Generalisations and applications of block bundles

By A. J. Casson

Introduction

Rourke and Sanderson [10] introduced the idea of a **block bundle**. They used block bundles in the PL (piecewise linear) category as a substitute for the normal bundles of differential topology. Their block bundles had fibre I^q (the unit cube in q-dimensional space).

We generalise the idea to allow any compact PL manifold F as fibre. Chapter I sets up the theory; in particular, it is shown that there is a classifying space $B\widetilde{PL}_F$ for block bundles with fibre F.

In Chapter II we compare block bundles with Hurewicz fibrations. Let F be a compact PL manifold with boundary ∂F, and let BG_F classify Hurewicz fibrations with fibre $(F, \partial F)$. We produce a map $\chi : B\widetilde{PL}_F \longrightarrow BG_F$, arising from a natural transformation of bundle functors.

We wish to obtain information about $B\widetilde{PL}_F$; in fact we can study BG_F (which is purely homotopy theoretic) and the fibre $G_F/\widetilde{PL}_F$ of χ. In Chapter III we construct a map $\theta : G_F/\widetilde{PL}_F \longrightarrow (G/PL)^F$, where G/PL is the space studied in Sullivan's thesis under the name F/PL, and $(G/PL)^F$ is the space of all <u>unbased</u> maps from F to G/PL. Theorems 5,6 show that, under suitable conditions, θ is almost a homotopy equivalence. For these results it is essential to work with block bundles rather than fibre bundles.

Sullivan shows in his thesis (see [13] for a summary) that G/PL is closely related to the problem of classifying PL manifolds homotopy equivalent to a given manifold. Therefore it is important to have information about the homotopy type of G/PL. In Chapter IV we apply Theorem 5 to show that $\Omega^4(G/PL)$ is homotopy equivalent to $\Omega^8(G/PL)$; it is almost true that G/PL is homotopy equivalent to $\Omega^4(G/PL)$.

In Chapter V (which is almost independent of the earlier chapters) we show that, with certain restrictions on the base-space, a block bundle with fibre $\mathbb{R}^q$ which is topologically trivial is necessarily piecewise linearly trivial. It follows from this (again using the results of [13]) that the Hauptvermutung is true for closed 1-connected PL manifolds M with $\dim M \geq 5$ and $H^3(M; \mathbb{Z}_2) = 0$.

I should like to thank Professor C.T.C.Wall for suggesting the study of generalized block bundles and for much encouragement. I am also very grateful to Dr. D.P.Sullivan for several conversations during the summer of 1966.

Origins of the ideas

Chapter I is based on §1 of [10]; the definitions and technical details are new, but the general plan is similar. The use of block bundles with arbitrary fibres was suggested to me by Professor Wall.

Chapter II is mainly technical, and new as far as I know.

Chapter III generalizes results in Sullivan's thesis (but the proofs are based on the references given rather than on Sullivan's work).

The result and method of proof in Chapter IV is new, as far as I know.

I believe that Sullivan* has a stronger result than Theorem 8 of Chapter V, but have not seen his proofs. I proved Theorem 8 before hearing of Sullivan's latest result. My proof is an extension of the idea of [16].

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* See D.P.Sullivan, **On the Hauptvermutung for manifolds**, Bull. Amer. Math. Soc. 73 (1967) 598-600. The theorem announced there includes ours, but the proof seems somewhat different.

I. Block Bundles

A **polyhedron** is a topological space together with a maximal family of PL related locally finite triangulations. A **cell complex** B is a collection of cells PL embedded in a polyhedron X such that:

(1) B is a locally finite covering of X,
(2) if $\beta, \gamma \in B$ then $\partial\beta, \beta \cap \gamma$ are unions of cells of B,
(3) if β, γ are distinct cells of B, then $\operatorname{Int}\beta \cap \operatorname{Int}\gamma = \emptyset$.

We write $|B|$ for X and do not distinguish between a cell β of B and the subcomplex it determines. A cell complex B' is a **subdivision** of B if $|B'| = |B|$ and every cell of B is a union of cells of B'. A **based polyhedron** is a polyhedron with a preferred base-point; a **based cell complex** is a cell complex with a preferred vertex. All base-points will be denoted by 'bpt'.

Let F be a polyhedron and let B be a based cell complex. A **block bundle** ξ over B with **fibre** F consists of a polyhedron $E(\xi)$ (the **total space** of ξ) with a closed sub-polyhedron $E_\beta(\xi)$ for each $\beta \in B$ and a PL homeomorphism $b(\xi) : F \longrightarrow E_{\mathrm{bpt}}(\xi)$, such that:

(1) $\{E_\beta(\xi) | \beta \in B\}$ is a locally finite covering of $E(\xi)$,
(2) if $\beta, \gamma \in B$ then

$$E_\beta(\xi) \cap E_\gamma(\xi) \;=\; \bigcup_{\delta \subset \beta\cap\gamma} E_\delta(\xi) \ ,$$

(3) if $\beta \in B$, there is a PL homeomorphism $h : F \times \beta \longrightarrow E_\beta(\xi)$ such that

$$h(F \times \gamma) \;=\; E_\gamma(\xi) \;\; (\gamma \subset \partial\beta) \ .$$

If ξ is a block bundle over B and B_0 is a subcomplex of B, the **restriction** $\xi|B_0$ is defined by

$$E(\xi|B_0) \;=\; \bigcup_{\beta \in B_0 \cup \{\mathrm{bpt.}\}} E_\beta(\xi) \ \ ,$$

$$E_\beta(\xi|_{B_0}) \;=\; E_\beta(\xi) \ \ , \ \ b(\xi|B_0) \;=\; b(\xi) \ .$$

Note that $\xi|B_0$ is a block bundle over $B_0 \cup \{\mathrm{bpt.}\}$, not necessarily over B_0 itself.

If ξ, η are block bundles over B, an **isomorphism** $h : \xi \longrightarrow \eta$ is a PL homeomorphism $h : E(\xi) \longrightarrow E(\eta)$ such that

$$hE_\beta(\xi) \;=\; E_\beta(\eta) \;\; (\beta \in B) \ \ , \ \ hb(\xi) \;=\; b(\eta) \ .$$

A particular block bundle ϵ over B is obtained by setting

$$E(\epsilon) \;=\; F \times B \ , \ E_\beta(\epsilon) \;=\; F \times \beta \ ,$$

$$b(\epsilon) \;=\; 1 \times \mathrm{bpt} \ : \ F \longrightarrow F \times \mathrm{bpt} \ .$$

A trivial block bundle is one isomorphic to ϵ; an isomorphism $h : \epsilon \longrightarrow \xi$ is a **trivialisation** of ξ. It follows from condition (3) that $\xi|\beta$ is trivial for each $\beta \in B$.

Let B, C be based cell complexes and let ξ be a block bundle over B. Define a block bundle $\xi \times C$ over $B \times C$ by

$$E(\xi \times C) \ = \ E(\xi) \times C \ , \ E_{\beta\times\gamma}(\xi \times C) \ = \ E_\beta(\xi) \times \gamma$$

(for cells $\beta \in B, \gamma \in C$) and $b(\xi \times C) = b(\xi) \times \mathrm{bpt}$.

Lemma 1. *Suppose $|B| = \beta$, where β is an n-cell of β, and let γ be an $(n-1)$-cell over B. If ξ and η are block bundles over B, any isomorphism $h : \xi|(\partial\beta - \gamma) \longrightarrow \eta|(\partial\beta - \gamma)$ can be extended to an isomorphism $h : \xi \longrightarrow \eta$.*

Proof. Since $\xi = \xi|\beta, \eta = \eta|\beta$, ξ and η are both trivial. Let k and l be trivialisations of ξ, η, respectively. Then a PL homeomorphism

$$l^{-1}hk \ : \ F \times (\partial\beta - \gamma) \longrightarrow F \times (\partial\beta - \gamma)$$

is defined.

Choose a PL homeomorphism

$$f : \ (\partial\beta - \gamma) \times I \longrightarrow B$$

such that $f_0 : (\partial\beta - \gamma) \longrightarrow B$ is the inclusion, and let

$$g \ = \ 1 \times f \ : \ F \times (\partial\beta - \gamma) \times I \longrightarrow F \times \beta.$$

The required extension of f is given by

$$h \ = \ lg(l^{-1}hk \times I)g^{-1}k^{-1} \ : \ E(\xi) \longrightarrow E(\eta) \ .$$

Lemma 2. *Let B be a based cell complex and take* $\mathrm{bpt} \times 0$ *as base-point for $B \times I$. If ξ, η are block bundles over $B \times I$, then any isomorphism*

$$h \ : \ \eta|(B \times 0) \cup (\mathrm{bpt} \times I) \longrightarrow \eta|(B \times 0) \cup (\mathrm{bpt} \times I)$$

can be extended to an isomorphism $h : \xi \longrightarrow \eta$.

Proof. Write B^r for the r-skeleton of B, and let $C^r = (B \times 0) \cup (B^r \times I)$. Suppose inductively that h can be extended to an isomorphism $h : \xi|C^r \longrightarrow \eta|C^r$; the induction starts trivially with $r = 0$. Let β be an $(r+1)$-cell of B. By Lemma 1,

$$h \ : \ \eta|(\beta \times 0) \cup (\partial\beta \times I) \longrightarrow \eta|(\beta \times 0) \cup (\partial\beta \times I)$$

can be extended to an isomorphism $h : \xi|(\beta \times I) \longrightarrow \eta|(\beta \times I)$. Thus we have defined an isomorphism

$$h \ : \ \xi|C^r \cup (\beta \times I) \longrightarrow \eta|C^r \cup (\beta \times I) \ .$$

Do this for all r-cells of B to obtain

$$h \ : \ \xi|C^{r+1} \longrightarrow \eta|C^{r+1}$$

extending the given isomorphism. The Lemma now follows by induction.

Let ξ be a block bundle over B and let B' be a subdivision of B. A block bundle ξ' over B' is a subdivision of ξ if $E(\xi') = E(\xi), E_{\beta'}(\xi') \subset E_\beta(\xi)$ (for all cells $\beta' \in B', \beta \in B$ with $\beta' \subset \beta$) and $b(\xi') = b(\xi)$.

Theorem 1. *Let B' be a subdivision of a cell complex B. Any block bundle over B' is a subdivision of some block bundle over B. Any block bundle ξ over B has a subdivision over B', and any two subdivisions of ξ over B' are isomorphic.*

Proof. First we prove the following propositions together by induction on n.

P_n : If $|B|$ is homeomorphic to an n-cell, then any block bundle over B is trivial.

Q_n : Let $\dim B \leq n$ and let B_0 be a subcomplex of B. Let B' be a subdivision of B, inducing subdivision B_0' of B_0. Let ξ be a block bundle over B and let ξ_0' be a subdivision of $\xi|B_0$ over B_0'. Then there is a subdivision ξ' of ξ over B' such that $\xi_0' = \xi'|B_0'$.

Observe that P_0 and Q_0 are both true. We shall prove that $Q_n \Longrightarrow P_n$ and $P_n \,\&\, Q_n \Longrightarrow Q_{n+1}$.

Proof that $Q_n \Longrightarrow P_n$. Suppose $|B|$ is homeomorphic to an n-cell, and let ξ be a block bundle over B. Since $|B|$ is collapsible, there is a simplicial subdivision B' of B which collapses simplicially to the base point [19]. Assuming Q_n, there is a subdivision ξ' of ξ over B'. It is enough to prove that ξ' is trivial.

Let

$$B' \;=\; K_k \;\searrow^s\; K_{k-1} \;\searrow^s\; \ldots \;\searrow^s\; K_0 \;=\; \{\text{bpt.}\}$$

be a sequence of elementary simplicial collapses. Suppose inductively that $\xi'|K_r$ is trivial; the induction starts with $r = 0$. Write

$$K_{r+1} \;=\; K_r \cup \triangle \;,\; K_r \cap \triangle \;=\; \Lambda \;,$$

where $\triangle$ is a simplex of K_{r+1} and Λ is the complement of a principal simplex in $\partial\triangle$. Let $h : F \times K_r \longrightarrow E(\xi'|K_r)$ be a trivialisation of $\xi'|K_r$. By Lemma 1, $h|F \times \Lambda$ extends to a trivialisation of $\xi|\triangle$. Thus we obtain a trivialisation of $\xi''|K_{r+1}$. By induction, ξ' is trivial, as required.

Proof that $P_n \,\&\, Q_n \Longrightarrow Q_{n+1}$. Suppose B, B_0, B', ξ, ξ_0' satisfy the hypotheses of Q_{n+1}. If A is any subcomplex of B, we write A' for the subdivision of A induced by B'. Let $B_1 = B_0 \cup B^n$, assuming Q_n there is a subdivision ξ_1' of $\xi|B_1$ over B_1' such that $\xi_0'|(B_0 \cap B^n)' = \xi_1'|(B_0 \cap B^n)'$. Let β be an $(n+1)$-cell of $B - B_0$, and let

γ be an n-cell of B contained in $\partial\beta$. Since $|\partial\beta-\gamma|$ is homeomorphic to an n-cell, $\xi_1'|(\partial\beta-\gamma)'$ is trivial by P_n. Let h be a trivialisation of $\xi_1'|(\partial\beta-\gamma)'$; a fortiori, h is a trivialisation of $\xi|(\partial\beta-\gamma)$.

By Lemma 1, h extends to a trivialisation of $\xi|\beta$. Let C be the cell complex consisting of β,γ and the cells of $(\partial\beta-\gamma)'$. Define a block bundle η over C by

$$E_\beta(\eta) \;=\; E_\beta(\xi)\ ,\ E_\gamma(\eta) \;=\; E_\gamma(\xi)$$

and $E_{\delta'}(\eta)=E_{\delta'}(\xi_1')$ for each cell δ' of $(\partial\beta-\gamma)'$. Then k is a trivialisation of η, so η satisfies condition (3) in the definition of block bundle.

Let δ' be an n-cell of $(\partial\beta-\gamma)'$, so $|\partial\beta'-\delta'|$ is homeomorphic to an n-cell. Assuming $P_n, \xi_1'|(\partial\beta'-\delta')$ is trivial; let h' be a trivialisation. A fortiori, h' is a trivialisation of $\eta|(\partial\beta'-\delta')$.

By Lemma 1, h' extends to a trivialisation k' of η. In fact, $k'|F\times\partial\beta'$ is a trivialisation of $\xi_1'|\partial\beta'$, because k' extends h' and $k'(F\times\delta')=E_{\delta'}(\xi_1')$. To extend $\xi_1'|\partial\beta'$ to a subdivision $\xi'|\beta'$ of $\xi|\beta$, we define $E_{\alpha'}(\xi')=k'(F\times\alpha')$ for each cell α' of β'.

Do this for all $(n+1)$-cells of $B-B_0$, and define $\xi'|\beta=\xi_0'|\beta$ for each $(n+1)$-cell β of B_0. We obtain a subdivision ξ' of ξ over B' such that $\xi_0'=\xi'|B_0'$, as required.

By induction, P_n and Q_n are true for all n. Let B be any based cell complex and let B' be a subdivision of B.

Let ξ' be a block bundle over B'. We define a block bundle ξ over B with $E(\xi)=E(\xi')$ by setting $E_\beta(\xi)=E(\xi'|\beta')$ (where β' is the subdivision of β induced by B') for each cell β of B. This clearly satisfies conditions (1),(2) in the definition of block bundle. By P_n, $\xi'|\beta'$ is trivial, so ξ also satisfies condition (3). Clearly, ξ' is a subdivision of ξ.

