A new max-plus based randomized algorithm for solving a class of HJB PDEs

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The work was conducted when I was a Ph.D. student at CMAP, École Polytechnique, France.

Infinite horizon linear quadratic optimal control

$$V(x) = \sup_{\mathbf{u} \in L_2^{\mathrm{loc}}([0,\infty);\mathbb{R}^k)} \int_0^\infty \frac{1}{2}\mathbf{x}(t)' D\mathbf{x}(t) - \frac{\gamma^2}{2}\|\mathbf{u}(t)\|^2 \, dt$$

where the state dynamics are given by

$$\begin{aligned} \dot{\mathbf{x}}(s) &= A\mathbf{x}(s) + \sigma \mathbf{u}(s) \\ \mathbf{x}(0) &= x \in \mathbb{R}^d. \end{aligned} \tag{1}$$

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Assumption (Existence assumption)

The matrices D and $-A-A'$ are positive definite and

$$\gamma^2/c_\sigma^2 > c_D/c_A^2$$

where $c_D = \lambda_{\max}(D)$, $c_A = \lambda_{\min}(-A-A')/2$ and $c_\sigma = \sqrt{\lambda_{\max}(\sigma\sigma')}$.

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• Under the existence assumption, the solution $V : \mathbb{R}^d \to \mathbb{R}$ is quadratic:

$$V(x) = \frac{1}{2}x'Px, \ \ \forall x \in \mathbb{R}^d,$$

where $P \in \mathbb{R}^{d\times d}$ is the solution of an Algebraic Riccati equation:

$$A'P + PA + P\Sigma P + D = 0$$

where $\Sigma = \sigma\sigma'/\gamma^2$.

Infinite horizon switched problem [McEneaney 07]

$$V(x) = \sup_{\substack{\mathbf{u}\in L_2^{\mathrm{loc}}([0,\infty);\mathbb{R}^k) \\ \mu:[0,\infty)\to[M]:=\{1,\dots,M\}}} \int_0^\infty \frac{1}{2}\mathbf{x}(t)' D^{\mu(t)}\mathbf{x}(t) - \frac{\gamma^2}{2}\|\mathbf{u}(t)\|^2 \, dt$$

where the state dynamics are given by

$$\dot{\mathbf{x}}(s) = A^{\mu(s)}\mathbf{x}(s) + \sigma^{\mu(s)}\mathbf{u}(s) \;\; \mathbf{x}(0) = x \in \mathbb{R}^d \;. \tag{2}$$

Infinite horizon switched problem [McEneaney 07]

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The matrices $\{D^m : m \in [M]\}$ *and* $\{-A^m - (A^m)' : m \in [M]\}$ *are positive definite and*

$$\gamma^2/c_\sigma^2 > c_D/c_A^2$$

where $c_D = \max_m \lambda_{\max}(D^m)$, $c_A = \min_m \lambda_{\min}(-A^m - (A^m)')/2$ *and* $c_\sigma = \max_m \sqrt{\lambda_{\max}(\sigma^m(\sigma^m)')}$.

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- nonconvex problem
- nonlinear robust H_∞ control (V is the available storage function)

Infinite horizon switched problem [McEneaney 07]

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$$\dot{\mathbf{x}}(s) = A^{\mu(s)}\mathbf{x}(s) + \sigma^{\mu(s)}\mathbf{u}(s) \;\; \mathbf{x}(0) = x \in \mathbb{R}^d \; . \qquad (2)$$

Under the existence assumption, the solution $V : \mathbb{R}^d \to \mathbb{R}$ is the (unique) viscosity solution of the following stationary HJB PDE:

$$H(x, \nabla V(x)) = 0, \;\; \forall x \in \mathbb{R}^d,$$

where $H(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the associated Hamiltonian:

$$H(x,p) = \sup_{m\in[M]} H^m(x,p) := \sup_{m\in[M]} (A^m x)' p + \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p$$

