Two-dimensional flows in slowly deforming domains

Adiabatic invariance and geometric angle

J. Vanneste

School of Mathematics

University of Edinburgh, UK

www.maths.ed.ac.uk/˜vanneste

with D. Wirosoetisno

University of Durham, UK

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Motivation: effect of slow, conservative perturbations on perfect fluids

Simplest model:

- 2D, incompressible fluid

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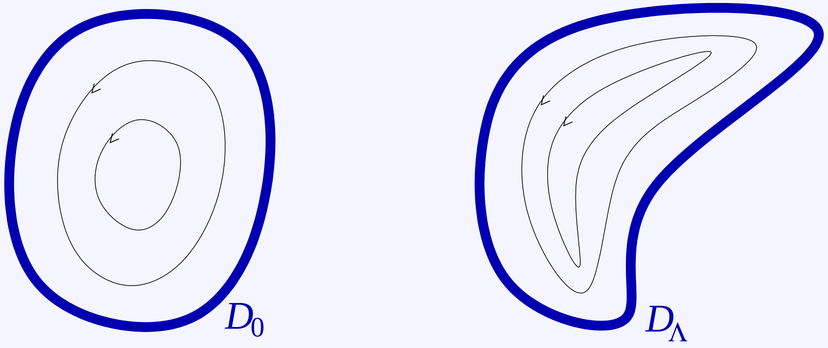

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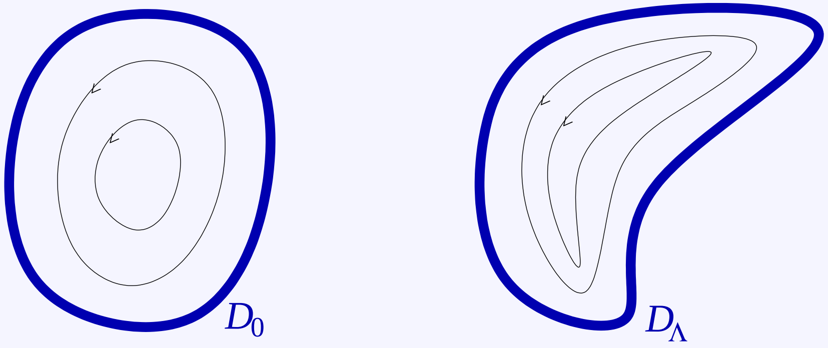


- 2D, incompressible fluid
- slowly deforming domain
- (almost) steady flow

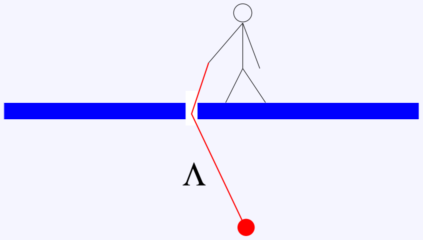
Two issues:

- Eulerian flow $u(x,t)$
- (Lagrangian) fluid-particle positions

for $O(1)$ deformations of the domain.

Flow in slowly deforming domain

Finite-dimensional analogue: Hamiltonian system with slowly variable parameters, e.g. pendulum with length $\Lambda(\epsilon t), \ \epsilon \ll 1$.



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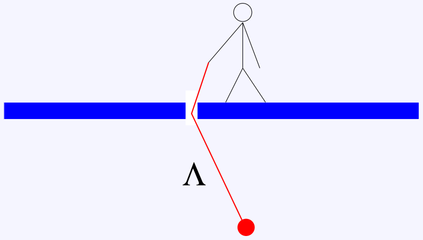

Amplitude (energy):

- determined by the adiabatic invariance of action I, $\Delta I = O(\epsilon)$.
- adiabatic invariance of I stems from invariance of $p\mathrm{d}q$.
- energy depends on Λ instantaneously.

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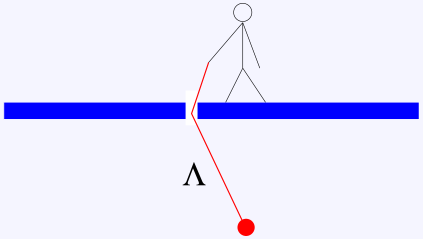

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Angle $\langle\theta\rangle$:

- dynamical angle $\int \omega\,\mathrm{d}t$
- + Hannay–Berry (geometric) angle
- depends on path in parameter space

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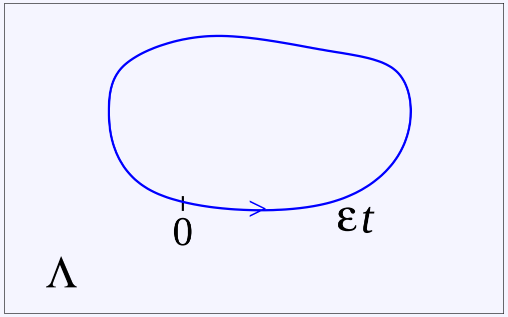
Flow in slowly deforming domains



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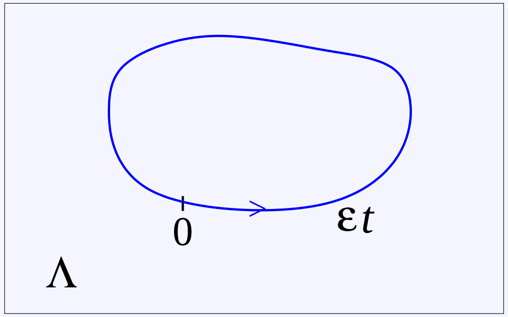

