Intermittency of a passive scalar advected by a quasifrozen velocity field

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We use a two-dimensional lattice model to study the intermittency problem of a passive scalar advected by a velocity field of finite correlation time. The stream function generating the incompressible velocity field is modeled by a random Gaussian noise that is identically independently distributed at each lattice point and is updated every certain finite time interval. A fixed scalar difference is maintained across one direction of the lattice. There are three time scales in the problem: the correlation or update time of the velocity field τ_c, the diffusion time of the scalar $\tau_{\rm diff}$, and the advection time of the velocity field $\tau_{\rm adv}$. Interesting behavior is observed when $\tau_{\rm diff} < \tau_c$. In this regime the passive scalar field is found to be intermittent while its dynamics between the updates of the velocity field is dominated by diffusion. The intermittency can be described by log-Poisson statistics and is independent of the ratio $\tau_c/\tau_{\rm adv}$. On the other hand, the passive scalar field exhibits dissipative scaling and is thus nonintermittent when $\tau_{\rm diff} \geqslant \tau_c$. [S1070-6631(99)00408-0]

I. INTRODUCTION

A major problem in turbulence theory is to understand the statistical properties of the inertial range, the range of length scales that are smaller than those of energy input and larger than those affected directly by molecular dissipation. Kolmogorov's seminal work in 1941[1] predicted simple power-law scaling for the velocity structure functions when the separating distance is within the inertial range, and the scaling exponent is $n/3$ for the structure function of order n. Power-law behavior has been confirmed by experimental measurements but there has been evidence that the scaling exponents are different from $n/3$. Such a deviation is known as anomalous scaling and reveals that a turbulent velocity field is intermittent. The intermittency manifests itself as a change in shape or form of the probability density function (pdf) of the velocity difference with the separating distance.

The inertial-range dynamics of an advected passive scalar is believed to be more tractable theoretically. Much recent effort[2–12] has been devoted to the study of the Kraichnan model[13] in which the advecting velocity field is incompressible and rapidly changing or delta-correlated in time. In this rapid-change limit, the effect of the velocity field on the passive scalar is described solely by an eddy diffusivity. Moreover, any equal-time multipoint correlation function of the scalar satisfies a closed linear differential equation,[14] which renders an analytical study of the inertial-range properties possible. In this model, the eddy diffusivity was taken to have a power-law scaling with exponent $0 < \zeta(\eta) < 2$. The second-order scalar structure function was found to exhibit power-law scaling with exponent $\zeta_2 = 2 - \zeta(\eta)$.[13] The scaling of the higher-order structure functions has been studied by perturbation theory around three limiting cases: (i) infinite space dimensionality,[2,9] (ii) smooth scalar field $\zeta(\eta) \to 0$,[5,6] and (iii) smooth velocity field $\zeta(\eta) \to 2$.[10] Anomalous scaling has been found which is understood to result from the dominance of the homogeneous solutions of the differential equation.

A velocity field that is delta-correlated in time is not physical. In any realistic turbulent flow, the velocity field has a finite correlation time. However, analytical simplicity would be lost when we move away from the rapid-change limit.

In this paper, we report our numerical study of the intermittency problem of a passive scalar when the advecting velocity field has a finite correlation time. We use a two-dimensional lattice model[15–17] in which the stream function generating the incompressible velocity field is modeled by a random Gaussian noise that is identically independently distributed at each lattice point and is updated every certain finite time interval. The velocity field remains the same within the time intervals between the updates and thus has a finite correlation time equals to the time between updates.

