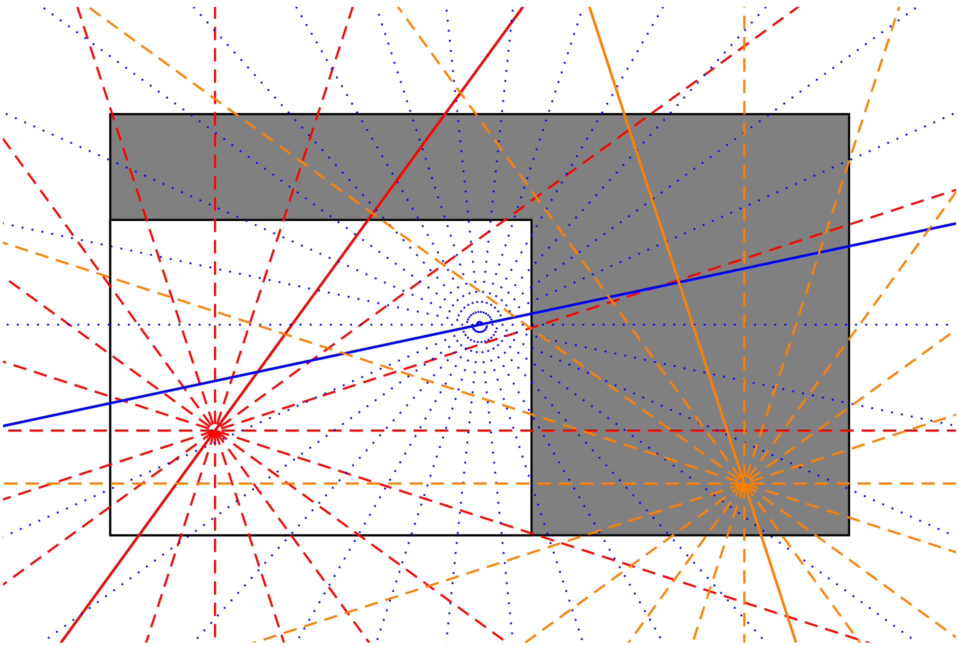
A Guide to Puzzle-based Learning in STEM Subjects



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A GUIDE TO
PUZZLE-BASED LEARNING
IN STEM SUBJECTS

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The picture on the front cover illustrates one technique for solving Puzzle 25.

Contents

1 Introduction

This report is part of a project funded by the Centre for Learning and Academic Development (CLAD) at the University of Birmingham and we gratefully acknowledge their support. The objective of the project was to develop new learning resources to enable staff working in Science, Technology, Engineering, and Mathematics (STEM) to incorporate puzzle-based learning in their teaching. This guide to puzzle-based learning accompanies a selection of mathematical and logic-based puzzles, grouped by mathematical topic and approximate 'level', as judged by our experiences. We shall comment more on this later. It is written to provide advice to staff on how to adapt such puzzles for use in their subject at the appropriate level(s).

Our motivation for this project is a belief, based on our experience here and elsewhere, that puzzle-based learning is under-exploited in the teaching of mathematics and problem solving to STEM students. In Section 2 we define our terms and provide examples. Our teaching experience strongly suggests that embedding puzzles in the curriculum enhances students' learning by developing their general problem-solving and independent learning skills. We also expect this will increase their motivation to learn mathematics, whether as a subject in its own right or as vital learning for other STEM disciplines. We expand on this theme in Section 3, and in Section 4 provide case studies of how such puzzles have been used with students.

2 What is a puzzle?

The phrase *puzzle-based learning* is taken from the title of Michalewicz and Michalewicz (2008), although it continues a long tradition within the mathematics, science, and engineering communities. We start by defining our terms. We use the word *task* as a catch-all for any activity given to a student. The educational literature contains many terms which describe more specific types of task including *exercise*, *problem* and *puzzle*. Unfortunately, there are no agreed definitions and these words are used, sometimes interchangeably, to encompass a wide variety of tasks. We shall discuss hallmarks and characteristics of tasks and also how and why they might be useful to students' education. Some of these characteristics refer to the mathematics of the task itself, others relate to common experiences of students and teachers when using the tasks.

Something that is technically complex, at least for the person undertaking it, but can be solved by a routine well-established technique is called an *exercise*.

▼ Example task 1

$$\textit{Show } \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x = \frac{22}{7} - \pi.$$

The answer to this task contains a mathematical joke. If students are curious about this strange and amusing result, then this is an exercise in polynomial long division and basic integration. Such exercises, often without any humour, form an important part of *direct instruction*. This is a form of teaching in which a teacher explains some theory and gives worked examples. A student is then given exercises to practise (imitate even) the techniques just shown. Such exercises certainly have their place and they characterise *traditional teaching*. One implicit part of the contract between the teacher and student is that such exercises relate closely to what has just been taught. The following, satirical, criticism of a strict diet of such instruction is a reminder to us that dissatisfaction with education is nothing new.

> I was at the mathematical school, where the master taught his pupils after a method scarce imaginable to us in Europe. The proposition and demonstration, were fairly written on a thin wafer, with ink composed of a cephalic tincture. This, the student was to swallow upon a fasting stomach, and for three days following, eat nothing but bread and water. As the wafer digested, the tincture mounted to his brain, bearing the proposition along with it. But the success has not hitherto been answerable, partly by some error in the *quantum* or composition, and partly by the perverseness of lads, to whom this bolus is so nauseous, that they generally steal aside, and discharge it upwards, before it can operate; neither have they been yet persuaded to use so long an abstinence, as the prescription requires. (Swift, 1726, Chapter 4)

A more substantial review of these issues was undertaken by Mason et al. (2010) and Schoenfeld (1992). However, part of the motivation for puzzle-based learning is a desire to widen the scope of activities from such exercises to problems and puzzles.

For us, a *problem* is more than an exercise. That is to say, it is more than a predictable task relating directly to work just taught. It will not be immediately apparent how to proceed and students need to try to understand what the problem is actually about. So, they have to take responsibility for making their own decisions.

Problems are often posed in words, somewhat dressed up. One of the skills we seek to develop in STEM students is the ability to un-dress tasks and isolate the essential details. This is *modelling* albeit in its most rudimentary form. It requires the student to make choices about how to represent a problem mathematically, even when there is a simple (e.g. linear) equation which represents the situation exactly. It is clearly a vital skill in engineering where problems are rarely presented in mathematical form.

▼ Example task 2

A large steel cylindrical tank is required to have a volume of 32 *m*^{3} *and to use the smallest amount of steel in its construction. What height will it have to be to satisfy these conditions?*

Once this process has been completed, students may then recognize the reformulated task as a routine exercise and hence be able to solve it. However, writing down equations poses some serious psychological challenges.

▼ Example task 3

Write an equation for the following statement: "There are six times as many students as professors at this university". *Use* S *for the number of students and* P *for the number of professors.*

When Clement et al. (1981) gave this task to 150 calculus level students, 37% answered incorrectly and $6S = P$ accounted for two thirds of the errors. There are genuine difficulties in moving from a word problem to a mathematical system which represents it. Similar conceptual difficulties occur with algebra story problems, particularly those concerning rates of work, concentration and dilution problems. A systematic analysis of story problems is given in Mayer (1981). It is precisely because these difficulties exist that we have a responsibility to address them explicitly in providing such tasks for our students. Some might argue they should be addressed at school.

> I hope I shall shock a few people in asserting that the most important single task of mathematical instruction in the secondary school is to teach the setting up of equations to solve word problems. [...] And so the future engineer, when he learns in the secondary school to set up equations to solve "word problems" has a first taste of, and has an opportunity to acquire the attitude essential to, his principal professional use of mathematics. (Polya, 1962, Vol. I, pg. 59)

Nevertheless, university students in all STEM subjects, including mathematics, appear to struggle with word problems. It is clear that practice of such tasks makes them less problematic and closer to exercises, and this is our point: the student's experience may have as important a bearing on the characteristics of the task as does the task itself. To some people a task is an exercise, to others it is a more challenging problem. Therefore we are unable to differentiate exercise from problem clearly, without the context of the students for whom the task is intended. Indeed, calculus and algebra are coherent systems of tools which enable a very wide range of problems to be framed in a way that they become exercises. Our point here is that exposure to genuine problem solving must accompany practice of exercises.

Problems *vs* exercises is a useful distinction and one we do not claim is novel.

> First, what is a *problem*? We distinguish between *problems* and *exercises*. An exercise is a question you know how to resolve immediately. Whether you get it right or not depends on how expertly you apply specific techniques, but you don't need to puzzle out which techniques to use. In contrast a problem demands much thought and resourcefulness before the right approach is found. (Zeitz, 2007)

It should be noted that there are many valuable problems which require *estimation* and *approximation*. These are valuable skills for all STEM students and the following task illustrates this.

▼ Example task 4

How many dumper trucks would be needed to cart away Mount Everest?

To defend a solution to this task a variety of choices need to be made and approximations used. *"How big is a dumper truck?"*, *"What do we mean by Mount Everest?"* (down to sea level or the plateau?), *"Can we approximate the mountain by a cone or a cube?"*. A key part of the task is identifying and estimating the missing information. By breaking tasks down into parts, it is often possible to arrive at an answer that is good to an *order of magnitude*. Estimation problems such as this are sometimes referred to as *Fermi problems*, after the Nobel prize-winning physicist Enrico Fermi (1901–1954). More comments on these kinds of tasks are given in Weinstein and Adam (2008). Estimation is a useful skill for all STEM students to develop, particularly engineers who can use estimates to check answers to *design problems* that have been found by more conventional means. Estimation does demand *"much thought and resourcefulness"*, Zeitz (2007), but the methods can still become, with practice, mainly routine.

Turning specifically to puzzles, Michalewicz and Michalewicz (2008) have said *"sometimes the difference is not clear between a puzzle and a real problem"*. However, for us *puzzles* have additional characteristics to other problems, which we try to articulate here. They also differ from estimation tasks in important ways.

2.1 Hallmarks of a puzzle

Michalewicz and Michalewicz (2008) state that (educational) puzzles satisfy four criteria: generality (explaining some universal mathematical problem-solving principle), simplicity, "Eureka" factor and entertainment factor. We believe that generality is a characteristic of problems, not just puzzles. As noted by Michalewicz and Michalewicz (2008) not all puzzles meet the simplicity criterion. However, the other two criteria are critical. Our contention is that a puzzle is a problem that is perplexing and either has a solution requiring considerable ingenuity, perhaps a lateral thinking solution, or possibly results in an unexpected, even a counter-intuitive or apparently paradoxical, solution. Solving the puzzle usually results in a "Eureka" moment, very satisfying for the solver and the process of finding a solution is both frustrating and entertaining. The application of ingenuity extends much further than being able to write down a correct model.

Puzzles constitute a significant intellectual challenge. Because of the difficulties this obviously presents when using such tasks with students we sought puzzles with a variety of fruitful approaches which lead to the correct answer. In particular, we sought puzzles where there was both a conventional solution (preferably a contrasting, particularly complex exercise) and a *lateral thinking* solution, de Bono (1967). Lateral thinking is a way of solving problems by by-passing traditional means, employing considerable ingenuity to reach a solution. A consequence of this lateral approach is that the correct answer should be more or less *obvious* once it has been seen. It should certainly be easy to check an answer. These combine, so that puzzles often have an elegant solution which has identifiable aesthetic value. Finding such tasks was, we found, very difficult.

▼ Example task 5

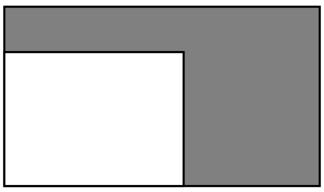
Diagonals of two faces of a cube meet at a vertex. What is the angle between the diagonals?

Clearly this puzzle (Puzzle 17) could be solved with routine trigonometry or perhaps vector methods, which would make it more like a problem than a puzzle. However, noticing that joining the other ends of the segments forms an equilateral triangle gives a lateral thinking solution. This allows us to classify this as a puzzle with both "Eureka" and entertainment factors. We liked puzzles, such as that above, with an element of surprise.

The following is one example of a puzzle (Puzzle 25) which students find difficult, but which has this characteristic.

▼ Example task 6

You own a rectangular piece of land such as that shown below. The 'L-shaped' grey part is woodland, the rectangular white part is pasture.



Explain, with justification, how to build a single straight fence which divides the pasture in half and the woodland in half.

The crucial observation is that a straight line cuts a rectangle in half if and only if it goes through the centre. This line need not, obviously, be restricted to diagonal, vertical or horizontal directions. The

"Aha!" moment is the observation that if we halve the whole rectangle *and* the pasture then we have also cut the woodland in half.

It is intriguing that this leads to an interesting question: what shapes apart from a rectangle have a point through which any line cuts the area in half? Beware, this is not the centre of mass and the point need not lie inside the shape itself. Not all shapes have such a point, e.g. some triangles do not. This opportunity to store up the observations which were crucial in puzzle solving, or which open the way to interesting areas, is something else we sought.

One characteristic of a puzzle and something that distinguishes them from estimation tasks, is that puzzles contain all the needed information; they are self-contained. When posing something as a puzzle this is implicit. For example, in Example task 6 there are no dimensions given. While some students might measure the diagram, the lack of this information signals its irrelevance. When first thinking about a puzzle it may appear impossible without assumptions or estimation. However, being self-contained is itself a very useful piece of information. Indeed, it may lead to the following reasoning, "*Because I know I have all the information needed, then this follows....*" We call such thinking "meta-inferences", but such confident logic is not apparent in the solutions most of our students provide, even when it is clear that "puzzle rules" rather than "estimation rules" are currently in play.

Sometimes problems are described as puzzles because they require deep knowledge of a specific discipline and require the student to work out the correct approach in a specific context. There are many such examples in "200 Puzzling Physics Problems" Gnädig et al. (2001). These may be "puzzling" and would seem to be of great pedagogic value but many are not puzzles as we define them here. This is because we believe the application of ingenuity in a puzzle has to extend much further than being able to write down a correct model. We admit the distinction is fine but in any case many of the problems in Gnädig et al. (2001) are more concerned with exploring physical principles rather than our purpose, which is the teaching of mathematics through puzzles.

The following is an example of a task which appears impossible because there seem to be too many unknowns. An experienced mathematician might be worried about this, but posing this as a puzzle (Puzzle 10) indicates it *must* have a solution.

▼ **Example task 7**

A man walked for 5 *hours, first along a level road, then up a hill, then he turned round and walked back to his starting point along the same route. He walked* 4 *miles per hour on the level,* 3 *uphill and* 6 *downhill. Find the distance walked.*

As described in Section 5, there is also a meta-inference solution to this puzzle. The characteristic that the mental moves needed to solve the puzzles are useful later counters the charge that such tasks are contrived and pointless. Yes, they are certainly contrived but that is the point. It is much more common for students to become disaffected by the problem of "trick questions".