If ξ is a block bundle over B, it follows from Q_n (by induction on the skeleton of B) that ξ has a subdivision over B'. Let ξ_0',ξ_1' be two such subdivisions. Recall that $\eta=\xi\times I$ is a block bundle over $B\times I$. Define a block bundle η_0' over $B'\times\partial I$ by $\xi_t'=\eta_0'|B'\times\{t\}$, $(t=0,1)$. Again it follows from Q_n that η has a subdivision η' over $B'\times I$ such that $\eta_0'=\eta'|B'\times\partial I$. Observe that $\eta'|\mathrm{bpt}\times I=\xi_0'\times I|\mathrm{bpt}\times I$. By Lemma 2, the identity isomorphism

$$\eta'|(B\times 0)\cup(\mathrm{bpt}\times I)\longrightarrow \xi_0'\times I|(B\times 0)\cup(\mathrm{bpt}\times I)$$

extends to an isomorphism $\eta'\longrightarrow\xi_0'\times I$; it follows that $\xi_1'\cong\xi_0'$. This completes the proof of Theorem 1.

Let X be a polyhedron and let B, C be cell complexes with $|B| = |C| = X$; suppose all three have the same base-point. Let ξ, η be block bundles over B, C respectively. We call ξ, η **equivalent** if, for some common subdivision D of B, C, the subdivision ξ over D is isomorphic to the subdivision of η over D. This relation is clearly reflexive and symmetric; by Theorem 1 it is also transitive.

Let $I_F(X)$ be the set of equivalence classes of block bundles over cell complexes B with $|B| = X$. It is easily checked that, if $|B| = X$, then each member of $I_F(X)$ is represented by a unique isomorphism class of block bundles over B.

Suppose X, Y are polyhedra and let $y \in I_F(Y)$. Let B, C be cell complexes with $|B| = X, |C| = Y$, and let η be a block bundle over C representing y. If $p_2 : X \times Y \longrightarrow Y$ is the projection, let $p_2^*(y) \in I_F(X \times Y)$ be the equivalence class of $B \times \eta$.

If $i : X \longrightarrow Y$ is a closed based PL embedding, let C' be a subdivision of C with a subcomplex D' such that $|D'| = i(X)$. Let η' be a subdivision of η over C', and let $\xi' = \eta' | D'$. It follows from Theorem 1 that the equivalence class $x' \in I_F(i(X))$ of ξ' depends only on y. Let $i^*(y) \in I_F(X)$ correspond to x' via the PL homeomorphism $i : X \longrightarrow i(X)$. The next lemma will enable us to define $f^* : I_F(Y) \longrightarrow I_F(X)$ for any based PL map $f : X \longrightarrow Y$.

Lemma 3. *Let X, Y, V, W be polyhedra and let $i : X \longrightarrow V \times Y$, $j : X \longrightarrow W \times Y$ be closed based PL embeddings such that $p_2 i = p_2 j : X \longrightarrow Y$. Then*

$$i^* p_2^* \;=\; j^* p_2^* \;:\; I_F(Y) \longrightarrow I_F(X)\ .$$

Proof. Let $k : X \longrightarrow V \times W \times Y$ be defined by

$$p_{13} k = \;i\; : \;X \longrightarrow V \times Y\ ,$$

$$p_{23} k \;=\; j\; : \;X \longrightarrow W \times Y\ .$$

In the diagram

$$\begin{tikzcd}
 & & I_F(V \times Y) \arrow[dl, "i^*"'] \arrow[d, "p_{13}^*"'] & \\
(1) & I_F(X) & I_F(V \times W \times Y) \arrow[l, "k^*"'] & I_F(Y) \arrow[ul, "p_2^*"'] \arrow[l, "p_3^*"'] \arrow[dl, "p_2^*"] \\
 & & I_F(W \times Y) \arrow[ul, "j^*"] \arrow[u, "p_{23}^*"'] &
\end{tikzcd}$$

the right-hand triangles are clearly commutative. We prove that the bottom left-hand triangle commutes. There is a contractible polyhedron Z and a closed based PL embedding $l : V \longrightarrow Z$. Consider the diagram

$$\begin{tikzcd}
I_F(X) & & I_F(V \times W \times Y) \arrow[ll, "k^*"'] \\
& I_F(Z \times W \times Y) \arrow[ul, "(\mathrm{bpt} \times j)^*"] \arrow[ur, "(l \times 1)^*"'] & \\
& I_F(W \times Y) \arrow[uul, "j^*"] \arrow[u, "p_{23}^*"] \arrow[uur, "p_{23}^*"'] &
\end{tikzcd}$$

The bottom two triangles clearly commute. Since Z is contractible to its base-point, $p_1(l \times 1)k \simeq p_1(\mathrm{bpt} \times j)$. But $p_{23}(l \times 1)k = j = p_{23}(\mathrm{bpt} \times j)$, so there is a closed based PL isotopy between $(l \times 1)k$ and $(\mathrm{bpt} \times j)$. It follows from Lemma 2 that $((l \times 1)k)^* = (\mathrm{bpt} \times j)^*$. Clearly $k^*(l \times 1)^* = ((l \times 1)k)^*$, so the top triangle commutes. Therefore the bottom left-hand triangle in diagram (1) commutes, so the Lemma is proved.

Let X and Y be based polyhedra and let $f : X \longrightarrow Y$ be a based PL map. There is a polyhedron V and a factorization $f = p_2 i$, where $i : X \longrightarrow V \times Y$ is a closed based PL embedding. For example, we can take $V = X$ and $i = 1 \times f$. By Lemma 3, the map $i^* p_2^* : I_F(Y) \longrightarrow I_F(X)$ depends only on f; we define $f^* = i^* p_2^*$.

Lemma 4. *I_F is a contravariant functor from the category of based polyhedra and based PL maps to the category of based sets.*

Proof. The base-point of $I_F(X)$ is the class of the trivial bundle. For any polyhedron $X, 1_X^*$ is the identity map. Let X, Y, Z be polyhedra, and let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be based PL maps. Let V, W be polyhedra and let $i : X \longrightarrow V \times Y$, $j : Y \longrightarrow W \times Z$ be closed based PL embeddings such that $f = p_2 i, g = p_2 j$. Consider the diagram

$$\begin{tikzcd}
 & & I_F(X) \\
 & & I_F(V\times Y) \arrow[u, "i^*"'] \\
I_F(V\times W\times Z) \arrow[urr, "(1\times j)^*"] & & I_F(Y) \arrow[u, "p_2^*"'] \\
 & & I_F(W\times Z) \arrow[u, "p_{23}^*"'] \arrow[ull, "p_{23}^*"'] \\
 & & I_F(Z) \arrow[u, "p_2^*"']
\end{tikzcd}$$

This clearly commutes; the right route defines f^*g^* and the left route defines $(gf)^*$. This proves that I_F is a contravariant functor.

Theorem 2. *If F is compact, then there is a based polyhedron $\widetilde{BPL}_F$ and an element $w_I \in I_F(\widetilde{BPL}_F)$ such that $f \mapsto f^*(w_I)$ defines a natural equivalence $[\quad, \widetilde{BPL}_F] \longrightarrow I_F$.*

Proof. First we show that I_F satisfies the following axioms:

(1) If X, Y are based polyhedra and $f_0 \simeq f_1 : X \longrightarrow Y$ by a based PL homotopy, then $f_0^* = f_1^* : I_F(Y) \longrightarrow I_F(X)$.
(2) If X_i is a based polyhedron ($i \in I$) and $u_j : X_j \longrightarrow \bigvee_{i\in I} X_i$ is the inclusion, then $\Pi_{i\in I} u_i^* : I_F(\bigvee_{i\in I} X_i) \longrightarrow \Pi_{i\in I} I_F(X_i)$ is an isomorphism.
(3) Suppose that X, X_0, X_1, X_2 are polyhedra with $X = X_1 \cup X_2, X_0 = X_1 \cap X_2$, and that the inclusions $u_i : X_0 \longrightarrow X_i$, $v_i : X_i \longrightarrow X$ are based maps. If $x_i \in I_F(X_i)$, ($i = 1,\ 2$) satisfy $u_1^*(x_1) = u_2^*(x_2)$, then there exists $x \in I_F(X)$ with $x_i = v_i^*(x)$, ($i = 1,\ 2$).
(4) $I_F(S^0)$ is a single point and $I_F(S^n)$ is countable where S^n denotes the boundary of an $(n+1)$-cell.

Proof of (1). This follows from Lemma 2 and a short argument about base-points.

Proof of (2). Let B_i be a cell complex with $|B_i| = X_i$. Let $x \in \Pi_{i\in I} X_i$ and let ξ_j be a block bundle over B_j representing $p_j(x)$. Let $A = \cup_{i\in I} E(\xi_i)$, $A_0 = \cup_{i\in I} E_{\mathrm{bpt}}(\xi_i)$, $b = \cup_{i\in I} b(\xi)^{-1} : A_0 \longrightarrow F$ and define $E(\eta) = A \cup_b F$. If β is a cell of

$\bigvee_{i\in I} B_i$, then β is a cell of some B_j, so we can define $E_\beta(\eta) = E_\beta(\xi_j) \subset E(\eta)$. Let $b(\eta) = b(\xi_j) : F \longrightarrow E_{\mathrm{bpt}}(\eta)$, which is independent of j. Then η is a block bundle over $\bigvee_{i\in I} B_i$; let $y \in I_F(\bigvee_{i\in I} X_i)$ be the class of η. Then $x \mapsto y$ defines an inverse to $\Pi_{i\in I} u_i^*$, so (2) is proved.

Proof of (3). Let B be a cell complex with $|B| = X$ and with subcomplexes B_0, B_1, B_2 such that $|B_i| = X_i$ $(i = 0, 1, 2)$. Let ξ_i be a block bundle over B_i representing x_i $(i = 1, 2)$. Since $u_1^*(x_1) = u_2^*(x_2)$, there is an isomorphism $h : \xi_1|B_0 \longrightarrow \xi_2|B_0$. Let $E(\xi) = E(\xi_1) \cup_h E(\xi_2)$, let $E_\beta(\xi) = E_\beta(\xi_i)$ if $\beta \in B_i$ and let $b(\xi) = b(\xi_1) = b(\xi_2) : F \longrightarrow E_{\mathrm{bpt}}(\xi)$. Then the class x of ξ has the required properties.

Proof of (4). Clearly $I_F(S^0)$ is a single point. Let B be a cell complex such that $S^n = |B|$, and let β be an n-cell of B. Any element $x \in I_F(S^n)$ can be represented by a block bundle ξ over B. Let k, l be trivialisations of $\xi|\beta$, $\xi|B - \beta$, and let $h = k^{-1}l : F \times \partial\beta \longrightarrow F \times \partial\beta$.

Since F is compact, there are finite simplicial complexes K, L with $|K| = F \times \beta, |L| = F \times (B - \beta)$ and such that h is simplicial. Clearly the simplicial isomorphism class of the triple $(K,\ L,\ h)$ determines x completely. But there are only countably many such classes (of triples), so $I_F(S^n)$ is countable.

Now we can apply Brown's Theorem on representable functors [4] to I_F. We deduce that there is a countable based CW complex W and a natural equivalence $R : [\quad, W] \longrightarrow I_F$. By a theorem of J. H. C. Whitehead [18], there is a polyhedron $\widetilde{BPL}_F$ and a homotopy equivalence $\phi : \widetilde{BPL}_F \longrightarrow W$. Let $w_I = R(\phi) \in I_F(\widetilde{BPL}_F)$; then the pair $(\widetilde{BPL}_F,\ w_I)$ has the required properties.

Remark. The compactness of F was only required to make the classifying space $\widetilde{BPL}_F$ a polyhedron. If F were an infinite discrete space (for example), then the space W constructed above would have uncountable fundamental group.

Our main concern is with block bundles having a compact PL manifold F as fibre. If ξ is such a bundle over a cell complex B, we can define a block bundle $\partial\xi$ over B with fibre ∂F as follows.

Let β be a cell of B, let k, l be trivialisations of $\xi|B$ and let $h = k^{-1}l : F \times \beta \longrightarrow F \times \beta$. Since $h(F \times \gamma)$ for each $\gamma \subset \partial\beta$, $h(F \times \partial\beta) = F \times \partial\beta$. Therefore

$$h(\partial F \times \beta) \ = \ h(\overline{\partial(F \times \beta) - F \times \partial\beta)} \ = \ \partial F \times \beta \ ;$$

it follows that $k(\partial F \times \beta) = l(\partial F \times \beta)$. Define

$$E_\beta(\partial\xi) \ = \ k(\partial F \times \beta) \ ,$$

where k is any trivialisation of $\xi|\beta$, and define

$$E(\partial\xi) \;=\; \bigcup_{\beta\in B} E_\beta(\partial\xi)\ ,\ b(\partial\xi) \;=\; b(\xi)|\partial F\ .$$

Then $\partial\xi$ is a block bundle over B with fibre ∂F.

Lemma 5. *Suppose that $|B|, F$ are compact PL manifolds, that $\partial\beta$ contains the base-point of B and let ξ be a block bundle over B with fibre F. Then $E(\xi)$ is a compact PL manifold and $\partial E(\xi) = E(\partial\xi)\cup E(\xi|\partial B)$.*

Proof. Let B' be a simplicial division of B and ξ' be a subdivision of ξ over B'. Clearly $\partial\xi'$ is then a subdivision of $\partial\xi$. If $p\in E(\xi)$, then

$$p\in P\ ,\ \text{where } P = \ E(\xi'|St(q,B')) - E(\xi'|Lk(q,B'))$$

for some vertex q of B'; we can choose $q\in \mathrm{Int}B$ unless $p\in E(\xi|\partial B)$. Let

$$Q \;=\; St(q,B') - Lk(q,B')\ ,$$

so Q is an open ball if $p\notin E(\xi|\partial B)$, and a half-open ball if $p\in E(\xi|\partial B)$.

A trivialisation k of $\xi'|St(q,B')$ defines a homeomorphism $k : F\times Q\longrightarrow P$ such that $k(\partial F\times Q) = P\cap E(\partial\xi)$. Let N be an open ball neighbourhood of $p_1k^{-1}(p)$ in F if $p\notin E(\partial\xi)$, or a half-open ball neighbourhood if $p\in E(\partial\xi)$.

If $p\notin E(\partial\xi)\cup E(\xi|\partial B)$, then $k(N\times Q)$ is an open ball neighbourhood of p in $E(\xi)$. If $p\in E(\partial\xi)\cup E(\xi|\partial B)$, then $k(N\times Q)$ is a half-open ball neighbourhood of p, and $p\in k(\partial(N\times Q))$. This proves that $E(\xi)$ is a PL manifold (obviously compact) with boundary $E(\partial\xi)\cup E(\xi|\partial B)$.

II. Homotopy Properties of Block Bundles

Let ξ be a block bundle over B with fibre F. A **block fibration** for ξ is a PL map $\pi : E(\xi)\longrightarrow|B|$ such that $E_\beta(\xi) = \pi^{-1}(\beta)$ for each $\beta\in B$. A **block homotopy** for ξ is a PL map $H : E(\xi)\times I\longrightarrow|B|$ such that, for all $t\in I$, $H_t : E(\xi)\longrightarrow|B|$ is a block fibration for ξ.

