Scalar decay in smooth random flows

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with P H Haynes (Cambridge)

Scalar decay

Concentration $C(\mathbf{x},t)$ of a passive scalar released in a flow obeys

$$\partial_t C + \mathbf{v}\cdot\nabla C = \kappa\Delta C, \quad \nabla\cdot\mathbf{v} = 0.$$

Focus on spatially smooth flows: $|\mathbf{v}(\mathbf{x},t) - \mathbf{v}(\mathbf{0},t)| \sim |\mathbf{x}|$ for small $|\mathbf{x}|$ at the scale of concentration fluctuations.

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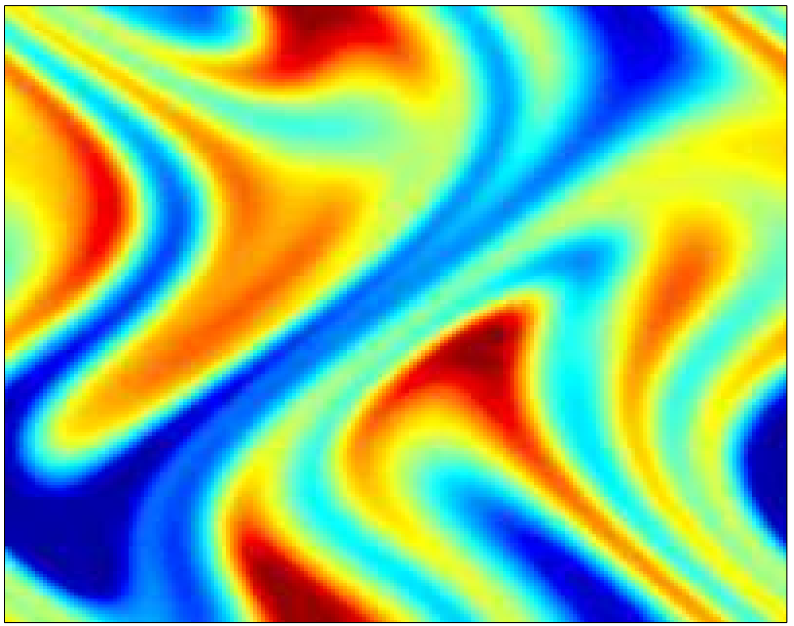
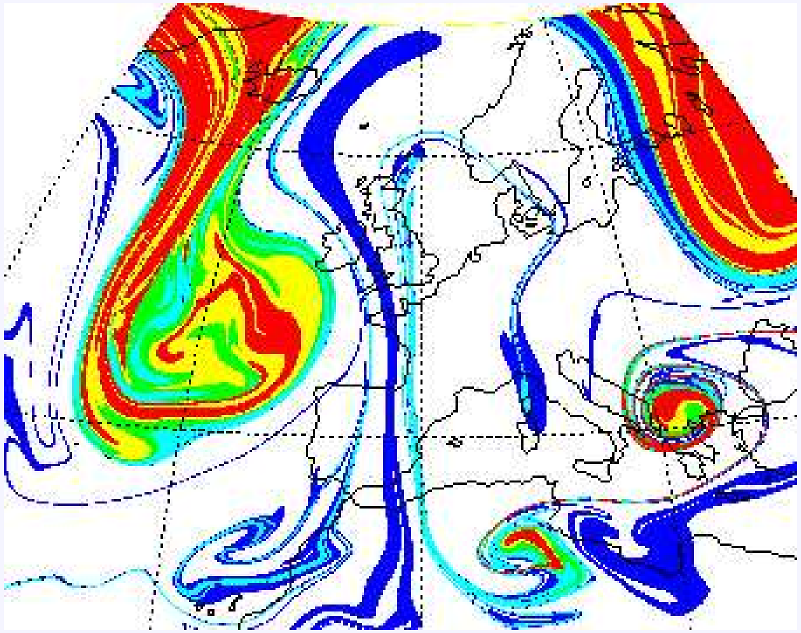
Large-scale-dominated flows:

- Batchelor regime: $\nu \gg \kappa$;
- Steep energy spectrum: $E(k) = o(k^{-3})$ as $k \to \infty$;
- 2D turbulence (enstrophy cascade);
- Large-scale geophysical flows.

