Wave radiation by slow flows

The breakdown of balance in a toy model of the atmosphere

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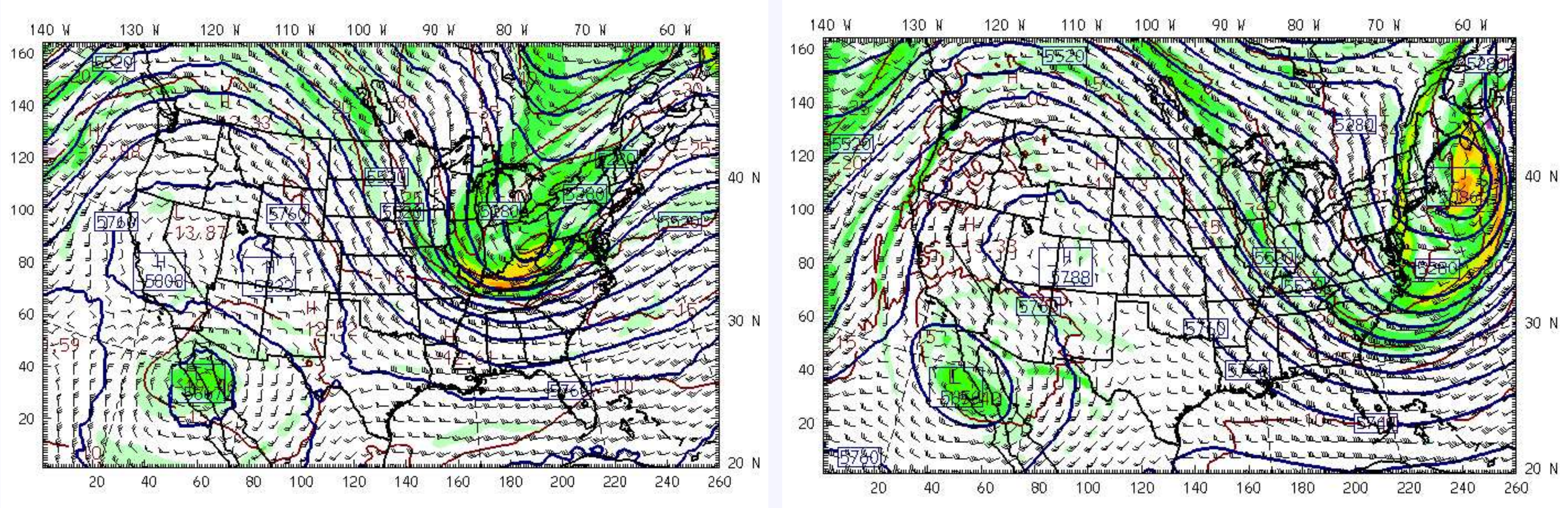

Weather maps at 1 day interval

Balance and the slow manifold (continued)

Mid-latitude atmosphere: rapidly rotating, strongly stratified fluid,

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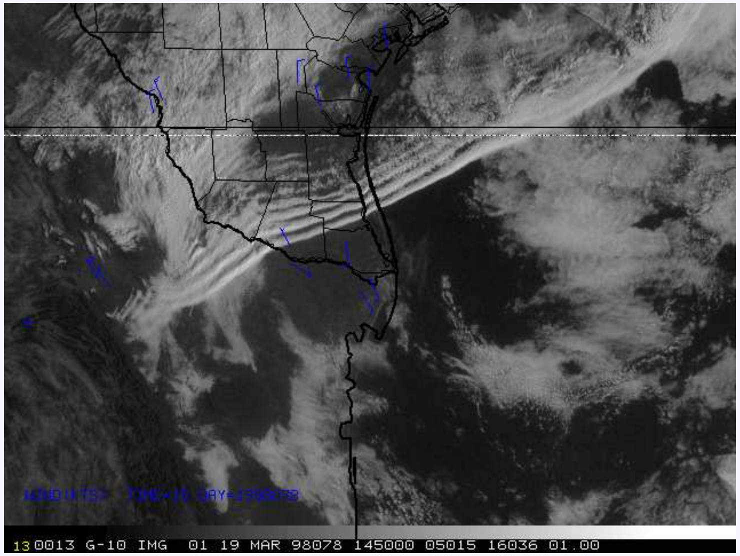

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$$\delta = \frac{T_{\mathrm{igw}}}{T_{\mathrm{a}}} \ll 1$$

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Dimensionless PDEs: with T_{a} as reference time,

$$\frac{\partial s}{\partial t} = N_s(s, f; \delta), \qquad \frac{\partial f}{\partial t} + \mathcal{L} f = N_f(s, f; \delta)$$

with $\operatorname{spec} \mathcal{L} = \{\mathrm{i}\omega : \omega \in \mathbf{R}, |\omega| > \delta^{-1}\}$.