This model is simple to study numerically and has been found to produce interesting results in earlier studies. It was found[15] that the passive scalar fluctuation becomes non-Gaussian for a certain range of parameters of the model. Moreover, such a change was shown[16] to be independent of the statistics prescribed for the velocity field. More recently, with the stream function suitably modified, the effects of a

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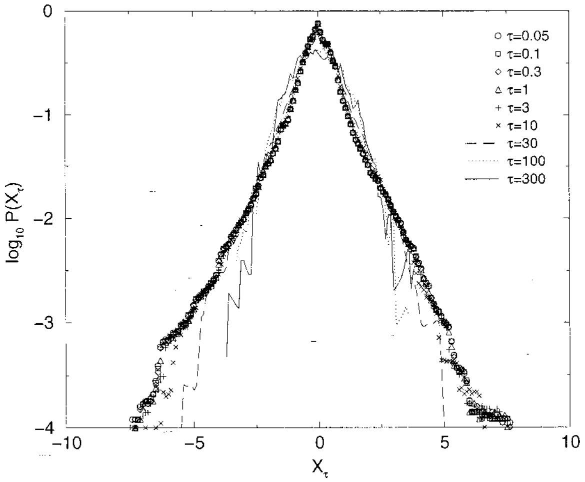

FIG. 1. The pdfs of normalized scalar difference $P(X_\tau)$ for $C=1$ and $K=1$. $P(X_\tau)$ is the same for $K=0.1$ and 1, and is independent of τ for $0<\tau<10$.

large-scale mean circulating flow on the passive scalar statistics have been studied.[17]

In the present study, we focus on the problem of intermittency and study the statistics of the passive scalar difference at the center of the lattice as the parameters of the model are varied. There are three time scales in the problem: the correlation or update time of the velocity field τ_c, the diffusion time of the passive scalar τ_{diff}, and the advection time of the velocity field τ_{adv}. We find that the passive scalar is intermittent when $\tau_{\mathrm{diff}}<\tau_c$ in that the pdf of the scalar difference changes its shape with the time separation τ for $\tau<\tau_c$. Moreover, the intermittency can be described by log-Poisson statistics and is independent of the ratio $\tau_c/\tau_{\mathrm{adv}}$. On the other hand, the passive scalar field exhibits dissipative scaling and is thus nonintermittent when $\tau_{\mathrm{diff}}\geqslant\tau_c$.

II. MODEL

The two-dimensional lattice model that we use was discussed in Refs. 15–17 so we shall only outline the main points here. We solve numerically the discrete advection-diffusion equation for the scalar field $T(i,j,t)$ on a $N\times N$ square lattice of spacing ξ:

$$\frac{\partial T(i,j,t)}{\partial t}+\mathbf{u}(i,j,t)\cdot\nabla_{ij}T(i,j,t)=D\nabla_{ij}^2T(i,j,t), \tag{1}$$

where D is an effective diffusivity, and $i,j=1,\ldots,N$. One should not identify D with the molecular diffusivity since, by construction, the smallest spatial scales are not resolved.

The velocity field $\mathbf{u}(i,j,t)$ is generated from the stream function $\phi(i,j,t)$ which is modeled by a random Gaussian noise with zero mean and standard deviation ϕ_0, identically independently distributed at each lattice point (i,j), and is updated every time interval τ_c. The typical size of the velocity fluctuation is given by $u_0=\phi_0/\xi$. The correlation length of the velocity field is ξ. Besides the velocity correlation or update time τ_c, the two other time scales in the problem are the diffusion time of the passive scalar $\tau_{\mathrm{diff}}=\xi^2/D$, and the advection time of the velocity field $\tau_{\mathrm{adv}}=\xi/u_0$. Thus, two independent dimensionless parameters can be constructed which are taken to be $C\equiv\tau_{\mathrm{diff}}/\tau_c$ and $K\equiv\tau_c/\tau_{\mathrm{adv}}$.