Lemma 6. *Any block bundle ξ has a block fibration, and any two block fibrations for ξ are block homotopic.*

Proof. Write B^r for the r-skeleton of B. There is a unique block fibration $\pi : E(\xi|B^0)\longrightarrow|B^0|$. Suppose inductively that π can be extended to a block fibration $\pi : E(\xi|B^r)\longrightarrow|B^r|$, and let β be an $(r+1)$-cell of B. Then $\pi : E(\xi|\partial\beta)\longrightarrow|\partial\beta|$ can be extended to a PL map $\pi : E(\xi|\beta)\longrightarrow|\beta|$ such that $\pi^{-1}(|\partial\beta|) = E(\xi|\partial\beta)$. Do this for all $(r+1)$-cells of B to obtain a block fibration $\pi : E(\xi|B^{r+1})\longrightarrow|B^{r+1}|$

extending the given block fibration. By induction, ξ has a block fibration; it is obvious that any two block fibrations for ξ are block homotopic.

Let ξ be a block bundle over B with fibre F, and let π be a block fibration for ξ. Let

$$\mathcal{E} \ = \ \{(x,\psi) : x \in E, \psi : I \longrightarrow |B| \text{ such that } \pi(x) = \psi(0)\}\ ,$$

with the compact open topology. Define $i : E(\xi) \longrightarrow \mathcal{E}$, $p : \mathcal{E} \longrightarrow |B|$ by $i(x) = (x, \text{constant})$ and $p(x,\psi) = \psi(1)$. Then i is a homotopy equivalence and p is a Hurewicz fibre map. Let $\mathcal{F} = p^{-1}(\mathrm{bpt})$ be the fibre of p.

Theorem 3. *The map* $ib(\xi) : F \longrightarrow \mathcal{F}$ *is a homotopy equivalence.*

Proof. By [9], $\mathcal{F}$ has the homotopy type of a CW complex. Choose a component $\mathcal{F}_1$ of $\mathcal{F}$; $\mathcal{F}_1$ lies in some component $\mathcal{E}_0$ of $\mathcal{E}$. Let E_0 be the corresponding component of $E(\xi)$, and let B_0 be the component of B containing the base-point. It is easy to see that $\pi|E_0 \longrightarrow B_0$ must be surjective, so $F_0 = E_0 \cap E_{\mathrm{bpt}}(\xi)$ is non-empty. Choose a base-point for $\mathcal{F}_0 = \mathcal{E}_0 \cap \mathcal{F}$.

If $n \geq 1$, there is a commutative diagram

$$\begin{tikzcd}
\pi_n(E_0, F_0) \arrow[rr, "i_*"] \arrow[dr, "\pi_*"'] & & \pi_n(\mathcal{E}_0, \mathcal{F}_0) \arrow[dl, "p_*"] \\
& \pi_n(|B_0|, \mathrm{bpt}) &
\end{tikzcd}$$

Since $p : \mathcal{E}_0 \longrightarrow |B_0|$ is a Hurewicz fibration, p_* is an isomorphism. Using the fact that $\pi : E_0 \longrightarrow |B_0|$ is a block fibration, we shall prove that π_* is an isomorphism. It will follow that i_* is an isomorphism; hence there is a unique component F_1 of F_0 with $i(F_1) \subset \mathcal{F}_1$. An application of the Five Lemma will show that $i_* : \pi_r(F_1) \longrightarrow \pi_r(\mathcal{F}_1)$ is an isomorphism for all $r \geq 1$, and the Theorem will follow by the Whitehead theorem.

To prove that π_* is surjective, consider an element $\alpha \in \pi_n(|B_0|, \mathrm{bpt})$. By subdividing, we may assume that B_0 is a simplicial complex (note that subdivision does not alter the homotopy class of $\pi : E_0, F_0 \longrightarrow |B_0|, \mathrm{bpt})$. Let D^n be a standard n-cell. There is a triangulation of D^n such that $D^n \searrow^s \mathrm{bpt} \in S^{n-1}$ and α is represented by a simplicial map $f : D^n, S^{n-1} \longrightarrow B_0, \mathrm{bpt}$. Let

$$D^n \ = \ K_k \searrow^s K_{k-1} \searrow^s \ldots \searrow^s K_0 \ = \ \mathrm{bpt}$$

be a sequence of elementary simplicial collapses.

Suppose inductively that there is a map $g : K_r \longrightarrow E_0$ such that, for all $x \in |K_r|$, $\pi g(x)$ is in the closed carrier of $f(x)$ in B_0. We can write $K_{r+1} = K_r \cup \Delta$, $K_r \cap \Delta = \Lambda$ for some simplex $\Delta \in K_{r+1}$. Let $\Delta_1 = \overline{\Delta - \Lambda}$, so Δ_1 is a principal simplex of $\partial\Delta$. Let $\beta = f(\Delta)$, $\beta_1 = f(\Delta_1)$ be the image simplices in B_0. Then

$$g \ : \ \Lambda, \partial\Lambda \longrightarrow E_\beta(\xi), E_{\beta_1}(\xi)$$

is defined. Since $E_{\beta_1}(\xi)$ is a deformation retract of $E_\beta(\xi)$, g can be extended to a map

$$g \ : \ \Delta, \Delta_1 \longrightarrow E_\beta(\xi), E_{\beta_1}(\xi) \ .$$

Thus we obtain an extension of g to $g : K_{r+1} \longrightarrow E_0$ such that, for all $x \in |K_{r+1}|$, $\pi g(x)$ is in the closed carrier of $f(x)$ in B_0.

Now we have completed our induction and have obtained a map $g : D^n \longrightarrow E_0$ such that for all $x \in D^n$, $\pi g(x)$ is in the closed carrier of $f(x)$ in B_0. In particular, $g(S^{n-1}) \subset F_0$, so g represents an element $\beta \in \pi_n(E_0, F_0)$. Clearly $\pi_*\beta = \alpha$, so π_* is injective as asserted. A similar argument shows that π_* is injective, and the Theorem is proved.

We now restrict F to be a compact PL manifold with boundary ∂F. Let X be a based polyhedron; a **Hurewicz fibration** over X with fibre $(F, \partial F)$ consists of a pair of topological spaces $(\mathcal{E}, \partial\mathcal{E})$, a map $p : \mathcal{E} \longrightarrow X$ and a homotopy equivalence of pairs

$$b : F, \partial F \longrightarrow p^{-1}(\mathrm{bpt}), p^{-1}(\mathrm{bpt}) \cap \partial\mathcal{E}$$

such that;

(1) For all $x \in X$, $(p^{-1}(x), p^{-1}(x) \cap \partial\mathcal{E}) \simeq (F, \partial F)$,

(2) Given a pair of topological spaces $A, \partial A$, a map $f : A, \partial A \longrightarrow \mathcal{E}, \partial\mathcal{E}$ and a homotopy $G : A \times I \longrightarrow X$ such that $G_0 = pf$, then there exists a homotopy $H : A \times I, \partial A \times I \longrightarrow \mathcal{E}, \partial\mathcal{E}$ with $H_0 = f$, $G = pH$.

Two Hurewicz fibrations $(\mathcal{E}, \partial\mathcal{E}, p, b)$, $(\mathcal{E}', \partial\mathcal{E}', p', b')$ are **fibre homotopy equivalent** if there are maps

$$h \ : \ \mathcal{E}, \partial\mathcal{E} \longrightarrow \mathcal{E}', \partial\mathcal{E}' \ \ , \ \ h' \ : \ \mathcal{E}', \partial\mathcal{E}' \longrightarrow \mathcal{E}, \partial\mathcal{E}$$

and homotopies $H : h'h \simeq 1$, $H' : hh' \simeq 1$ such that, for all $t \in I$,

$$pH_t \ = \ p \ \ , \ \ H_t b \ = \ b \ \ , \ \ p'H'_t \ = \ p' \ \ , \ \ H'_t b' \ = \ b' \ .$$

We write $H_F(X)$ for the set of fibre homotopy equivalence classes of Hurewicz fibrations over X with fibre $(F, \partial F)$. The well-known construction for induced fibrations makes H_F into a contravariant functor from the category of based polyhedra and based PL maps to the category of based sets. A proof that H_F is representable is indicated in [4]; the step which is given without proof can be dealt with by the methods of Theorem 3 above. We summarise the conclusion as follows.

Proposition. *If F is a compact PL manifold, then there is a based polyhedron BG_F and an element $w_H \in H_F(BG_F)$ such that $f \mapsto f^*(w_H)$ defines a natural equivalence* $[\ \ , BG_F] \longrightarrow H_F$.

Lemma 7. *There is a natural transformation* $S : I_F \longrightarrow H_F$.

Construction of S. Let X be a based polyhedron and let $x \in I_F(X)$. Let B be a cell complex with $|B| = X$, and let ξ be a block bundle over B representing x. By Lemma 6, ξ has a block fibration $\pi : E(\xi) \longrightarrow X$. Construct $p : \mathcal{E} \longrightarrow X$ as above, and let $\partial\mathcal{E} = \{(x, \psi) \in \mathcal{E} : x \in E(\partial\xi)\}$. It is easily proved that $p : \mathcal{E}, \partial\mathcal{E} \longrightarrow X$ satisfies part (2) of the definition of Hurewicz fibration. By Theorem 3, part (1) is also satisfied, and

$$ib(\xi)\ :\ F, \partial F \longrightarrow p^{-1}(\text{bpt}), p^{-1}(\text{bpt}) \cap \partial\mathcal{E}$$

is a homotopy equivalence. Therefore $(\mathcal{E}, \partial\mathcal{E}, p, ib(\xi))$ defines an element $S(\xi, \pi) \in H_F(X)$.

Let π' be another block fibration for ξ. Construct $(\mathcal{E}', \partial\mathcal{E}', p', i')$ from π' as above. Define $j' : \mathcal{E}', \partial\mathcal{E}' \longrightarrow E(\xi), E(\partial\xi)$ by $j'(x, \psi) = x$; then j' is a homotopy inverse to i'. Thus $ij' : \mathcal{E}', \partial\mathcal{E}' \longrightarrow \mathcal{E}, \partial\mathcal{E}$ is a homotopy equivalence of pairs and $p' \simeq p.ij'$ via a homotopy H with $H_t.i'b(\xi) = ib(\xi)$. It follows from Theorem 6.1 of [5] (modified to take account of base-points and pairs of fibres) that $(\mathcal{E}', \partial\mathcal{E}', p', i'b(\xi))$ is fibre homotopy equivalent to $(\mathcal{E}, \partial\mathcal{E}, p, ib(\xi))$. Therefore $S(\xi, \pi)$ depends only on ξ.

If ξ' is a subdivision of ξ, then $S(\xi, \pi) = S(\xi, \pi') = S(\xi', \pi')$ for any block fibrations π, π' of ξ, ξ'. Therefore $S(\xi, \pi)$ depends only on the equivalence class x of ξ; we write $S(x) = S(\xi, \pi)$.

Naturality of S. It is enough to prove that S is natural

1. with respect to projections $p_2 : Y \times X \longrightarrow X$,
2. with respect to closed based PL embeddings $j : Y \longrightarrow X$.

Proof of (1). Let B, C be cell complexes with $|B| = X$, $|C| = Y$. Let ξ be a block bundle over B representing $x \in I_F(X)$, and let π be a block fibration for ξ. Then $\pi \times 1 : E(\xi) \times Y \longrightarrow X \times Y$ is a block fibration for $\xi \times C$ (which represents $p_2^*(x)$). Construct $(\mathcal{E}, \partial\mathcal{E}, p, ib(\xi))$ representing $S(x)$. Let Y^I be the space of **unbased** maps $\psi : I \longrightarrow Y$ and define $e_1 : Y^I \longrightarrow Y$ by $e_1(\psi) = \psi(1)$. Then

$$(\mathcal{E} \times Y^I, \partial\mathcal{E} \times Y^I, p \times e_1, (i \times \text{bpt})b(\xi))$$

represents $S(p_2^*(x))$. But this fibration is equivalent to

$$(\mathcal{E} \times Y, \partial\mathcal{E} \times Y, p \times 1, (i \times \text{bpt})b(\xi))\ ,$$

which represents $p_2^*(S(x))$.

Proof of (2). Let B be a cell complex with $|B| = X$ and with a subcomplex C such that $|C| = j(Y)$. Let ξ be a block bundle over B representing $x \in I_F(X)$, and let π be a block fibration for ξ. then $\pi|E(\xi|C)\longrightarrow|C|$ is a block fibration for $\xi|C$ (which represents $j^*(x)$). Construct $(\mathcal{E}, \partial\mathcal{E}, p, ib(\xi))$ representing $S(x)$. Write i' for the restriction

$$i| \ : \ E(\xi|C), E(\partial\xi|C) \longrightarrow p^{-1}|C|, p^{-1}|C| \cap \partial\mathcal{E} \ ;$$

clearly $\pi|E(\xi|C) = pi'$.

Identify F with $b(\xi)F$ and write $\mathcal{F}$ for $p^{-1}(\text{bpt})$. Consider the commutative diagram

$$\begin{tikzcd}
\pi_n(E(\xi|C), F) \arrow[rr, "i'_*"] \arrow[dr, "\pi_*"'] & & \pi_n(p^{-1}|C|, \mathcal{F}) \arrow[dl, "p_*"] \\
& \pi_n(|C|, \text{bpt}) &
\end{tikzcd}$$

As in Theorem 3, π_* is an isomorphism; since p_* is an isomorphism, i'_* is also an isomorphism. Therefore $i' : E(\xi|C)\longrightarrow p^{-1}|C|$ is a homotopy equivalence. Similarly $i' : E(\partial\xi|C)\longrightarrow p^{-1}|C| \cap \partial\mathcal{E}$ is a homotopy equivalence, so i' is a homotopy equivalence of pairs. It follows from Theorem 6.1 of [5] that

$$(p^{-1}|C|, p^{-1}|C| \cap \partial\mathcal{E}, p|p^{-1}|C|, ib(\xi))$$

represents $S(j^*x)$, so $j^*S(x) = S(j^*x)$. This completes the proof of Lemma 7.

Recall that $w_I \in I_F(B\widetilde{PL}_F)$, $w_H \in H_F(BG_F)$ are the universal elements. There is a based map $\chi : B\widetilde{PL}_F\longrightarrow BG_F$ such that $S(w_I) = \chi^*(w_H)$. This defines the based homotopy class of χ uniquely.

Consider the topological space $L = \{(x, \psi)\}$ of pairs with $x \in B\widetilde{PL}_F$, $\psi : I\longrightarrow BG_F$ such that $\chi(x) = \psi(0)$, $\psi(1) = \text{bpt}$, with (bpt,constant) as base-point. There is a based map $\chi' : L\longrightarrow B\widetilde{PL}_F$ defined by $\chi'(x, \psi) = x$. By theorems of Milnor [9] and J. H. C. Whitehead [18], there is a based polyhedron $G_F/\widetilde{PL}_F$ and a homotopy equivalence $i : G_F/\widetilde{PL}_F\longrightarrow L$. Define $\chi_1 = \chi' i : G_F/\widetilde{PL}_F\longrightarrow B\widetilde{PL}_F$.