Lax-Oleinik semigroup [McEneaney 07]

The semigroup $\{S_t\}_{t\geq 0}$ associated with the switched problem:

$$S_t[\phi](x) := \sup_{\substack{\mathbf{u}\in\dot{=}L_2^{\mathrm{loc}}([0,\infty);\mathbb{R}^k) \\ \mu:[0,\infty)\to[M]:=\{1,\dots,M\}}} \int_0^t \frac{1}{2}\mathbf{x}(t)' D^{\mu(t)}\mathbf{x}(t) - \frac{\gamma^2}{2}\|\mathbf{u}(t)\|^2 dt + \phi(x(t)).$$

where

$$\dot{\mathbf{x}}(s) = A^{\mu(s)}\mathbf{x}(s) + \sigma^{\mu(s)}\mathbf{u}(s), \ \ \mathbf{x}(0) = x \in \mathbb{R}^d \ .$$

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- order-preserving property: $\phi_1 \leq \phi_2 \Rightarrow S_t[\phi_1] \leq S_t[\phi_2]$
- group property: $S_{t_1+t_2}[\phi] = S_{t_1}[S_{t_2}[\phi]]$
- max-plus linearity:
  $S_t[\sup(\phi_1,\phi_2)+\lambda] = \sup(S_t[\phi_1], S_t[\phi_2]) + \lambda$

Lax-Oleinik semigroup [McEneaney 07]

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where

$$\dot{\mathbf{x}}(s) = A^{\mu(s)}\mathbf{x}(s) + \sigma^{\mu(s)}\mathbf{u}(s), \;\; \mathbf{x}(0) = x \in \mathbb{R}^d \;\; .$$

- For all $t \geq 0$, $S_t[V] = V$
- For all $V_0 \leq V$, $V = \lim_{T\to+\infty} S_T[V_0]$.
- $V \simeq S_T[V_0] = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0]$.

Semigroup approximation

- Semigroup approximation:

$$S_\tau \simeq \tilde{S}_\tau := \sup_m S_\tau^m$$

- $\{S_t^m\}_{t\geq 0}$ is the semigroup associated to the mth linear quadratic control problem:

$$S_t^m[\phi](x) = \sup_{\mathbf{u}} \int_0^t \frac{1}{2}\mathbf{x}(t)' D^m \mathbf{x}(t) - \frac{\gamma^2}{2}\|\mathbf{u}(t)\|^2 dt + \phi(\mathbf{x}(t)).$$

$$\dot{\mathbf{x}}(s) = A^m \mathbf{x}(s) + \sigma^m \mathbf{u}(s); \;\; \mathbf{x}(0) = x \in \mathbb{R}^d \quad .$$

- $S_t^m[\phi]$ is a quadratic function if ϕ is. (Riccati)

$$V \simeq \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \quad .$$

Arborescent propagation

$$V \simeq V_T = \{S_\tau\}^N[V_0] \simeq \{\tilde{S}_\tau\}^N[V_0] = \sup_{i_N,\dots,i_1} S_\tau^{i_N} \circ \dots S_\tau^{i_1}[V_0] \ .$$

V_0

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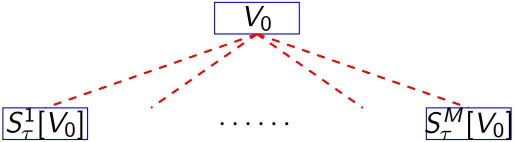

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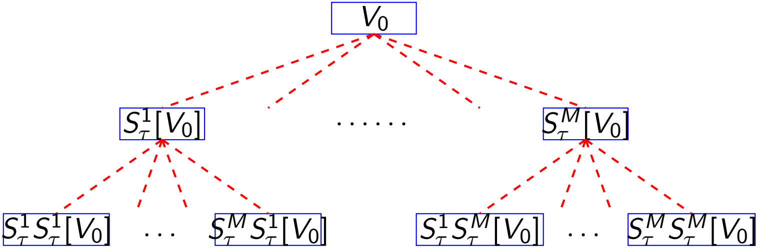