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- ○ slow (nonlinear) dynamics, s, $T = O(1)$,
- ○ fast inertia-gravity waves, f, $T < O(\delta)$.

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Observations:

- ◦ fast oscillations are weak, $f \ll s$

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Balance and slow manifold

Reduce the dynamics to a slow manifold: submanifold of state space which is approximately invariant and on which the motion is slow (van Kampen 1985, MacKay 2004).

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At higher orders: find the slow manifold by slaving

$$f = F(s) = \delta F_1(s) + \delta^2 F_2(s) + \cdots,$$

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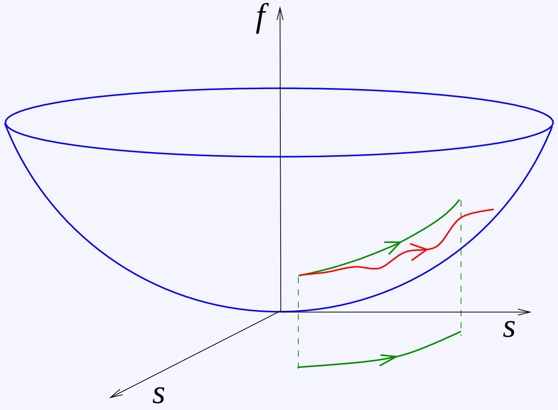

Balance, slow manifold (continued)

Interest:

- Balanced models: describe dynamics on a slow manifold, with reduced dimension ($\dim s = \dim f\ /2$) and large time steps,

$$\frac{\partial s}{\partial t} = N_s(s, F(f, \delta), \delta)$$

(quasi-, semi-geostrophic, L1, BE...)

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Questions:

- Can balance be perfect? Is there an invariant slow manifold?
- If not, describe the spontaneous generation of inertia-gravity waves. What is the wave amplitude?

Lorenz’s 5-component model

Use toy ODE model to examine these questions.

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Lorenz’s less-famous equations:

Shallow-water on the sphere, spectral expansion, truncation to 3 modes → 9-component model

Neglect 2 IGWs → 5-component model (Lorenz 1986):

$$\begin{aligned} \dot{u} &= -vw + bvy \\ \dot{v} &= uw - buy \\ \dot{w} &= -uv \\ \delta\dot{x} &= -y \\ \delta\dot{y} &= x + \delta buv \end{aligned}$$

Slow variables: $s = (u, v, w)$ Rossby-wave triad

Fast variables: $f = (x, y)$ inertia-gravity-wave pair.

Lorenz’s 5-component model (continued)

2 parameters:

- $b = f_C L/\sqrt{gH}$ = rotational Froude number,
- $\epsilon = U/(f_C L)$ = Rossby number.

They combine in $\delta = T_{\mathrm{a}}/T_{\mathrm{igw}} = \epsilon/\sqrt{1 + 1/b^2}$ = (oscillation frequency)$^{-1}$.

(Compare with shallow-water dispersion relation: $\omega = \epsilon^{-1}\sqrt{1 + k^2/b^2}$.)

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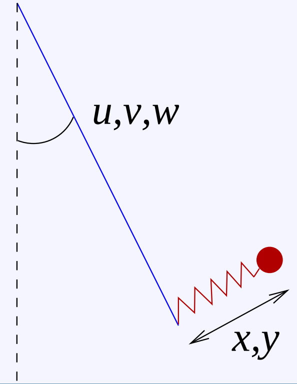
Lorenz's model: a pendulum coupled to a stiff spring



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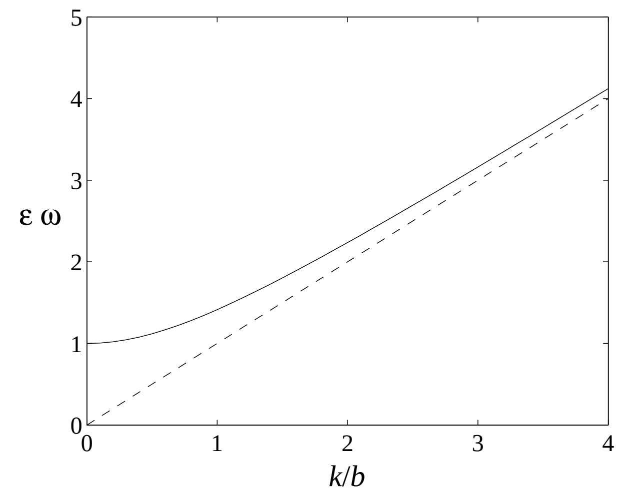
- These oscillations have exponentially small amplitudes $\asymp \exp(-\alpha/\delta)$, and appear through a Stokes phenomenon.
- They can be captured by exponential asymptotics (V 2004, V & Yavneh 2004, Olfasdottir, Olde Daalhuis & V 2005).