Let B be a based cell complex and let F be a compact PL manifold. A $G_F/\widetilde{PL}_F$**-bundle** over B consists of a block bundle ξ over B with fibre F and a PL map

$$t \ : \ E(\xi), E(\partial\xi) \longrightarrow F, \partial F$$

such that $tb(\xi) = 1$. Two $G_F/\widetilde{PL}_F$-bundles (ξ, t) and (η, u) over B are **isomorphic** if there is an isomorphism $h : \xi\longrightarrow\eta$ such that $uh \simeq t$ (rel $b(\xi)(F)$). Define the **equivalence** of $G_F/\widetilde{PL}_F$-bundles over a polyhedron X as in Chapter I, and let $J_F(X)$ be the set of equivalence classes.

Lemma 8. *Let (ξ, t) be a $G_F/\widetilde{PL}_F$-bundle over B and let $\pi : E(\xi)\longrightarrow|B|$ be a block fibration for ξ. Then*

$$t \times \pi \ : \ E(\xi), E(\partial\xi) \longrightarrow F \times |B|, \partial F \times |B|$$

is a homotopy equivalence of pairs.

Proof. Apply Theorem 3 as in the proof of Lemma 7.

We make J_F into a contravariant functor as follows. Let $f : X\longrightarrow Y$ be a based PL map, and suppose $f = p_2 j$, where $j : X\longrightarrow V \times Y$ is a closed based PL embedding. Let B, C, D be cell complexes with

$$|B| \ = \ X \ \ , \ \ |C| \ = \ Y \ \ , \ \ |D| \ = \ Z \ ,$$

and let (η, u) be a $G_F/\widetilde{PL}_F$-bundle over C representing $y \in J_F(Y)$. Let $(D \times C)'$ be a subdivision of $D \times C$ with $j(B)$ as a subcomplex, and let $(D \times \eta)'$ be a subdivision of $D \times \eta$ over $(D \times C)'$. Then $f^*(y)$ is represented by

$$((C \times \eta)'|j(B) \ \ , \ \ up_2|E((C \times \eta)'|j(B))) \ .$$

The proof of Lemma 3 shows that $f^* : J_F(Y)\longrightarrow J_F(X)$ is well-defined, and that J_F is a contravariant functor.

Theorem 4. *If F is a compact PL manifold, then there is an element $w_J \in J_F(G_F/\widetilde{PL}_F)$ such that $f \mapsto f^*(w_J)$ defines a natural equivalence $[\ \ , G_F/\widetilde{PL}_F]\longrightarrow J_F$.*

Proof. Let C_I, C_J, C_H be cell complexes with

$$|C_I| \ = \ B\widetilde{PL}_F \ \ , \ \ |C_J| \ = \ G_F/\widetilde{PL}_F \text{ and } |C_H| \ = \ BG_F \ .$$

Let η_I be a block bundle over C_I representing w_I, and let η_H be a Hurewicz fibration over $|C_H|$ representing w_H. Let η_J be a block bundle over C_J representing $\chi_1^*(w_I)$, and let π_I, π_J be block fibrations for η_I, η_J.

Recall that $S(w_I) = \chi^*(w_H)$; let $h : S(\eta_I, \pi_I)\longrightarrow\chi^*(\eta_H)$ be a fibre homotopy equivalence. The proof of naturality of S (Lemma 7) provides a fibre homotopy equivalence

$$S(\eta_J, \pi_J) \longrightarrow \chi_1^* S(\eta_I, \pi_I) \ .$$

Compose this with

$$\chi_1^* h : \chi_1^* S(\eta_I, \pi_I) \longrightarrow \chi_1^* \chi^*(\eta_H)$$

to obtain a fibre homotopy equivalence

$$h_1 \ : \ S(\eta_J, \pi_J) \longrightarrow \chi_1^* \chi^*(\eta_H) \ .$$

Now $\chi\chi_1 = \chi\chi' i$, where $\chi' : L \longrightarrow BG_F$ sends (x, ψ) to x. There is an obvious null-homotopy $H : L \times I \longrightarrow BG_F$ of $\chi\chi'$, so $H' = H(i \times 1)$ is a null-homotopy of $\chi\chi_1$. Let $h' : \chi_1^* \chi^*(\eta_H) \longrightarrow \epsilon$ be the trivialisation defined by H'. The composite

$$E(\eta_J) \xrightarrow{\ i\ } S(\eta_J, \pi_J) \xrightarrow{\ h_1\ } \chi_1^* \chi^*(\eta_H) \xrightarrow{\ h'\ } \epsilon \xrightarrow{\ p_1\ } F$$

defines a map

$$u_J \ : \ E(\eta_J), E(\partial\eta_J) \longrightarrow F, \partial F$$

such that $u_J b(\eta_J) = 1$. Let w_J be the equivalence class of (η_J, u_J) in $J_F(G_F/\widetilde{PL}_F)$.

Clearly $f \mapsto f^*(w_J)$ defines a natural transformation from $[\ \ , G_F/\widetilde{PL}_F]$ to J_F. Let B be a cell complex and let (ξ, t) be a $G_F/\widetilde{PL}_F$-bundle over B; we have to prove that the equivalence class of (ξ, t) corresponds to a unique element of $[|B|, G_F/\widetilde{PL}_F]$. Let π be a block fibration for ξ.

There is a map $g : |B| \longrightarrow B\widetilde{PL}_F$ such that ξ represents $g^*(w_I)$; g is unique up to homotopy. The proof of naturality of S (Lemma 7) provides a fibre homotopy equivalence $S(\xi, \pi) \longrightarrow g^* S(\eta_I, \pi_I)$. Compose this with $g^* h : g^* S(\eta_I, \pi_I) \longrightarrow g^* \chi^*(\eta_H)$ to obtain a fibre homotopy equivalence $k : S(\xi, \pi) \longrightarrow g^* \chi^*(\eta_H)$.

Now $tk^{-1} : g^* \chi^*(\eta_H) \longrightarrow F$ defines a fibre homotopy trivialisation of $g^* \chi^*(\eta_H)$, unique up to fibre homotopy. Let $K : |B| \times I \longrightarrow BG_F$ be the corresponding null-homotopy of $\chi g : |B| \longrightarrow BG_F$. Then (g, K) defines the unique homotopy class of maps $f : |B| \longrightarrow G_F/\widetilde{PL}_F$ such that (η, t) represents $f^*(w_J)$. This completes the proof of Theorem 4.

III. Tangential Properties of Block Bundles

Let I^n denote the product of n copies of the unit interval; we write $G_n/\widetilde{PL}_n$ for $G_{I^n}/\widetilde{PL}_{I^n}$. The obvious natural transformation $J_{I^n} \longrightarrow J_{I^{n+1}}$ (multiply the fibre of each bundle by I) defines a homotopy class of maps $G_n/\widetilde{PL}_n \xrightarrow{i_n} G_{n+1}/\widetilde{PL}_{n+1}$. Write G/PL for the direct limit of the sequence

$$\xrightarrow{\ i_{n-1}\ } G_n/\widetilde{PL}_n \xrightarrow{\ i_n\ } G_{n+1}/\widetilde{PL}_{n+1} \xrightarrow{\ i_{n+1}\ } \ \ldots\ .$$

More precisely, for $n = 1, 2, 3, \ldots$ replace $G_{n+1}/\widetilde{PL}_{n+1}$ by a homotopy equivalent polyhedron in such a way that i_n is an injection, and identify $G_n/\widetilde{PL}_n$ with $i_n(G_n/\widetilde{PL}_n)$. Now define G/PL to be the nested union of the $G_n/\widetilde{PL}_n$; it can be shown that the homotopy type of G/PL is independent of the choices made (see Lemma 1.7 of [3]).

G/PL was studied by Sullivan in his thesis (but he called it F/PL). The aim of this chapter is to obtain a map $\theta : G_F/\widetilde{PL}_F \longrightarrow (G/PL)^F$, where $(G/PL)^F$

is the space of all **unbased** maps from F into G/PL (with the compact open topology). Let $\mathcal{C}$ be the category of based, compact, stably parallelizable PL manifolds and based PL maps. Our first step is to define a natural transformation

$$T \; : \; [\;\;, G_F/\widetilde{PL_F}] \longrightarrow [\;\;, (G/PL)^F] \; ,$$

where the functors are defined on $\mathcal{C}$.

Let N be an object of $\mathcal{C}$, with boundary ∂N, and let B be a cell complex with $|B| = N$. Let β be a principal cell of B with the base-point as one vertex. Let $x \in [N, G_F/\widetilde{PL_F}]$ be represented by a $G_F/\widetilde{PL_F}$-bundle (ξ, t) over B. Extend $b(\xi)p_1 : F \times \mathrm{bpt} \longrightarrow E(\xi|\mathrm{bpt})$ to a homeomorphism $b : F \times \beta \longrightarrow E(\xi|\beta)$. Change t by a homotopy (rel $b(\xi)(F)$) until $tb = p_1 : F \times \beta \longrightarrow F$.

We write $E = E(\xi)$, so E is a PL manifold with $\partial E = E(\partial\xi) \cup E(\xi|\partial B)$. We write W for $F \times \beta$ and identify W with $b(W)$. Let $\pi : E \longrightarrow N$ be a block fibration such that $\pi|F \times \beta = p_2$. Let $Q = F \times N$, so by Lemma 8, $t \times \pi : E, \partial E \longrightarrow Q, \partial Q$ is a homotopy equivalence of pairs. Note that $t \times \pi|W = 1$ and $(t \times \pi)^{-1}(W) = W$. Let $g : Q, \partial Q \longrightarrow E, \partial E$ be a homotopy inverse to $t \times \pi$ such that $g|W = 1$ and $g^{-1}(W) = W$.

Let k be large, and choose embeddings

$$e \; : \; E, \partial E \longrightarrow D^k, S^{k-1} \quad , \quad q \; : \; Q, \partial Q \longrightarrow D^k, S^{k-1}$$

such that $e|W = q|W$. By [6], there exists normal bundles ν_Q, ν_E of Q, E in D^k. Choose ν_Q, ν_E so that $\nu_Q|W = \nu_E|W$ (using the uniqueness theorem of [6] and regular neighbourhood theory). Let Q^ν, E^ν, W^ν be Thom spaces for ν_Q, ν_E, $\nu_Q|W$, and let

$$\gamma \; : \; Q^\nu/\partial Q^\nu \longrightarrow W^\nu/\partial W^\nu \quad , \quad \gamma \; : \; E^\nu/\partial E^\nu \longrightarrow W^\nu/\partial W^\nu$$

be the collapsing maps. Let $\bar\nu_Q = g^*(\nu_E)$ have Thom space $Q^{\bar\nu}$ and collapsing map

$$\bar\gamma \; : \; Q^{\bar\nu}/\partial Q^{\bar\nu} \longrightarrow W^{\bar\nu}/\partial W^{\bar\nu} \; = \; W^\nu/\partial W^\nu \; .$$

There is a homotopy equivalence $\bar h : E^\nu/\partial E^\nu \longrightarrow Q^{\bar\nu}/\partial Q^{\bar\nu}$ covering $t \times \pi : E \longrightarrow Q$ and such that $\bar\gamma \bar h = \gamma$.

There is a map $D^k \longrightarrow Q^\nu/\partial Q^\nu$ which collapses

$$S^{k-1} \cup (\text{complement of total space of } \nu_Q)$$

to a point. If we identify S^k with D^k/S^{k-1}, we obtain a map $\phi : S^k \longrightarrow Q^\nu/\partial Q^\nu$; let $\psi : S^k \longrightarrow E^\nu/\partial E^\nu$ be defined similarly. Let $\bar\phi = \bar h\psi : S^k \longrightarrow Q^{\bar\nu}/\partial Q^{\bar\nu}$; then $\bar\gamma\bar\phi = \gamma\phi = \gamma\psi$.

By theorems of Atiyah [1] and Wall [15, Th 3.5] there is a fibre homotopy equivalence $\bar f : \bar\nu_Q \longrightarrow \nu_Q$ such that $\bar f \bar\phi \simeq \phi$. It follows from Wall's theorem that $\bar f$ is unique up to fibre homotopy. Consider $\tilde f = \bar f|(\bar\nu_Q|W) \longrightarrow (\nu_Q|W)$; this has the property that

$$\gamma\phi \simeq \gamma\bar{f}\bar{\phi} \;=\; \tilde{f}(\bar{\gamma}\bar{\phi}) \;=\; \tilde{f}(\gamma\phi) \ .$$

By the uniqueness clause in Wall's theorem, $\tilde{f}$ is fibre homotopic to the identity. Therefore we can alter $\bar{f}$ by a fibre homotopy until it is the identity on $\bar{\nu}_Q|W$.

Let G be defined as in [8] (this agrees with the definition used in [15]), so G is an H-space. Since W is a retract of Q, the map $[Q/W, G]{\longrightarrow}[Q, G]$ is injective. It follows that two fibre equivalences $\bar{f}_0, \bar{f}_1 : \bar{\nu}_Q{\longrightarrow}\nu_Q$ which are the identity on $\bar{\nu}_Q|W$ are fibre homotopic (rel $\bar{\nu}_Q|W$) if and only if they are fibre homotopic. Therefore the fibre homotopy equivalence $\bar{f} : \bar{\nu}_Q{\longrightarrow}\nu_Q$ obtained above is unique up to fibre homotopy (rel $\bar{\nu}_Q|W$).

Let τ_Q be the tangent bundle on Q, and choose a fixed trivialisation $\kappa : \tau_Q \oplus \nu_Q{\longrightarrow}\epsilon$. Then

$$f \;=\; \kappa(1 \oplus \bar{f}) \;:\; \tau_Q \oplus \bar{\nu}_Q \longrightarrow \epsilon$$

is a fibre homotopy equivalence, which agrees with κ on $\tau_Q \oplus \bar{\nu}_Q|W$. The pair $(\tau_Q \oplus \bar{\nu}_Q, f)$ represents an element

$$T(x) \in [Q/W, G/PL] \cong [N, (G/PL)^F] \ .$$

Since the normal invariants ϕ, ψ are unique up to homotopy and PL bundle automorphisms, $T(x)$ depends only on x. Thus we have defined a map

$$T \;:\; [N, G_F/\widetilde{PL_F}] \longrightarrow [N, (G/PL)^F] \ .$$

Lemma 9. *T is a natural transformation (between functors from $\mathcal{C}$ to the category of based sets).*

Proof. Let $f : M{\longrightarrow}N$ be a based PL map. Express f as a composite

$$M \xrightarrow{\times 0} M \times D^r \xrightarrow{u} N \times D^s \xrightarrow{p_1} N \ ,$$

where u is a codimension 0 embedding. We prove that T is natural

(1) with respect to ×0 and p_1,
(2) with respect to codimension 0 embeddings.