But not 3D turbulence ($E(k) \sim k^{-5/3}$).

Scalar decay





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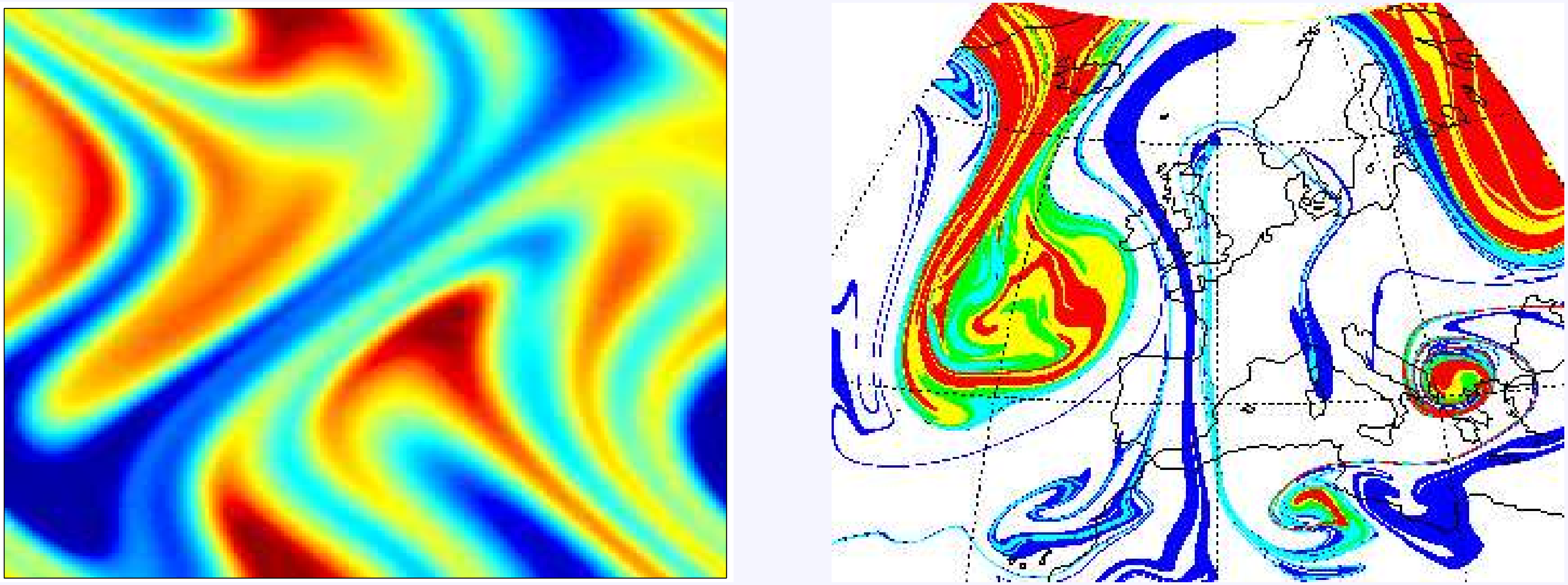

Assume $\bar{C} = 0$; in bounded domains, C decays exponentially as $t \to \infty$:

$$C \sim \exp(-\gamma_C t), \quad \gamma_C > 0.$$

What controls γ_C, in particular for $\kappa \to 0$?

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For mixing flows, $\gamma_C \to \gamma_C^0 \neq 0$ as $\kappa \to 0$.

What controls γ_C^0? What is the diffusive correction for $\kappa \neq 0$?

Scalar decay

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Here: we show that either case is possible, depending on flow and domain size.
Focus on variance decay rate γ_2,

$$\langle \overline{C^2} \rangle \sim \exp(-\gamma_2 t)$$

(In numerical simulations, it appears that $\gamma_2 \approx 2\gamma_C$.)