Lorenz's 5-component model (continued)

The reduction from PDEs to ODEs has some drawbacks:

- ODEs cannot describe wave generation in the small-Froude number regime $b \ll 1$, $\epsilon = O(1)$.

  It is caused by resonance of large-scale waves, with $k = O(b)$: Lighthill radiation (also gravitational waves)



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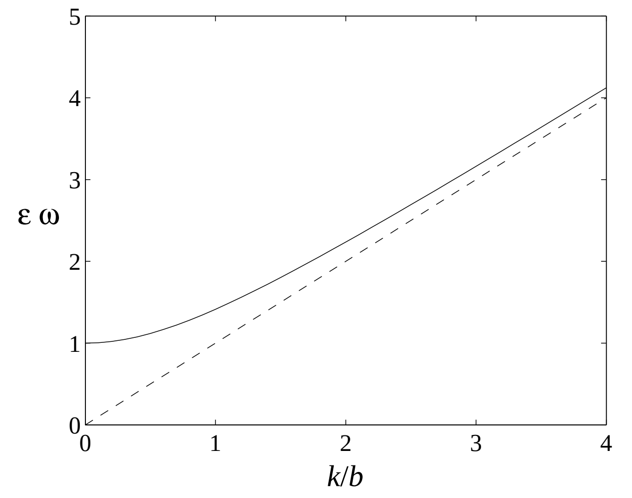

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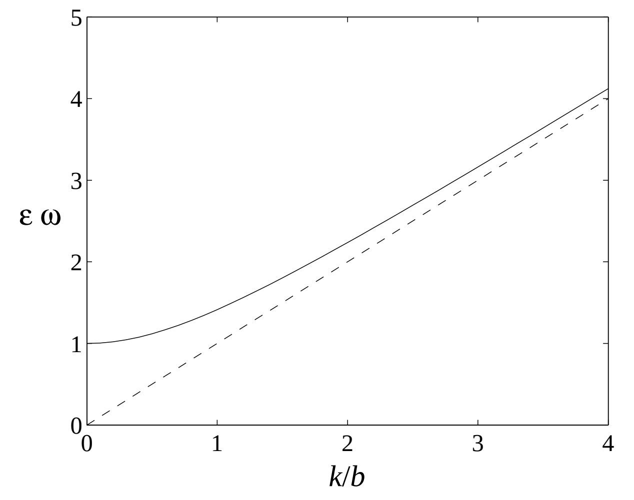

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Introduce a new toy model that retains a PDE component

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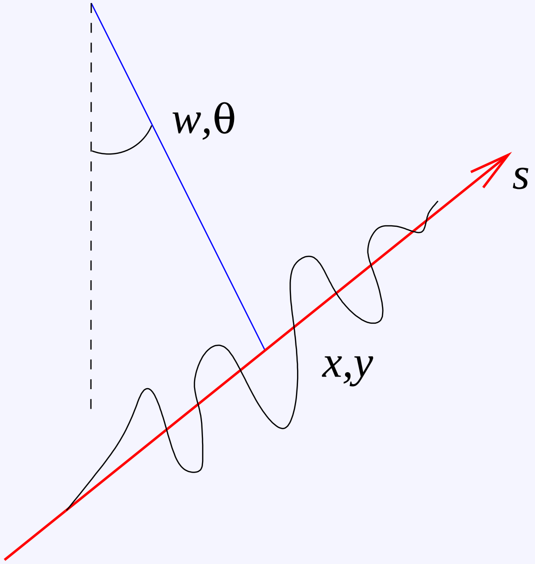
Extended Lorenz model

Keep the pendulum for the slow variables, replace the oscillator by a 1D linear Klein–Gordon equation, with dispersion relation

$$\omega^2 = \epsilon^{-2}(1 + k^2/b^2).$$

The equations of motion are:

$$\begin{aligned}
\dot{\theta} &= w \\
\dot{w} &= -\sin\left[2\theta + 2\epsilon \int f(s)x(s,t)\,\mathrm{d}s\right] \\
\epsilon x_t &= -y \\
\epsilon y_t &= x - x_{ss}/b^2 \\
&\quad +\epsilon f(s)\sin\left[2\theta + 2\epsilon \int f(s)x(s,t)\,\mathrm{d}s\right]
\end{aligned}$$



ODE/PDE coupling through the function $f(s) = \dfrac{a}{(2\pi)^{1/2}}\dfrac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-s^2/2}$.