Proof of 1. Consider $p_1 : N \times D^s{\longrightarrow}N$; let B be a cell complex with $|B| = N$. Let (ξ, t) be a $G_F/\widetilde{PL_F}$-bundle over B representing $x \in [N, G_F/\widetilde{PL_F}]$, so that $(\xi \times D^s, tp_1)$ represents $p_1^*(x)$. Let

$$Q \ , \ W \ , \ \nu_Q \ , \ \bar{\nu}_Q \ , \ \phi \,:\, S^k \longrightarrow Q^\nu/\partial Q^\nu \ , \ \bar{\phi} \,:\, S^k \longrightarrow Q^{\bar{\nu}}/\partial Q^{\bar{\nu}}$$

be defined for (ξ, t) as above. The corresponding objects for $(\xi \times D^s, tp_1)$ are

$$\begin{aligned} Q_s &= Q \times D^s \quad , \quad W_s = W \times D^s \ , \\ \nu_{Q_s} &= \nu_Q \times D^s \quad , \quad \bar{\nu}_{Q_s} = \bar{\nu}_Q \times D^s \ , \\ S^s\phi &: S^{s+k} \longrightarrow Q^\nu_s/\partial Q^\nu_s \ , \ S^s\bar{\phi} : S^k \longrightarrow Q^{\bar{\nu}}_s/\partial Q^{\bar{\nu}}_s \end{aligned}$$

(note that $Q^\nu_s/\partial Q^\nu_s \cong S^s(Q^\nu/\partial Q^\nu)$, $Q^{\bar{\nu}}_s/\partial Q^{\bar{\nu}}_s \cong S^s(Q^{\bar{\nu}}/\partial Q^{\bar{\nu}})$).

Let $\bar{f} : \bar{\nu}_Q \longrightarrow \nu_Q$ be a fibre homotopy equivalence such that $\bar{f}\bar{\phi} \simeq \phi$ and $\bar{f}$ is the identity on $\bar{\nu}_Q|W$. Then $\bar{f}_s = \bar{f} \times 1 : \bar{\nu}_{Q_s} \longrightarrow \nu_{Q_s}$ is the identity on $\bar{\nu}_{Q_s}|W$, and $\bar{f}_s(S^s\bar{\phi}) \simeq S^s\phi$. Therefore $(\tau_{Q_s} \oplus \bar{\nu}_{Q_s}, 1 \oplus \bar{f}_s)$ represents $T(p_1^*(x))$. It follows that $T(p_1^*(x)) = p_1^*(T(x))$, as required. Since $\times 0 : N \longrightarrow N \times D^s$ is a homotopy inverse to p_1, T is also natural with respect to $\times 0$.

Proof of 2. Let $u : M \longrightarrow N$ be a codimension 0 embedding. Let B be a cell complex with $|B| = N$ and with a subcomplex A such that $|A| = u(M)$. Choose β to be a cell of A containing the base-point, as above. Let (ξ, t) be a $G_F/\widetilde{PL_F}$-bundle over B representing $x \in [N, G_F/\widetilde{PL_F}]$; then $(\xi|A, t|E(\xi|A))$ represents $u^*(x)$. Let

$$E = E(\xi) \quad , \quad D = D(\xi|A) \quad , \quad Q = F \times N \quad , \quad P = F \times M \ .$$

Identify W with $b(W) \subset D \subset E$, as above. Let

$$g : Q, P, \partial Q, \partial P \longrightarrow E, D, \partial E, \partial D$$

be a homotopy inverse to $t \times \pi$ such that $g|W$ is the identity.

Choose embeddings

$$g : Q, \partial Q \longrightarrow D^k, S^{k-1} \quad , \quad e : E, \partial E \longrightarrow D^k, S^{k-1}$$

agreeing on W, as above. Let ν_Q, ν_E be normal bundles with $\nu_Q|W = \nu_E|W$, and let $\nu_P = \nu_Q|P$, $\nu_D = \nu_E|D$. We obtain collapsing maps

$$\eta : Q^\nu/\partial Q^\nu \longrightarrow P^\nu/\partial P^\nu \quad , \quad \eta : E^\nu/\partial E^\nu \longrightarrow D^\nu/\partial D^\nu \ .$$

Let $\bar{\nu}_Q = g^*(\nu_E)$, let $\bar{\nu}_P = \bar{\nu}_Q|P$ and let $\bar{h} : E^\nu/\partial E^\nu \longrightarrow Q^{\bar{\nu}}/\partial Q^{\bar{\nu}}$ be a homotopy equivalence covering $t \times \pi : E \longrightarrow Q$, such that $\bar{\gamma}\bar{h} = \gamma$ (where γ, $\bar{\gamma}$ are as above).

If

$$\phi : S^k \longrightarrow Q^\nu/\partial Q^\nu \quad , \quad \psi : S^k \longrightarrow E^\nu/\partial E^\nu$$

are collapsing maps for Q, E, then $\eta\phi$, $\eta\psi$ are collapsing maps for P, D. Let $\bar{\phi} = \bar{h}\psi : S^k \longrightarrow Q^{\bar{\nu}}/\partial Q^{\bar{\nu}}$; the corresponding map for P is $\bar{h}\eta\psi : S^k \longrightarrow P^{\bar{\nu}}/\partial P^{\bar{\nu}}$. Let $\bar{f} : \bar{\nu}_Q \longrightarrow \nu_Q$ be a fibre homotopy equivalence such that $\bar{f}$ is the identity on $\bar{\nu}_Q|W$ and $\bar{f}\bar{\phi} \simeq \phi$.

Now $\tilde{f} = \bar{f}|\bar{\nu}_P \longrightarrow \nu_P$ is a fibre homotopy such that

$$\tilde{f}(\bar{h}\eta\psi) = \tilde{f}(\eta\bar{\phi}) = \eta\bar{f}\bar{\phi} \simeq \eta\phi$$

and $\tilde{f}$ is the identity on $\bar{\nu}_P|W$. Therefore $T(x)$, $T(u^*(x))$ are represented by $(\tau_Q \oplus \bar{\nu}_Q, 1 \oplus \bar{f})$, $(\tau_P \oplus \bar{\nu}_P, 1 \oplus \tilde{f})$ respectively. It follows that $T(u^*(x)) = u^*(T(x))$, as required. This proves the Lemma.

Since $G_F/\widetilde{PL}_F$ and $(G/PL)^F$ have the homotopy type of countable CW complexes, it follows from Lemma 1.7 of [3] that there is a map $\theta : G_F/\widetilde{PL}_F \longrightarrow (G/PL)^F$ such that $T = \theta_*$. Unfortunately, the homotopy class of θ is not uniquely determined by this condition.

Theorem 5. *Let F^n be a closed 1-connected PL manifold with $n \geq 4$. Let $F^* = \overline{F - D^n}$, and let $\rho : (G/PL)^F \longrightarrow (G/PL)^{F^*}$ be the restriction map. Then the composite $\rho\theta$ induces isomorphisms*

$$(\rho\theta)_* \ : \ \pi_r(G_F/\widetilde{PL}_F) \longrightarrow \pi_r((G/PL)^{F^*})$$

for $r \geq 1$.

Remark. For any based space X let X_0 be the component of X containing the base-point. Then $(G_F/\widetilde{PL}_F)_0$ is homotopy equivalent to $((G/PL)^{F^*})_0$, but $(G/PL)^{F^*}$ usually has more components than $G_F/\widetilde{PL}_F$.

Proof. First we prove that $(\rho\theta)_*$ is surjective; we defer the case $n = 4, r = 1$ until after Theorem 7. Let B be a cell complex with $|B| = S^r$, and let β be a principal cell of B. Let $f : S^r, \beta \longrightarrow (G/PL)^{F^*}, \text{bpt}$ represent an element of $x \in \pi_r((G/PL)^{F^*})$. Let

$$g \ : \ F^* \times S^r, F^* \times \beta \longrightarrow (G/PL), \text{bpt}$$

be the adjoint map. Extend g over $(F^* \times S^r) \cup (F \times \beta)$ by defining $g(F \times \beta) = \text{bpt}$. Let $Q = F \times S^r$, $W = F \times \beta$ and let Q^* be obtained from Q by deleting the interior of an $(n+r)$-disc in $Q - W$. Then Q^* deformation retracts onto $(F^* \times S^r) \cup (F \times \beta)$, so g defines a homotopy class of maps $h : Q^*, W \longrightarrow G/PL, \text{bpt}$.

Let k be large, identify D^k with the northern hemisphere of S^k and identify $2D^k$ with the closed region to the north of the Antarctic circle. Let $q : Q \longrightarrow S^k$ be an embedding such that $q^{-1}(D^k) = W$, $q^{-1}(2D^k) = Q^*$. Let ν_Q be a normal bundle of Q in S^k such that $\nu_Q|W$, $\nu_Q|Q^*$ are normal bundles of W, Q^* in D^k, $2D^k$ respectively. Let $\phi^* : S^k \longrightarrow Q^{*\nu}/\partial Q^{*\nu}$ be the collapsing map.

Choose a piecewise linear bundle $\bar{\nu}_{Q^*}$ over Q^* and a fibre homotopy equivalence $\bar{f} : \bar{\nu}_{Q^*} \longrightarrow \nu_{Q^*}$ such that $\bar{\nu}_{Q^*}|W = \nu_{Q^*}|W$, $\bar{f}$ is the identity on $\bar{\nu}_{Q^*}|W$ and $(\tau_{Q^*} \oplus \bar{\nu}_{Q^*}, 1 \oplus \bar{f})$ represents h. By the theorem of Wall quoted above, there is a map $\bar{\phi} : S^k \longrightarrow Q^{*\bar{\nu}}/\partial Q^{*\bar{\nu}}$ such that $\bar{f}\bar{\phi} \simeq \phi^*$. Let $\eta : Q^{*\bar{\nu}} \longrightarrow Q^{*\bar{\nu}}/\partial Q^{*\bar{\nu}}$ be the collapsing map; if k is large enough then there is a map $\psi' : 2D^k, 2S^{k-1} \longrightarrow Q^{*\bar{\nu}}/\partial Q^{*\bar{\nu}}$ such that $\eta\psi'$ and $\bar{\phi}$ represent the same element of $\pi_k(Q^{*\bar{\nu}}/\partial Q^{*\bar{\nu}})$.

Adjust ψ' by a homotopy until $\psi'|D^k = \phi|D^k$, and ψ' is transverse regular on $Q^* \subset Q^{*\bar{\nu}}$; let $E' = \psi'^{-1}(Q^*)$, so $W \subset E'$. We shall modify $E', \partial E'$ by surgery (keeping W fixed), attempting to make $\psi'| : E', \partial E' \longrightarrow Q^*, \partial Q^*$ a homotopy equivalence of pairs.

Since the inclusion induces an isomorphism $\pi_1(\partial Q^*) \longrightarrow \pi_1(\overline{Q^* - W})$ (in fact both groups are zero) and $n+r \geq 6$, we can use Theorem 3.3 of [17] to the manifold $\overline{E' - W}$. This has two boundary components, namely ∂W and $\partial E'$; we wish to do surgery on $\operatorname{Int}(\overline{E' - W})$ and $\partial E'$, but **not** on ∂W.

We obtain a map $\psi^* : 2D^k, 2S^{k-1} \longrightarrow Q^{*\bar{\nu}}, \partial Q^{*\bar{\nu}}$, which is transverse regular on Q^* and is homotopic to ψ' (rel D^k), with the following property. Let

$$E^* \;=\; \psi^{*-1}(Q^*) \;\; ;$$

then

$$\psi^*| \;:\; \overline{E' - W}, \partial E^* \longrightarrow \overline{Q^* - W}, \partial Q^*$$

is a homotopy equivalence of pairs. It follows that $\psi^*| : E^*, \partial E^* \longrightarrow Q^*, \partial Q^*$ is a homotopy equivalence of pairs.

Since $\partial Q^* \cong S^{n+r-1}$ and $n + r - 1 \geq 5$, ∂E^* is homeomorphic to S^{n+r-1}. Let $E = E^* \cup_{\partial E^*} D^{n+r}$, and extend the embedding $E^* \subset 2D^k$ to an embedding $E \subset S^k$. Let ν_E be a normal bundle of E in S^k such that $\nu_E|W$, $\nu_E|E^*$ are normal bundles of W, E^* in D^k, $2D^k$ respectively. Extend $\psi^* : 2D^k \longrightarrow Q^{*\bar{\nu}}$ to a map $\psi : S^k \longrightarrow Q^{\bar{\nu}}$, transverse regular on $Q \subset Q^{\bar{\nu}}$ and with $E = \psi^{-1}(Q)$. Then $\psi|E \longrightarrow Q$ is a homotopy equivalence, and $\psi|W$ is the identity.

Recall that B is a cell complex with $|B| = S^r$, and β is a principal cell of B. Let γ be an $(r-1)$-cell of B contained in $\partial\beta$. Choose a PL homeomorphism $k : |\partial\beta - \gamma| \times I \longrightarrow |B - \beta|$ such that k_0 is the inclusion. Recall that $Q = F \times |B|$. Since $n + r \geq 6$, we can use the relative h-cobordism theorem [12] to extend

$$\psi^{-1}(1 \times k)| \;:\; F \times |\partial\beta - \gamma| \times 0 \longrightarrow \partial\overline{E - W}$$

to a homeomorphism $H : F \times |\partial\beta - \gamma| \times I \longrightarrow \overline{E - W}$.

Define a block bundle ξ over B with $E(\xi) = E$ by $E_\beta(\xi) = W$ and, for each cell δ in $(B - \beta)$, $E_\delta(\xi) = H(1 \times k^{-1})(F \times \delta)$. Then ξ satisfies the local triviality condition in the definition of a block bundle. Let

$$b(\xi) \;=\; 1 \times \mathrm{bpt} \;:\; F \longrightarrow F \times \mathrm{bpt} \;=\; E_{\mathrm{bpt}}(\xi) \;.$$

Let $t = p_1\psi : E \longrightarrow F$; then (ξ, t) is a $G_F/\widetilde{PL}_F$-bundle over S^r, representing an element $y \in \pi_r(G_F/\widetilde{PL}_F)$. It is easily checked that $p_*(T(y)) \in \pi_r((G/PL)^{F^*})$ is represented by $(\tau_{Q^*} \oplus \bar{\nu}_{Q^*}, 1 \oplus \bar{f})$ so $p_*(T(y)) = x$. Therefore $(\rho\theta)_*(y) = x$, so $(\rho\theta)_*$ is surjective, as required (provided $n + r \geq 6$).

Similar arguments prove that $(\rho\theta)_*$ is injective; we have to consider $G_F/\widetilde{PL}_F$-bundles (ξ_0, t_0), (ξ_1, t_1) over $S^r \times 0$, $S^r \times 1$. We prove that they are isomorphic by extending them to a $G_F/\widetilde{PL}_F$-bundle (ξ, t) over $S^r \times I$. Since $n + \dim(S^r \times I) \geq 6$, we can always carry out surgery and use the h-cobordism theorem. Thus the Theorem is established, except for surjectivity of $(\rho\theta)_*$ when $n = 4, r = 1$.

Theorem 6. *Let F^n be a compact PL manifold with $\pi_1(\partial F)$ isomorphic to $\pi_1(F)$ by inclusion and $n \geq 6$. Then θ induces isomorphisms*

$$\theta_* \ : \ \pi_r(G_F/\widetilde{PL}_F) \longrightarrow \pi_r((G/PL)^F)$$

for $r \geq 1$.

Proof. Since the proof is essentially the same as the proof of Theorem 5, we shall not give the details. To prove that θ_* is surjective, let B, β, ξ, Q, W be as above. Since Q has a boundary ∂Q such that $\pi_1(\partial Q) \longrightarrow \pi_1(\overline{Q - W})$ is an isomorphism, it is unnecessary to cut out a disc from Q. We can use Theorem 3.3 of [17] to construct a manifold $E \supset W$ with boundary ∂E and a **simple** homotopy equivalence $\psi : E, \partial E \longrightarrow Q, \partial Q$ with $\psi | W$ equal to the identity.