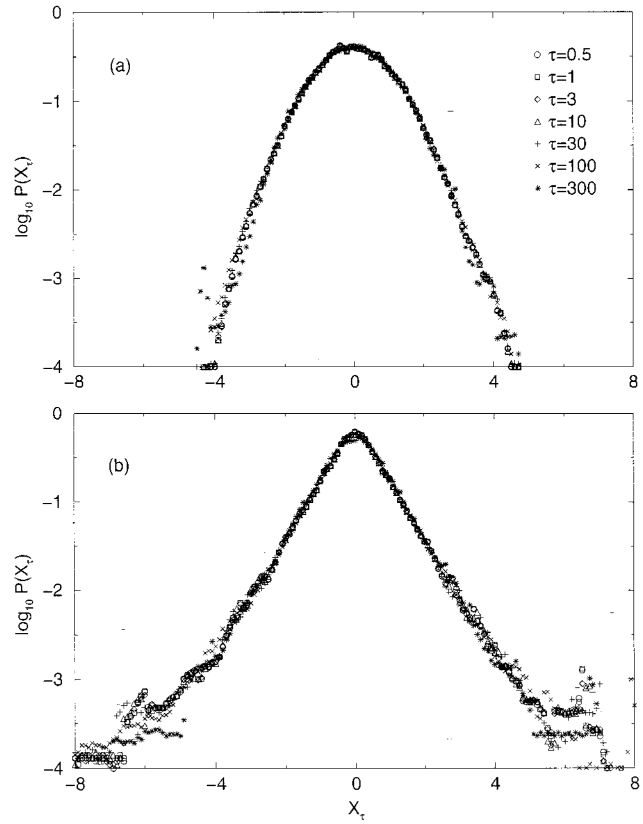

FIG. 2. The pdfs of normalized scalar difference $P(X_\tau)$ for: (a) $C=10$, $K=0.1$ and (b) $C=10$, $K=1$. $P(X_\tau)$ is independent of τ for $0<\tau\leqslant300$. The symbols used in (b) are the same as those in (a).

Equation (1) is integrated in time using the finite difference method with a small time step $\Delta t=0.005$. We take $\xi=1$ and $N=31$. Such a relatively small N is sufficient as we shall evaluate the statistics by averaging over time. The stream function is updated every $m=20\,000$ time steps at each lattice site so that $\tau_c=m\Delta t=100$. The boundary condition for both the velocity and the scalar fields is periodic in the i direction. In the j direction, the velocity field is no-slip on both the ''top'' and ''bottom'' boundaries, while the scalar field satisfies a fixed-difference condition: $T(i,j=0,t)=0$ and $T(i,j=N+1,t)=1$. We measure $T(t)$ at the center of the lattice as a function of time after the system reaches the steady state. The scalar difference $T_\tau(t)$ is defined as $T(t+\tau)-T(t)$. Long time series with at least 10^7 data points are used for calculating the statistics.

III. RESULTS AND DISCUSSION

As discussed in earlier work,[15–17] the mean scalar profile is almost linear in the j direction and has a slight dependence on i. The one-point scalar pdf changes from Gaussian to exponential, and to stretched exponential as C increases when K is fixed. The change from Gaussian to non-Gaussian statistics occurs at a smaller value of C as K increases. Moreover, the pdf is found to be the same at every lattice point

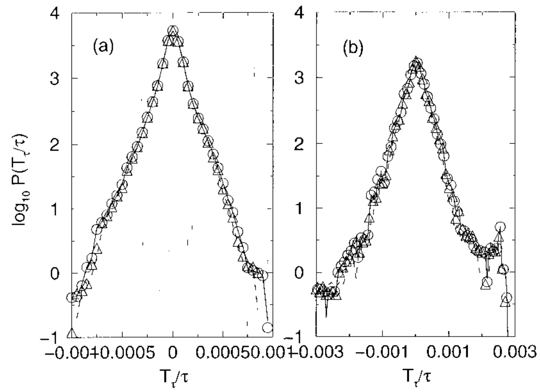

FIG. 3. $P(T_\tau/\tau)$ for (a) $C=1$ with $\tau=0.05$ (solid line), $\tau=0.3$ (circles), $\tau=3$ (triangles), and $\tau=5$ (dashed line); (b) $C=10$ with $\tau=0.05$ (solid line), $\tau=1$ (circles), $\tau=10$ (triangles), and $\tau=50$ (dashed line).

within the bulk of the lattice, but becomes positively (negatively) skewed near the bottom (top) boundary.