Lagrangian stretching

The distance $\mathbf{d}$ between nearby trajectories in $\mathbf{v}$ evolves according to

$$\dot{\mathbf{d}} = \mathbf{d} \cdot \nabla \mathbf{v},$$

and grows exponentially for $t \to \infty$.

The growth is characterized by finite-time stretching factors

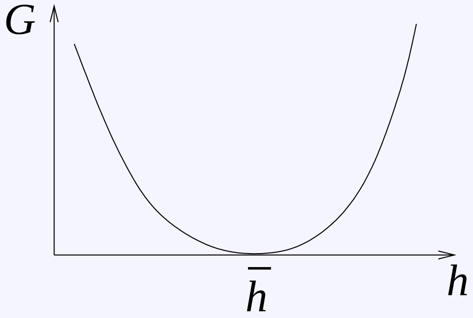
$$h := \frac{1}{t} \log \frac{d(t)}{d(0)}, \quad \text{where} \quad d = |\mathbf{d}|.$$

The pdf of h has the large-deviation approximation,

$$p(h;t) \propto \exp[-tG(h)],$$

for some Cramer function $G(h)$.

Lagrangian stretching



(Stretching) Liapunov exponent $\bar{h}$ satisfies $G(\bar{h}) = G'(\bar{h}) = 0$.

Moments of d evolve according to

$$\langle d^m \rangle = \int_0^\infty \mathrm{e}^{mht} p(h;t)\,\mathrm{d}h \propto \exp[F(m)t]$$

where the free energy

$$F(m) = \sup_h [mh - G(h)].$$

(Intermittency: $F(m) \neq m\bar{h}$.)

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Lagrangian stretching

Regard C as a superposition of simple functions, experiencing different stretching histories with pdf $p(h;t)$, then

$$\langle C^2\rangle \propto \int_0^\infty \mathrm{e}^{-ht} p(h;t)\,\mathrm{d}h \propto \int_0^\infty \mathrm{e}^{-t[h+G(h)]}\,\mathrm{d}h.$$

(Antonsen et al, Balkovsky & Fouxon).

Approximating the integral gives the decay rate:

$$\gamma_2^{\mathrm{L}} = \begin{cases} -F(-1) & \text{if } \; F'(-1) \geq 0 \\ G(0) & \text{if } \; F'(-1) \leq 0 \end{cases}$$

Attempts to confirm this result numerically have given ambiguous results.

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Covariance equation

Consider classes of flows for which the covariance

$$\Gamma(\mathbf{x},t) := \langle C(\mathbf{x}+\mathbf{y},t)C(\mathbf{y},t)\rangle$$

satisfies a closed equation.

- Kraichnan flows: short correlation time, white noise;
- renewing flows: complete decorrelation in finite time, independent random maps.

The results are identical in both cases. Describe the Kraichnan case:

$$\langle v_i(\mathbf{x},t)v_j(\mathbf{x}',t')\rangle = 2B_{ij}(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

Decay in Kraichnan flows

The covariance satisfies the linear equation

$$\partial_t \Gamma = 2\kappa \Delta \Gamma + 2 D_{ij}(\mathbf{x}) \partial^2_{ij} \Gamma, \quad D_{ij}(\mathbf{x}) := B_{ij}(0) - B_{ij}(\mathbf{x}).$$

The variance decay rate γ_2 is the smallest eigenvalue λ of

$$2 \left[\kappa \Delta + D_{ij}(\mathbf{x}) \partial^2_{ij} \right] \Gamma = -\lambda \Gamma$$

Study the spectrum of a linear operator.

Boundary conditions: doubly periodic domain $[0, 2P\pi] \times [0, 2P\pi]$.