In the construction of the block bundle ξ above, we used the h-cobordism theorem to construct a homeomorphism $F \times |B - \beta| \longrightarrow \overline{E - W}$. Here we can use the s-cobordism theorem [7] twice (first for $\partial F \times |B - \beta|$, then for $F \times |B - \beta|$), since ψ is a simple homotopy equivalence and $\dim(\partial F \times |B - \beta|) \geq 6$. The rest of the proof proceeds as above.

IV. Periodicity of G/PL

In his thesis, Sullivan interpreted $[M, G/PL]$ in terms of PL structures on manifolds homotopy equivalent to M. Thus it is useful to have information about G/PL which facilitates computation of $[M, G/PL]$. It has been known for some time that $\pi_r(G/PL) \cong \mathbb{Z}, 0, \mathbb{Z}_2, 0$ according as $r \equiv 0, 1, 2, 3 \pmod 4$; in particular, $\pi_r(G/PL) \cong \pi_{r+4}(G/PL)$.

Theorem 7. *There is a map $\lambda : G/PL \longrightarrow \Omega^4(G/PL)$ such that $\lambda_* : \pi_r(G/PL) \longrightarrow \pi_{r+4}(G/PL)$ is an isomorphism if $r \neq 0, 4$ and a monomorphism onto a subgroup of index 2 if $r = 4$.*

Proof. Let F^n be a closed 1-connected PL manifold with $n \geq 4$. If X is a based space we write X_0 for the component of X containing the base-point. Consider the diagram

(1)

$$\begin{tikzcd}
& & (G_F/\widetilde{PL}_F)_0 \arrow[dl, "\theta"'] \arrow[dr, "\rho\theta"] & \\
\Omega^n(G/PL)_0 \arrow[r, "\alpha"] & (G/PL)_0^F \arrow[rr, "\rho"] & & (G/PL)_0^{F^*}
\end{tikzcd}$$

where α is induced by a map $F \longrightarrow S^n$ of degree 1.

Suppose first that $n \geq 5$. Then $\rho\theta$ is a homotopy equivalence by Theorem 5 (we are not using the unproved case!). Let γ' be a homotopy inverse to $\rho\theta$. Let $\gamma = \theta\gamma'$, so $\rho\gamma \simeq 1 : (G/PL)_0^{F^*} \longrightarrow (G/PL)_0^{F^*}$.

The Whitney sum construction gives a multiplication map $\mu : G/PL \times G/PL \longrightarrow G/PL$. If K is a finite CW complex, μ defines Abelian group structures on

$$[K, \Omega^n(G/PL)_0] \ , \ [K, (G/PL)_0^F] \ , \ [K, (G/PL)_0^{F^*}]$$

such that α_*, ρ_* are homeomorphisms. Let $x \in [K, (G/PL)_0^F]$ and let $y = (1 - \gamma_*\rho_*)(x)$, so $\rho_*(y) = 0$. Therefore $y = \alpha_*(z)$ for some $z \in [K, \Omega^n(G/PL)_0]$. Since ρ has a right homotopy inverse, α_* is injective and z is unique. Define a natural transformation

$$S \ : \ [\ \ , (G/PL)_0^F] \longrightarrow [\ \ , \Omega^n(G/PL)_0]$$

on finite CW complexes by $S(x) = z$. By Lemma 1.7 of [3], there is a map

$$\sigma \ : \ (G/PL)_0^F \longrightarrow \Omega^n(G/PL)_0$$

with $S = \sigma_*$. Observe that

$$\alpha_*\sigma_*\alpha_* \ = \ \alpha_* - \gamma_*\rho_*\alpha_* \ = \ \alpha_* \ ,$$

since $\rho\alpha \simeq$ bpt. Since α_* is injective, $\sigma_*\alpha_* = 1$.

Let $r \geq 1$ and consider the homomorphism

$$\sigma \ : \ \pi_r((G/PL)^F) \longrightarrow \pi_{n+r}(G/PL) \ .$$

This is an epimorphism (with right inverse α_*). Let $x \in \pi_r((G/PL)^F)$ be represented by

$$g \ : \ F \times S^r, F \times \beta \longrightarrow G/PL, \mathrm{bpt}$$

(where β is a cell of S^r containing the base-point). Let $Q = F \times S^r$, $W = F \times \beta$, as above.

Let k be large and identify D^k with the northern hemisphere of S^k. Let $q : Q \longrightarrow S^k$ be an embedding such that $q^{-1}(D^k) = W$. Let ν_Q be a normal bundle of Q in S^k such that $\nu_Q|W$ is a normal bundle of W in D^k. Let $\phi : S^k \longrightarrow Q^\nu$ be the collapsing map.

Choose a piecewise linear bundle $\bar\nu_Q$ over Q and a fibre homotopy equivalence $\bar f : \bar\nu_Q \longrightarrow \nu_Q$ such that $\bar\nu_Q|W = \nu_Q|W$, $\bar f$ is the identity on $\bar\nu_Q|W$ and $(\tau_Q \oplus \bar\nu_Q, 1 \oplus \bar f)$ represents g. As in Chapter III, there is a map $\psi' : S^k \longrightarrow Q^{\bar\nu}$ such that $\bar f \psi' \simeq \phi$. Adjust ψ' by a homotopy until $\psi'|D^k = \phi|D^k$ and ψ' is transverse regular on $Q \subset Q^{\bar\nu}$; let $E' = \psi'^{-1}(Q)$, so $W \subset E'$. We attempt to modify E' by surgery (keeping W fixed), to make $\psi'|E' \longrightarrow Q$ a homotopy equivalence.

We seek a map $\psi : S^k \longrightarrow Q^{\bar\nu}$ which is transverse regular on Q and is homotopic to $\psi'(\text{ rel } D^k)$, and with the following property. Let $E = \psi^{-1}(Q)$; then $\psi|E \longrightarrow Q$ is a homotopy equivalence. Let $P_r = \mathbb{Z}, 0, \mathbb{Z}_2, 0$ according as $r \equiv 0, 1, 2, 3 \pmod 4$ (as in [8]). By [14,§4], since Q is 1-connected and $\dim Q \geq 5$, there is an obstruction $\bar\sigma(x) \in P_{n+r}$ to performing the surgery. Note that $\bar\sigma(x)$ depends only on x.

Using the homotopy group addition in $\pi_r((G/PL)^F)$ (**not** the H-structure on G/PL) and the interpretation of $\bar\sigma(x)$ as a signature or Arf invariant, we see that $\bar\sigma : \pi_r((G/PL)^F) \longrightarrow P_{n+r}$ is a homomorphism. Consider the homomorphism $\bar\sigma\alpha_* : \pi_{n+r}(G/PL) \longrightarrow P_{n+r}$. This coincides with the canonical homomorphism obtained in [13], and is therefore an isomorphism. It follows that $\bar\sigma$ is an epimorphism.

If $x \in \pi_r((G/PL)^F)$, then $\gamma_*\rho_*(x)$ is represented by $\bar g : Q, W \longrightarrow G/PL$, bpt, where $\bar g$ agrees with g on $Q^* = \overline{Q - D^{n+r}}$, and $\bar g|D^{n+r}$ is chosen so that $\bar\sigma(\bar g) = 0$ (because the surgery problem for $\gamma_*\rho_*(x) \in \operatorname{im}(\theta_*)$ is clearly soluble). Since $\bar\sigma\alpha_*$ is a monomorphism, these conditions characterise the homotopy class of $\bar g$. Therefore $x = \gamma_*\rho_*(x)$ if and only if $\bar\sigma(x) = 0$, so $\ker\sigma_* = \ker\bar\sigma$. If we identify $\pi_{n+r}(G/PL)$ with P_{n+r} via the canonical isomorphism, we see that $\bar\sigma(x) = \sigma_*(x)$.

Let $\epsilon : G/PL \longrightarrow (G/PL)^F$ be induced by the map $F \longrightarrow$point, and let λ^F denote the composite

$$G/PL \xrightarrow{\epsilon} (G/PL)^F \xrightarrow{\sigma} \Omega^n(G/PL) .$$

If $\dim F = n = 4$, this construction fails as

$$(\rho\theta)_* : \pi_1(G_F/\widetilde{PL_F}) \longrightarrow \pi_1((G/PL)^{F^*})$$

is not yet known to be surjective. However, we can construct a map $\bar\lambda^F : \Omega^4(G/PL)_0 \longrightarrow \Omega^{n+4}(G/PL)_0$; simply apply the functor Ω^4 to diagram (1) and argue as above.

Let $x \in \pi_r(G/PL)$ be represented by $g : S^r \longrightarrow G/PL$. Then $\epsilon_*(x)$ is represented by $gp_2 : F \times S^r \longrightarrow G/PL$. Note that $x = \bar\sigma(x)$ and

$$\bar\sigma(\epsilon_*(x)) = \sigma_*\epsilon_*(x) = \lambda^F_*(x) .$$

Now $\bar{\sigma}(x)$ is the obstruction to making a certain map $\psi'| : V' \longrightarrow S^r$ a homotopy equivalence by surgery (where V' is a certain framed r-manifold). Similarly $\bar{\sigma}(\epsilon_*(x))$ is the obstruction to making $1 \times \psi'| : F \times V' \longrightarrow F \times S^r$ a homotopy equivalence.

Take $F = \mathbb{CP}^2 \times \mathbb{CP}^2$. Suppose $r \equiv 0 \pmod 4$; then by [14], $\bar{\sigma}(x) = \frac{1}{8}$(signature of V') if $r \geq 8$; but $\bar{\sigma}(x) = \frac{1}{16}$(signature of V') if $r = 4$. Similarly,

$$\begin{aligned} \bar{\sigma}(\epsilon_*(x)) &= \tfrac{1}{8}(\text{signature of } F \times V' - \text{signature of } F \times S^r) \\ &= \tfrac{1}{8}(\text{signature of } V') \ \text{ for all } r. \end{aligned}$$

Thus $\bar{\sigma}(\epsilon_*(x)) = \bar{\sigma}(x)$ unless $r = 4$, when $\bar{\sigma}(\epsilon_*(x)) = 2\bar{\sigma}(x)$.

If $r \equiv 2 \pmod 4$, then it follows from Theorem 9.9 of [17] that $\bar{\sigma}(x) = \bar{\sigma}(\epsilon_*(x))$. (The theorem is stated for $r \geq 5$, but the argument seems to work when $r = 2$.) Since $\pi_r(G/PL) = \pi_{r+8}(G/PL) = 0$ if r is odd, we have proved that $\lambda_*^F : \pi_r(G/PL) \longrightarrow \pi_{r+8}(G/PL)$ is an isomorphism if $r \neq 0, 4$, and a monomorphism onto a subgroup of index 2 if $r = 4$.

Similar arguments show that, if $F = \mathbb{CP}^2$ and $r \geq 1$, then $\bar{\lambda}_*^F : \pi_{r+4}(G/PL) \longrightarrow \pi_{r+8}(G/PL)$ is an isomorphism. Therefore $\bar{\lambda}^F : \Omega^4(G/PL)_0 \longrightarrow \Omega^8(G/PL)_0$ is a homotopy equivalence. Let $\lambda : G/PL \longrightarrow \Omega^4(G/PL)$ be the composite of $\lambda^{\mathbb{CP}^2 \times \mathbb{CP}^2}$ with a homotopy inverse to $\bar{\lambda}^{\mathbb{CP}^2}$; then λ has the desired properties.

Now we can complete the proof of Theorem 5 by showing that, if $\dim F = 4$, then

$$(\rho\theta)_* \ : \ \pi_1(G_F/\widetilde{PL_F}) \longrightarrow \pi_1((G/PL)^{F^*})$$

is surjective. Consider the following diagram:

$$\begin{tikzcd}
\pi_1((G/PL)^F) \arrow[r, "\rho_*"] \arrow[d, "\lambda_*"] & \pi_1((G/PL)^{F^*}) \arrow[r, "\partial"] \arrow[d, "\lambda_*"] & \pi_0(\Omega^4(G/PL)) \arrow[d, "\lambda_*"] \\
\pi_1((\Omega^4(G/PL))^F) \arrow[r, "\rho_*"] & \pi_1((\Omega^4(G/PL))^{F^*}) \arrow[r, "\partial"] & \pi_0(\Omega^8(G/PL)) .
\end{tikzcd}$$

The rows are taken from the homotopy exact sequences of the Hurewicz fibrations

$$(G/PL)^F \longrightarrow (G/PL)^{F^*} \quad , \quad (\Omega^4(G/PL))^F \longrightarrow (\Omega^4(G/PL))^{F^*} .$$

The proof of Theorem 7 shows that, in the bottom row, ρ_* is surjective so $\partial = 0$. But

$$\lambda_* \ : \ \pi_0(\Omega^4(G/PL)) \longrightarrow \pi_0(\Omega^8(G/PL))$$

is injective, so $\partial = 0$ in the top row. Therefore

$$\rho_* \;:\; \pi_1((G/PL)^F) \longrightarrow \pi_1((G/PL)^{F^*})$$

is surjective.

Let $x \in \pi_1((G/PL)^{F^*})$, and choose an element

$$\bar{x} \in \pi_1((G/PL)^F)$$

such that $\rho_*(\bar{x}) = x$. Let β be an interval in S^1 containing the base-point, let $Q = F \times S^1$, $W = F \times \beta$. Let ν_Q, ψ' be as in the proof of Theorem 7. Since $\bar{\sigma}(x) \in P_5 = 0$, we can do surgery to find a map $\psi : S^k \longrightarrow Q^{\bar{\nu}}$ which is transverse regular to ψ' (rel D^k), with the following property. Let $E = \psi^{-1}(Q)$; then $\psi| : E \longrightarrow Q$ is a homotopy equivalence.

Let b_0, b_1 be the end-points of β, and let B be the cell complex $\{b_0, b_1, \beta, \overline{S^1 - \beta}\}$. Then $\overline{E - W}$ is an h-cobordism between $F \times b_0$ and $F \times b_1$, and the PL homeomorphism $1 \times b_1 : F \times b_0 \longrightarrow F \times b_1$ is in the preferred homotopy class. By Barden's h-cobordism theorem for 5-manifolds [2], there is a PL homeomorphism $H : F \times |B - \beta| \longrightarrow \overline{E - W}$ with $H(F \times b_i) = F \times b_i$. Now we can define a block bundle ξ over B with $E(\xi) = E$, and a map $t : E \longrightarrow F$, as in the proof of Theorem 5. We obtain a $G_F/\widetilde{PL}_F$-bundle (ξ, t) over B, representing an element $y \in \pi_1(G_F/\widetilde{PL}_F)$ such that $\theta_*(y) = \bar{x}$. Therefore $x = (\rho\theta)_*(y)$, so

$$(\rho\theta)_* \;:\; \pi_1(G_F/\widetilde{PL}_F) \longrightarrow \pi_1((G/PL)^{F^*})$$

is surjective. This completes the proof of Theorem 5.