Extended Lorenz model

The model is Hamiltonian, with energy

$$H = \frac{w^2}{2} - \frac{1}{2}\cos\left[2\theta + 2\epsilon \int f(s)x(s,t)\,\mathrm{d}s\right] + \frac{1}{2}\int \left(x_s^2/b^2 + x^2 + y^2\right)\mathrm{d}s$$

and symplectic form $\Omega = \mathrm{d}\theta \wedge \mathrm{d}\omega + \epsilon \mathrm{d}y \wedge \mathrm{d}x$.

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Use the model to study wave generation in the regimes:

- $b \ll 1,\ \epsilon = O(1)$: Lighthill-type radiation,
- $\epsilon \ll 1,\ b = O(1)$: exponentially small radiation

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Small-Froude-number regime

For $b \ll 1$, long waves with $k = O(b)$ are slow, and generated resonantly.

We can use matched asymptotics to derive

- the form of the long waves generated (cf. Einstein 1918, Lighthill 1952),
- a post-balanced model: closed, dissipative model for (θ, w) that accounts for wave radiation (cf. Blanchet 2002, Ford, McIntyre & Norton 2000).

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To leading order, the waves satisfy the linear Klein–Gordon equation

$$\epsilon^2 x_{tt} - x_{SS} + x = ab^2 \epsilon \delta'(S) \sin(2\theta), \quad \text{with} \quad S = bs$$

Dipolar generation, leading to $O(b^2)$ waves.

Small-Froude-number regime (continued)

The wave feedback appears at $O(b^3)$, leading to the post-balanced model

$$\begin{aligned} \dot{\theta} &= w \\ \dot{w} &= -\sin\left[2\theta - b^2 a^2 \epsilon \sin(2\theta)/\pi^{1/2} - 2b^3 a \epsilon B(t)\right] \end{aligned}$$

with $B(t) = -a\epsilon^2 w \cos(2\theta) - \dfrac{a\epsilon J_1(t/\epsilon)}{2t} \star \sin(2\theta)$.

The post-balanced model consists of integro-differential equations; it reduces to differential equations for non-dispersive waves (in odd dimensions).

Compare numerical solutions for initial condition on unperturbed ($b = 0$) separatrix:

$$\cos\theta = \operatorname{sech}\left[\sqrt{2}(t - t_0)\right] \quad \text{and} \quad \sin\theta = -\tanh\left[\sqrt{2}(t - t_0)\right]$$