To study whether the passive scalar field exhibits intermittency, we consider the pdf of the scalar difference and investigate whether there is a change in its shape as the time separation changes. Since we are interested in the shape of the pdf, it is more convenient to evaluate the pdf of the normalized scalar difference $X_\tau \equiv T_\tau/\langle T_\tau^2\rangle^{1/2}$, i.e., the temperature difference normalized by its standard deviation. We shall see that the result depends crucially on whether the parameter C is larger than 1 or not.

We first present and discuss the results for $C \geqslant 1$. In this regime, the pdfs of the normalized scalar difference $P(X_\tau)$ are found to be independent of τ for τ ranging from 0 to about τ_c. We plot $P(X_\tau)$ for $C=1$ and $C=10$, respectively in Figs. 1 and 2. For $C=1$, $P(X_\tau)$ is the same for $K=0.1$ and $K=1$, whereas for $C=10$, $P(X_\tau)$ is Gaussian for $K=0.1$ and exponential for $K=1$. Since $P(X_\tau)$ is related to the one-point pdf $P(T)$ when τ is large, the latter result is consistent with the previous finding[15] that $P(T)$ is Gaussian and exponential, respectively, for $K=0.1$ and $K=1$ when $C=10$. The independence of $P(X_\tau)$ on τ leads to

$$\langle T_\tau^{2n}\rangle = C_{2n}\langle T_\tau^2\rangle^n, \tag{2}$$

where $C_{2n}=\langle \dot{T}^{2n}\rangle/\langle \dot{T}^2\rangle^n$ is independent of τ. Here, the overdot indicates time derivative. When scaling behavior exists, that is, $\langle T_\tau^{2n}\rangle \sim \tau^{\zeta_{2n}}$, (2) implies simple scaling: $\zeta_{2n}=n\zeta_2$.

We evaluate $P(T_\tau/\tau)$ in Fig. 3 and find that it remains the same for $0<\tau<\tau_d$, where $\tau_d \approx 3$ and 30, respectively, for $C=1$ and 10. Therefore,