Decay in Kraichnan flows

$$2\left[\kappa\Delta + D_{ij}(\mathbf{x})\partial^2_{ij}\right]\Gamma = -\lambda\Gamma$$

For $\kappa = 0$, the operator is singular at $\mathbf{x} = 0$:

- continous spectrum $[\ell, \infty)$, with ℓ determined by $D_{ij}(\mathbf{x})$ as $\mathbf{x} \to 0$;

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Two different regimes of variance decay:

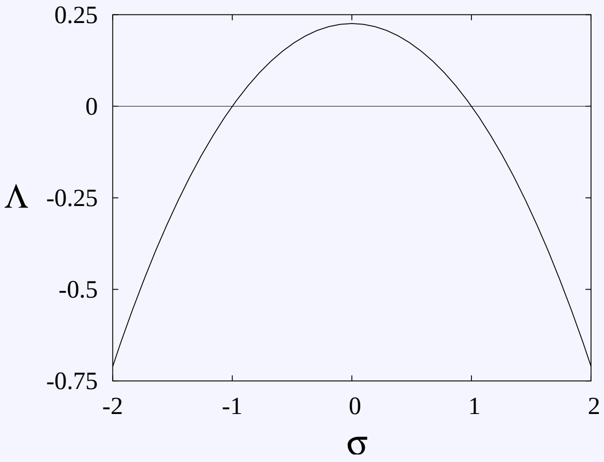
- Local control: in the absence of discrete eigenvalue, $\gamma_2 \approx \ell$;
- Global control: in the presence of discrete eigenvalues $\lambda < \ell$, $\gamma_2 \approx \lambda$.

Local eigenvalue problem

For $\kappa = 0$ and $\mathbf{x} \to 0$, the eigenvalue problem reduces to

$$S_{ijkl} x_k x_l \, \partial^2_{ij} \Gamma = \lambda \Gamma, \quad S_{ijkl} := -\, \partial^2_{kl} D_{ij}(\mathbf{x})\big|_{\mathbf{x}=0},$$

and admits simple solutions $\Gamma(\mathbf{x}) = r^{\sigma-1} f_\sigma(\theta)$.



This defines a real function $\Lambda(\sigma) = \lambda$.

For $\lambda > \Lambda_{\text{max}}$, solution oscillates as $\mathbf{x} \to 0$: the limit of the continuous spectrum is $\ell = \Lambda_{\text{max}}$.

In 2D, $\Lambda(\sigma) = \Lambda(-\sigma)$, hence $\ell = \Lambda(0)$.

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Matched asymptotics

For $\kappa \neq 0$, matched asymptotics can be used to find approximations to the eigenvalues.
Near ℓ, there is an eigenvalue

$$\lambda = \ell - \frac{2\pi^2 \Lambda''(0)}{\log^2 \kappa} + o(1/\log^2 \kappa).$$

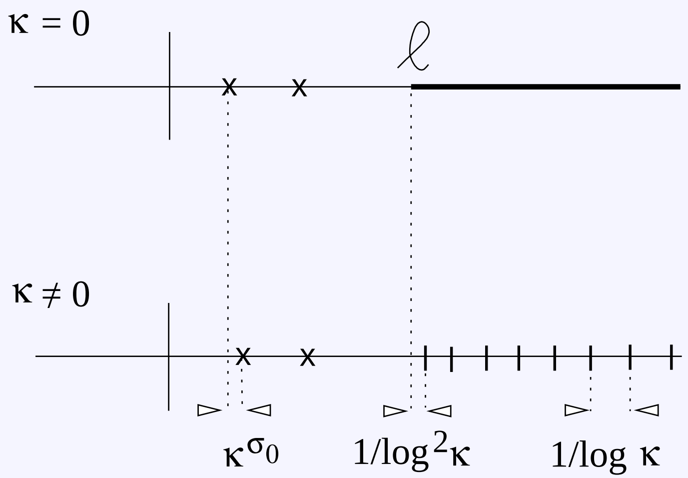
Other eigenvalues emerging from the continuous spectrum are separated by $O(1/\log \kappa)$ gaps.