2.2 "Tricks" vs lateral thinking

We have already commented on our desire to choose puzzles for which there is a lateral thinking solution as well as a longer routine solution. This risks us choosing tasks for which a *trick* is needed.

The word trick is also hard to define. Here, by trick we mean an intellectual move which is key to solving a task but which is unique to that task, or to very few disparate tasks. Actually, most insightful intellectual moves are worth remembering for use in future problem solving and so we have struggled to find convincing examples for this guide. One candidate would be writing

$$\int \ln(x)\,\mathrm{d}x = \int 1 \times \ln(x)\,\mathrm{d}x \qquad (1)$$

to facilitate integration by parts. Even here, *multiplication by one* and *addition of zero* find uses in many other mathematical proofs. It is a trick work remembering. Note that a student is unlikely to re-invent (1), even with lateral thinking. We wanted to avoid questions where there could be a legitimate charge of it being a trick. This is perhaps best achieved by choosing tasks which have both elegant lateral thinking and more prosaic solutions. In this case the trick is not necessary. Actually, an alternative lateral thinking solution begs the question "what constitutes a solution?". Contemporary students may not be familiar with some forms of arguments, e.g. purely geometrical reasoning. These forms of arguments can be just as rigourous as an algebraic calculation, and one of the values of puzzles lies in expanding the range of ways a problem can be tackled and in the subsequent discussion about the legitimacy of a particular argument.

Lateral thinking is to be encouraged, but before that we need to encourage *thinking*! Example task 8 below might be considered an unfair trick by some but it certainly teaches a lesson about thinking first and problem solving later. It is also a case in which drawing a diagram is helpful, as in so much problem and puzzle solving.

▼ Example task 8

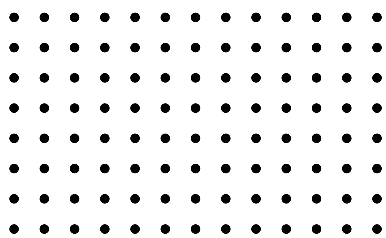
There are two telephone poles, perpendicular to level ground. Each one is 30 *m tall. The poles are an unknown distance apart. A* 50 *m cable is to be strung from the top of one pole to the top of the other. Because the cable is heavy, it will of course droop and take up the shape of a catenary. What must the distance between the two poles be so that the lowest point of the cable touches the ground?*

Drawing a diagram makes it immediately clear that the cable will never touch the ground, even if the poles are adjacent to each other (or indeed, coincident).

The next example is a trick of another kind (Puzzle 46).

▼ Example task 9

Below is part of an infinite integer lattice. A lattice triangle *is a triangle where the coordinates of all vertices are integers.*



What is the size of the smallest equilateral lattice triangle?

The trick here is that is impossible to draw an equilateral triangle[1] on the lattice. We have used this problem with undergraduates during problem solving sessions and with postgraduates and staff during

[1]A triangle must have three distinct finite vertices.

teacher training events. Proving that no triangles exist is an interesting task in its own right. More experienced problem solvers expect problems without a solution and hence this becomes seen as less of a trick as experience increases. Notice that ultimately “the solution” here is the argument as to why no equilateral triangles exist on the integer lattice. Disciplines differ in the extent to which such issues as irrationality are important and this problem is likely to appeal to more mathematically minded students.

2.3 Cultural artifacts

Mathematics constitutes an intellectual sub-culture. Indeed, mathematics has its own history, folklore and humor, see Renteln and Dundes (2005). Mathematics, like sport and music, has international student competitions, see (Djukić et al., 2011).

This sub-culture is not new. Indeed, the first recorded use of algebra story problems occurs during the mathematical training of scribes from around 2500BCE in ancient Iraq, see Robson (2008) and Høyrup (1990). Since then puzzles have always been traded and shared throughout the world and there is a continuous history of use. Many can be found in the historical record. For example *Propositiones ad acuendos juvenes* (Problems to Sharpen the Young) by Alcuin of York (732–804), is an early European collection of tasks, many of which we retain in recognizable form today. See Hadley and Singmaster (1992) for more details and a translation. During the Edo period (1603–1867) the Japanese developed a distinctive form of geometric puzzles called *Sangaku*. These were written on wooden tablets and hung in temples as offerings or challenges (Hidetoshi and Rothman, 2008). Where we are aware of the provenance of a particular task we have recorded it, and more information is available from Swetz (2012).

Clearly a mathematician might be interested in this aspect of puzzles. We go further and claim that an educated scientist and engineer should also engage with puzzles as part of their broader education. Just as people appreciate poetry and music, so a puzzle can also be savored. Clearly this is not our primary motivation for asking students to solve puzzles. We believe they also have more practical aims. However, we have included some puzzles mainly because they have such historical interest. They are cultural artifacts in their own right.

3 What do students learn by engaging with puzzles?

It has long been known that students must struggle to solve problems independently and construct their own meaning. Mathematics education is the art of helping students to reinvent the wheel. For example, as early as 1543 in one of the first English textbooks on arithmetic, Robert Recorde acknowledges this as follows.

> Scholar. Sir, I thanke you, but I thynke I might the better doo it, if you did showe me the woorkinge of it.
> Master. Yea but you muste prove yourselfe to do som thynges that you were never taught, or els you shall not be able to doo any more then you were taught, and were rather to learne by rote (as they cal it) than by reason. (Recorde, 1543, Ground, Sig.F, i, v)[2]

This fundamental tension between *telling students the correct method* to solve a problem and *requiring them to solve for themselves* is particularly marked in the STEM disciplines. More recent scholars echo this sentiment:

[2]This highly influential textbook had over 25 editions between 1543 and 1700, see Howson (2008).

> One of the fundamental contributions of modern *didactique* consists of showing the importance of the rôle played in the teaching process by the learning phases in which the students works almost alone on a problem or in a situation for which she assumes the maximum responsibility. (Brousseau, 1997, pg. 229)

When a problem is posed the student trusts that this will be both interesting and lead to useful insights. This is fundamental to what Brousseau (1997) calls the *didactic contract*. The teacher has the responsibility of choosing problems which are sufficiently novel to be a worthwhile challenge, but which students still have a realistic prospect of solving. The phrase *zone of proximal development* is used to refer to problem solving processes that have not yet matured but are in the process of maturation. It is

> the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers. (Vygotsky, 1978, pg.86)

These ideas are explored more full in, for example, Mason et al. (2007).

What are the learning objectives and educational purpose of using puzzles? This is not a simple question and does not have a unique answer. The hallmarks and characteristics of puzzles cited in our earlier discussion leads us to suggest that puzzles are helpful to students in several ways. It is clear that in solving problems and puzzles a student needs to

- take personal responsibility;
- adopt novel and creative approaches, making choices;
- develop modelling skills;
- develop tenacity;
- practice recognition of cases, reducing problem situations to exercises.

As discussed earlier, the additional hallmark of a puzzle is that students often have to apply considerable ingenuity to solve a perplexing problem and that in doing so they will be frustrated and entertained and in reaching a solution may experience that "Eureka" moment.

Solving puzzles is often a solitary activity and enjoyed as such, but in a teaching context, group work is prevalent. Although any student in a group might be the first to a solution, the pleasure of the "Eureka" moment can be shared, with benefits in team building and student engagement.

We believe the outcomes for students of successful problem-based courses include an increased confidence in problem solving. This is tautological of course, because if you practice solving problems you would expect to become better at it. Problem-based learning addresses a programme level aim in all STEM courses. In the Quality Assurance Agency (QAA) for Higher Education's benchmark statement for mathematics this is explicit:

> 2.20 Programmes in mathematics typically involve continuous mathematics, discrete mathematics, logical argument, problem solving and mathematical modelling. (Lawson et al., 2007)

Similar statements exit in the QAA subject benchmarks for all STEM subjects and are reinforced by the accreditation criteria issued by all STEM professional societies. Problem solving is a key skill in all STEM subjects. Furthermore, there are other important affective outcomes, including real challenge and therefore satisfaction, a sense of achievement and enjoyment. They provoke curiosity and help students to refine their intuition. Nunn (1911) claims that *"the point of immediate importance here is that mathematics is conceived not as a static body of 'truths' but in the dynamic form of an activity"*. In this context, with appropriately chosen puzzles, puzzle-based learning is a good opportunity for students to discover that there may be more than one solution to a problem. As a subclass of problems, puzzles can provide additional challenges, insight and entertainment, all of which can increase student engagement and promote independent learning.

4 How can puzzles be used and adapted?

One way of using puzzles in teaching is through bespoke puzzle-based courses, e.g. Michalewicz and Michalewicz (2008) propose this approach. However our intention is different; we propose that puzzles should be redrafted into an appropriate STEM context and embedded alongside exercises and problems in traditional teaching. In order to do this, it seemed that it would be ideal if the core content of puzzles could be identified, stripped as far as possible of any superfluous context, for example farmers' fields and grazing horses, which commonly appear in puzzle books for lay people. It would then be possible to rebuild the puzzles for specific use and preferably with a clear relevance to a chosen STEM discipline. However, this is more difficult than it reads and in some cases may not actually be desirable. For example, subject specificity may conflict with desirable simplicity, as discussed in Section 5. It is also possible that trivial contextualization might actually be annoying. Before we consider this further, we provide three brief cases of activities in which problems play a leading role but in which puzzle-based learning has already been incorporated. They have all been used successfully within the University of Birmingham for many years and are an integral part of our programmes. The first is being used in Engineering. The second highlights group work, whereas the third focuses on developing coherent mainstream mathematics topics through a sequence of related puzzles and problems.

If the reader prefers to consider some specific puzzles immediately, our selection is in Section 5, page 18.

4.1 Modelling Concepts and Tools

Modelling Concepts and Tools is a first year module currently taught to students from three Engineering disciplines (and previously to four). This module includes modelling techniques, Engineering mathematics, estimation, Excel and MATLAB programming, all taught in an Engineering context. Mathematics is learned primarily through guided study, although a limited number of lectures describes both the scope and the context of the intended learning. In regular Mathematical Problem Classes, this mathematical knowledge is assumed and the bulk of the study is based on problem solving. Sometimes the problems are little more than exercises with an appropriate context, especially early in the academic year.

▼ **Example task 10**

The flow of water through a pipe to a heat-treat quench furnace is given by

$$Q = \sqrt{(3d)^5 \frac{H}{L}}$$

where Q is the flow of water through a pipe of length L and diameter d, with an associated head loss of H. If d decreases by 1% and H by 2%, use the binomial theorem to estimate the decrease in Q. *(Ans.:$\approx 3.5\%$)*

However, other problems are true word problems and reflect more closely the type of problem that students will face later in their progress to professional practice. For example, the following is adapted from Evans (1997).

▼ **Example task 11**

A given volume of a dangerous chemical has to be stored in a closed cylindrical container, which must be filled completely. The cylinder is to stand on one flat end in an open space. If the surface area exposed to the atmosphere (i.e. excluding the area of the base) is to be the minimum possible, calculate the relationship between the diameter and the height of the cylinder. *(Ans.: diameter $= 2\times$ height)*

Mathematical modelling techniques are taught formally and Modelling Problem Classes introduce students to a very difficult class of word problems in which the initial steps to developing a model and its solution can be very challenging. For a pre-calculus example:

▼ **Example task 12**

Reverse osmosis can be used to recover drinking water from sea water. A large unit is treating 140000 m^3 of sea water every day. The sea water contains 36000 ppm of salts and the drinking water is effectively salt free. The plant returns waste brine to the sea at 120000 ppm. What is the volume of drinking water produced per day in acre-feet?
*(*Hint: An acre is roughly 4000 m^2 and a foot about 30 cm.*)*
If an American family uses 1 acre-foot of water per year, roughly how many families will the desalination plant support? *(Ans.: ca. 30000)*

Group work is encouraged, particularly in Modelling Problem Classes. It is vital that Engineers learn to work in teams as they will be most likely to be required to do in professional practice. This is a requirement of the UK Standard for Professional Engineering Competence (UK-SPEC), which has been adapted as the Quality Assurance Agency (QAA) benchmark statement for Engineering. Advantages of group work are described later.

One aspect of modelling, which is also important in mathematics although usually less emphasised, is the need for rigorous checking of solutions. In Engineering problems this usually includes a check on dimensional consistency, consideration of extreme behaviour of equations and the realism of model predictions (for example compared to experimental data). One useful tool for checking models is estimation. Whereas Fermi problems or the "guestimation" tasks given in Weinstein and Adam (2008) tend to be general in nature, many of the estimation tasks in the Modelling Concepts and Tools module are Engineering specific. For example:

▼ **Example task 13**

In your teams, estimate the height of the tallest possible building.

This problem is interesting in having many solutions, the choice of a particular solution depending on the student's view of factors limiting the height. These could be mechanical i.e. considering stresses and strains in the structure, economic, architectural e.g. lift access to the higher floors, psychological e.g. asking if people want to live or work so far from the ground, or indeed some other factor not identified here.

Puzzle-based learning has been embedded within the module for some years, primarily as extension activities in Mathematics Problem Classes for more able students. However, it was found that few students actually attempted these problems because they were "bolted on" to other problems. Nevertheless, the answer to one (Example task 14) is so counter-intuitive that it inspired one diligent student to a significant self-guided investigation.

▼ **Example task 14**

A railway track is exactly 1 km long. It sits on a piece of ground that is flat. One day, under intense heat from the sun, the track expands 1 m in length. Its ends remain fixed to the ground, so the track bows up to form a circular arc of length 1001 m. At the centre of the arc, how high is the track above the ground? What do you think about your answer?

This problem has a surprising answer (20 m), which can be found in several conventional ways. We contend that the surprise alone is not enough to call this a puzzle.

To overcome this lack of engagement, many puzzles are now embedded alongside other problems in Mathematics and Modelling Problem Classes. In most cases, they are presented in an Engineering context. A typical example, adapted from Cooper (2010), would be Puzzle 19:

▼ **Example task 15**

An Icelandic civil engineer is in charge of laying a pipe between a geothermal power plant A and a town B. Between A and B there is a small mountain range of uniform width 3 km. The pipe must go through a straight tunnel through the mountains perpendicular to the edges of the latter. The perpendicular distance of A from the mountains is 3 km, and B is 6 km away. The distance between the town and the power plant, as the crow flies, is 15 km. Where should the tunnel be built to minimise the pipe length?

(Ans: 3 km from the point nearest to A on the mountains)

Recent experience suggests that this deep embedding has increased student engagement.