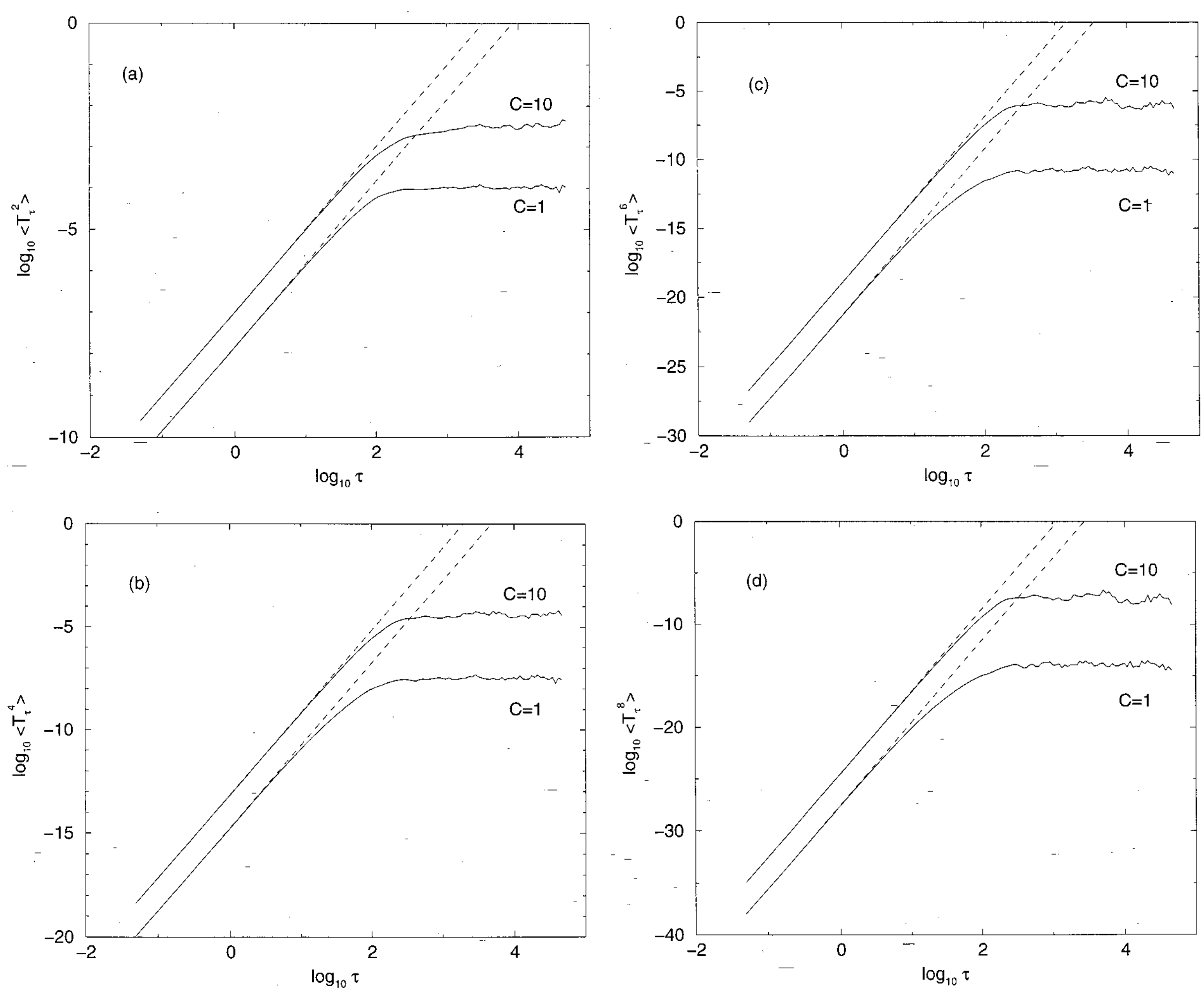

FIG. 4. The scalar structure functions $\langle T_\tau^n\rangle$ for (a) $n=2$, (b) $n=4$, (c) $n=6$, and (d) $n=8$. The dashed lines are $\tau^n\langle \dot{T}^n\rangle$ [see (6)].

$$P\left(\frac{T_\tau}{\tau}\right)=P(\dot{T}) \quad \text{for} \quad \tau<\tau_d. \tag{3}$$

Equation (3) suggests that the scalar field changes approximately linear in time between each update of the velocity field. Another relation between $P(T_\tau/\tau)$ and $P(\dot{T})$ can be obtained via their respective relations to $P(X_\tau)$:

$$\frac{\sqrt{\langle T_\tau^2\rangle}}{\tau}P\left(\frac{T_\tau}{\tau}\right)=P(X_\tau)=\lim_{\tau\to 0}P(X_\tau)$$
$$=\sqrt{\langle \dot{T}^2\rangle}\,P(\dot{T}) \quad \text{for} \quad \tau<\tau_c. \tag{4}$$

The second equality holds since $P(X_\tau)$ remains the same for τ ranges from 0 to close to τ_c. Comparing (3) and (4), we find

$$\langle T_\tau^2\rangle=\tau^2\langle \dot{T}^2\rangle \quad \text{for} \quad \tau<\tau_d. \tag{5}$$

Equations (2) and (5) imply that for $C\gg 1$, the passive scalar exhibits simple dissipative scaling:

$$\langle T_\tau^{2n}\rangle=\langle \dot{T}^{2n}\rangle\,\tau^{2n} \quad \text{for} \quad \tau<\tau_d. \tag{6}$$

This result is confirmed by the plots of the scalar structure function, some of which are shown in Fig. 4. The correlation time of the scalar field, corresponding to the time at which the flat region starts, is also seen to be approximately τ_c as one would have guessed.

Result (6) for $C\gg 1$ could be put in perspective using the perturbation result around the $\zeta(\eta)\to 0$ limit. The velocity field in our model is not correlated in space so we may take $\zeta(\eta)=0$, and it becomes rapidly changing in time when $C\gg 1$. In the $\zeta(\eta)\to 0$ limit, the anomalous scaling exponents $n\zeta_2-\zeta_{2n}$ have been found[6,7] to be proportional to $\zeta(\eta)$. Hence, for $\zeta(\eta)=0$ in the rapid-change limit, we anticipate $\zeta_{2n}=n\zeta_2=2n$, which is in agreement with (6).