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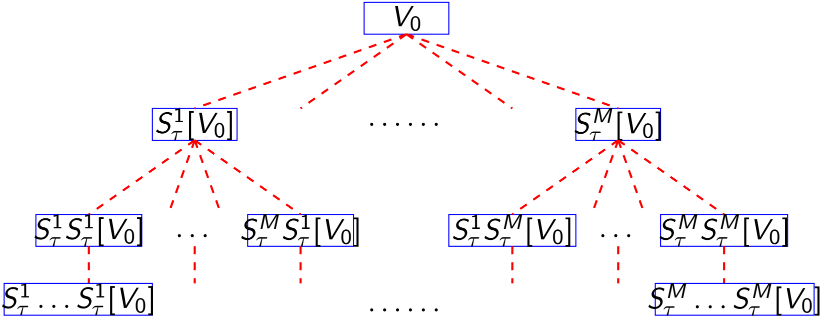

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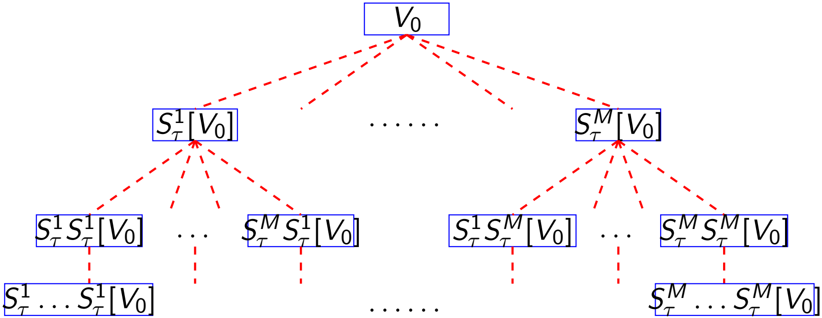
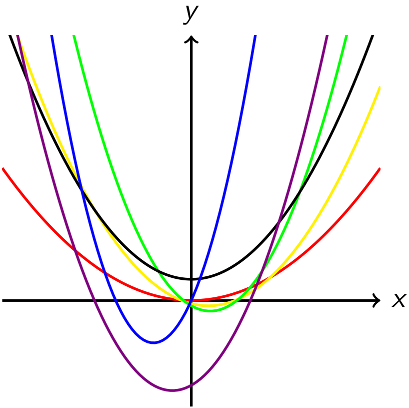


Computational complexity: $O(M^N d^3)$ $\Rightarrow$ curse of dimensionality free

Pruning operation:

$$V \simeq \sup(\phi_{green}, \phi_{red}, \phi_{violet}, \phi_{yellow}, \phi_{black}, \phi_{blue})$$



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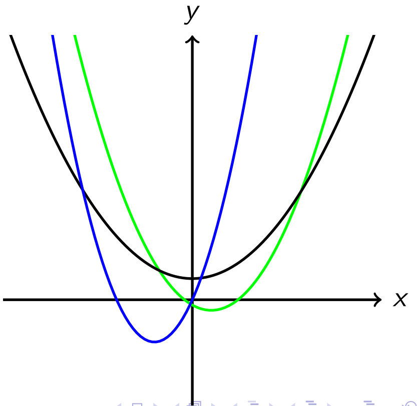

SDP based pruning algorithms

Suppose that the approximation at current step is:

$$\sup_{i\in I} \phi_i(x) = \sup_{i\in I} (x' \ \ 1) Q_i \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

SDP approach ([McEneaney,Deshpande,Gaubert 08]): for every quadratic function ϕ_i, compute the *maximum lost* caused by removing it from the basis set, keep those functions with largest maximum lost.