Finally, it should be noted that the Engineering context should not be just a trivial change to a puzzle. For example a classic puzzle was rewritten as:

▼ **Example task 16**

"Each day," said the demanding boss to the metallurgist, "you must fill some casting moulds with molten titanium and you will continue to do this until all the moulds are full. Moreover, each day your work will become more strenuous. On each day after the first, you must fill double the number of moulds that you have so far filled. For example, if you fill 3 *moulds with titanium on the first day, you will fill* 6 *on the second,* 18 *on the third and so on. Clear?" "Perfectly clear," said the metallurgist who summoned his team and with great skill and dedication, the moulds began to fill. After a week, a third of the available moulds were full. How long did it take them to do the job? Prove this mathematically. (Ans.: 8 days)*

It was pointed out by a colleague that no metallurgist would recognise this situation, which made the adaptation useless; this is not proper discipline specific contextualisation. This will be discussed further in Section 5 where examples of puzzles are given.

4.2 Workshops

The School of Mathematics at the University of Birmingham runs *Workshops* for first year students. The purpose, stated in the course description, is explicitly to develop problem solving skills:

> The material covered here is not on any syllabus and is not needed later in your degree course. Instead, it is designed to improve vital skills essential to all courses:
>
> - understanding and solving problems;
> - tackling problems unlike ones you have met before;
> - thinking clearly and logically;
> - communicating solutions clearly, concisely and convincingly.
>
> [...] The Workshops are designed to make you think for yourself rather than being told what to do; you may need to explore various approaches before you find one that works for you. For the most part, there is no one right way to solve the problems and no solutions are handed out.

The Workshops take place in the even weeks of the term in the first year. Each week students are assigned to a group of three or four students who will work together. Each group has 2 hours to produce its solution to the week's problem on at most two sides of A4 paper. An important part of the Workshops is to be able to submit precise, concise and well argued solutions on which all members of the group agree. Marks are awarded for mathematical presentation and clarity of exposition. Lastly, during the last session of the Spring term each student gives a short talk (4–5 min) on a mathematical topic of their own choosing. The assessment includes mathematical content, presentation and the talk.

Notice during these workshops the students work in a group and produce a joint report. The social dynamic here is an important part of the activity.

One of the tasks used is the Monty Hall problem (Puzzle 42), another is the game of NIM (Puzzle 45). Both of these can certainly be puzzling. The following puzzle, common in problem solving books, is also used.

▼ **Example task 17**

How many squares, of all sizes, with vertical and horizontal sides can you draw with all four corners on the dots of a 5-dot by 5-dot square lattice?
(+ further problems which generalize.)

Other workshops are much more mathematical, these include geometry, formal logical systems or classical topics such as the following.

▼ **Example task 18**

We want to find $\sum_{k=1}^{n} k^d$ *for various powers* $d \in \mathbb{N}$.

1. *What is* $\sum_{k=1}^{n} 1$?
2. *Simplify* $(k+1)^2 - k^2$. *Sum the equation you get from* $k = 1$ *to* n.
   Hence find the formula for $\sum_{k=1}^{n} k$.
3. *Now consider* $(k+1)^3 - k^3$. *Find a formula for* $\sum_{k=1}^{n} k^2$.

3. How far can you go?

This example, as posed in this form, is more like a structured sequence of exercises than a puzzle. However, we argue that we could adapt some of the mathematics underlying the task into a puzzle in the following way. We don't claim that these puzzles will reveal the general power of finite difference methods, as students are instructed to do in the task above. The point of puzzles is that a particular method is not prescribed. We also note that physical artifacts[3] are being used to motivate a puzzle.

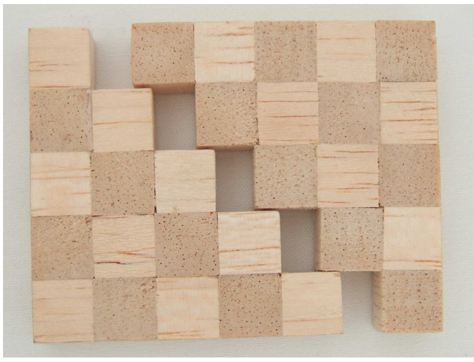


Figure 1: Illustrating the sum from 1 to n

[3] Figure 1 and Figure 2 are reproduced from Bryant and Sangwin (2008) with permission.

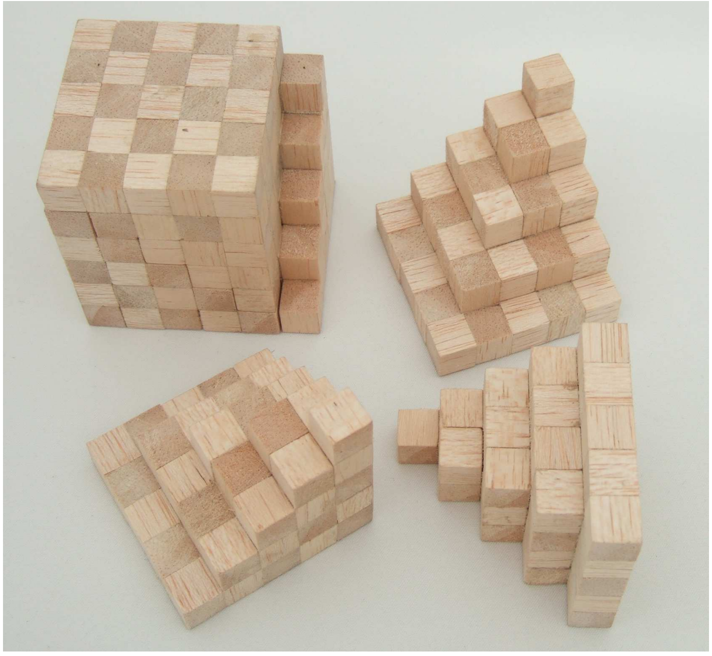


Figure 2: Summing the squares of the numbers from 1 to n

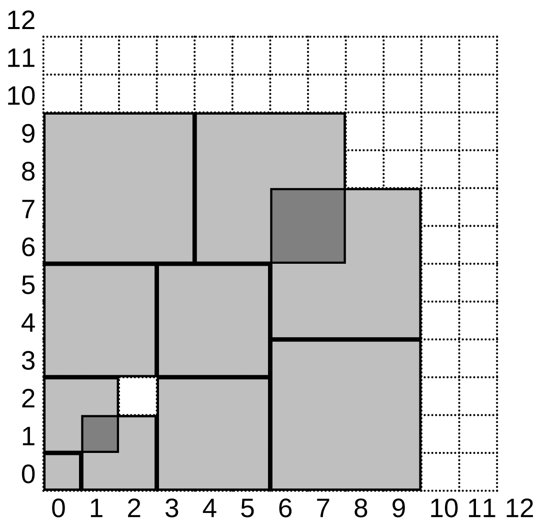


Figure 3: Summing the cubes of the first four integers.

▼ **Example task 19**

---

1. Examine the first model (shown in Figure 1). What does this tell us about $1+2+3+\cdots+n$*?*
2. Examine the second model (shown in Figure 2). What does this tell us about $1^2+2^2+3^2+\cdots+n^2$*?*
3. What does Figure 3 tell us about $1^3+2^3+3^3+\cdots+n^3$*?*

One of the key points about workshops is the development of group working skills. We believe that embedding puzzles in such activities leads to improved group work as it is less likely (than with more conventional problems) that one student would jump immediately to a solution or a solution method. Indeed, some students who are excellent at solving routine tasks may lack the ingenuity to solve puzzles easily. In any case, a puzzle solution should and does engender considerable group discussion as it is explained by the solver to his or her group members.

4.3 Moore Method

The Moore Method is a type of enquiry based learning (EBL) developed by the influential Texan topologist Robert Lee Moore (1882–1974) for university mathematics courses and it has been used widely within a variety of STEM subjects and at a number of levels. A biography of Moore, together with a discussion of his contribution to education, is given by Parker (2004). A Moore Method class works in the following way

1. Tasks, which might be puzzles, are posed by the lecturer to the whole class.
2. Students solve these independently of each other.
3. Students present their solutions to the class, on the board.
4. Students discuss solutions to decide whether they are correct and complete.

Solutions are not imposed by the lecturer, who chairs discussion before offering their own comments. Moore is quoted as saying *"That student is taught the best who is told the least"*, (Parker, 2004, vii).

Moore has a reputation for running his classes in an authoritarian way. For example, he required that students worked alone; those who sought help from their peers or the published literature were expelled (Parker, 2004, pg. 267). One misconception regarding Moore's Method is that he simply stated axioms and theorems and expected students to expound a complete theory. W. Mahavier (see Parker (2004)) said of Moore

> Moore helped his students a lot but did it in such a way that they did not feel that the help detracted from the satisfaction they received from having solved a problem. He was a master at saying the right thing to the right student at the right time.

Moore was particularly successful in attracting and encouraging graduate students, many of whom adopted his teaching approach. As a result, this method is still used widely. Naturally, each teacher varies the precise approach, with some colleagues encouraging students to work as a group, both answering questions and formulating research topics of their own. Alternative solutions are sometimes encouraged, presented and discussed, helping students refine their sense of aesthetics and providing other strategies. In all forms, a key aspect is that *the students' take responsibility for their activity*. Furthermore, in all versions *the group* criticises these solutions and ultimately, together with the teacher, decides if a solution is complete and correct. Given such variations, (Coppin et al., 2009, pg. 13) lists six principles common to versions of Moore's Method:

1. The goal of elevating students from recipients to creators of knowledge.
2. The commitment to teaching by letting students discover the power of their minds.
3. The attitude that every student can and will do mathematics.
4. The time for students to discover, present, and debate mathematics.
5. The careful matching of problems and materials to students.
6. The material, varying widely in difficulty, to cover a significant body of knowledge.

It is important to note that the Method does not aim to transmit coherent bodies of knowledge in a polished professional and pre-defined format. The teacher does not normally provide model solutions to problems. Hence, the method itself can be used with a very wide range of tasks and clearly the choice of these by the teacher is key.

We have found the Moore Method to be a particularly productive way of using puzzles with students. It has both the benefits of individual work and the social dynamic. The course was set up as an optional 10 credit module outside the main discipline (MOMD) in 2003 by Dr Chris Good. Since then it has been taught by two other colleagues and from 2011 all students on the Mathematics MSci are expected to choose this MOMD in preference to the other MOMD offered by Mathematics. After four years of using one set of problems there is a surprising consistency and stability of the way the class runs. Indeed, each year we have ended up ± 2 problems from the same place with little or no effort to set a particular pace for the work. The following list is a caricature of the cycle of the class.

Week 1: Anticipation.
"What is this class going to be about?"

Week 2: Excitement and enthusiasm.
"Someone is going to take me seriously and this sounds like fun!"

Week 3: Frustration.
"Actually I'm finding these problems a bit difficult!" "So-and-so's presentation was awful. What a waste of time!"

Week 4-5: Despondency, Doldrums and Despair.
"I can't do these!" / *"They can't do these!"*

Week 6-7: Re-build confidence.
"Actually, I can do some of them."

Week 8-9: Adjust expectations.
"Problem-solving takes time, so how many problems do we expect to do?"

Week 10-11: Collegiate conviviality.
"Ok, so let's get on with it..."

This collegiate conviviality remains after the class with students forming close working friendships which are seen to persist throughout their degree programme. We are currently undertaking a substantial follow-up analysis of the effectiveness of this course in a separate research project.

One drawback of the Moore Method is that it becomes very unproductive with groups of fewer than 5 or more than 20 students. This is a serious flaw which limits its use in institutions which rely on large class sizes and lectures. Furthermore, if classes are run in parallel then each needs different tasks to avoid potential collusion.

Puzzles have many characteristics suited to a Moore method class. For example, puzzles should constitute a significant intellectual challenge, require considerable ingenuity and in some cases, succumb to a variety of fruitful approaches. That said, Moore's original approach was to structure tasks that led students through a major mathematical topic in a coherent way. This is much more difficult to achieve using puzzles and creating class materials represents a serious intellectual challenge for teachers.

Serious attempts have been made to provide sequences of tasks in mainstream mathematics suitable for a Moore Method course. There following are examples.

1. Calculus of a single variable. Wall (1969)
2. Classical real analysis. Burn (2000)
3. Number theory. Burn (1996)
4. Group theory through geometry. Burn (1987)
5. Axiomatic systems in geometry. Yates (1949)

There are many problem sets in geometry, such as Gutenmacher and Vasilyev (2004), Hubbard (1955) and Yates (1949). Contemporary students' unfamiliarity with geometry makes such tasks puzzling, even at an elementary level. Hence geometry is likely to be a fruitful choice of topic for a puzzle-based course, independent of the mainstream curriculum, but with few prerequisites. The solution we have given to puzzle 32 is typical of the kind of geometric reasoning we mean here. The *Journal of Inquiry-based Learning in Mathematics* (`www.jiblm.org`) contains peer-reviewed course notes which have been tested in classes for a variety of Moore Method courses. Some of these are closer to puzzle-based approaches than others. These are freely available for download and use. It is perfectly sensible, and reassuring, for colleagues new to puzzle-based learning to adopt or adapt tasks which other colleagues have found to work well with similar groups of students. As we have already commented, expecting each teacher to write entirely novel puzzles is unrealistic and wasteful. However, when selecting and using existing puzzles, the teacher needs to consider carefully the prior knowledge and experience of the students, as mentioned in Section 2.1.

Even the goal of finding coherent problems can be achieved, it is not clear that the careful structure in the tasks is evident to students. To the participants of the class the tasks may *appear to them* to be disconnected puzzles!

> It is then assumed that if learners 'work through' the particular cases, they will emerge with a sense of the generalised whole. This assumption is contradicted by the observation that 'one thing we do not seem to learn from experience, is that we rarely learn from experience alone'. Something more is required. Mason et al. (2007)

Such a development of some deeper structure may appeal particularly to mathematics students. The best way to use puzzles with the Moore method is not clear to us, nor is it obvious how this might be investigated, given that there are so many variables that the coherence of the tasks is only one aspect of many. As with Workshops (section 4.2), a key aspect of the Moore method is a development of the group working skills that are so important to the STEM disciplines.

5 The Puzzles

This section contains the puzzles that we have gathered and selected[4] during the course of this work. Our criteria in selecting puzzles is eclectic. Most we have actually used with students. Some we experienced ourselves as students and others are simply classics. The vast majority already appear, under various guises, in many other books, including the following.

1. Mathematical puzzles, Zeitz (2007)
2. For mechanics and physics, Gnädig et al. (2001).
3. School competition mathematics, Haese et al. (1995) and Haese et al. (1998).
4. Classic puzzle books, Dudeney (1907), Dudeney (1917), Dudeney (1932).