Next, we consider $C<1$. In this regime, the update time of the velocity field is longer than the diffusion time of the scalar. In the limit $C\ll 1$, the velocity field is quasifrozen with infrequent updates. Between these infrequent updates of the velocity field, the dynamics of the scalar is dominant by diffusion. One might then expect smooth nonintermittent scalar statistics. However, the occasional updates of the velocity field instill discontinuous randomness into the dynamics, which become rare events when the update time is very long compared to the diffusion time. The interplay of the two effects can lead to interesting results. Indeed, we find that $P(X_\tau)$ changes with τ in this regime. In Fig. 5, we plot $P(X_\tau)$ for $C=0.01$ and $C=0.1$. In both cases, the pdf changes from stretched exponential to Gaussian as τ increases. For a given value of C, it is found that $P(X_\tau)$ is the same for K = 0.1, 1, and 10. The change in shape or form of the pdf with τ indicates that the statistics are different at different time scales and the passive scalar field is thus intermittent. It is interesting to note that this change in shape of the pdf is similar to that of an active scalar observed in turbulent convection experiments.[18]

To quantify the intermittency, we study the τ dependence of the moments $\langle|X_\tau|^n\rangle$. Some of the plots for $C=0.1$ are shown in Fig. 6. As can be seen, only a short power-law region over about a decade can be fitted. The exponent μ_p, defined by $\langle|X_\tau|^p\rangle\sim\tau^{\mu_p}$, is a measure of the intermittency. From Fig. 7, we see that μ_p is a nonlinear function of p. An interesting feature is the linear dependence of μ_p for large p. This feature, as noted in Ref. 13, gives the lowest possible rate of growth of intermittency towards smaller scale.

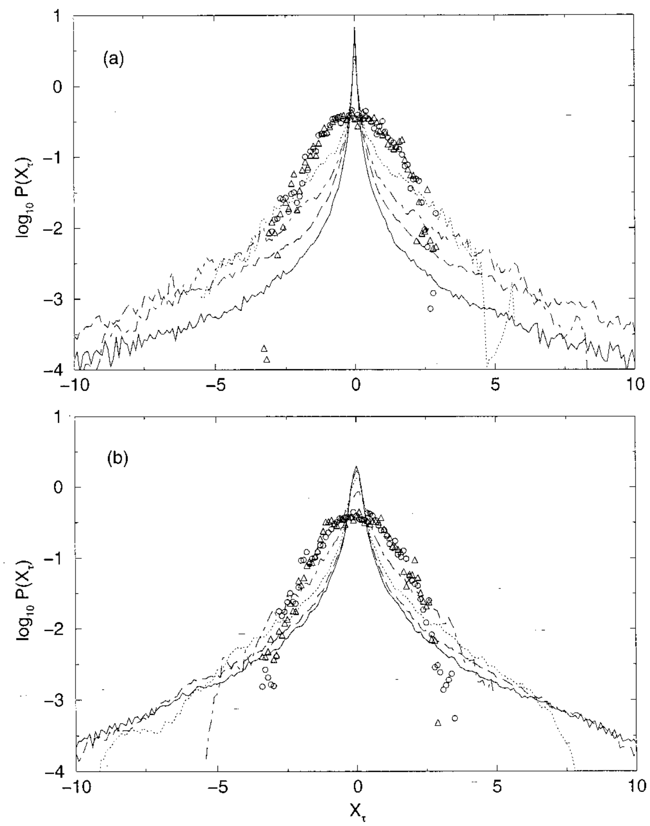

FIG. 5. The pdfs of normalized scalar difference $P(X_\tau)$ for (a) $C=0.01$, $K=1$ and (b) $C=0.1$, $K=1$. The symbols used are: $\tau=0.05$ (solid line), $\tau=3$ (dotted line), $\tau=30$ (dot-dashed line), $\tau=100$ (circles), and $\tau=300$ (triangles). For a given value of C, $P(X_\tau)$ is found to be independent of K for K = 0.1, 1, and 10, and changes from stretched-exponential to Gaussian as τ increases.