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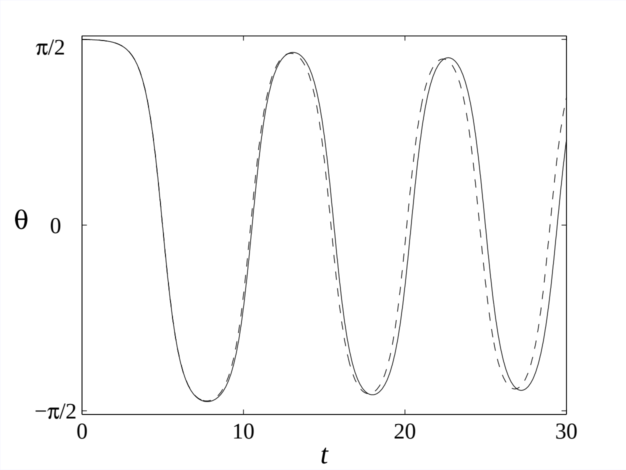
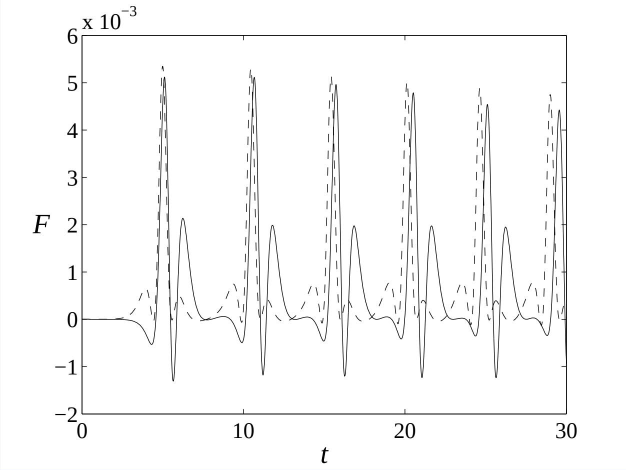
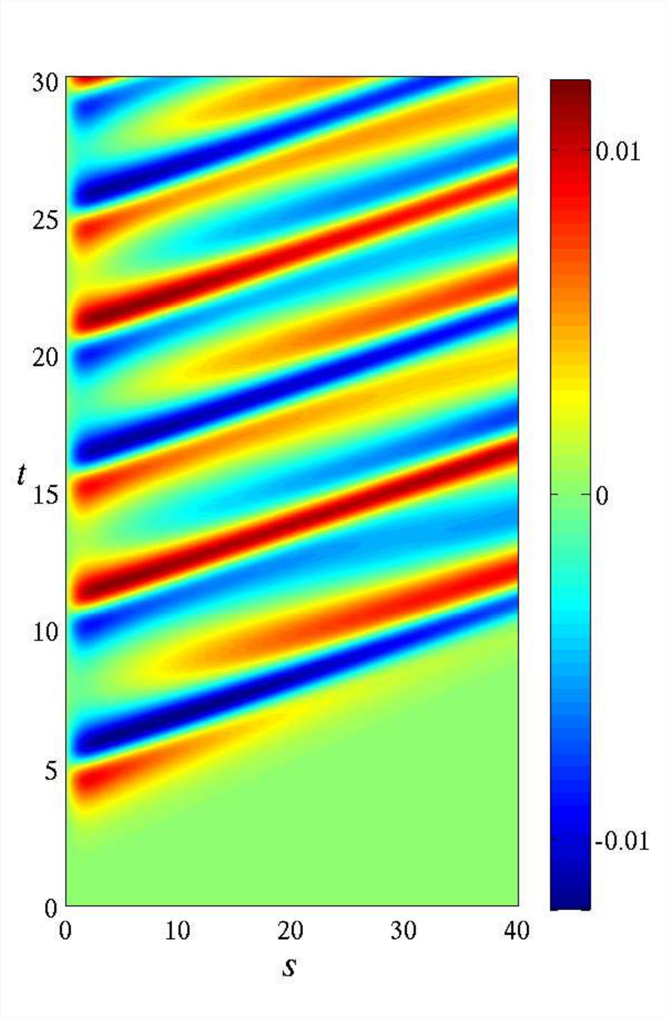
Small-Froude-number regime (continued)







Results for $b = 0.15,\ a = \epsilon = 1,\ t_0 = 5$.

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Small-Rossby-number regime

For $\epsilon \ll 1$, a slow manifold can be found to any accuracy $O(\epsilon^N)$ by slaving:

$$x_{\mathrm{bal}}(s,\theta,w) = \sum_{n=0}^{\infty} \epsilon^{2n+1} x_{\mathrm{bal}}^{(n)}(s,\theta,w),$$

with $x_{\mathrm{bal}}^{(0)} = -\frac{ab^2 \mathrm{e}^{b^2/2}}{4}\left[\mathrm{e}^{bs}\mathrm{erfc}\,\frac{b+s}{\sqrt{2}} - \mathrm{e}^{-bs}\mathrm{erfc}\,\frac{b-s}{\sqrt{2}}\right]\sin(2\theta)$, etc.

Can add wave solution:

$$x_{\mathrm{w}} = \int C(k)\exp[\mathrm{i}(ks-\omega t)]\,\mathrm{d}k, \quad \text{with} \quad \omega^2 = (1+k^2/b^2)/\epsilon^2 \gg 1.$$

but interest in balanced solutions: $C(k) = 0$ at $t = 0$.

No wave generation visible for $t \in \mathbb{R}$: $C(k) = o(\epsilon^N), \forall N$.

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Small-Rossby-number regime (continued)

For $t \in \mathbb{C}$, near singularities of $(\theta(t), w(t))$, the slow-manifold expansion breaks, and wave generation appears at leading order.

Examine equations near singularities, and match (Kruskal–Segur): $C(k)$ jumps from 0 to exponentially small values when $t \in \mathbb{R}$ crosses Stokes lines ($\theta = 0$).