Since it is unlikely that students will exhaust the potential of more than one or two puzzles per hour session, our selection is brief. We have included a variety of levels of difficulty and topics. Our contribution here is to make a selection suitable for undergraduates in the STEM subjects, particularly those learning mathematics in early years of study. As mentioned in Section 2.1, we sought puzzles where there was both a conventional solution (preferably a contrasting, particularly complex, exercise) and a lateral thinking solution. In some cases, we have been able to strip puzzles down to essential details and have then provided variants for subject specific use. However, in most cases we have given a generic version of the puzzle and whatever useful subject specific variants we could find or develop ourselves or with colleagues. In each case we provide solutions and a brief commentary. We welcome correspondence on STEM subject specific variants to the puzzles, and especially any tested examples.

It should be noted that a STEM context is not always appropriate. For example, take the following (Puzzle 27).

▼ Example task 20

Alice and Bob take two hours to dig a hole. Bob and Chris take three hours to dig the hole, while Chris and Alice would take four hours. How long would they take working together?

This has been used in a modelling class by one of the authors. It led to a long and fruitful discussion about formulating equations, using rate equations and defining (and checking) units. It would of course be possible to rewrite this problem in a STEM context e.g.

▼ Example task 20 variant

There are three construction companies: A, B and C. Working together, A and B take two days to erect a building. B and C would take three days to build a similar building whilst A and C would take four days. How long would A, B and C take working together?

This may well detract, however, from the usual (and incorrect) first attempt of many students (and staff) to solve this puzzle i.e. writing equations such as

$$A + B = 2$$

[4] We acknowledge preliminary work on this problem by L. Holyfield.

The simplicity of the original formulation is valuable. A similar situation is found with Puzzle 22. In other cases, such as that described in Exercise task 16, Section 4.1, the context might be too contrived to have any real value.

Our original intention was to “classify the puzzles by level of difficulty”. However, on reflection this seems like an erroneous goal. The difficulty a particular person, student or colleague, has when solving a puzzle will be determined as much by their prior experience as the characteristics of the puzzle itself. For example, we have provided a purely geometric solution to Puzzle 32 which involves recognizing that all the points on the pitch with a particular property lie on a circle. Familiarity with similar puzzles make this quite a natural move, but otherwise we have to fall back on general tools such as coordinates, algebra and trigonometry. In practical situations there is often a homogeneity within a particular group of students which enable an appropriate puzzle to be used. Hence, we have ordered the puzzles in a broadly increasing level of challenge.

It is important to note that most of the puzzles have a provenance and a heritage that is difficult fully to discover – they have been passed down from teacher to student over centuries, and in the case of Puzzle 28, millennia, changing to suit the zeitgeist. We have modified or adapted puzzles to our purpose here. However, we have recorded the immediate source of the puzzles, where known to us, and investigated the history of the puzzles and puzzle-solving as an activity. It follows that the sources we offer for each puzzle are unlikely to be original, and are either where the puzzles were found or give an example close to a puzzle already known by one of the researchers on the project. However, historical concerns were not a primary aim of the project.

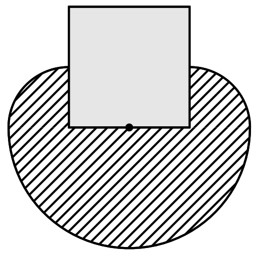
Grazing Horse

▼ Puzzle 1

A horse is tied with a 10 *m rope to the middle of one side of a square barn with side length* 10 *m. What area of grass does it have to graze?*

Solution

On the side of the barn to which the horse is tethered, the area that can be grazed will be a semicircle of radius 10 m. However, the rope will also allow the horse to graze around the corner of the barn, as demonstrated in the picture.



5 m of the rope will run along the side of the square and so the length remaining to reach around the corner will be 5 m. On each side the perimeter of the accessible area will trace out a quarter circle and so the total area above the line of the side of the square will be a semicircle of radius 5 m.

Thus to total area will be $\pi(12.5 + 50)$ m^2.

Extensions and Commentary

This puzzle is something of a cultural artefact in mathematics education, but its origins are unknown. One example is Mason et al. (2010), pg. 27.

This question was co-author Matthew Badger's EdExcel GCSE coursework in 2000, posed in terms of a horse and a barn. The coursework encouraged students to extend the problem and there are many options for doing so. Two are to move the point at which the horse is tethered and to increase the length of the rope so that is more than half the perimeter of the barn. It is possible to strip this puzzle to its core content.

▼ Puzzle 1 variant

One end of a 10 *m rope is tied in the middle of one side of a square with side length* 10 *m. What area is enclosed by the set of points which the other end of the rope may meet?*

This variant has the apparent advantage of not referring to a grazing horse, which is outside the STEM disciplines. However, the original formulation makes it clear that all the points that the non-tethered end of the rope can reach are of interest (because the horse can graze there). This variant might be less accessible to many students and therefore not as enjoyable.

Mile markers

▼ Puzzle 2

There are 12 *successive mile markers on a road. A motorist takes* 10 *min to drive from the first to the sixth. If she continues to drive at the same speed, how long will she take to reach the last marker?*

Solution

After leaving the first marker, the motorist has to drive past 5 markers to reach the sixth marker, so it takes 2 min from a marker to the next marker. From the sixth marker to the twelfth marker, there are a further 6 markers, which will take 12 min.

Extensions and Commentary

This is a classic “fence and gateposts” puzzle of which there are many variants.

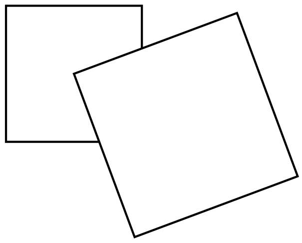
▼ Puzzle 2 variant

A civil engineer has been instructed to design a security fence for one side of a building site. Whilst his design had the posts for such a fence 6 metres apart, the number of posts delivered was actually 5 *fewer than he needed. However, after some quick recalculation, he found he could do it if the posts were* 8 *m apart. How long was the side of the building site?*

Intersecting squares

▼ Puzzle 3

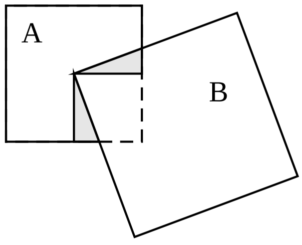
Two squares intersect as shown in the diagram. The smaller square has side length 30 *cm, the larger* 40 *cm, and the top left corner of the larger square sits at the centre of the smaller square.*



Find the area of the intersection of the two squares.

Solution

This can either be answered with basic trigonometry, or by noticing that the triangle cut off by B in the top half of A is congruent to the triangle left by B in the bottom-right quadrant of A. Thus the area removed is independent of the angle between the squares and so its value is 15^2 cm^2, or 225 cm^2.



The lateral thinking solution is to note that the angle that B is rotated with respect to A is not given. By meta-inference (Example task 6, Section 2.1) it can be concluded that the angle does not matter. One may therefore assume that the sides of the squares are parallel to one-another; it is then clear that the area of intersection is a quarter of the area of the smaller square.

Extensions and Commentary

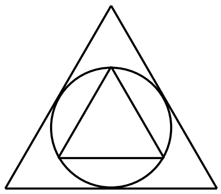
Source: Townsend (1994).

For a STEM context, this puzzle could be written about square buildings or other facilities. Realising that one might assume the sides of the squares are parallel to each other provides students with a real "Eureka" moment.

Beware of the Road Sign

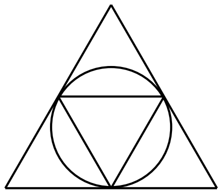
▼ Puzzle 4

This road sign means "Beware of the Road Sign". What is the ratio of the area of the smaller triangle to that of the larger?



Solution

Rotate the inner triangle by 180^o. Its vertices bisect the sides of the larger triangle from which it can be seen that the ratio is $\frac{1}{4}$.



Alternatively, let the radius of the circle be 1 unit. The height of the large triangle is 3 units because the incentre of an equilateral triangle is $\frac{1}{3}$ of the length of the bisector from the base. Similarly, the height of the smaller triangle is 1.5 units. As the areas are proportional to the squares of the heights, the ratio is $\frac{1}{4}$.

Extensions and Commentary

Source: Maslanka (1990). Either solution to this puzzle may lead to further discussion as neither in its current form is entirely mathematically rigorous.

It is also possible to argue that the smaller triangle is in the same proportion by area to the incircle as the larger is to the excircle, which has a radius of 2 units, again leading to the answer that the ratio is $\frac{1}{4}$.

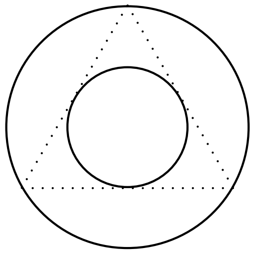
▼ Puzzle 4 variant

A gyrangle is a structure built from hollow triangles, some flat, some folded. See (`http://www.georgehart.com/DC/index.html`*). If the geometry of a flat hollow triangle is as shown in the figure, what fraction by area of a solid triangle is removed to make the hollow version?*

Holey Road Sign

▼ Puzzle 5

This holey road sign has a circular hole in its middle, cut using the equilateral triangle inscribed into the larger circle as a guide. What is the ratio of the area of the smaller circle to that of the larger?



Solution

Let the radius of the inner circle be 1 unit. The radius of the larger circle will be 2 units and therefore the ratio of the area of the smaller circle to that of the larger will be a $\frac{1}{4}$.

Using the result of Puzzle 4, inscribing an equilateral triangle in the smaller circle and noting that the smaller triangle is in the same proportion by area to the smaller circle as the larger triangle is to its excircle, the ratio must be $\frac{1}{4}$.

Extensions and Commentary

Source: Maslanka (1990).

A Packing Puzzle

▼ Puzzle 6

You have some objects that need packing in containers. If you pack 24 objects per container, you will have one of the objects left over. If you pack 25 objects per container, you will have one container left over. How many objects and containers do you have?

Solution

Consider the situation when you have 25 objects per container and one container left over. Take one object from each (packed) container leaving 24 objects per container. You would need 24 of those objects to pack the spare container and 25 if you are to have one object left over. You must therefore have taken objects from 25 containers and with the spare, you must have had 26 containers altogether. It follows there were $26 \times 24 + 1 = 625$ apples.

Algebraically: let the number of objects be n and the number of containers be N. Then

$$24N + 1 = n$$

and

$$25(N - 1) = n$$

Solving these simultaneous equations gives $N = 26$ and $n = 625$.

Extensions and Commentary

Source: Maslanka (1990).

In this case, the algebraic solution might be considered easier than the lateral thinking solution and likely to be less error prone. However, this could be an opportunity to emphasise to students the need to check solutions.

The original puzzle concerned apples and boxes but can be re-written with any objects and any containers. In a realistic case, the objects would be identical as would be the containers but this is not essential given the formulation of the problem. The number of containers could also be changed. This can therefore lead to many puzzles with a STEM context. For example:

▼ Puzzle 6 variant

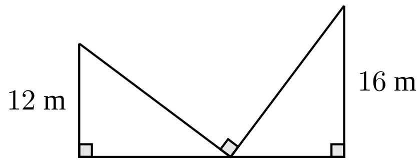
Some columns are to be loaded with chromatography beads. If 9 kg of beads are packed in each column, one column will be left unfilled. However, if only 8 kg of beads are put in each column, 1 kg of beads will be left over. What weight of beads must be put in each column so that every column contains the same amount of beads and there are no left over beads?

Lasers

▼ Puzzle 7

Two lasers are mounted above a detector in such a way that their beams are orthogonal. The beams are switched on simultaneously. The light from both reaches a particular point on the surface of a flat detector, also simultaneously. The perpendicular distance from the plane of the detector to one of the lasers is 12 m and to the other 16 m. What is the distance between the lasers?

Solution



The velocity of light is the same for both laser beams and therefore the distance from laser to detector along the beams must also be the same. As the beams are orthogonal, the two triangles in the figure must be congruent, each with two (perpendicular) sides of lengths 12 m and 16 m. The path length for either beam is $\sqrt{12^2 + 16^2} = 20$ m. It follows that the distance between the lasers is $20\sqrt{2}$ m ≈ 28.3 m.

Extensions and Commentary

Source: Maslanka (1990).

This is a puzzle rather than just a problem because it is not intuitively obvious that the triangles are congruent. Clearly more mathematically experienced students will find this less puzzling.

Buckets

▼ Puzzle 8

You have two buckets; one of 5 *litres and one of* 3 *litres, and a tap for water. How can you measure* 4 *litres?*

Solution

Fill the 5 litre bucket and pour it into the 3 litre bucket. Empty the 3 litre bucket and pour the remaining 2 litres from the 5 litre bucket into it. Fill the 5 litre bucket again, and use the water in it to top up the 3 litre bucket. As there was already 2 litres in it, only one litre will be removed from the 5 litre bucket, leaving four litres.

Extensions and Commentary

This puzzle was made famous in the 1995 film Die Hard with a Vengeance, though variations on it have appeared in many puzzle books and on the web. The earliest reference we could find was in O'Beirne (1965).

It is interesting to see which volumes of water one can measure using two jugs; it can be proved that any multiple of the highest common factor of the two jugs can be measured, up to the capacity of the largest jug.

The essence of this puzzle is the containers of different volumes. It could easily be adapted to other contexts. For example, there might be two fermentation vessels and a supply of fermentation medium.

Moisture Content

▼ Puzzle 9

Fresh apricots have a moisture content of 80%. *When left in the sun to dry they lose* 75% *of their moisture content. What is the moisture content of dried apricots?*

Solution

Let w indicate a unit of water and a indicate a unit of apricot flesh, then a (non-dimensional!) fresh apricot consists of $\frac{80}{100}w + \frac{20}{100}a$. If the water content is reduced by 75% then the dried apricot is

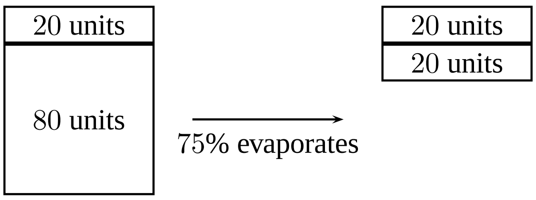
$$\frac{1}{4} \times \frac{80}{100}w + \frac{20}{100}a = \frac{20}{100}w + \frac{20}{100}a.$$

Hence the moisture content is now 50% of the remaining units.

Extensions and Commentary

This puzzle is classical and there are many variations on this theme. Such problems with ratios are notoriously difficult. Except for the difficulty many students have in formulating the correct equation to solve this puzzle, it might be considered just a problem.

A diagrammatic solution is helpful here.