Near discrete eigenvalues λ_0, the eigenvalues for $\kappa = 0$ are

$$\lambda = \lambda_0 + c\kappa^{\sigma_0} + o(\kappa^{\sigma_0}),$$

where $\lambda_0 = \Lambda(\sigma_0)$.

Spectrum



- **Local control**: no isolated discrete eigenvalues, $\gamma_2 \approx \ell$;
- **Global control:** isolated eigenvalues, $\gamma_2 \approx \lambda_0 < \ell$.

Expect local control when domain size $\approx$ flow size, global control when domain size $\gg$ flow size.

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Connection with Lagrangian stretching theories

$$\gamma_2^{\mathrm{L}} = \begin{cases} -F(-1) & \text{if } F'(-1) \geq 0 \\ G(0) & \text{if } F'(-1) \leq 0 \end{cases}$$

The Fokker–Planck equation for $\mathbf{d}$ is identical to the local eigenvalue problem for Γ: as a result

$$F(m) = -\Lambda(-m-1).$$

It follows that

$$\ell = G(0).$$

In 2D, $\Lambda(\sigma) = \Lambda(-\sigma)$ implies $F(m) = F(-m-2)$ and

$$\ell = G(0) = -F(-1).$$

Lagrangian stretching theories correctly predict γ_2 in the locally controlled case.

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Example

We take $\mathbf{v} = (-\psi_x, \psi_y)$, with

$$\psi = \cos(x + \phi_1(t)) + \cos(y + \phi_2(t)),$$

and ϕ_i independent, short-correlation phases.
The corresponding eigenvalue problem

$$-2\kappa\Delta\Gamma - 4\left[\sin^2(y/2)\partial^2_{xx} + \sin^2(x/2)\partial^2_{yy}\right]\Gamma = \lambda\Gamma$$

can be solved numerically for small κ for different domain sizes P.
The local version

$$-\left(y^2\partial^2_{xx} + x^2\partial^2_{yy}\right)\Gamma = \lambda\Gamma$$

and solutions $r^{\sigma-1}f_\sigma(\theta)$ gives $\Lambda(\sigma)$, with $\ell = \Lambda(0) = 0.226$.
A Rayleigh quotient argment indicates that there are isolated discrete eigenvalues $\lambda_0 < 2/P^2$: global control for $P \geq 3$.

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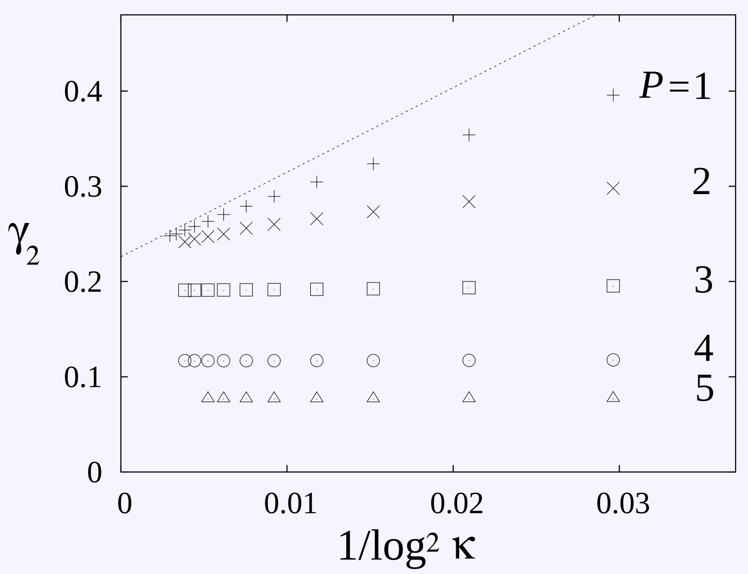
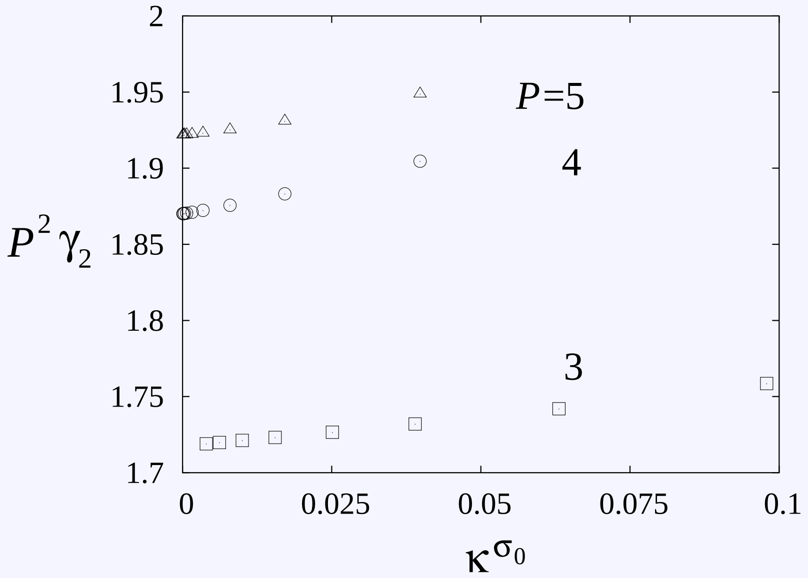
Decay rate γ_2 as a function of κ:





- **Local scaling**: $\gamma_2 \approx 0.226 + 0.88/\log^2\kappa$;
- **Global scaling**: $\gamma_2 \approx \lambda_0 + c\kappa^{\sigma_0}$.
- **Homogenization**: $\gamma_2 \approx 2/P^2 + c\kappa$.

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- The different control mechanisms can be identified with different parts of spectrum of 'transfer operator' for $\kappa = 0$.
- Results obtained for Kraichnan flows and renewing flows: presumably also hold for flows with finite correlation time.
- In the locally controlled case, the convergence to zero diffusivity limit is slow ($1/\log^2\kappa$).

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  $$\gamma_n \approx n\gamma_C.$$

  Equivalently, what is the distribution of finite-time decay rates $\gamma_t = \frac{1}{t}\log(||C(t)||/||C(0)||)$?
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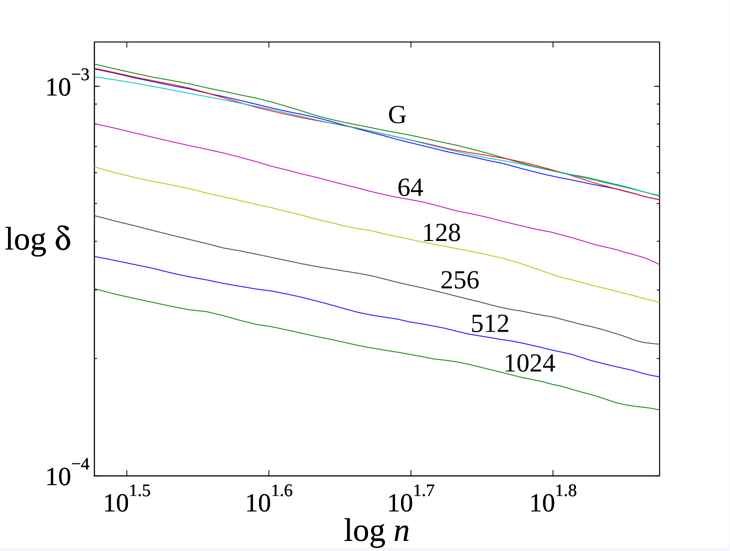

Open issues

- Conditions on $\mathbf{v}$ for $\gamma_C \to \gamma_0 \neq 0$ as $\kappa \to 0$?
- Relationship to ergodic theory for $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t) + \sqrt{2\kappa}\dot{\mathbf{W}}$.
- Scalar decay in flows with transport barriers (invariant curves):

$$\gamma_C \to 0 \quad \text{as} \quad \kappa \to 0.$$

Scaling?