslant 4$,

$$[P^n, \mathrm{G/PL}] = Z_4 \oplus \sum_{i=6}^{n} \pi_i(\mathrm{G/PL}) \otimes Z_2 \quad . \tag{16.1}$$

This follows easily from (15.2). For G/TOP the calculation is only simpler. One gets

$$[P^n, \mathrm{G/TOP}] = \sum_{i=2}^{n} \pi_i(\mathrm{G/TOP}) \otimes Z_2 \quad . \tag{16.2}$$

Calculation of $\mathcal{S}_{\mathrm{PL}}(P^n) \equiv I_n$ is non-trivial [54] [94]. One gets (for $i \geqslant 1$)

$$I_{4i+2} = I_{4i+1} = [P^{4i}, \mathrm{G/PL}] \,; I_{4i+3} = I_{4i+2} \oplus Z \,; I_{4i+4} = I_{4i+2} \oplus Z_2 \,. \tag{16.3}$$

The result for $\mathcal{S}_{\mathrm{TOP}}(P^n)$ is similar, when one uses TOP surgery. Then

$$\mathcal{S}_{\mathrm{PL}}(P^n) \to \mathcal{S}_{\mathrm{TOP}}(P^n)$$

is described as the direct sum of an isomorphism with the map

$$Z_4 = [P^4, \mathrm{G/PL}] \to [P^4, \mathrm{G/TOP}] = Z_2 \oplus Z_2 \quad ,$$

which sends Z_4 onto $Z_2 = \pi_2 \mathrm{G/TOP}$.

Remark 16.4. — When two distinct elements of $\mathcal{S}_{\mathrm{PL}}(P^n)$, $n \geqslant 5$, are topologically the same, we know already from 15.1 that their PL normal invariants are distinct since $H^3(P^n; Z_2)$ is not reduced integral. This facilitates detection of examples.

Consider the fixed point free involution T on the Brieskorn-Pham sphere in C^{m+1}

$$\Sigma_d^{2m-1} : z_0^d + z_1^2 + z_2^2 + \cdots z_m^2 = 0 \quad , \quad |z| = 1 \quad ,$$

given by $T(z_0, z_1, \ldots, z_m) = (z_0, -z_1, \ldots, -z_m)$. Here d and m must be odd positive integers, $m \geq 3$, in order that Σ_d^{2m-1} really be topologically a sphere [61].

As T is a fixed point free involution the orbit space $\Pi_d^{2m-1} \equiv \Sigma_d^{2m-1}/T$ is a DIFF manifold. And using obstruction theory one finds there is just one oriented equivalence $\Pi_d^{2m-1} \to P^{2m-1}$ (Recall $P^\infty = K(Z_2, 1)$). Its class in $\mathcal{S}_{\mathrm{CAT}}(P^{2m-1})$ clearly determines the involution up to equivariant CAT isomorphism and conversely.

THEOREM 16.5. — *The manifolds Π_d^5, d odd, fall into four diffeomorphism classes according as $d \equiv 1, 3, 5, 7$ mod 8, and into two homeomorphism classes according as $d = \pm 1, \pm 3$ mod 8. Π_1^5 is diffeomorphic to P^5.*

Remark 16.6. — With Whitehead C^1 triangulations, the manifolds Π_d^5 have a PL isomorphism classification that coincides with the DIFF classification (§5, [9] [64]). Hence we have here rather explicit counterexamples to the Hauptvermutung. One can check that they don't depend on Sullivan's complete analysis of $(G/PL)_{(2)}$. The easily calculated 4-stage suffices. Nor do they depend on topological surgery.

PROBLEM. — Give an explicit homeomorphism $P^5 \approx \Pi_7^5$.

Remark 16.7. — Giffen states [23] that (with Whitehead C^1 triangulations) the manifolds Π_d^{2m-1}, $m = 5, 7, 9, \ldots$ fall into just four PL isomorphism classes $d \equiv 1, 3, 5, 7$ mod 8. In view of theorem 16.4., these classes are already distinguished by the restriction of the normal invariant to P^5 (which is that of Π_d^5). So Giffen's statement implies that the homeomorphism classification is $d \equiv \pm 1$, ± 3 mod 8.