Whilst processing of apricots may be considered part of agricultural or food engineering, it seems likely that this could be put into several STEM discipline specific contexts by choice of a material to be dried and the change in the method of drying. For example:

▼ Puzzle 9 variant 1

A wet precipitate of calcium carbonate has a moisture content of 80%. *When heated in an oven at* 105^o *C for* 2 *h, the precipitate loses* 75% *of its moisture content. What is the moisture content of the dried precipitate?*

▼ Puzzle 9 variant 2

Wet cells have a moisture content of 80%. *When heated in an oven at* 105^o *C for* 2 *h, the cells lose* 75% *of their moisture content. What is the moisture content of the dried cells?*

Walking

▼ Puzzle 10

A man walked 5 hours, first along a level road, then up a hill, then he turned round and walked back to his starting point along the same route. He walks 4 miles per hour on the level, 3 uphill, and 6 downhill. Find the distance walked.

Solution

Let x be the total distance walked and y be the distance uphill. The walk has four parts: level, uphill, downhill, level. The time taken can be written as

$$\frac{x/2 - y}{4} + \frac{y}{3} + \frac{y}{6} + \frac{x/2 - y}{4} = 5.$$

One equation in two unknowns — appears insufficient! But, collect like terms to get

$$\frac{x}{4} = 5.$$

Hence $x = 20$ miles.

Extensions and Commentary

Source: “Knot I” of “A Tangled Tale”, by Lewis Carroll Carroll (1936).

This puzzle appears not to have all the required information in it and indeed most variations on the speeds render the puzzle insoluble. It could be re-written for any moving object. The discovery that this puzzle can be solved with apparently insufficient information should lead students to question how this can work and to consider what combination of speeds make a solution possible.

There is a “meta-inference” solution to this puzzle. The length of the road on the hill is not specified so we can assume the solution is independent of this length. If the length is zero, the man walks 5 h at 4 miles per hour i.e. 20 miles.

Buying Pipes

▼ Puzzle 11

A pipe engineer had a budget of $£1000$*. She could order* 10 *m lengths of pipe at* $£50$ *each,* 6 *m lengths at* $£30$ *each, and* 2 *m lengths at* $£5$ *each, but she could only buy whole numbers of each pipe length. She ordered at least one of each pipe length, and used the entire budget. If she bought* 100 *lengths of pipe, how many of each length did she buy to ensure she bought the longest total length of pipe possible, and what length was that?*

Solution

Let us assume the engineer bought m 10 m pipe lengths, n 6 m lengths and p 2 m lengths, where n, m and p are integers.

$$m + n + p = 100, \quad n \geq 1,\ m \geq 1,\ p \geq 1. \tag{2}$$

Let the total cost of the pipes be $£c$.

$$c = 50m + 30n + 5p, \tag{3}$$

where

$$c \leq 1000. \tag{4}$$

Let the total length of length of pipe be l m.

$$l = 10m + 6n + 2p. \tag{5}$$

l has to be as large as possible.

There doesn't seem to be enough information to solve this problem. We need to find c, n, m and p but only have 3 equations, (2), (3) and (5). From (3) and (4)

$$50m + 30n + 5p \leq 1000$$

and from (2) $p = 100 - m - n$ so that

$$50m + 30n + 5(100 - m - n) \leq 1000$$

or

$$9m + 5n \leq 100. \tag{6}$$

Now 10 m pipes are the same cost per m as 6 m pipes and given the overall limit on the number of pipes it is best to buy as many 10 m pipes as possible. However, n has to be at least 1 as the pipe engineer buys at least 1 of each type. This gives $m = 10$ and $n = 2$. From equation (2), $p = 88$. From equation (5), $l = 100 + 12 + 176 = 288$ m. Therefore, the cost is $£(500 + 60 + 440) = £1000$ as required.

Extensions and Commentary

Source: Snape and Scott (1991). This problem is easily adapted to STEM scenarios, as we have done here.

Fermenters

▼ Puzzle 12

When asked about the laboratory fermenters she had in stock, the vendor jokingly replied that they were all 5 *L working volume except for* 17*, all* 10 *L except* 11*, and all* 20 *L except* 20*. How many of each working volume did she have?*

Solution

In adding together all the fermenters that are not 5 L, not 10 L and not 20 L, each fermenter has effectively been counted twice. Hence the total number is $48/2 = 24$. There are therefore seven 5 L fermenters, thirteen 10 L fermenters and four 20 L fermenters. As a check: $7 + 13 + 4 = 24$, which is OK.

Alternatively, let the number of fermenters be N. Then the number of 5 L fermenters is $N - 17$, of 10 L fermenters $N - 11$, and 20 L fermenters $N - 20$. Adding these together should give the total number of fermenters N.

$$N - 17 + N - 11 + N - 20 = N$$

so $2N = 48$ and $N = 24$.
5 L fermenters: $N - 17 = 7$.
10 L fermenters: $N - 11 = 13$.
20 L fermenters: $N - 20 = 4$.
As a check: $7 + 13 + 4 = 24$, which is OK.

Extensions and Commentary

Source: Maslanka (1990). This problem is easily adapted to STEM scenarios, as we have done here.

Frustum Cone

▼ Puzzle 13

A large (right circular) cone required 3 L of paint to cover all its surfaces, 2 L for the curved surface and 1 L for the base. It was then decided that the conical top of the cone would be removed and discarded, leaving a frustum of half the height of the original cone. All surfaces of the frustum were then painted or repainted. How much paint was needed?

Solution

Consider the discarded conical top. Being half the height, this had a surface area $\frac{1}{4}$ of that of the original cone. The curved surface of this smaller cone would have required 0.5 L to paint. Subtracting this from the original requirement leaves 2.5 L of paint needed for the curved surface of the frustum and its base. However, the frustum also has a top surface. This has the same area as the bottom surface of the discarded top, i.e. an area $\frac{1}{4}$ of that of the base of the original cone. That would require 0.25 L to paint. Total paint needed $= 2.75$ L.

Extensions and Commentary

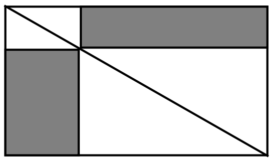
Source: Maslanka (1992).

This puzzle can be solved by tedious calculation. The solution above provides scope for discussions of geometric similarity and the concept of a frustum.

Rectangle 2

▼ Puzzle 14

Which of the two shaded rectangles is biggest?



Solution

The diagonal cuts the shape in half. Identical white triangles are removed, so the areas are the same. Algebra is also possible.

Extensions and Commentary

Source: Borovik and Gardiner (2005).

This puzzle has both an uninspired algebraic solution and the neat lateral one given above. Further, given that the problem is posed without defining the sizes of the white rectangles, one may assume by meta-inference (Example task 6, Section 2.1) that they are the same size and meet at the centre of the large rectangle. From this one concludes that the two grey rectangles have the same area as, in this particular instance, they are the same size and shape.

This could also be expressed as a puzzle concerning a sheet of metal.

Wall Slide

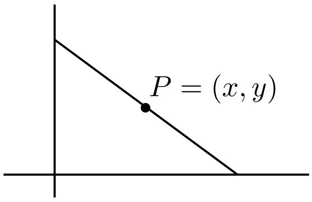
▼ Puzzle 15

A ladder stands on the floor and against a wall. It slides along the floor and down the wall. What curve does the midpoint of the ladder move along?

Solution

The midpoint moves along the arc of a quarter circle.

There are two ways to approach this puzzle. The first is to define coordinates with the origin at the intersection of the wall and floor, which respectively define the x and y axes. Then assign coordinates $P = (x, y)$ to the point in the middle of the ladder and to define the ladder to be length $2l$.



A straightforward application of similar triangles and the Pythagorean Theorem leads to

$$(2x)^2 + (2y)^2 = (2l)^2,$$

or $x^2 + y^2 = l^2$ which is the equation for a circle, centred at the origin.

The other, lateral thinking solution, is to place an identical ladder to form an X-shape with the ladders crossing at P. As the ladders move, this X opens and closes. Since one end of the new ladder is fixed at the corner formed by the wall and the floor, the distance from the midpoint is a constant distance from the corner and it is clear that the midpoint must indeed move along a curve centred at the origin.

Extensions and Commentary

Source: Gutenmacher and Vasilyev (2004).

This problem would be recognized by most STEM students and the mechanism which is implicit here is widely used. The solution appears to many students to be counter-intuitive and therefore can engender useful discussions and investigations.

Life Jacket

▼ Puzzle 16

Two identical motor boats set off from the same pier heading in opposite directions along a river. A lifejacket, which is dropped off the end of the pier, floats downstream just as the two boats set off. An hour later both boats reverse their courses in pursuit of the lifejacket. Which boat gets to the lifejacket first?

Solution

If the river is flowing at r km h^{-1} and the boats travel at s km h^{-1} on calm water then the downstream boat will be travelling at $s+r$ km h^{-1} for the first hour and $s-r$ km h^{-1} until it reaches the lifejacket. Thus it will be travelling away from the lifejacket at $s+r-r=s$ km h^{-1} until it turns around and $s-r+r=s$ km h^{-1} on the return journey, therefore taking an hour to reach the lifejacket. The equations are reversed for the upstream boat; hence the boats arrive at the lifejacket at the same time.

Extensions and Commentary

Source: Michalewicz and Michalewicz (2008).

A meta-inference solution (see the discussion of Example task 6, Section 2.1) could be that because no speeds are specified, zero speed is an acceptable case. In that case, the lifejacket and boats all float downstream together for an hour and both boats “reach” the lifejacket together. This is effectively a “reduce to rest” solution.

Cube Faces

▼ Puzzle 17

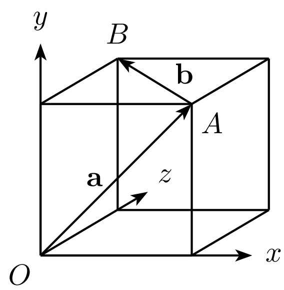
Diagonals of two faces of a cube meet at a vertex. What is the angle between the diagonals?

Solution

$60°$. The block is a cube so its faces are all the same size and thus the diagonals of each face are the same length. Thus the diagonals will form an equilateral triangle.

As a vector solution, let the cube sides be unit length. Set up 3D Cartesian coordinates with one vertex of the cube at the origin O and the axes along the sides meeting at the vertex. Consider the angle between lines AO and AB. The coordinates of A are $(1, 0, 1)$ and of B are $(0, 1, 1)$. The vector $\mathbf{a}$ from A to O is $[-1\ 0\ -1]$ and $\mathbf{b}$ from A to B is $[-1\ 1\ 0]$. It follows that the angle between these vectors is

$$\cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \cos^{-1}(1/2) = 60^o.$$



Extensions and Commentary

Source: Townsend (1994).

This puzzle has been used in a problem class of one of the authors. For Chemical Engineering students, for example:

▼ Puzzle 17 variant 1

As part of her plant layout, as shown in Figure 1, a chemical engineer has a pipe going from point A on a cubical tank up to point B and then across to point C. What angle does the pipe have to be bent to fit the tank?

Figure 1: A schematic of the tank design (omitted)

For Mechanical Engineering students, this might read:

▼ Puzzle 17 variant 2

As part of his latest engine block design shown in Figure 1, a mechanical engineer has a pipe going from point A in his cubical block up to point B and then across to point C. What angle does the pipe have to be bent to fit the block?

Figure 1: A schematic of the engine block (omitted)

For Chemistry students we might have:

▼ Puzzle 17 variant 3

Sodium chloride crystallises into a face centred cubic structure. The unit cell has sodium ions at each corner. Diagonals on two faces of the unit cell meet at one of the corner ions. What is the angle between the diagonals?

Bees and trains

▼ Puzzle 18

Two model trains are travelling toward each other at 5 km h^{-1} on the same track. When they are 50 metres apart, a bee sets off at 10 km h^{-1} from the front of one train, heading toward the other. If the bee reverses its direction every time it meets one of the trains, how far will it have travelled before it must fly upwards to avoid a grisly demise?

Solution

The trains are travelling at the same speed, so each will cover a distance of 25 m before they collide. The bee is travelling twice as fast as one of the trains and so will cover twice as far, i.e. 50 m.

Let the speed of the trains be v and the bee u. For a typical train to train leg of the bee's flight, let the initial distance between the trains be l_n. Then the time taken for that leg is $t_n = \frac{l_n}{u+v}$. Given the bees is flying at a speed u, the distance the bee flies in this typical leg is $d_n = ut_n = \frac{ul_n}{u+v}$. In the time taken for the leg, the trains move $2vt_n$ closer, so for the next leg $l_{n+1} = l_n - 2vt_n = t_n(u - v)$. Hence $d_{n+1} = ut_n\frac{u-v}{u+v} = d_n\frac{u-v}{u+v}$. The first leg when the trains start L apart is $d_1 = \frac{uL}{u+v}$. This gives a geometric series, which can be summed to get the total distance travelled. Therefore the bee travels

$$\sum_{n=1}^{\infty} d_n = \frac{uL}{u+v}\frac{1}{1-\frac{u-v}{u+v}} = \frac{uL}{2v}.$$

In the puzzle, $v = 5$ km h^{-1}, $u = 10$ km h^{-1}, $L = 50$ m so the bee travels 50 m.

Extensions and Commentary

Allegedly posed by a dinner party guest to either Norbert Weiner (see Gilkey (1990)) or John von Neumann (see Nalebuff (1990)) who, it is claimed, summed the infinite series in their head so quickly that the questioner believed them to have settled on the easier method.

The difficulty in formulating a practical version of this puzzle is the need to find a creature or object (e.g. a bee) that can move between the two approaching objects (e.g. trains) at a greater speed than the objects are moving (hence model trains in the version given here). There is a version of this puzzle in which a dog runs between a couple out for a walk. This leads to a possible STEM version such as:

▼ Puzzle 18 variant

Two XXXs are out for a walk with a dog. At a particular moment there are 50 m apart and are walking at 5 km h^{-1} towards each other. At that moment, the dog sets out from one of the XXXs towards the other. If the dog reverses its direction every time it reaches one of the XXXs, how far will it have travelled before they meet?

XXX could be chemists, physicists, engineer or any other (STEM) discipline. This customisation is, of course, trivial but may nevertheless make the puzzle look more discipline specific.

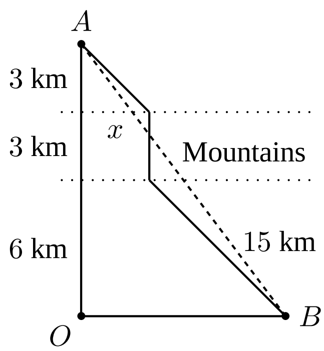
This puzzle has to be idealised to generate the infinite series or indeed to reach the lateral thinking solution. In particular, it must be assumed the bee or dog has negligible size and that it can reverse its direction instantaneously. Which way is the bee facing at the end? This can lead to an interesting discussion with students about the use of abstraction in practical problem solving.