Importance metric of ϕ_j

$$\begin{aligned} \nu_j = \max \ & \nu \\ & \nu \le z'(Q_j - Q_i)z, \forall i \neq j \\ & z^T z = 1. \end{aligned}$$

SDP relaxation

$$\begin{aligned} \overline{\nu}_j = \max \ & \nu \\ & \nu \le \langle (Q_j - Q_i), Z \rangle, \forall i \neq j \\ & Z \geq 0, \ \ \mathrm{trace}(Z) = 1. \end{aligned}$$

Motivation

- SDP based COD free method:

$$V \simeq \sup_{i_1,\cdots i_N} S^{\tau}_{i_N} \cdots S^{\tau}_{i_1}[V_0] \ .$$

add all the possible basis functions + prune

- SDP based COD free algorithm works but is time-consuming:

τ=0.2, N=25	Total time	Propagation	SDP	Pruning
sort lower	1.04h	1.85%	98.15%	0.00%
sort upper	1.34h	1.52%	98.43%	0.05%
J-V p-d	1.38h	1.45%	89.47%	9.08%
greedy	1.43h	1.63%	97.84%	0.53%

- New algorithm:

add "*useful*" basis functions (no need to prune)

Idea

Suppose that at current step we have a sub-approximation of V by:

$$V \simeq \sup_{i \in \{1,\dots,k\}} \phi_i \leq V.$$

Instead of computing $S_\tau^m[\phi_i]$ for every $i \in \{1, \dots, k\}$ and every $m \in \{1, \dots, M\}$ and then prune by solving kM SDP programs:

$$V \simeq \sup_{i \in \{1,\dots,k\}} \phi_i$$

$$S_\tau^1[\phi_1] \quad \cdots\cdots \quad S_\tau^1[\phi_k] \quad \cdots\cdots \quad S_\tau^M[\phi_1] \quad \cdots\cdots \quad S_\tau^M[\phi_k]$$

we will find out efficiently a propagation time $t_0 > 0$, an index $i_0 \in \{1, \dots, k\}$ and an index $m_0 \in \{1, \dots, M\}$ such that

$$\sup_{i \in \{1,\dots,k\}} \phi_i(x_0) < \sup(\sup_{i \in \{1,\dots,k\}} \phi_i(x_0), S_{t_0}^{m_0}[\phi_{i_0}](x_0))$$

Key observation

Recall that:

$$H^m(x,p) = (A^m x)' p + \frac{1}{2} x' D^m x + \frac{1}{2} p' \Sigma^m p$$

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a quadratic function, $x_0 \in \mathbb{R}^d$ and $m_0 \in \{1, \ldots, M\}$. If

$$H^{m_0}(x_0, D\phi(x_0)) > 0 \quad ,$$

then for all sufficiently small $t_0 > 0$,

$$S_{t_0}^{m_0}[\phi](x_0) > \phi(x_0) \quad .$$

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$$S_{t_0}^{m_0}[\phi](x_0) > \phi(x_0) \ .$$

Key observation

Let $\phi_1, \ldots, \phi_k : \mathbb{R}^d \to \mathbb{R}$ be quadratic functions such that:

$$\sup_{i \in \{1,\ldots,k\}} \phi_i(x) \leq V(x), \ \forall x \in \mathbb{R}^d \ .$$

If
$$\phi_{i_0}(x_0) = \sup_{i \in \{1,\ldots,k\}} \phi_i(x_0) \ ,$$
$$H(x_0, D\phi_{i_0}(x_0)) > 0 \ ,$$

then there is $m_0 \in \{1, \ldots, M\}$ such that for all sufficiently small $t_0 > 0$,

$$S_{t_0}^{m_0}[\phi_{i_0}](x_0) > \sup_{i=1,\ldots,k} \phi_i(x_0) \ .$$

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then there is $m_0 \in \{1, \dots, M\}$ such that for all sufficiently small $t_0 > 0$,

$$S_{t_0}^{m_0}[\phi_{i_0}](x_0) > \sup_{i=1,\dots,k} \phi_i(x_0) \ .$$

We find proper i_0 and m_0 in $O(k + M)$.