V. Topologically Trivial Block Bundles

Let ξ be a block bundle over B with fibre F. A **proper trivialisation** of ξ is a proper map

$$h \;:\; E(\xi) \longrightarrow F \times |B|$$

such that

$$h(E_\beta(\xi)) \subset F \times \beta \quad \text{for each } \beta \in B$$

(base-points will be irrelevant in this chapter). Two proper trivialisations h_0, h_1 of ξ are **properly homotopic** if there is a proper map

$$H \;:\; E(\xi) \times I \longrightarrow F \times |B|$$

such that

$$H(E_\beta(\xi) \times I) \subset F \times \beta$$

for each $\beta \in B$ and $H_t = h_t$ $(t = 0, 1)$. A **topological trivialisation** of ξ is a proper trivialisation which is a topological homeomorphism; a PL **trivialisation** is defined similarly.

Theorem 8. *Let ξ be a block bundle over B with fibre $\mathbb{R}^q$ ($q \geq 3$). Let $h : E(\xi) \longrightarrow \mathbb{R}^q \times |B|$ be a topological trivialisation of ξ. Then there is an obstruction $w \in H^3(B;\mathbb{Z}_2)$ which vanishes if and only if h is properly homotopic to a PL trivialisation of ξ.*

Proof. Let V, W be PL manifolds and let N be a compact submanifold of W with $\partial N = N \cap \partial W$. A map $\phi : V \longrightarrow W$ is **h-regular** on N if it is transverse regular on N and $\phi| : \phi^{-1}(N) \longrightarrow N$ is a homotopy equivalence. Let Q denote $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2$. Our first objective is to construct the following:

(1) A proper map $f : E(\xi) \times Q \longrightarrow \mathbb{R}^q \times |B| \times Q$ such that, for each $\beta \in B$,

$$f| \ : \ E_\beta(\xi) \times Q \longrightarrow \mathbb{R}^q \times \beta \times Q$$

is h-regular on $0 \times \beta \times Q$.

(2) A proper homotopy F from $h \times 1$ to f such that, for each $\beta \in B$,

$$F(E_\beta(\xi) \times Q \times I) \subset \mathbb{R}^q \times \beta \times Q \ .$$

We shall eventually use f and F to construct a PL trivialisation of ξ. The factor Q is introduced to avoid difficulties with low-dimensional manifolds.

Let $T = \partial\Delta^2$ and write T^r for the product of r copies of T. Note that the universal covering space $\widetilde{T}^r$ of T is PL homeomorphic to $\mathbb{R}^r$. Choose a PL embedding $\mathbb{R} \times T^{q-1} \subset \mathbb{R}^q$ and a PL homeomorphism $\mathbb{R}^q \longrightarrow \mathbb{R} \times \widetilde{T}^{q-1}$ such that the composite

$$e \ : \ \mathbb{R}^q \longrightarrow \mathbb{R} \times \widetilde{T}^{q-1} \longrightarrow \mathbb{R} \times T^{q-1} \subset \mathbb{R}^q$$

is the identity on a neighbourhood of the origin.

Let A be a subcomplex of B. Let $W_{A,r}$ denote $\mathbb{R}^r \times T^{q-r} \times |A| \times Q$ and let $N_{A,r} = 0 \times T^{q-r} \times A \times Q \subset W_{A,r}$. We have an embedding $W_{A,1} \subset \mathbb{R}^q \times A \times Q$ and there is a covering map $p : W_{A,r} \longrightarrow W_{A,r-1}$. Define $V_{A,1} = (h \times 1)^{-1}(W_{A,1})$ and let $g_{A,1} = h \times 1| : V_{A,1} \longrightarrow W_{A,1}$. Define $V_{B,r}$, $g_{B,r}$ ($r \geq 2$) inductively as follows. Let $p : V_{B,r} \longrightarrow V_{B,r-1}$ be the covering map induced from $p : W_{B,r} \longrightarrow W_{B,r-1}$ by the homeomorphism $g_{B,r-1} : V_{B,r-1} \longrightarrow W_{B,r-1}$. Let $g_{B,r} : V_{B,r} \longrightarrow W_{B,r}$ be a homeomorphism such that $pg_{B,r} = g_{B,r-1}p$. Finally let $V_{A,r} = p^{-1}(V_{A,r-1})$ and let $g_{A,r} = g_{B,r}|V_{A,r}$. We write W^n_r, N^n_r, V^n_r, g^n_r for $W_{B^n,r}$, $N_{B^n,r}$, $V_{B^n,r}$, $g_{B^n,r}$ respectively, and abbreviate $W_{B,r}$, $N_{B,r}$, $V_{B,r}$, $g_{B,r}$ to W_r, N_r, V_r, g_r.

Suppose inductively that we have constructed the following, for some integer n:

(1) A proper map $f^{n-1}_1 : V^{n-1}_1 \longrightarrow W^{n-1}_1$ such that, for each $\beta \in B^{n-1}$, $f^{n-1}_1|V_{\beta,1} \longrightarrow W_{\beta,1}$ is h-regular on $N_{\beta,1}$.

(2) A proper homotopy F^{n-1}_1 from g^{n-1}_1 to f^{n-1}_1 such that, for each $\beta \in B^{n-1}$, $F^{n-1}_1(V_{\beta,1} \times I) \subset W_{\beta,1}$.

Suppose also that f_1^{n-1}, F_1^{n-1} are extensions of f_1^{n-2}, F_1^{n-2}.

Now let $\beta \in B^n - B^{n-1}$. Let $f_{\partial\beta,1} = f_1^{n-1}|V_{\partial\beta,1}$ and let $F_{\partial\beta,1} = F_1^{n-1}|V_{\partial\beta,1} \times I$. The inductive hypothesis ensures that $f_{\partial\beta,1}$ is transverse regular on $N_{\partial\beta,1}$. Thus $M_{\partial\beta,1} = f_{\partial\beta,1}^{-1}(N_{\partial\beta,1})$ is a submanifold of $V_{\partial\beta,1}$ of codimension 1.

Lemma 10. *$f_{\partial\beta,1}$ is h-regular on $N_{\partial\beta,1}$.*

Proof. Let B be a cell complex. A **blocked space** E over B consists of a topological space E and, for each $\beta \in B$, a subspace E_β of E such that the following conditions are satisfied:

1. $\{E_\beta : \beta \in B\}$ is a locally finite covering of E.
2. If $\beta, \gamma \in B$, then $E_\beta \cap E_\gamma = \bigcup_{\delta \subset \beta\cap\gamma} E_\delta$.
3. If β is a face of $\gamma \in B$, then the inclusion $E_\beta \subset E_\gamma$ is a homotopy equivalence.
4. If $\beta \in B$ and $E_{\partial\beta} = \bigcup_{\gamma\subset\partial\beta} E_\gamma$, then the pair $(E_\beta, E_{\partial\beta})$ has the absolute extension property.

If $E^{(1)}$, $E^{(2)}$ are blocked spaces over B, a **blocked equivalence** $\phi : E^{(1)} \longrightarrow E^{(2)}$ is a continuous map such that $\phi(E_\beta^{(1)}) \subset E_\beta^{(2)}$ and $\phi| : E_\beta^{(1)} \longrightarrow E_\beta^{(2)}$ is a homotopy equivalence for each $\beta \in B$. Observe that $M_{\partial\beta,1}$ and $N_{\partial\beta,1}$ are blocked spaces over $\partial\beta$, and $f_{\partial\beta,1}| : M_{\partial\beta,1} \longrightarrow N_{\partial\beta,1}$ is a blocked equivalence.

Suppose inductively that, if $E^{(1)}$, $E^{(2)}$ are blocked spaces over B^{s-1}, then any blocked equivalence $\phi : E^{(1)} \longrightarrow E^{(2)}$ is a homotopy equivalence. Now let $\phi : E^{(1)} \longrightarrow E^{(2)}$ be a blocked equivalence over B^s.

Let $C^{(i)} = \bigcup_{\beta\in B^{s-1}} E_\beta^{(i)}$ and let $D^{(i)}$, $\partial D^{(i)}$ be the disjoint unions of

$$\{E_\beta^{(i)} : \beta \in B^s - B^{s-1}\}, \quad \{E_{\partial\beta}^{(i)} : \beta \in B^s - B^{s-1}\}.$$

Then $\partial D^{(i)} \subset D^{(i)}$ and there are maps $\lambda^{(i)} : \partial D^{(i)} \longrightarrow C^{(i)}$ such that $E^{(i)} = C^{(i)} \cup_{\lambda^{(i)}} D^{(i)}$. By induction, $\phi : C^{(1)} \longrightarrow C^{(2)}$ is a homotopy equivalence.

Now ϕ defines a homotopy equivalence $\psi : D^{(1)} \longrightarrow D^{(2)}$ such that $\phi\lambda^{(1)} = \lambda^{(2)}\psi|\partial D^{(1)}$. By induction, $\psi|\partial D^{(1)} \longrightarrow \partial D^{(2)}$ is a homotopy equivalence. The pairs $(D^{(i)}, \partial D^{(i)})$ satisfy the absolute extension condition; using a result in homotopy theory we deduce that $\phi : E^{(1)} \longrightarrow E^{(2)}$ is a homotopy equivalence. By induction, any blocked equivalence over a finite-dimensional complex is a homotopy equivalence, and the Lemma follows.

Now the PL manifold $V_{\beta,1}$ has two tame ends (for definition see [11]) with free Abelian fundamental groups. Since $M_{\partial\beta,1} \subset V_{\partial\beta,1}$ is a homotopy equivalence (by Lemma 10), $M_{\partial\beta,1}$ bounds collars of the ends of $V_{\partial\beta,1}$. Since $\dim V_{\beta,1} \geq 8$, we can apply Siebenmann's theorem [11,§5] to construct a compact submanifold $M_{\beta,1}$ of

$V_{\beta,1}$ with boundary $M_{\partial\beta,1}$ and such that $M_{\beta,1} \subset V_{\beta,1}$ is a homotopy equivalence. As in [16], we can extend $f_{\partial\beta,1}$ to $f_{\beta,1} : V_{\beta,1} \longrightarrow W_{\beta,1}$, transverse regular on $N_{\beta,1}$ and with $M_{\beta,1} = f_{\beta,1}^{-1}(N_{\beta,1})$. We can also extend $F_{\partial\beta,1}$ to a proper homotopy $F_{\beta,1}$ from $g_{\beta,1}$ to $f_{\beta,1}$.

Do this for all n-cells β of B to obtain extensions f_1^n, F_1^n of f_1^{n-1}, F_1^{n-1} satisfying the inductive hypotheses. This completes our induction on n; we have defined the following:

(1) A proper map $f_1 : V_1 \longrightarrow W_1$ such that for each $\beta \in B$, $f_1| : V_{\beta,1} \longrightarrow W_{\beta,1}$ is h-regular on $N_{\beta,1}$.

(2) A proper homotopy F_1 from g_1 to f_1 such that, for each $\beta \in B$, $F_1(V_{\beta,1} \times I) \subset W_{\beta,1}$.

Suppose inductively that we have defined the following, for some integer $r \geq 1$:

(1) A proper map $f_r : V_r \longrightarrow W_r$ such that for each $\beta \in B$, $f_r| : V_{\beta,r} \longrightarrow W_{\beta,r}$ is h-regular on $N_{\beta,r}$.

(2) A proper homotopy F_r from g_r to f_r such that, for each $\beta \in B$, $F_r(V_{\beta,r} \times I) \subset W_{\beta,r}$.

Let

$$\widetilde{N}_r \; = \; 0 \times \mathbb{R} \times T^{q-r-1} \times |B| \times Q \subset W_{r+1} \; .$$

If $p : W_{r+1} \longrightarrow W_r$ is the covering map then $\widetilde{N}_r = p^{-1}(N_r)$. Lift F^r to a proper homotopy $\widetilde{F}_r$ from g_{r+1} to a map $\tilde{f}_r : V_{r+1} \longrightarrow W_{r+1}$. Let $\widetilde{M}_r = \tilde{f}_r^{-1}(\widetilde{N}_r)$ and let $M_r = f_r^{-1}(N_r)$. Since $p| : \widetilde{M}_r \longrightarrow M_r$ is a covering map and $f_r : M_r \longrightarrow N_r$ is a homotopy equivalence, $\tilde{f}_r| : \widetilde{M}_r \longrightarrow \widetilde{N}_r$ is a proper homotopy equivalence. Let A be a subcomplex of B. Let $\tilde{W}_{A,r} = p^{-1}(N_{A,r})$, $\tilde{f}_{A,r} = \tilde{f}_r|V_{A,r+1}$, $\widetilde{F}_{A,r} = \widetilde{F}_r|V_{A,r+1} \times I$, $\widetilde{M}_{A,r} = \widetilde{M}_r \cap V_{A,r+1}$ and $M_{A,r} = M_r \cap V_{A,r}$.

We construct the following:

(1) A proper map $\phi_r : \widetilde{M}_r \longrightarrow \widetilde{N}_r$ such that for each $\beta \in B$, $\phi_r| : \widetilde{M}_{\beta,r} \longrightarrow \widetilde{N}_{\beta,r}$ is h-regular on $N_{\beta,r+1}$.

(2) A proper homotopy Φ_r from $\tilde{f}_r|\widetilde{M}_r$ to ϕ_r such that, for each $\beta \in B$, $\Phi_r(\widetilde{M}_{\beta,r} \times I) \subset \widetilde{N}_{\beta,r}$.

The construction is exactly the same as the one given above for f_1 and F_1. We apply Siebenmann's theorem to $\widetilde{M}_{\beta,r}$ instead of $V_{\beta,1}$; the details will be omitted.

Using the product structure on a neighbourhood of $\widetilde{M}_r$ in V_{r+1}, we can construct the following:

(1) A proper map $f_{r+1} : V_{r+1} \longrightarrow W_{r+1}$ such that for each $\beta \in B$, $f_{r+1}| : V_{\beta,r+1} \longrightarrow W_{\beta,r+1}$ is h-regular on $N_{\beta,r+1}$.

(2) A proper homotopy F_{r+1} from g_{r+1} to f_{r+1} such that, for each $\beta \in B$, $F_{r+1}(V_{\beta,r+1} \times I) \subset W_{\beta,r+1}$.

This completes the induction on r. When $r = q$ we obtain a proper map $f_q : V_q \longrightarrow W_q = \mathbb{R}^q \times |B| \times Q$ and a proper homotopy F_q from g_q to f_q, satisfying the inductive hypotheses.

Consider the commutative diagram:

$$\begin{tikzcd}
V_q \arrow[r, "g_q"] \arrow[d, "\epsilon"] & W_q \arrow[r, equal] \arrow[d, "\epsilon"] & \mathbb{R}^q \times |B| \times Q \arrow[d, "e \times 1"] \\
E(\xi) \times Q \arrow[r, "h \times 1"] & \mathbb{R}^q \times |B| \times Q \arrow[r, equal] & \mathbb{R}^q \times |B| \times Q
\end{tikzcd}$$

where ϵ denotes a covering map followed by an inclusion. Recall that $e : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is the identity on an open disc neighbourhood U of the origin.

Let A be a subcomplex of B, let X_A denote

$$h^{-1}(U \times |A|) \times Q - h^{-1}(0 \times |A|) \times Q \subset E(\xi) \times Q \ ,$$

and let $X = X_B$, $X^n = X_{B^n}$. Suppose inductively that we have constructed the following, for some integer n.