Proof of 16.5. (**)

The first means of detecting exotic involutions on S^5, was found by Hirsch and Milnor 1963 [32]. They constructed explicit(*) involutions (M_{2r-1}^7, β_r), r an integer ≥ 0, on Milnor's original homotopy 7-spheres, and found invariant spheres $M_{2r-1}^7 \supset M_{2r-1}^6 \supset M_{2r-1}^5$. They observed that the class of M_{2r-1}^7 in $\Gamma_{28}/2\Gamma_{28}$ is an invariant of the DIFF involution (M_{2r-1}^5, β_r) — (consider the suspension operation to retrieve (M_{2r-1}^7, β) and use $\Gamma_6 = 0$). Now the class of M_{2r-1}^7 in $Z_{28} = \Gamma_7$ is $r(r-1)/2$ according to Eells and Kuiper [18], which is odd iff $r \equiv 2$ or 3 mod 4. So this argument shows (M_{2r-1}^5, β_r) is an exotic involution if $r \equiv 2$ or 3 mod 4.

Fortunately the involution (M_{2r-1}^5, β_r) has been identified with the involution (Σ_{2r+1}^5, T).

(*) β_r is the antipodal map on the fibers of the orthogonal 3-sphere bundle M_{2r-1}^7.

(**) See major correction added on pg. 337.

There were two steps. In 1963 certain examples (X^5, α_r) of involutions were given by Bredon, which Yang [101] explicitly identified with (M^7_{2r-1}, β_r). Bredon's involutions extend to O(3) actions, α_r being the antipodal involution in O(3). And for any reflection α in O(3) , α has fixed point set diffeomorphic to $L^3(2r+1, 1) : z_0^{2r+1} + z_1^2 + z_2^2 = 0 ; |z| = 1$. This property is clearly shared by (Σ^5_{2r+1}, T), and Hirzebruch used this fact to identify (Σ^5_{2r+1}, T) to (X, α_r) [33, §4] [34]. The Hirsch-Milnor information now says that Π^5_d is DIFF exotic if $d \equiv 5,7 \bmod 8$.

Next we give a TOP invariant for Π_d in Z_2. Consider the normal invariant ν_d of Π_d in $[P^5, G/O] = Z_4$. Its restriction $\nu_d | P^2$ to P^2 is a TOP invariant because $[P^2, G/O] = [P^2, G/TOP] = Z_2$.

Now Giffen [22] shows that $\nu_d | P^2$ is the Arf invariant in Z_2 of the framed fiber of the torus knot $z_0^d + z_1^2 = 0, |z_0|^2 + |z_1|^2 = 1$ in $S^3 \subset C^2$. This turns out to be 0 for $d \equiv \pm 1 \bmod 8$ and 1 for $d \equiv \pm 3 \bmod 8$, (Levine [53], cf. [61, § 8]).

We have now shown that the diffeomorphism and homeomorphism classifications of the manifolds Π^5_d are *at least* as fine as asserted. But there can be at most the four diffeomorphism classes named, in view of 16.3. (Recall that the PL and DIFF classifications coincide since $\Gamma_i = \pi_i(PL/O) = 0 , i \leqslant 5$). Hence, by Remark 16.4, there are exactly four – two in each homeomorphism class.

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Correction to proof of 16.5 : Glen Bredon has informed me that [101] is incorrect, and that in fact (X^5, α_r) can be identified to $(M^5_{2r+1}, \beta_{r+1})$. Thus, a different argument is required to show that the DIFF manifolds Π^5_d , $d \equiv 1, 3, 5, 7 \bmod 8$, respectively, occupy the four distinct diffeomorphism classes of DIFF 5-manifolds homotopy equivalent to P^5 . The only proof of this available in 1975 is the one provided by M. F. Atiyah in the note reproduced overleaf. So many mistakes, small and large, have been committed with these involutions that it would perhaps be wise to seek *several* proofs.