Inner problem: for $\tau = (t - t_*)/\epsilon = O(1)$, expand

$$w = \epsilon^{-1} \sum_{n=1}^{\infty} \frac{w_n}{\tau^{2n-1}}, \quad x = \epsilon^{-1} \sum_{n=1}^{\infty} \frac{X_n(s)}{\tau^{2n}}, \quad \text{etc.}$$

to find recurrence relations for w_n, $X_n(s)$, etc. For $n \gg 1$, observe that

$$\hat{X}_n \sim \frac{(-1)^{n+1}(2n-1)!\lambda(k)}{2\pi\Omega^{2n}} k \mathrm{e}^{-k^2/2}, \quad \Omega^2 = 1 + k^2/b^2.$$

for some $\lambda(k)$, provided that $k \ll 1$.

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Small-Rossby-number regime (continued)

Borel-summation of the divergent series

$$\hat{x} = \epsilon^{-1} \sum_{n=1}^{\infty} \frac{\hat{X}_n(k)}{\tau^{2n}},$$

shows that the late behaviour is associated with the wave part

$$\hat{x}_{\mathrm{w}}(k,t) \sim \frac{-\mathrm{i}k\lambda(k)}{\epsilon}\,\mathrm{e}^{-\alpha\omega/\epsilon - k^2/2}\cos[\omega(t-\beta)/\epsilon], \quad k \ll 1,$$

with $\alpha = \Im t_* > 0$ and $\beta = \Re t_*$.

Inverting the Fourier transform by steepest descent gives

$$x_{\mathrm{w}}(s,t) \sim (2\pi)^{1/2} b^3 \lambda(0)\,\mathrm{e}^{-\alpha/\epsilon} S\,\Re \frac{\mathrm{e}^{\mathrm{i}(t-\beta)/\epsilon - b^2 S^2/[2(\alpha - \mathrm{i}(t-\beta))]}}{[\alpha - \mathrm{i}(t-\beta)]^{3/2}},$$

for $S = \epsilon^{1/2} s = O(1)$.

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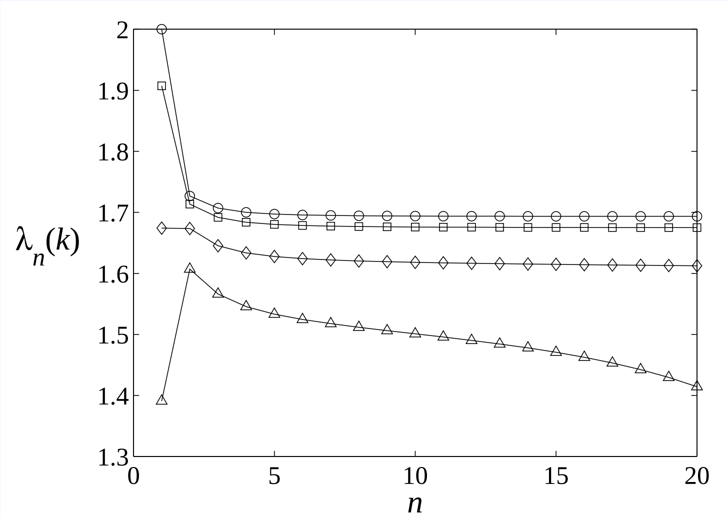
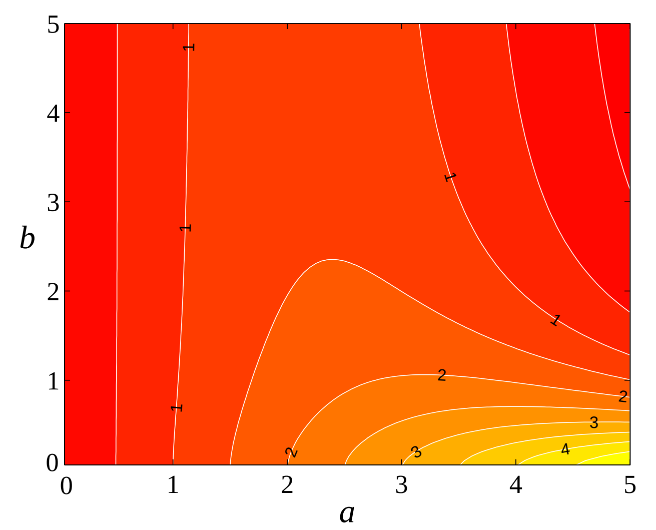
Small-Rossby-number regime (continued)

Expression for the exponentially small waves generated by the balanced motion when a Stokes line is crossed.
It depends on a single parameter $\lambda(0)$ to be determined numerically for each a and b. This is done by solving recurrence relations for $\hat{X}_n(k)$ discretised (Gauss–Hermite), and

$$\frac{(-1)^{n+1} 2\pi k \Omega^{2n}}{(2n-1)!} \hat{X}_n \mathrm{e}^{k^2/2} \to \lambda(k) \quad \text{as} \quad n \to \infty.$$