Tunnel

▼ Puzzle 19

An Icelandic civil engineer is in charge of laying a pipe between a geothermal power plant A and a town B. Between A and B there is a small straight mountain range of uniform width 3 km. The pipe must go through a straight tunnel through the mountains perpendicular to the edges of the latter. The perpendicular distance of A from the mountains is 3 km and B is 6 km away. The distance between the town and the power plant, as the crow flies, is 15 km. Where should the tunnel be built to minimise the pipe length?

Solution



Let O be the point on a line through A (the power plant) perpendicular to the mountains and on a line through B (the town) parallel to the mountains. $OA = 12$ km and as $AB = 15$ km and so $OB = 9$ km. Let the distance from OA along the mountains to the tunnel be x km. Then the length of the pipe from A to B, l km, is given by:

$$l = \sqrt{3^2 + x^2} + 3 + \sqrt{(9 - x)^2 + 6^2}.$$

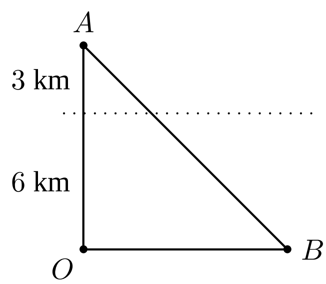
The minimum length will be when $\frac{\mathrm{d}l}{\mathrm{d}x} = 0$, i.e.

$$\frac{x}{\sqrt{x^2 + 9}} - \frac{9 - x}{\sqrt{(9 - x)^2 + 36}} = 0.$$

from which $x = 3$ is the only positive root. The tunnel should be 3 km from OA.

The following is an alternative lateral thinking solution.

Call the axis through O and B the X axis and through O and A the Y axis. As the pipe crosses the mountains, there is no change in its X coordinate. Therefore imagine the situation with no mountains. This is equivalent to moving A 3 km down the Y axis. The shortest route for the pipeline would then be the straight line from A to B. OB is still 9 km so by symmetry, the tunnel should be 3 km along the mountain from OA.



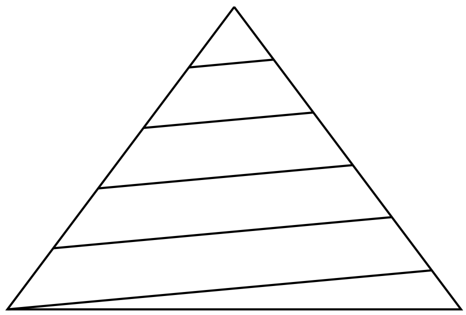
Extensions and Commentary

Source: Cooper (2010).

Pyramid

▼ Puzzle 20

A 100 m high pyramid with a square base of side length 150 m has a straight line running around the pyramid diagonally across each face as shown in the diagram.



If the slope of the line is 1 metre gained for every 10 metres horizontal travel, how long is the line?

Solution

This puzzle would be very tedious to answer by summing the lengths of the line over each of the individual sides. However, if we note that the pyramid is 100 m high and the slope of the line is 0.1, the length of the line is $(100/0.1)(1 + 0.1)^2 \approx 1005$ m.

Extensions and Commentary

This is a variation on a puzzle in Townsend (1994).

STEM variants might include asking for the amount of asphalt needed to pave a path up the pyramid. As an extension one might ask if the shape of the structure affects the length of the line and/or what happens if the structure is a triangular pyramid or a cone?

Gears

▼ Puzzle 21

There are two gears of the same size and number of teeth rotating around one-another. Relative to the second gear, the first gear rotates around the second until it returns to its starting point. How many revolutions does the first gear make whilst doing this?

Solution

Because both gears are the same size, they have the same circumference. Relative to the surface of the second gear the first gear rotates once. However, the surface of the second gear makes a complete revolution about its centre (as it is a circle) and so the first gear rotates twice as it moves around the second.

Extensions and Commentary

Source: Bolt (1984). Most STEM students would recognize the relevance of this puzzle.

Marriage

▼ Puzzle 22

Alice looks at Bob and Bob looks at Clare. Alice is married but Clare is not. Prove that a married person looks at an unmarried person.

Solution

Bob is either married or unmarried. If Bob is married then Bob looks at Clare. If Bob is unmarried then Alice looks at Bob. In both situations a married person looks at an unmarried person.

Extensions and Commentary

The point of this puzzle is that you can solve it without knowing *which* person looks at a married or unmarried person.

There is an interesting more mathematical use of this kind of logic: prove that an irrational power of an irrational number can be rational.

$\sqrt{2}$ is irrational. Consider $\sqrt{2}^{\sqrt{2}}$. If this is rational we are done. Assume this is irrational and consider

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = (\sqrt{2})^2 = 2$$

which is rational.

The simplicity of this puzzle means that trying to put it into STEM discipline specific context could destroy much of its value. See also Puzzle 27 and a discussion at the beginning of Section 5.

Balancing balls

▼ **Puzzle 23**

You are given a set of 27 ball bearings and a balance. One of the bearings is known to be heavier than the others, which weigh the same. In how few balances can the heavy ball be determined?

Solution

Three. Divide the bearings into three sets of 9; then balance two sets. If they are equal, the third set has the heavy ball, otherwise it is the heavy set. Take the heavy set and divide it in three again, balancing two sets. Take the heavy set again and balance two balls. Either one is the heavy ball or the remaining ball is the heaviest.

Extensions and Commentary

Source: Bolt (1984).

Variations of this puzzle are seen in several mathematics puzzle books; a variation using 9 balls and two weighings is Problem 55 of Eastaway and Wells (1995), pg. 43. The solution we provide here is essentially a careful and systematic *enumeration of cases*. Ball bearings are sufficiently familiar to all STEM students that this puzzle does not need modification to make it STEM specific. However, a suitable variant is given below.

▼ **Puzzle 23 variant**

You have 27 samples containing mixtures of oil and water. You know that all the samples have the same volume but that one weighs less than the others because it contains a higher mass fraction of oil. Using a beam balance, in how few balances can the anomalous sample be identified?

Burning Rope

▼ Puzzle 24

You have two lengths of rope, each of which will burn for an hour. However, each rope will not burn at a constant rate along its length; so you cannot assume that half a rope burns in half an hour. You may have as many lighters as you want, how can you measure 45 minutes?

Solution

If you light one length of rope at both ends simultaneously it will burn out in half an hour. Thus, if you light one length of rope at both ends and *at the same time* light the other rope at one end only, half an hour of that rope will have burnt. Thus, when the first rope has run out, light the second rope from the other end and it will have burnt out in another 15 minutes.

Extensions and Commentary

It is very difficult to put this into a specific STEM context, although rope is of course familiar to all students. One possibility for chemists or mining engineers might be:

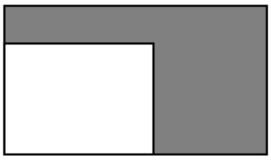
▼ Puzzle 24 variant

William Bickford (1774–1834) invented the safety fuse for use in mining. These fuses had a core of gunpowder wrapped in jute and then varnished for waterproofing. You have two such fuses, each of which will burn for an hour...

Rectangle

▼ Puzzle 25

You have two rectangles, one within the other, as shown below. There are no dimensions given, and you must not measure the diagram.



Explain, with justification, how to draw a single straight line which divides both the smaller rectangle and the 'L'-shaped area in half.

Solution

Any line which cuts a rectangle in half goes through the centre. Hence, the line goes through the centre of the large rectangle and the small rectangle.

Extensions and Commentary

Source: Hubbard (1955).

In practice many students initially attempt to impose coordinates and use algebra, which is usually fruitless. A STEM version of this puzzle might involve two joined pieces of metal. A variant of this puzzle if given in Example Task 6 on page 4.

There are various extensions to this where the *method* will still work. Any two shapes which have a *centre of area* can be used in place of the rectangles. An extension to this would be require students to place a third rectangle that is also bisected by the straight line. There is a much deeper problem here, with arbitrary sets. Given n measurable sets of finite measure in n-dimensional space, it is possible to divide all of them in half (with respect to their measure) with a single $(n-1)$-dimensional hyperplane. This is sometimes called the Stone-Tukey theorem, see Stone and Tukey (1942), or the "ham sandwich theorem".

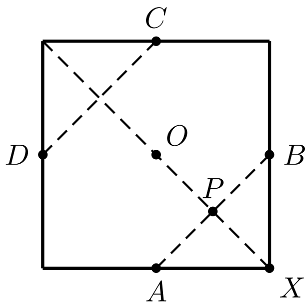
Table

▼ Puzzle 26

Imagine a square table, with legs placed half way along each side, rather than at the corners. What is the maximum mass which can be placed on one corner before the table tips over?

Solution

Imagine we have the table viewed from above, with legs at A, B, C and D, as shown in the diagram below. We assume that a mass is placed at the corner X in order to topple the table.



This can be solved most easily by thinking laterally, reducing it to a one-dimensional problem by taking moments along the diagonal through X.

The weight of the table including the legs, w, acts through O. Let the weight causing the table to tip over be W. When the table is about to tip over there is no reaction (force exerted by the floor) on the legs at C and D. Taking moments along the diagonal through X about the point P, gives $W = w$, as the distance from O to P equals the distance from P to X.

Therefore the maximum weight that can be placed at X is the weight of the table, or (as the acceleration due to gravity is a cancellable constant) the maximum mass that can be placed at X is the mass of the table.

Extensions and Commentary

Source: Austen (1880).

This puzzle appears not to have all the required information in it, but it is relatively straightforward to solve. It could be used as an introduction to moments and the concept of the centre of mass. Alternatives for civil engineers could be when looking at a structure or scaffolds in three dimensions. This puzzle should lead to a discussion of the difference between “weight” and “mass”, the appropriate (SI) units of each and how non-technical language is sometimes scientifically imprecise.

Working Together

▼ Puzzle 27

Alice and Bob take two hours to dig a hole. Bob and Chris take three hours to dig the hole, while Chris and Alice would take four hours. How long would they take working together?

Solution

Let A, B, and C represent the number of holes dug per hour by Alice, Bob and Chris respectively. Assuming the holes are the same and independence of work we have the following system of equations. Note that A represents the *rate* of Alice's work, etc.

$$\begin{aligned} 2A + 2B &= 1, \\ 3B + 3C &= 1, \\ 4A + 4C &= 1. \end{aligned}$$

These equations can be solved in a number of ways giving

$$B = \frac{7}{24},\ A = \frac{5}{24},\ C = \frac{1}{24}.$$

Extensions and Commentary

The difficulty of this puzzle is not the algebra, although this might cause problems, but rather the modelling step in which the student needs to recognise that A is a rate, not an amount of work. It helps if they consider units of the variables they use.

This puzzle has a built in check. If students form equations incorrectly as $A + B = 2$, etc., then working together takes longer than when working in pairs! There is further discussion of this puzzle at the beginning of Section 5.

STEM variants may be possible with a variety of zero order processes, i.e. where the process rates can be considered to be independent. Such a variant was presented in the introduction to Section 5 (Example task 20 variant), although we do not believe this is an improvement on the original, because of the loss of simplicity.

Alternatives using chemical kinetics:

▼ Puzzle 27 variant

X produces Y catalyser by $A + B$ in 2 hours, $B + C$ in 3 hours and $A + C$ in 4 hours. All processes are first order. What would be the rate if we combine A, B and C?

The Pursuit

▼ Puzzle 28

A dog starts in pursuit of a hare at a distance of 30 *of his own leaps from her. He takes* 5 *leaps while she takes* 6 *but covers as much ground in* 2 *as she in* 3*. In how many leaps of each will the hare be caught?*

Solution

We solve this by finding an equation to represent the situation and then solve it. If we say that the dog takes x leaps to catch the hare, then we know that the hare will have travelled $\frac{2}{3} \times \frac{6}{5}x$ dog leaps in the same time, since the hare covers two-thirds the amount of ground but does so 20% more quickly than the dog. The hare begins thirty of the dog's leaps ahead so we add those to the hare's side of the equation. The result is the solution of the equation

$$x = 30 + \frac{2}{3} \times \frac{6}{5}x.$$

We solve the equation to find when the dog and the hare have covered an equal distance (including the hare's head-start); giving us 150 leaps.

Extensions and Commentary

Source: Hadley and Singmaster (1992).

The puzzle as we have given it is a classic problem in European mathematics teaching whose origin is Alcuin of York's *Propositiones Alcuini Doctoris Caroli Magni Imperatoris ad Acuendes Juvenes*, more briefly titled *Problems to Sharpen the Young*, written around 775 (see Hadley and Singmaster (1992) for an annotated translation). Alcuin's hare and hounds problem is thought to be, see Swetz (1972), a version of a problem in 九章算術 (*The Nine Chapters on the Mathematical Art*), a book compiled in the first century AD from texts dated between 1000BC and 200BC. This problem too regaled the reader with the story of a hound in pursuit of a hare.

The authors have not yet thought of a non-trivial STEM discipline specific example of this puzzle.

Trains

▼ Puzzle 29

In a railway journey of 90 *kilometres an increase of* 5 *kilometres per hour in the speed of the train decreases the time taken by* 15 *minutes. What is the speed of the slow train?*

Solution

We define position d to be the arc length along the track from start to destination. Then we form a system of equations using the formula $d = vt$, i.e.

$$90 = tv,$$

$$90 = (v + 5)(t - \frac{15}{60}).$$

Seeing that $t = \frac{90}{v}$ we can substitute t in the second equation to give us

$$90 = (v + 5)\left(\frac{90}{v} - \frac{15}{60}\right),$$

which we can rewrite as

$$0 = v^2 + 5v - 1800.$$

This quadratic has solutions -45 and 40. The negative solution represents a train moving in the other direction, which we reject. As our answer must be positive we conclude that the slower train was travelling at 40 km h^{-1}.

Extensions and Commentary

This puzzle was created by the authors though it is unlikely to be unique. Except for the difficulty many students have in formulating the correct equation to solve this puzzle, it might be considered just a problem. Notice the units trap, with the velocity in kilometres per hour, but the time in minutes. This sort of puzzle should be easily adapted to other moving objects or indeed rates of reaction as below.

▼ Puzzle 29 variant

A chemical process is controlled to occur at a constant rate as the reactant concentration decreases from 0.1 *mol dm*$^{-3}$ *to* 0.01 *mol dm*$^{-3}$*. Due to a process upset, the reaction rate increased by* 5×10^{-3} *mol dm*$^{-3}$ *and the time taken by the reaction decreased by* 15 *minutes. What was the rate of the reaction before the process upset?*

Bricks

▼ Puzzle 30

A brick has faces with areas 110 *cm*^{2}, 52.5 *cm*^{2} *and* 231 *cm*^{2}. *What is its volume?*

Solution

If the brick has linear dimensions A cm, B cm and C cm, the volume is ABC cm^3. The volume squared $= A^2B^2C^2$ cm^6 $= (AB)(BC)(AC)$ cm^6. Regardless of the choice of which dimensions match which area, the volume squared $= 110 \times 52.5 \times 231$ cm^6 $= 55 \times 105 \times 231$ cm^6 $= 5 \times 11 \times 3 \times 5 \times 7 \times 3 \times 7 \times 11$ cm^6 $= (3 \times 5 \times 7 \times 11)^2$ cm^6 $= 11552$ cm^6. The volume is therefore 1155 cm^3.