(1) A subset Y^{n-1} of X^{n-1} such that, for each $\beta \in B^{n-1}$, $Y_\beta = Y^{n-1} \cap X_\beta$ is a compact submanifold of X_β of codimension one and $Y_\beta \subset X_\beta$ is a homotopy equivalence. Then $E(\xi|B^{n-1}) \times Q - Y^{n-1}$ has two components; let Z^{n-1} be the closure of the bounded component. Let $(Z')^{n-1}$ be the component of $\epsilon^{-1}(Z^{n-1})$ which lies in $g_q^{-1}(U \times |B^{n-1}| \times Q)$, and let $(Y')^{n-1} = (Z')^{n-1} \cap \epsilon^{n-1}(Y^{n-1})$.

(2) PL homeomorphisms

$$\begin{aligned} \gamma^{n-1} : Y^{n-1} \times [0,\infty) &\longrightarrow \overline{E(\xi|B^{n-1}) \times Q - Z^{n-1}} \ , \\ (\gamma')^{n-1} \ : \ (Y')^{n-1} \times [0,\infty) &\longrightarrow \overline{V_q^{n-1} - (Z')^{n-1}} \end{aligned}$$

such that γ_0^{n-1}, $(\gamma')_0^{n-1}$ are the inclusions.

Suppose further that γ^{n-1}, $(\gamma')^{n-1}$ are extensions of γ^{n-2}, $(\gamma')^{n-2}$.

Now let $\beta \in B^n - B^{n-1}$. Let

$$\begin{aligned} Y_{\partial\beta} &= Y^{n-1} \cap X_{\partial\beta} \ , \\ \gamma_{\partial\beta} &= \gamma^{n-1} | Y_{\partial\beta} \times [0,\infty) \ , \\ \gamma'_{\partial\beta} &= (\gamma')^{n-1} | Y'_{\partial\beta} \times [0,\infty) \ . \end{aligned}$$

Then $Y_{\partial\beta}$ bounds a collar of the end of $E(\xi|\partial\beta) \times Q$. It follows that $Y_{\partial\beta} \subset X_{\partial\beta}$ is a homotopy equivalence; since $\dim X_{\partial\beta} \geq 8$, $Y_{\partial\beta}$ bounds a collar of the ends of

$X_{\partial\beta}$.

Since the ends of X_β are tame and have trivial fundamental groups, Siebenmann's theorem shows that there is a compact submanifold Y_β of X_β with boundary $Y_{\partial\beta}$ and such that $Y_\beta \subset X_\beta$ is a homotopy equivalence. It follows that Y_β bounds a collar of the end of $E(\xi|\beta) \times Q$. Let

$$\gamma_\beta \ : \ Y_\beta \times [0,\infty) \longrightarrow \overline{E(\xi|\beta) \times Q - Z_\beta}$$

be a PL homeomorphism such that $(\gamma_\beta)_0$ is the inclusion and $\gamma_\beta|Y_{\partial\beta} \times [0,\infty) = \gamma_{\partial\beta}$. Do this for all n-cells β of B to obtain Y^n, γ^n satisfying the inductive hypotheses.

Define $(Z')^n$, $(Y')^n$ as in (1) above, and note that $\epsilon \ : (Z')^n \longrightarrow Z^n$ is a PL homeomorphism. Then, for each $\beta \in B^n - B^{n-1}$, $Y'_\beta \subset \overline{V_{\beta,q} - Z'_\beta}$ is a homotopy equivalence, so Y'_β bounds a collar of the end of $V_{\beta,q}$. Let

$$\gamma'_\beta \ : \ Y'_\beta \times [0,\infty) \longrightarrow \overline{V_{\beta,q} - Z'_\beta}$$

be a PL homeomorphism such that $(\gamma'_\beta)_0$ is the inclusion and $\gamma'_\beta|Y'_{\partial\beta} \times [0,\infty) = \gamma'_{\partial\beta}$. Then the γ'_β fit together to define an extension $(\gamma')^n$ of $(\gamma')^{n-1}$ satisfying the inductive hypotheses. This completes the induction on n.

Let

$$Y \ = \ \bigcup_{n=1}^{\infty} Y^n \quad , \quad Z \ = \ \bigcup_{n=1}^{\infty} Z^n \quad , \quad \gamma \ = \ \bigcup_{n=1}^{\infty} \gamma^n \quad , \quad \gamma' \ = \ \bigcup_{n=1}^{\infty} (\gamma')^n \ .$$

Define a PL homeomorphism $\psi \ : \ E(\xi) \times Q \longrightarrow V_q$ by $\psi \ = \ \epsilon^{-1}$ on Z and $\psi \ = \ \gamma^{-1}(\epsilon^{-1} \times 1)\gamma$ elsewhere. Define a proper homotopy Ψ from $g_q\psi$ to $h \times 1$ as follows. If $x \in \mathbb{R}^q$, $y \in |B|$, $z \in Q$ and $t \in [0,1)$, let $g_q\psi(h^{-1}(tx,y),z) = (x',y',z')$, and define

$$\Psi(h^{-1}(x,y),z,t) \ = \ (t^{-1}x',y',z') \ .$$

Define

$$\Psi(h^{-1}(x,y),z,0) \ = \ (x,y,z) \ ;$$

this makes Ψ continuous since $(x',y',z') = (tx,y,z)$ provided t is sufficiently small.

Now we can define the proper map $f : E(\xi) \times Q \longrightarrow \mathbb{R} \times |B| \times Q$ and proper homotopy F from $h \times 1$ to f, as promised at the beginning of the proof. Let $f = f_q\psi$ and let $F = \Psi * (F_q\psi)$ be defined by

$$F(x,t) \ = \ \begin{cases} \Psi(x,2t) & 0 \le t \le \frac{1}{2} \\ F_q\psi(x,2t-1) & \frac{1}{2} \le t \le 1 \ . \end{cases}$$

Then f and F have the required properties (1) and (2).

Suppose inductively that we have constructed the following, for some integer

n.

(1) A proper trivialisation $j^{n-1} : E(\xi|B^{n-1}) \longrightarrow \mathbb{R}^q \times |B^{n-1}|$.
(2) A proper homotopy J^{n-1} from h^{n-1} to j^{n-1}.
(3) A proper homotopy L^{n-1} from f^{n-1} to $j^{n-1} \times 1$ such that, for each $\beta \in B^{n-1}$, $L^{n-1}|E(\xi|\beta) \times Q$ is h-regular on $0 \times \beta \times Q$.
(4) A proper homotopy $\mathcal{L}^{n-1}$ from $\bar{F}^{n-1} * (J^{n-1} \times 1)$ to L^{n-1} (rel $\mathbb{R}^q \times |B^{n-1}| \times Q \times \partial I$).

Suppose further that j^{n-1}, J^{n-1}, L^{n-1}, $\mathcal{L}^{n-1}$ are extensions of j^{n-2}, J^{n-2}, L^{n-2}, $\mathcal{L}^{n-2}$ respectively.

Let $\beta \in B^n - B^{n-1}$. If A is a subcomplex of B^{n-1}, then j_A, J_A, L_A, $\mathcal{L}_A$ will have the usual meanings. As in Lemma 10 we see that

$$L_{\partial\beta} \;:\; E(\xi|\partial\beta) \times Q \times I \longrightarrow \mathbb{R}^q \times \partial\beta \times Q$$

is h-regular on $0 \times \partial\beta \times Q$. Note that $L_{\partial\beta}$ is a proper homotopy from $f_{\partial\beta}$ to $j_{\partial\beta} \times 1$. Extend $L_{\partial\beta}$ to a proper homotopy K_β from f_β to a proper map $k_\beta : E_\beta(\xi) \times Q \longrightarrow \mathbb{R}^q \times \beta \times Q$. We can arrange for K_β to be h-regular on $0 \times \beta \times Q$.

Now $J_{\partial\beta}$ is a proper homotopy from $h_{\partial\beta}$ to $j_{\partial\beta}$. Extend $J_{\partial\beta}$ to a proper map I_β from h_β to a proper map $i_\beta : E_\beta(\xi) \longrightarrow \mathbb{R} \times \beta$. Using the homotopies $(I_\beta \times 1) * F_\beta * K_\beta$ and $\mathcal{L}_{\partial\beta}$, we see that $i_\beta \times 1$ is properly homotopic (rel $\mathbb{R}^q \times \partial\beta \times Q$) to k_β.

The obstruction to deforming i_β properly (rel $E(\xi|\partial\beta)$) to a PL homeomorphism $j'_\beta : E_\beta(\xi) \longrightarrow \mathbb{R}^q \times \beta$ is an element $x \in \pi_n(G/PL)$. Let $\lambda_* : \pi_n(G/PL) \longrightarrow \pi_{n+8}(G/PL)$ be the periodicity homomorphism discussed in Chapter IV. Then $\lambda_*(x)$ is the obstruction to deforming $i_\beta \times 1$ properly (rel $E(\xi|\partial\beta) \times Q$) to a map k'_β which is h-regular on $0 \times \beta \times Q$. The previous paragraph shows that $\lambda_*(x) = 0$; since λ_* is a monomorphism, $x = 0$. Choose a PL homeomorphism $j'_\beta : E_\beta(\xi) \longrightarrow \mathbb{R} \times \beta$ and a proper homotopy J'_β from h_β to j'_β extending $J_{\partial\beta}$.

Now $\mathcal{L}_{\partial\beta}$ is a proper homotopy from $\bar{F}_{\partial\beta} * (J_{\partial\beta} \times 1)$ to $L_{\partial\beta}$. Extend $\mathcal{L}_{\partial\beta}$ to a proper homotopy $\mathcal{G}_\beta$ (rel $E_\beta(\xi) \times Q \times \partial I$) from $\bar{F}_\beta * (J'_\beta \times 1)$ to a proper homotopy G_β between f_β and $j'_\beta \times 1$. Let $y \in \pi_{n+9}(G/PL)$ be the obstruction to deforming G_β properly (rel $\partial(\mathbb{R}^q \times \beta \times Q \times I)$) to a homotopy G'_β which is h-regular on $0 \times \beta \times Q$.

If we vary (j'_β, J'_β) by an element $z \in \pi_{n+1}(G/PL)$, we replace y by $y + \lambda_*(z)$. If $n \neq 3$ then λ_* is surjective, so we can choose z so that $y + \lambda_*(z) = 0$. In other words, we can replace (j'_β, J'_β) by a pair (j_β, J_β) for which y vanishes. Then there is a proper homotopy L_β from f_β to $j_\beta \times 1$ which is h-regular on $0 \times \beta \times Q$, and a proper homotopy $\mathcal{L}_\beta$ (rel $\mathbb{R}^q \times \beta \times Q \times \partial I$) from $\bar{F}_\beta * (J_\beta \times 1)$ to L_β; L_β and $\mathcal{L}_\beta$ are extensions of $L_{\partial\beta}$ and $\mathcal{L}_{\partial\beta}$ respectively.

Do this for all n-cells β of B to obtain j^n, J^n, L^n, $\mathcal{L}^n$ satisfying conditions (1)–(4). This completes the induction provided $n \neq 3$. In case β is a 3-cell of B, let $c(\beta) \in \mathbb{Z}_2$ be the mod 2 reduction of $y \in \pi_{12}(G/PL) = \mathbb{Z}$. This defines a cochain $c \in C^3(B; \mathbb{Z}_2)$. The above argument enables us to construct j^3, J^3, L^3, $\mathcal{L}^3$ provided $c = 0$.

We consider the effect of varying L^2. Suppose j^1, J^1, L^1, $\mathcal{L}^1$, j^2, J^2, L^2, $\mathcal{L}^2$ are constructed, and let β be a 3-cell in B. Observe that, if the cells $\alpha \subset \partial\beta$ are oriented suitably, then $\partial\beta = \sum_{\alpha\subset\partial\beta} \alpha \in C_2(B; \mathbb{Z})$. If we vary L_α by an element $u_\alpha \in \pi_{12}(G/PL) = \mathbb{Z}$, it can be seen that $c(\beta)$ is replaced by $c(\beta) + (\sum_{\alpha\subset\partial\beta} u_\alpha)_2$. Let $u \in C^2(B; \mathbb{Z}_2)$ be the cochain defined by $u(\alpha) = u_\alpha$; then we have replaced c by $c + \delta u$.

Now let γ be a 4-cell of B, so $\partial\gamma = \sum_{\beta\subset\partial\gamma} \beta \in C_3(B; \mathbb{Z})$. For each 3-cell $\beta \subset \partial\gamma$, define j'_β, J'_β as above, and define

$$J'_{\partial\gamma} \;:\; E(\xi|\partial\gamma) \times I \longrightarrow \mathbb{R}^q \times \partial\gamma$$

by $J'_{\partial\gamma}|E(\xi|\beta) \times I = J'_\beta$. It is easy to adjust (j'_β, J'_β) on one cell $\beta \subset \partial\gamma$ until $J'_{\partial\gamma}$ extends to a proper homotopy J'_γ from h_γ to a PL homeomorphism $j'_\gamma : E(\xi|\partial\gamma) \longrightarrow \mathbb{R}^q \times \partial\gamma$.

Define

$$G_{\partial\gamma} \;:\; E(\xi|\partial\gamma) \times Q \times I \longrightarrow \mathbb{R}^q \times \partial\gamma \times Q$$

by $G_{\partial\gamma}|E(\xi|\beta) \times Q \times I = G_\beta$. Let $v_\gamma \in \pi_{12}(G/PL) = \mathbb{Z}$ be the obstruction to deforming $G_{\partial\gamma}$ properly (rel $E(\xi|\partial\gamma) \times Q \times \partial I$) to a proper homotopy G' which is h-regular on $0 \times \partial\gamma \times Q$. Then it can be seen that $v_\gamma = \sum_{\beta\subset\partial\gamma} y_\beta$, so $(\delta c)(\gamma) = c(\partial\gamma)$ is equal to the mod 2 reduction of v_γ.

On the other hand, v_γ is the obstruction to deforming $\bar{F}_{\partial\gamma} * (J'_{\partial\gamma} \times 1)$ properly (rel $E(\xi|\partial\gamma) \times Q \times \partial I$) to a proper homotopy $G'_{\partial\gamma}$ which is h-regular on $0 \times \partial\gamma \times Q$. But $\bar{F}_{\partial\gamma} * (J'_{\partial\gamma} \times 1)$ extends to a proper homotopy $\bar{F}_\gamma * (J'_\gamma \times 1)$ from f_γ to $j'_\gamma \times 1$, both of which are h-regular on $0 \times \gamma \times Q$. Now it follows from Wall's surgery theorem [14] that $v_\gamma = 0$. Therefore $(\delta c)(\gamma) = 0$, so c is a cocycle.

Let $w \in H^3(B; \mathbb{Z}_2)$ be the cohomology class of c; we have shown that our construction can be carried out if $w = 0$. Assume now that $w = 0$, and let $j = \bigcup_{n=1}^\infty j^n$, $J = \bigcup_{n=1}^\infty J^n$. Then j is a PL trivialisation of ξ and J is a proper homotopy from h to j, as required. It is not hard to see that $w = 0$ whenever h is properly homotopic to a PL trivialisation, so Theorem 8 is proved.

Theorem 8 implies a result on the Hauptvermutung by fairly well-known arguments, given in Sullivan's thesis.

Corollary. *Let* M^n, N^n *be closed,* 1*-connected PL manifolds with* $n \geq 5$ *and let* $h : M \longrightarrow N$ *be a topological homeomorphism. If* $H^3(M;\mathbb{Z}_2) = 0$*, then* h *is homotopic to a PL homeomorphism.*

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University of Liverpool
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