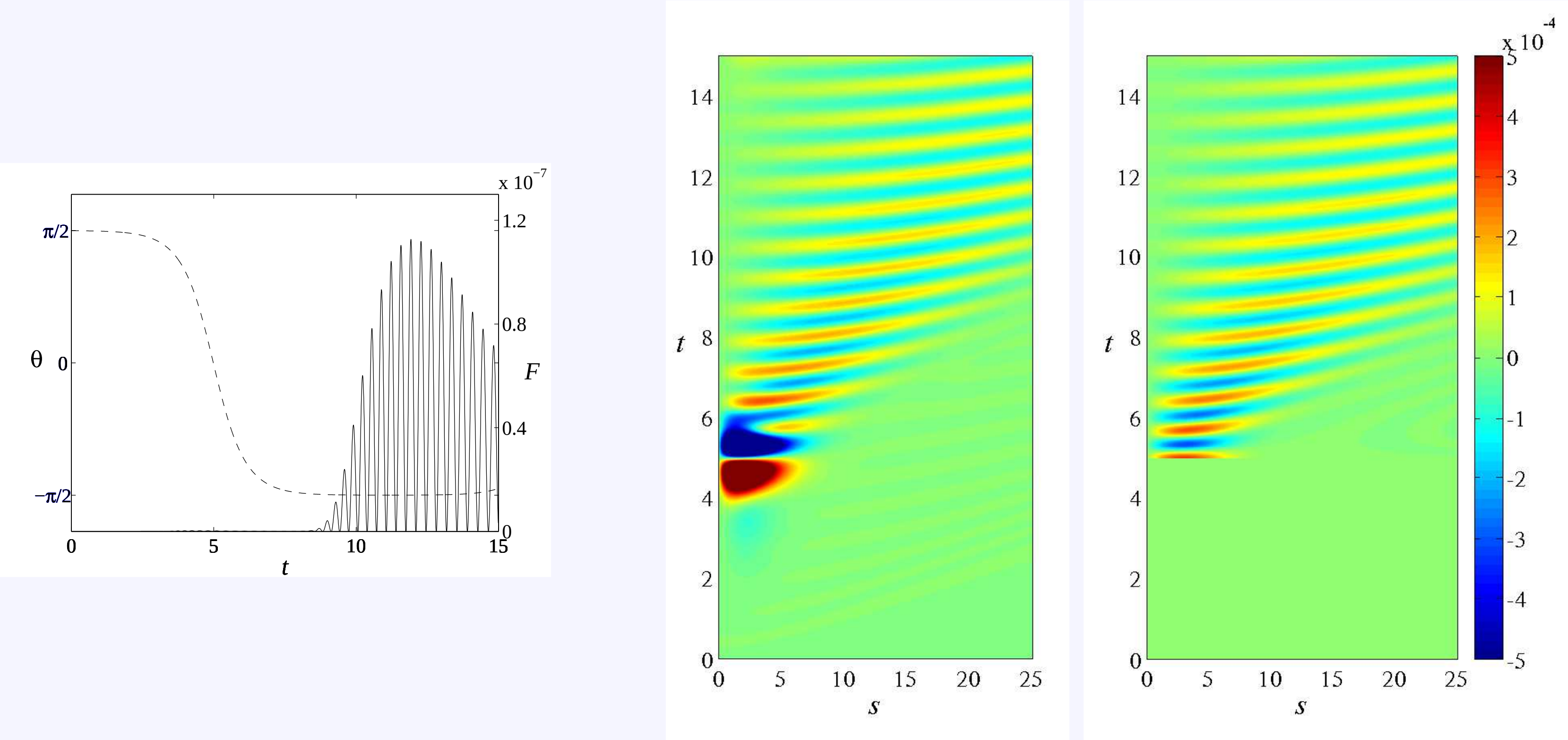




For $a \ll 1$ or $b \ll 1$, $\lambda(0) \sim a$: weak coupling limits.