Randomized max-plus algorithm

Algorithm 1 Randomized max-plus algorithm

INPUT: $V^0 = \sup_i \phi_i^0 \leq V$, compact $X \ni 0$, threshold $\vartheta > 0$

for $k = 0, 1, 2, \ldots$ **do**

  Randomly choose a point $x_0 \in X$;

  if $H(x_0, DV^k(x_0)) > \vartheta$, **then**

$$V^{k+1} = \sup(V^k, \Psi(V^k, x_0));$$

  else

$$V^{k+1} = V^k.$$

  end if

end for

Convergence of the algorithm

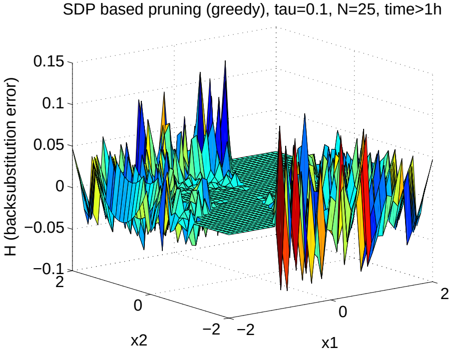
Theorem (Convergence result)

Under the existence assumption, for every threshold $\vartheta > 0$, the randomized algorithm stops surely after a finite number of iterations. There is a constant $L > 0$ such that for all $\vartheta > 0$, almost surely we have:

$$\lim_{k \to +\infty} \sup_x |V(x) - V^k(x)|/|x|^2 \leq L\vartheta$$

Experimental results

$M=6, \quad d=6$



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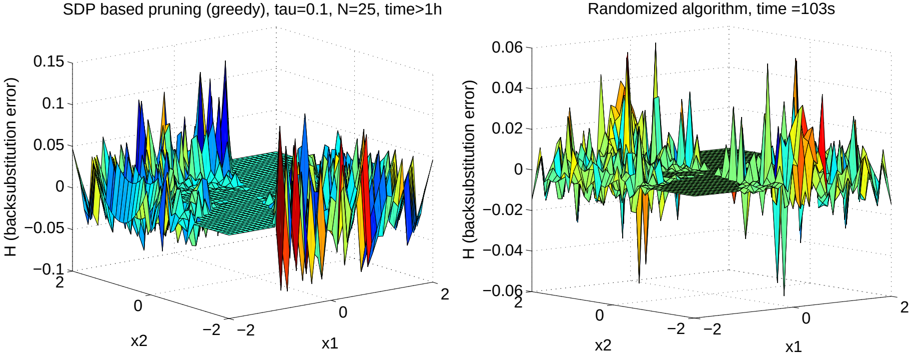
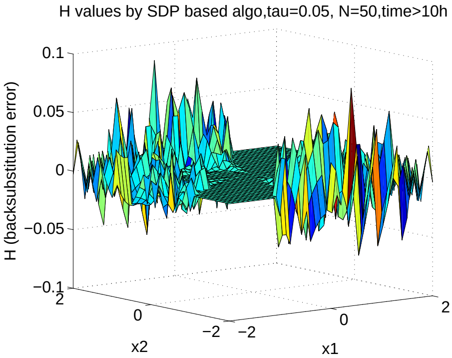