The idea of the infrequent update of the velocity field being a rare event in the discontinuous random dynamics prompts us to suspect that the intermittency might be described by a log-Poisson distribution. Thus, we fit μ_p by the form $a+bp+\alpha\beta^p$[19–21] which also automatically gives the linear asymptotics when $|\beta|<1$. By definition, $\mu_0=\mu_2=0$, so there are only two independent fitting parameters which we choose to be a and β, and express α and b as $\alpha=-a$ and $b=-a(1-\beta^2)/2$, respectively. The fitting form is thus

$$\mu_p^{(\mathrm{fit})}=a\left[1-\frac{(1-\beta^2)}{2}p-\beta^p\right] \tag{7}$$

and can be seen to describe the exponent μ_p quite well (see Fig. 7). The good fit of μ_p by (7) confirms that the statistics of π_τ, defined by

$$\pi_\tau\equiv\frac{|T_\tau|}{|T_\tau|^{(\infty)}} \tag{8}$$

with

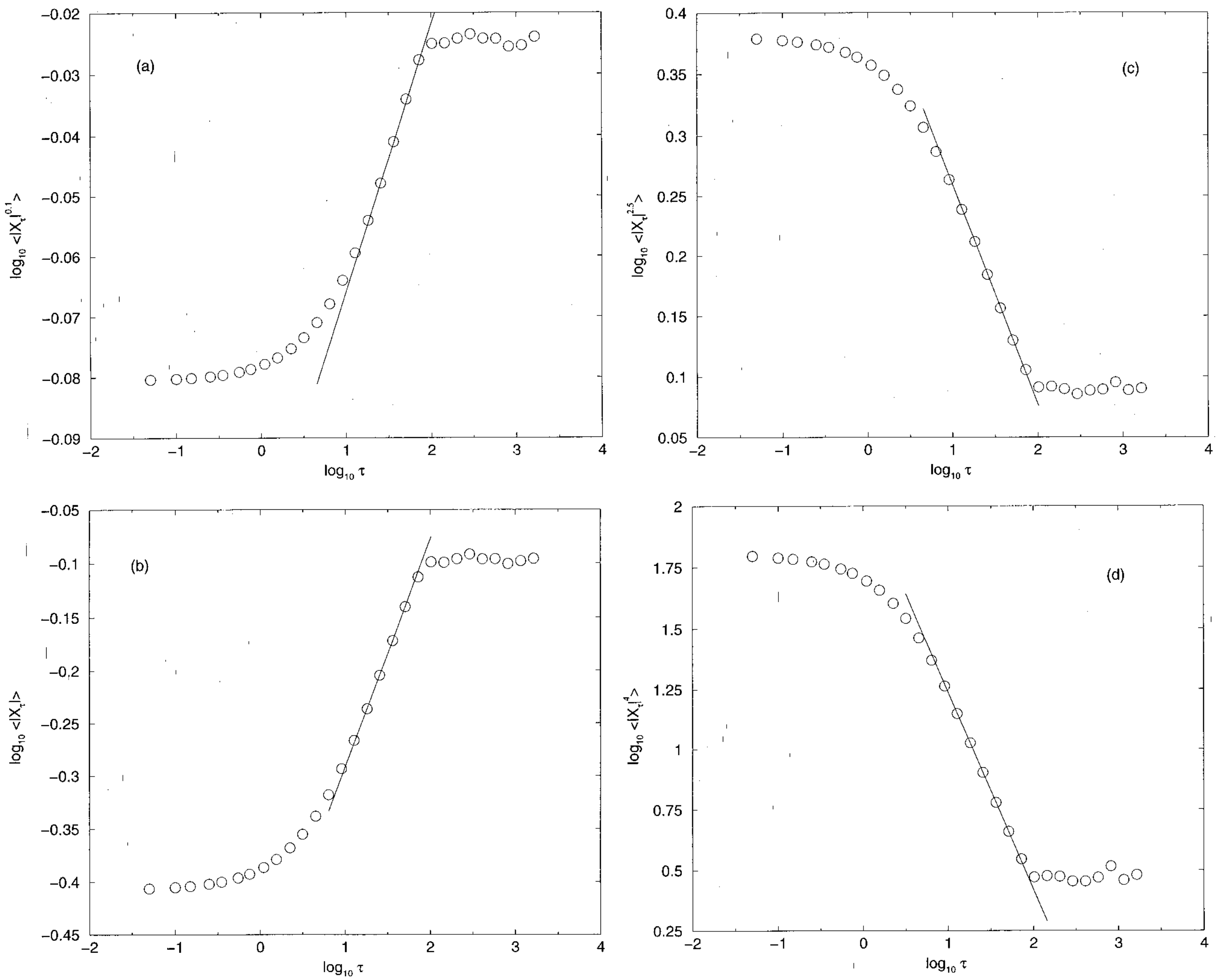

FIG. 6. The τ dependence of the moments $\langle |X_\tau|^p\rangle$ for $C=0.1$: (a) $p=0.1$, (b) $p=1$, (c) $p=2.5$, and (d) $p=4$. A short power-law region over about a decade is fitted (solid line).

$$|T_\tau|^{(\infty)} \equiv \lim_{p\to\infty} \frac{\langle |T_\tau|^{p+1}\rangle}{\langle |T_\tau|^{p}\rangle}, \tag{9}$$

can be given by a log-Poisson distribution.[20] We note that log-Poisson statistics have been reported for a passive scalar in realistic flows.[22]

IV. SUMMARY