Alternatively, let $AB = 110$, $BC = 52.5$ and $AC = 231$. Then $B/A = 52.5/231$ or $B = 52.5A/231$. Therefore, $52.5A^2/231 = 110$ and $A = \sqrt{25410/52.5} = 22$. It follows that $B = 5$ and $C = 10.5$. The volume of the brick is therefore $22 \times 5 \times 10.5$ cm^3 $= 1155$ cm^3.

Extensions and Commentary

Source: Maslanka (1990).

The surprise in this puzzle is that the volume can be found without knowing the linear dimensions of the brick. Bricks are of course familiar to all (STEM) students.

Socrates and Meno

▼ Puzzle 31

Socrates and Meno each receive box-shaped presents. Each is tied with three loops of string - one in each of the three possible directions. Socrates's package has loops of lengths 40 cm, 60 cm, 60 cm, while Meno's package has loops of lengths 40 cm, 60 cm, 80 cm. Decide whose package has the larger volume and find the volumes of the two packages.

Solution

Let Socrates's package be x cm by y cm by z cm. The first loop has length $2x + 2y = 40$, second $2y + 2z = 60$, and the third $2z + 2x = 60$. Adding gives $4x + 4y + 4z = 160$; so

$$x = (x + y + z) - (y + z) = 40 - 30 = 10.$$

Similarly

$$y = (x + y + z) - (z + x) = 40 - 30 = 10;$$

$$z = (x + y + z) - (x + y) = 20.$$

So Socrates's cuboid has volume $10 \times 10 \times 20 \times = 2000$ cm^3. If we do the same with Meno's p cm by q cm by r cm, we get $2p + 2q = 40$, $2q + 2r = 60$, $2r + 2p = 80$, so $p + q + r = 45$; so

$$p = (p + q + r) - (q + r) = 45 - 30 = 15;$$

$$q = (p + q + r) - (r + p) = 45 - 40 = 5;$$

$$r = (p + q + r) - (p + q) = 25.$$

So Meno's cuboid has volume $15 \times 5 \times 25 = 1875$ cm^3.

Extensions and Commentary

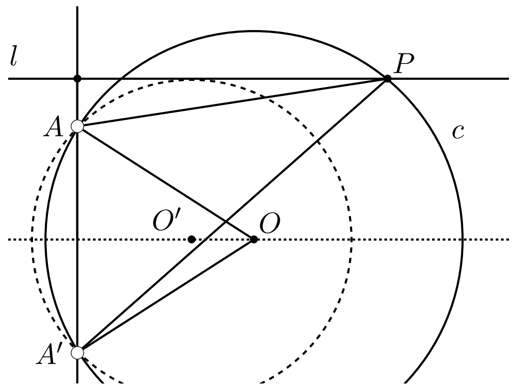
This is an interesting puzzle with a highly counter intuitive solution, because "larger" in the sense of volume does not mean "larger" in the sense of amount of string needed.

Rugby

▼ Puzzle 32

In a rugby pitch you wish to kick a ball to convert a try. You can place the ball any distance from the try line, but must place it on the line perpendicular to the try line through the point at which the try was scored. What is the position to ensure the angle between the lines connecting the goal posts to the ball is maximised?

Solution



Let A, A' be the goal posts and let P lie on a line, l, perpendicular to AA' outside segment AA'. Maximize $\angle APA'$.

Draw the circle c through the A, A' and P, centered at O on the perpendicular bisector of AA'.

For any other point P' on c to the right of the goal line we note that (i) $\angle APA' = \angle AP'A'$ and (ii) $\angle AOA' = 2\angle APA'$. Hence to maximize $\angle APA'$ we need to move O as close as possible to the line AA'. This happens when circle c is tangent to l, shown in the diagram as a dashed circle with centre O'.

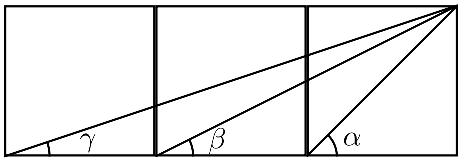
From these observations a formula for the position of P can be derived and the argument can be examined when the line lies between AA'.

Extensions and Commentary

This puzzle is an equivalent formulation of Regiomontanus' Maximum Problem, originally posed in 1471, see Dorrie (1965). We note that many similar problems occur in ancient texts on gunnery. However, it may be that this is not really a puzzle in the sense described in Section 2.1. For less experienced students, it may indeed be perplexing while for those familiar with geometry of circles it may be routine.

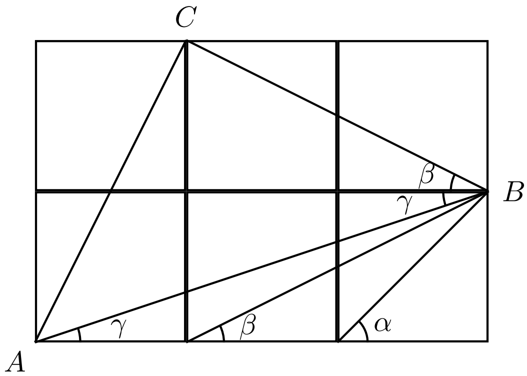
Sums of angles

▼ Puzzle 33



Assume these are squares. Prove that $\alpha = \beta + \gamma$.

Solution



Note that ABC is a right angled isosceles triangle, so that $\angle ABC = \alpha$.

Extensions and Commentary

This puzzle has the "thinking out of the box" solution described above.

The more prosaic solution notes that

$$\tan(\alpha) = 1, \quad \tan(\beta) = \frac{1}{2}, \quad \tan(\gamma) = \frac{1}{3},$$

and then uses the identity

$$\tan(\beta + \gamma) = \frac{\tan(\beta) + \tan(\gamma)}{1 - \tan(\beta)\tan(\gamma)} = 1 = \tan(\alpha).$$

▼ Puzzle 33 variant

The figure shows part of the (001) surface of Mg0 (with Mg^{2+} and O^{2-} ions lying at alternate vertices of a square lattice). Prove that the marked angles α, β and γ between the ions satisfy $\alpha = \beta + \gamma$.

Diagonals

▼ Puzzle 34

A rectangle grid is covered with 19×91 identical squares. How many squares are touched by a line going from diagonally opposite corners of the grid? [Repeat this problem and devise a general rule for a grid covered by $n \times m$ square tiles.]

Solution

The line leaves one square and enters the next by crossing an edge of the square, either horizontally or vertically, or through the corner of a square. If n and m are *co-prime* then the line does not cross a corner of any square. However, if n and m have a common factor there will be a corner.

19 (is prime) and is co-prime to numerical $91 = 7 \times 13$. So in our first example there are no corners to cross. To get from the top to the bottom we cross $19 - 1 = 18$ edges. To get from the left to the right the line crosses numerical $91 - 1 = 90$ edges. So, in total 108 edges are crossed. If there are 108 crossings, the line crosses 109 squares.

When n and m are co-prime the formula for the number of squares crossed is $n + m - 1$. In general, if a line crosses a corner it touches either no squares or two. To find a general formula we need to agree a convention here.

Extensions and Commentary

Source: Borovik and Gardiner (2005).

The puzzle as it is stated makes certain assumptions about the pipe and the tiles that remove it from the real world. If the tiles have a gap between them for grout and the pipe is not infinitely thin, the problem requires a more pragmatic approach involving modelling.

A STEM version of this puzzle might concern a pipe crossing a tiled floor. Formulating the puzzle in this way makes it appear less abstract and therefore it may engage better non-mathematicians. However, it would be necessary (and interesting) to discuss with students that a real pipe has thickness and to discuss how in some situations we might want to idealise a real situation to obtain a problem solution.

Fly on the wall

▼ Puzzle 35

How do you find the shortest path between two points on opposite walls of a room, travelling without leaving the walls?

Solution

The essence of the solution is to open up the net of the room and flatten it. The fly takes a straight line.

Extensions and Commentary

This is the simple lateral solution.

An interesting extension activity is to ask whether it is possible to open the net of the room in different ways and whether this would affect the apparent "shortest" path. If so, how is the "correct" net opening to be chosen. Note that the shortest path need not be unique.

It is challenging to prove mathematically that the correct opening of the net leads to the shortest path.

Handshaking

▼ Puzzle 36

Prove that, for any graph G, the number of vertices of odd degree is even.

Solution

Each edge requires two vertices. Thus the sum of the vertex degrees, summed over all vertices, is even. Any number of "even vertices" can be ignored, however only an *even* number of "odd vertices" can have an even sum of vertex degrees, as an odd number would result in an odd total. Hence the number of "odd vertices" is even.

Extensions and Commentary

This is a classic graph theory problem. Its extension to handshaking is shown below.

▼ Puzzle 36 variant 1

127 chemical engineers attend an event at their Institution to discuss the use of puzzles in teaching. Prove that the number of them who shook hands an odd number of times is even.

Such adaptations, although perhaps somewhat trivial, make the puzzle look more discipline specific, adding to student engagement particularly for non-mathematicians. An interesting variant for chemistry students is given below.

▼ Puzzle 36 variant 2

Fullerenes are closed polyhedral clusters of carbon, in which each carbon atom is bonded to three other carbons. Explain why fullerenes of different sizes always have an even number of carbon atoms e.g. buckminsterfullerene (C_{60}).

Wason selection test

▼ Puzzle 37

Imagine you have a four cards on a table and every card has a letter on one side and a number on the other.

With the cards placed on a table, you see

D 3 K 7

Turn over the fewest cards to establish the truth of the following statement “Every card which has a D on one side has a 7 on the other.”

Solution

Turn over D and 3 only.

Extensions and Commentary

This puzzle is a classic and well studied logic test, devised in 1966 by Peter Wason, see Wason (1968). When put in the context of social relations, such as “If you are drinking alcohol then you must be over 18” people perform much better overall (Griggs and Cox, 1982). This suggests that it would not be advantageous to put this puzzle into a STEM discipline specific context.

Circles inside circles

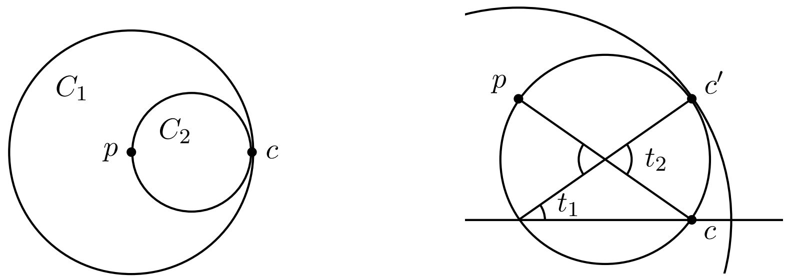
▼ Puzzle 38

Take a stationary circle and inside it put another of half the diameter, touching the larger one from within. Roll the smaller along the inside of the larger circle without sliding. Describe the path of a point K, on the small circle.

Solution

Every point on the small circles moves along a diameter of the larger circle. I.e. it moves on a straight line segment. To solve this you need to match up the arc length of the small circle with that of the large one.

Let C_1 have radius r_1 and C_2 have half the radius, so $2r_2 = r_1$. Begin with both horizontal diameters coinciding. On C_2 mark two points, c where C_1 and C_2 touch and p the point on the perimeter of C_2 which is also the centre of C_1.



Assume that C_2 has rolled around the inside of C_1, as shown on the right hand above. We now have a new contact point, c' of C_1 with C_2 and we consider the angle t_2 which is the angle between the two radii in C_2 connecting the horizontal to c'. Assume there is no slipping between the two circles C_1 and C_2 as they roll the *arc length* from c to c' on C_1 must equal that from c to c' on C_2. Arc length $l = tr$ so

$$r_1 t_1 = l = r_2 t_2.$$

Since $2r_2 = r_1$,

$$2r_2 t_1 = r_2 t_2,$$

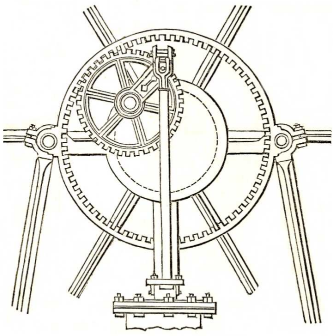
so that $t_2 = 2t_1$. Hence, c remains on the horizontal diameter of C_1 as C_2 rolls.

Extensions and Commentary

Source: Gutenmacher and Vasilyev (2004).

Note that this puzzle is quite hard to solve. Key is the choice of coordinates. The solution is so simple that one suspects that there must be a a lateral thinking solution but this has eluded the authors so far.

This observation was really used as a mechanism with a small cog wheel inside a larger one to generate straight line motion. For example, in 1801 James White patented the following mechanical device. However, in practice this mechanism places great strain on the central bearing and it was not particularly widely used. See, e.g. Bourne (1846).



The puzzle could be redrafted for mechanical engineering students using reference to this mechanism, possibly asking the students to consider why it was not particularly widely used.

Pick's Theorem

▼ Puzzle 39

A piece of metal has small holes drilled in it on a square lattice. By cutting in straight lines from hole to hole, polygon shaped pieces can be created, of any size and various shapes. The holes at the ends of cuts do not have to be adjacent to each other on the lattice and the polygons can be concave in parts. What is the area of a polygon expressed in terms of the number of small holes remaining in the interior of the piece of metal, the number of holes on the perimeter of the piece through which cuts were made and the distance between adjacent holes.

Solution

Stripped of its context this might be rephrased as follows.

▼ Puzzle 39 variant

A simple grid-polygon *is a closed chain of line segments constructed on a grid of integer coordinates in the plane which do not have points in common other than the common vertices of pairs of consecutive segments. Find the area of a simple grid-polygon in terms of the number of lattice points in the interior of polygon and the number of lattice points on the boundary.*

Pick's theorem provides a simple formula for calculating the area A of this polygon in terms of the number i of lattice points in the interior located in the polygon and the number b of lattice points on the boundary placed on the polygon's perimeter

$$A = i + \frac{b}{2} - 1$$

This is an example where some experimentation will enable students for form a conjecture. Also, by proving this formula for simple shapes, e.g. squares, rectangles and triangles, the student can devise a strategy for justifying why the formula holds in general. A detailed worked solution is available online at
`http://www.geometer.org/mathcircles/pick.pdf`.