Experimental results

M=6, d=6



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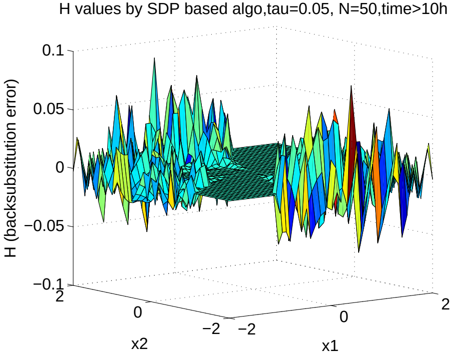
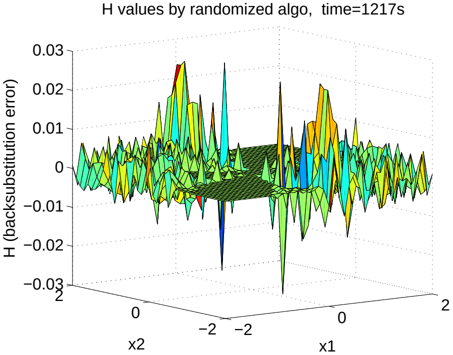
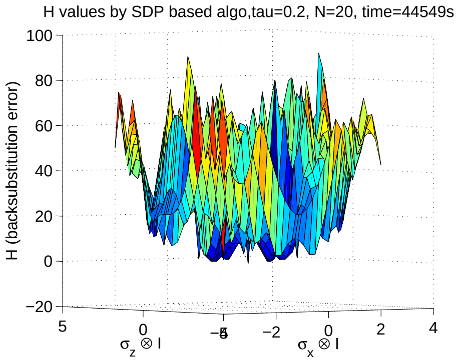




Experimental results

M=6, d=15 (an optimal control problem on SU(4) –quantum optimal gate synthesis)



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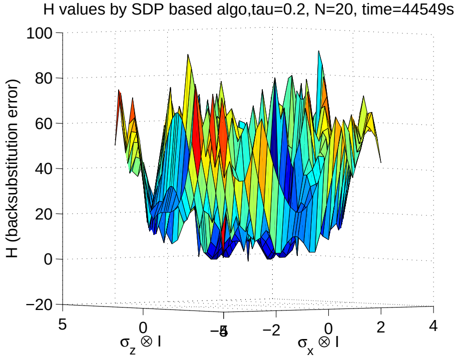

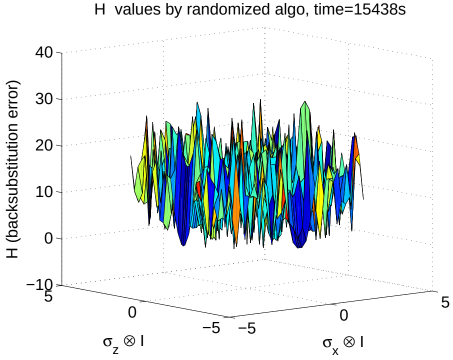

Instance nb.	d	M	initial backsub. error	backsub. error	SDP based		randomized algo	
					cpu time	basis nb.	cpu time	basis nb.
1	4	10	1.2	0.8	280s	100	10s	345
1	4	10	1.2	0.007	-	-	527s	26605
2	4	20	4.5	0.14	-	-	184s	9925
2	4	20	4.5	0.024	-	-	2900s	75831
3	4	50	3.8	1.5	5923s	80	41s	120
3	4	50	3.8	0.1	-	-	155s	5938
3	4	50	3.8	0.034	-	-	3650s	99848
4	6	6	2.3	0.21	188s	170	27s	1231
4	6	6	2.3	0.12	1.5h	170	37s	1636
4	6	6	2.3	0.08	$>$10h	320	103s	3281
4	6	6	2.3	0.0257	-	-	1217s	10227
5	8	20	0.13	0.1	163s	50	16s	56
5	8	20	0.13	0.01	-	-	90s	8108
5	8	20	0.13	0.0045	-	-	1887s	32860
6	8	30	0.16	0.012	-	-	151s	13698
6	8	30	0.16	0.012	-	-	2994s	76393
7	8	50	0.21	0.01	-	-	120s	10011
7	8	50	0.21	0.0044	-	-	3124s	67515
8	12	20	0.7	0.01	-	-	1485s	44894
8	12	20	0.7	0.0052	-	-	5368s	93292
9	12	30	0.83	0.012	-	-	1322s	39987

End

Thank you!

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