1. Uniqueness properties of the Kerr space-times

1.1. **Explicit solutions.** The simplest examples of Einstein vacuum space-times which contain a black hole are the *Schwarzschild spaces*. The Schwarzschild spaces are spherically symmetric spaces that depend on a parameter m (the mass of the black hole). In the *domain of outer communication*, the metric can be written explicitly in the form

$$ds^2 = -\big(1 - \frac{2m}{r}\big)dt^2 + \big(1 - \frac{2m}{r}\big)^{-1}dr^2 + r^2(d\theta^2 + (\sin\theta)^2 d\phi^2), \tag{1.1}$$

where $(r, t, \theta, \phi) \in (2m, \infty) \times \mathbb{R} \times (0, \pi) \times \mathbb{S}^1$. The Schwarzschild space of mass m is asymptotically flat (the metric approaches the Minkowski metric as $r \to \infty$) and static (it admits a timelike Killing vector field $\xi = \partial_t$). Moreover, it is well known that the Schwarzschild spaces are (locally) the only spherically symmetric Einstein vacuums.

The apparent singularity in the metric (1.1) at $r = 2m$ is removable (it is a coordinate singularity). In fact, the domain of outer communication described in (1.1) can be isometrically imbedded into a larger, locally inextendible manifold, called the Kruskal extension. The domain of outer communication described in (1.1) is isometric to an open subset of the Kruskal extension; the boundary of this open subset (which corresponds to $r \to 2m$ in (1.1)) is the union of two smooth manifolds on codimension 1. This boundary is called the *event horizon*.

A more general family of vacuum space-times which contain a black hole are the *Kerr spaces*. The Kerr spaces are the only known explicit solutions that model rotating black holes in vacuum. They depend on two parameters: m (the mass of the black hole) and J (the angular momentum of the black hole, which is 0 in the case of the Schwarzschild spaces). We assume $m > 0$ and $a = J/m \in [0, m)$. In standard Boyer-Lindquist coordinates $(r, t, \theta, \phi) \in (r_+, \infty) \times \mathbb{R} \times (0, \pi) \times \mathbb{S}^1$, $r_+ = m + (m^2 - a^2)^{1/2}$, the metric in the domain of outer communication is

$$-\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\Big(d\phi - \frac{2amr}{\Sigma^2}dt\Big)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2, \tag{1.2}$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases} \tag{1.3}$$

The Kerr space of mass m and angular momentum J is asymptotically flat and *stationary*, i.e. it admits a Killing vector field $\xi = \partial_t$ which is timelike in the asymptotic region. The Killing vector field ∂_t, however, is not timelike in the entire domain of outer communication described in (1.2), which is an important difference between the Kerr spaces and the Schwarzschild spaces. The region in which ξ is spacelike is called the *ergoregion*.

As in the case of the Schwarzschild spaces, the apparent singularity in the metric (1.2) at $r = r_+$ is removable. The Kerr domain of outer communication described in (1.2) can be isometrically imbedded into a larger, locally inextendible manifold; the boundary of the domain of outer communication in this larger manifold is called the event horizon of the black hole.

1.2. **Description of the main problem.** A fundamental conjecture in General Relativity asserts that the domain of outer communication of a regular, stationary, four dimensional, vacuum black hole is isometrically diffeomorphic to the domain of outer communication of a Kerr black hole. One expects, due to gravitational radiation, that general, asymptotically flat solutions of the Einstein-vacuum equations[1] settle down, asymptotically, into a stationary regime. Thus the conjecture, if true, would characterize all possible asymptotic states of the general evolution.

So far the conjecture has been resolved, by combining results of Hawking [2], Carter [1] and Robinson [3], under the additional hypothesis of non-degenerate event horizons and *real analyticity* of the space-time. The assumption of real analyticity is both hard to justify on physical grounds and difficult to dispense of. One can show, using standard elliptic theory, that stationary solutions are real analytic in regions where the corresponding Killing vector field ξ is timelike, but there is no reason to expect the same result to hold true in the ergoregion. One of the main steps in the current proof, due to Hawking [2], depends heavily on analyticity. Roughly speaking Hawking's argument is based on the observation that, though a general stationary space may seem quite complicated, its behavior along the event horizon is remarkably simple. Thus Hawking has shown that in addition to the original, stationary, Killing vector field, which has to be tangent to the event horizon, there must exist, infinitesimally along the horizon, an additional Killing vector field. To extend this information, from the event horizon to the domain of outer communication, requires one to solve a boundary value problem, with data on the horizon, for a linear differential equation. Such problems are typically *ill posed*[2]. In the analytic category, however, the problem can be solved by a straightforward Cauchy-Kowalewsky type argument. Thus, by assuming analyticity for the stationary metric, Hawking bypasses this fundamental difficulty, and thus is able to extend this additional Killing field to the entire domain of outer communication. As a consequence, the space-time under consideration is not just stationary but also axi-symmetric, situation for which Carter-Robinson's uniqueness theorem [1], [3] applies. It is interesting to remark that this final step does not require analyticity.

Though ill posed problems do not, in general, admit solutions, one can, when a solution is known to exist, often prove uniqueness. This fact has led us to develop a different strategy for proving uniqueness based on a characterization of the Kerr solution, due to Mars, and geometric Carleman estimates applied to covariant wave equations on a general, stationary, black hole background. Our main result (joint work with S. Klainerman) proves uniqueness of the Kerr family among all, smooth, appropriately regular, stationary solutions, with a regular, bifurcate, event horizon, under an additional assumption which has to be satisfied along the bifurcate sphere S_0 of the event horizon. More precisely we

[1] A similar scenario is supposed to hold true in the presence of matter.

[2] Thus solutions may fail to exist.

assume a pointwise complex scalar identity relating the Ernst potential σ and the Killing scalar $\mathcal{F}^2$ on the bifurcate sphere S_0.

References

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