Extensions and Commentary

Source: Pick (1899). This puzzle would not be appealing to the majority of non-mathematicians in its current form, mainly because the language. Pick's Theorem can be used to solve Puzzle 46, because the area of an equilateral triangle of base A is an irrational multiple of A^2, whereas an equilateral triangle drawn on the lattice would have an integral area.

Double weighings

▼ Puzzle 40

Four weights, A, B, C and D, were weighed in pairs. However, the two largest weights, C and D, were too heavy for the scales to be weighted together. Only the following five weights were recorded: 31.2 *kg,* 35.6 *kg,* 37.8 *kg,* 44.4 *kg and* 46.6 *kg.*

What are the individual weights?

Solution

We have the following pairs of weighings: $A+B$, $A+C$, $A+D$, $B+C$, $B+D$. If we order the weights by weight $A < B < C < D$, we do not know which of the pairs $A+D$ or $B+C$ will be heaviest, though we do know that they will be the third and fourth heaviest. If we therefore add together these to weighings, giving us $A+D+B+C$ we can subtract the lightest pair, $A+B$, to determine the weight of $C+D$. Hence

$$A+B+C+D = 37.8+44.4 = 82.2,$$

$$C+D = 82.2-21.2 = 51.0.$$

Now we have a situation which reduces to standard simultaneous equations, an exercise.

Deciding that $A+D < B+C$ or vice-versa is not a valid answer because it is not known beforehand which is the case.

Extensions and Commentary

This puzzle was an example posed to attendees to the Moore Legacy Conference, Washington D.C., June 2011.

This is usable in most STEM contexts. However, it should lead to a discussion of the difference between "weight" and "mass", the appropriate (SI) units of each and how non-technical language is sometimes scientifically imprecise.

Dice

▼ Puzzle 41

I have a set of three fair dice, Red, Green and Blue. The numbers on their faces are

$$\begin{array}{ll} R & 1,\ 4,\ 4,\ 4,\ 4,\ 4 \\ B & 3,\ 3,\ 3,\ 3,\ 3,\ 6 \\ G & 2,\ 2,\ 2,\ 5,\ 5,\ 5 \end{array}$$

Two people play the following game which begins by each player choosing one of the die. They then play an agreed number of rounds in which the highest roll wins. Which dice is best?

Solution

On average, G beats R, R beats B and B beats G, so there is no "best" die for pairwise comparison games. However, the winning strategy is clearly to choose second.

We consider the three possible combinations of die and the associated probabilities of each player winning. First consider R verses B. The symbol ■ indicates B wins.

				R			
		1	4	4	4	4	4
	3	■					
	3	■					
B	3	■					
	3	■					
	3	■					
	6	■	■	■	■	■	■

So B wins $\frac{11}{36}$ while R wins $\frac{25}{36}$. Next consider R verses G. The symbol ■ indicates G wins.

				R			
		1	4	4	4	4	4
	2	■					
	2	■					
G	2	■					
	5	■	■	■	■	■	■
	5	■	■	■	■	■	■
	5	■	■	■	■	■	■

So G wins $\frac{21}{36}$ while R wins $\frac{15}{36}$. Last consider B verses G. The symbol ■ indicates G wins.

				B			
		3	3	3	3	3	6
	2						
	2						
G	2						
	5	■	■	■	■	■	
	5	■	■	■	■	■	
	5	■	■	■	■	■	

So B wins $\frac{21}{36}$ while G wins $\frac{15}{36}$.

Extensions and Commentary

With this particular set of dice there is a very interesting extension game. Each player takes two identical die. Each time they throw they add the numbers together and the highest total in each round wins. What happens now?

The dice described in this problem can be bought at
`http://www.grand-illusions.com/acatalog/Non_Transitive_Dice_-_Set_2.html`

Other sets are available from
`http://www.mathsgear.co.uk/`

A lateral thinking solution is to compare the times which one face number will win or lose against the numbers on the other die, then multiply up by the number of these face numbers. For a similar problem with different values on the faces of the dice, see Bolt (1984), pg. 57.

It is really difficult to see how at STEM version of this puzzle could be created, at least one that was not clearly an artifice.

Monty Hall

▼ Puzzle 42

You are a contestant on a game show and are faced with the choice of opening one of three doors to choose your prize. You know that behind two of the doors the prize is a goats and behind the other a new car, but you do not know which prize is behind which door.

You pick a door, but before opening it the host opens one of the other two doors to reveal a goat. They then give you the choice of changing to the third, unchosen and unopened door. Should you?

Solution

Yes, when faced with three doors the chances that the car is behind any one of them is $1/3$. Therefore, the chances that the car is behind a door that you did not pick are $2/3$. When the game show host reveals that there is a goat behind one of the two remaining doors, it does not change the original odds on those two doors having a car behind them. Thus the chances that the car is behind the remaining door are $2/3$.

Extensions and Commentary

This is a classic puzzle based on the U.S. game show *Let's Make a Deal*; the puzzle itself was first posed in a letter by Steve Selvin to the American Statistician in Selvin (1975). Its usual formulation was given by Marilyn vos Savant in vos Savant (1990) as

> Suppose you're on a game show and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1 and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

This is such a classic puzzle, which in its original form generates so much discussion, that it is not worthwhile considering STEM discipline specific examples.

Snails

▼ Puzzle 43

Three small snails are each at the vertex of an equilateral triangle of side 60 cm. The first sets out toward the second, the second toward the third, and the third toward the first, each with a uniform speed of 5 cm/min. During their motion, each snail always heads toward its respective target. How much time elapses and how far does each snail travel before they all meet?

Solution

The way to find the time taken is to begin to resolve one snail's speed towards either the snail chasing it, or towards the centre. For this solution we will go back towards the previous snail. Resolving v_1 in the horizontal place (forming v_2):

$$v_2 = v_1 \sin(60°) = \frac{v_1}{2}.$$

The $60°$ is from the equilateral triangle and thus snail 3 is approaching snail 1 along the bottom line of the triangle at the relative speed of

$$v_1 + \frac{v_1}{2} = \frac{3v_1}{2}.$$

We know that the snail speed v_1 is 5 cm/min, so

$$\frac{3v_1}{2} = \frac{3(5)}{2} = 7.5 \text{ cm/min.}$$

It can now be said the time taken for the snails to meet is the distance of the bottom side of the triangle divided by the relative approach speed, so

$$\text{time } = \frac{60}{7.5} = 8 \text{ minutes}.$$

Given the snails' speed of 5 cm/min the distance each travels is 40 cm.

Extensions and Commentary

Source: Gnädig et al. (2001). We are unsure if this can be converted to a puzzle rooted in STEM. This is because each snail is attracted to only one of the other snails whereas objects or particles in similar physical situations would be mutually attractive or would not have three-fold symmetry.

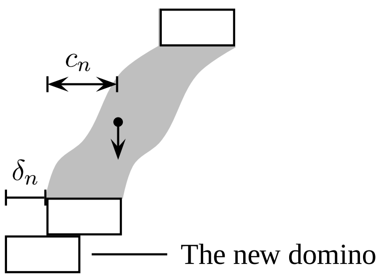
Stacking Blocks

▼ Puzzle 44

You have an unlimited supply of identical blocks. You stack one on top of the other and make the stack lean in one direction. What is the maximum horizontal distance you can cover before the stack collapses? (No glue, no nails etc...)

Solution

We construct a balancing stack by induction, assuming that the width of each block is 2 "units". Our strategy is this: at each stage we consider an existing balancing stack of n blocks which has its centre of mass a distance c_n from its left hand edge. Obviously $c_n \leq 2$ for all n as the centre of mass is to be above the bottom domino! We then place this stack on *top* of a new block a distance δ_n from the left of the domino.



There will clearly be no toppling if

$$\delta_n + c_n \leq 2 \quad \text{for all } n. \tag{7}$$

The new centre of mass of the whole stack of $n+1$ blocks will be c_{n+1} from the left of the bottom block where

$$c_{n+1} = \frac{(\delta_n + c_n)n + 1}{n+1} \quad \text{with} \quad c_1 = 1 \quad \text{(one domino)}. \tag{8}$$

Using (7), the maximum displacement without toppling is $\delta_n := 2 - c_n$. Combining this with (8) and solving for δ_n (the displacements) gives $\delta_1 = 1$ and

$$\delta_{n+1} = 2 - c_{n+1} = 2 - \frac{(\delta_n + c_n)n + 1}{n+1} = 2 - \frac{(\delta_n + 2 - \delta_n)n + 1}{n+1} = \frac{1}{n+1}.$$

So that for all n, $\delta_n = \frac{1}{n}$. The question becomes, what is the value of

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} = \sum_{n=1}^{N} \frac{1}{n}, \tag{9}$$

for large N? This is the *harmonic series*, which diverges. I.e. it is possible to make the sum (9) as large as one would wish so in theory we can produce an arbitrarily large horizontal displacement.

Extensions and Commentary

Source: A classic, going back to 1850, see Winkler (2007). Extensions to this problem are given by Paterson and Zwick (2009) and Paterson et al. (2009). This puzzle can be made more STEM discipline specific by giving a context, for example build a structure across a river. One could also consider what is possible if the stack has to support a load as would be the case with a real bridge.

NIM

▼ Puzzle 45

This is a competitive game, to be played between two players. Start with three heaps of any number of objects. The two players alternate taking any number of objects from any one heap. The goal is to be the last to take an object. What is a winning strategy and why?

Solution

This is a classical game, which has been solved for any number of initial heaps and objects. It is possible to determine which player, first or second to move, will win and what winning moves are open to that player.

The key is the binary digital sum, also known as “exclusive or” (xor), of the heap sizes. This is the sum (in binary) neglecting all carries from one digit to another. The winning strategy is to finish every move with a binary digital sum of zero.

Extensions and Commentary

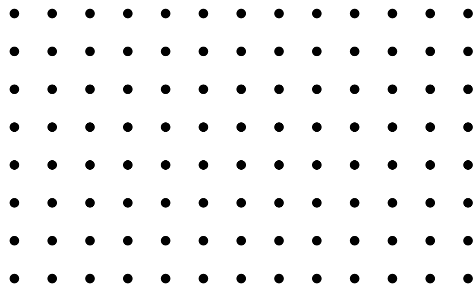
Clearly the game, as stated, uses only three heaps. Any number of heaps is possible, without changing the theory. A full explanation is given by, for example, Rouse Ball (1960).

We note in passing that games such as this are enjoyable and many are susceptible to a complete analysis. Such games are a fruitful source of puzzles. Another particularly simple game with interesting puzzling questions is known as “Hex”, see Browne (2000). This is a puzzle for which the simplicity provides clarity that would be lost by artificially putting it into a specific STEM context.

Lattice triangle

▼ Puzzle 46

Below is part of an infinite integer lattice. A lattice triangle *is a triangle where the coordinates of all vertices are integers.*



What is the size of the smallest equilateral lattice triangle?

Solution

There are no equilateral lattice triangles. For a justification of why see Puzzle 39.

Extensions and Commentary

This puzzle may be considered unfair as students are likely to assume that there is such a triangle to be found. Notice that a lattice triangle in three dimensions is essentially given in Puzzle 17. Extensions to other triangles and higher dimensions are found in Beeson (1992).

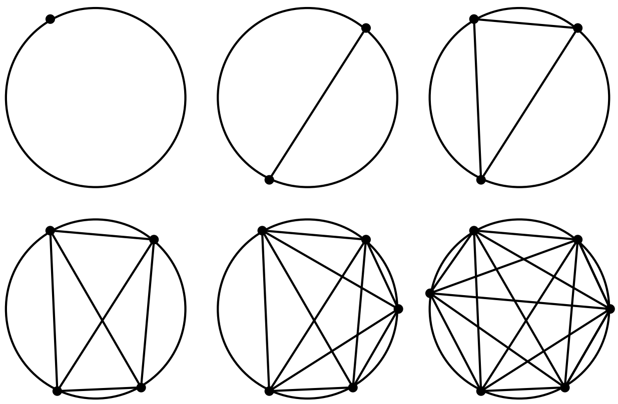
A STEM discipline specific version of this might be too artificial e.g. cutting out an equilateral triangle from a piece of metal, with each side of the triangle beginning and ending at pre-drilled holes arranged in a lattice pattern.

Why Bother with Proof?

▼ Puzzle 47

Draw an integral number of points on the circumference of a circle and join every pair of points with a segment. What is the greatest number of regions into which the circle can be divided?

Solution



This is a well-known puzzle. You will get the sequence 1, 2, 4, 8, 16 for the number of regions. Now consider 6 dots. It would seem only reasonable that there should be 32 regions. In fact there are 31. For 7 dots there are 57 regions instead of the 64 we might have expected.

If N is the number of regions and n the number of points then the number of regions r is given by

$$N = C_4^n + C_2^n + 1 = \frac{1}{24}\left(n^4 - 6n^3 + 23n^2 - 18n + 24\right).$$

There is a real challenge in finding the correct formula and understanding why the first 5 instances agree with the formula $N = 2^{n-1}$.

For many puzzles, especially those with a lateral solution, the solver will be guided by intuition or prior experience, with little or no attempts at proof. This puzzle is useful to remind students why mathematicians often insist on formal and rigorous proofs.

VC10

▼ Puzzle 48

As the sun was setting in a clear African sky, it was noticed in a Super VC10 flying north that the outline of the westward windows was projected on the other side of the cabin about 6 inches above the window on that side. Estimate roughly the height of the aircraft.

Solution

This problem cannot be solved with the given information.

Extensions and Commentary

Source: Brian Thwaites, SMP A-level mathematics exam.

This problem needs extra information. It may therefore not be a proper puzzle as with puzzles there is implied contract with the puzzle-setter is that the puzzle can be solved (see the discussion in Section 2.1). As an estimation problem it might be solved by estimating the width of the aircraft and using prior knowledge of the radius of the Earth. However, it is entertaining and with no solution without extra information, perplexing.

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Endnote

Higher Education Academy Workshop, 7th May 2012.
Embedding Puzzle-based learning in STEM Teaching.

There were 36 attendees at this meeting including:

Matthew Badger	University of Coventry
Andrew Csizmadia	Newman University College
Neil Currie	University of Salford
Sarah Hart	Birkbeck, University of London
Phil Harvey	De Montfort University
Eleanor Lingham	De Montfort University
Ken McKelvie	University of Liverpool
Susan Moron-Garcia	University of Birmingham
Natalie Rowley	University of Birmingham
Chris Sangwin	University of Birmingham
Jon Scaife	University of Sheffield
Duc Tham	University of Birmingham
Colin Thomas	University of Birmingham
Esther Ventura-Medina	University of Manchester
Nik Whitehead	Swansea Metropolitan University
Nicola Wilkin	University of Birmingham

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