Small-Rossby-number regime (continued)

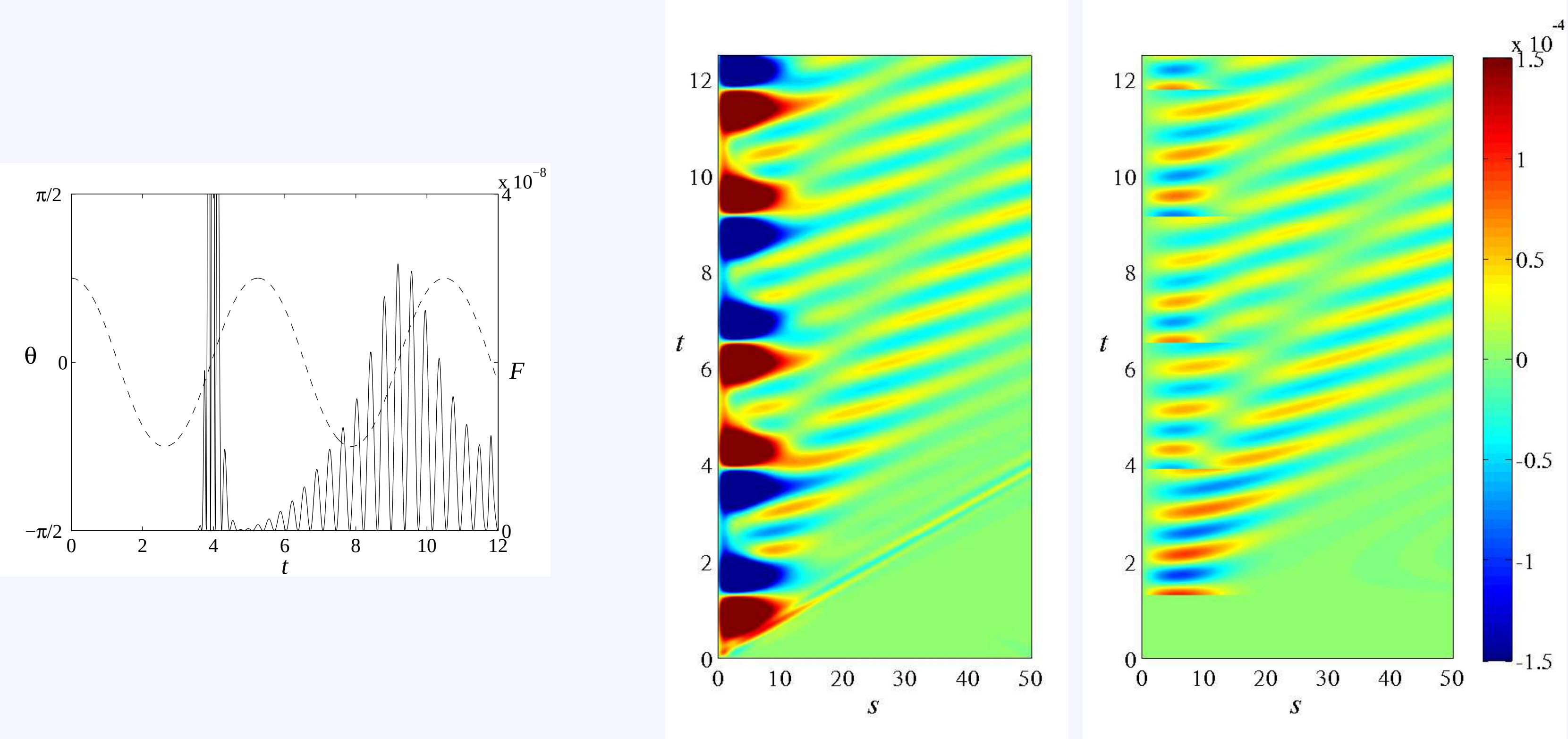
Comparison of asymptotic results with numerical simulations: initial conditions on unperturbed ($\epsilon = 0$) separatrix.



Plots of $\theta(t)$, $F(t)$ for $s = 25$, $x(s,t) - x_{\mathrm{bal}}^{(0)}(s, \theta(t)) = O(\epsilon^3)$ and $x_{\mathrm{w}}(s,t)$, for $\epsilon = 0.15$, $a = 2$ and $b = 1$.

Small-Rossby-number regime (continued)

Initial conditions on unperturbed ($\epsilon = 0$) periodic orbit $\theta(0) = \pi/4$, $w(0) = 0$.



Plots of $\theta(t)$, $F(t)$ for $s = 25$, $x(s,t) - x^{(0)}_{\mathrm{bal}}(s, \theta(t)) = O(\epsilon^3)$ and $x_{\mathrm{w}}(s,t)$ for $\epsilon = 0.125$, $a = 2.5$ and $b = 0.5$.

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- A toy model to study wave generation by slow motion
  - with time-scale separation for $k = O(1)$ but not $k = O(b)$ (acoustic waves, gravitational waves),
  - with time-scale separation for all k: exponentially small radiation.

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