We have studied numerically the intermittency problem of a passive scalar advected by a random incompressible velocity field using a two-dimensional lattice model. To investigate the effect of an advecting velocity field that has a finite correlation time, the random stream function is updated only every certain finite time interval. We find that when the diffusion time is comparable to or longer than the update time, the passive scalar exhibits dissipative scaling that could be understood using the perturbation result for the Kraichnan model around a smooth scalar field.

The more intriguing result of our work is the discovery of the intermittency exhibited by the passive scalar field

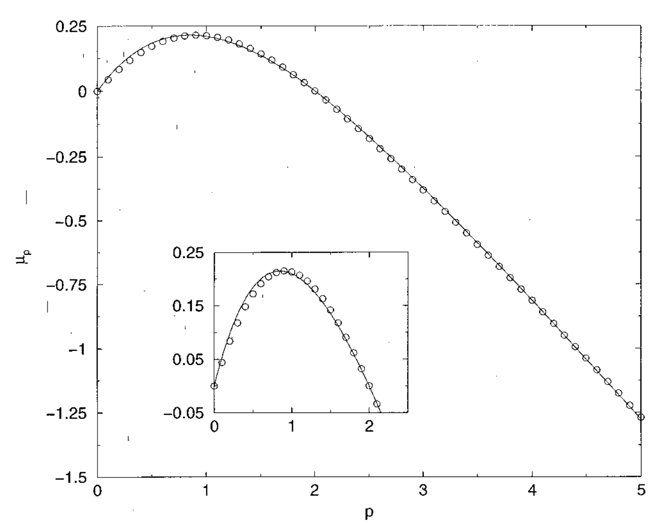

FIG. 7. A plot of μ_p as a function of p for $C=0.1$. The solid line is a fit by the form (7). The fitted values of a and β are, respectively, 1.13 and 0.39. The inset shows an expanded view for $0<p<2$.

when the update time is long compared to the diffusion time. At first sight, this seems paradoxical[23] as diffusion dominates the dynamics of the scalar field between the updates of the velocity field. However, the updates of the velocity field, which instill discontinuous randomness into the dynamics, become rare events in this regime. Interestingly, it turns out that this latter effect leads to an intermittent passive scalar field whose intermittency is described by log-Poisson statistics, and is independent of the ratio of the update time to the advection time. It is interesting to understand this mechanism of generating log-Poisson intermittency and also its relevance to physical turbulent flows. This will be explored and reported in future publications.

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