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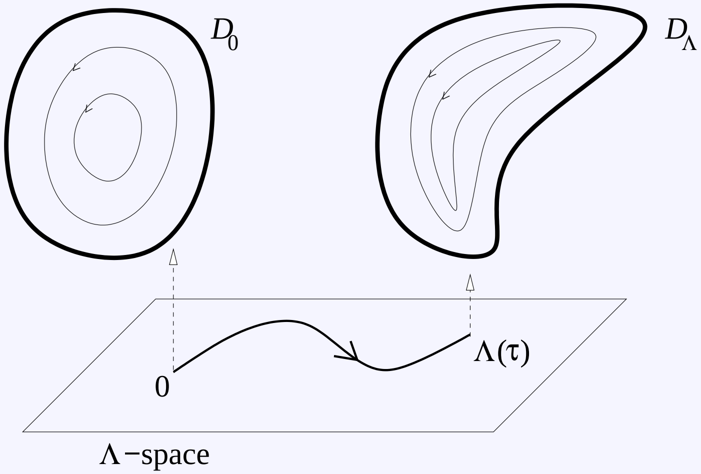
Analogy f-d system/fluid:

f-d system	fluid
amplitude	Eulerian flow
angle	particle position

Formulation

Domain defined by parameters: $\Lambda = \Lambda(\epsilon t)$, $\epsilon \ll 1$.



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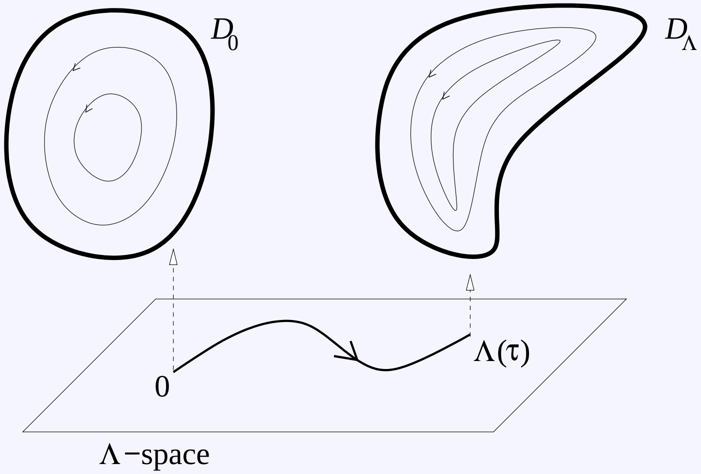

Perfect, incompressible flow in 2D: $\partial_t u + u \cdot \nabla u = -\nabla p, \quad \operatorname{div} u = 0$.

Vorticity formulation: $\omega = \operatorname{curl} u$, $u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi)$,

$$\partial_t \omega + [\psi, \omega] = 0, \quad \Delta \psi = \omega,$$

where $[\psi, \omega] = \partial_{x_1} \psi \partial_{x_2} \omega - \partial_{x_2} \psi \partial_{x_1} \omega$.

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Formulation

Boundary condition: ∂D_Λ is a material curve

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The vorticity is rearranged: $\omega(x,t) = \omega_0(g_t^{-1}x)$, i.e.

$$\omega = \omega_0 \circ g_t^{-1},$$

where g_t is an area-preserving diffeomorphism, with $\dot{g}_t = v$.

Steady Flows: $[\psi, \omega] = 0$

$$\Rightarrow \quad \psi = F_\Lambda(\omega) \qquad \text{in } D_\Lambda\,.$$

Vorticity and streamfunction are functionaly related.

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Perturbation expansion

Expand in power series:

$$\omega = \omega^{(0)} + \epsilon\omega^{(1)} + \cdots, \quad \psi = \psi^{(0)} + \epsilon\psi^{(1)} + \cdots$$

and substitute into 2D Euler.

Assuming all coefficients depend on ϵt, we find:

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At the next order, $\partial_{\epsilon t}\omega^{(0)} + [\psi^{(1)}, \omega^{(0)}] + [\psi^{(0)}, \omega^{(1)}] = 0$.

Rewriting as

$$\partial_{\epsilon t}\omega^{(0)} + [\phi, \omega^{(0)}] = 0, \quad \text{with} \quad \phi = \psi^{(1)} - F'(\omega^{(0)})\omega^{(1)}$$

shows that $\omega^{(0)}$ is rearranged by velocity $\nabla^{\perp}\phi$.

Eulerian flow

The leading-order flow $\omega^{(0)}$ can be found by imposing that it is:

- steady, $\psi^{(0)} = \Delta^{-1}\omega^{(0)} = F(\omega^{(0)})$ for some $F(\cdot)$.
- a rearrangement, $\omega^{(0)}(x,t) = \omega_0(g_\Lambda^{-1}x)$ for an area-preserving diffeomorphism g_Λ.

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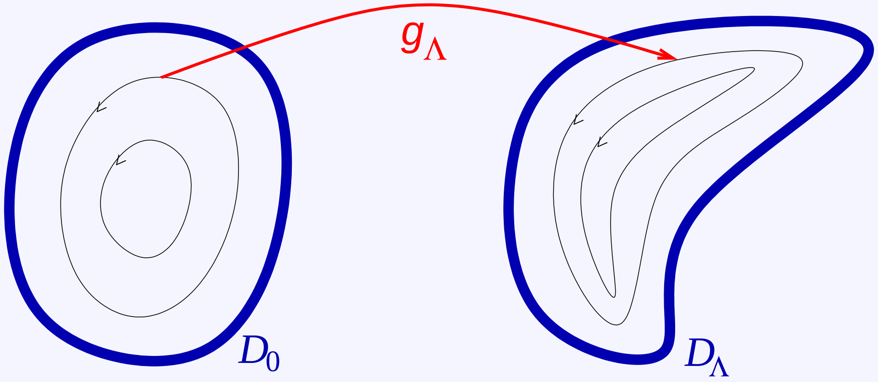

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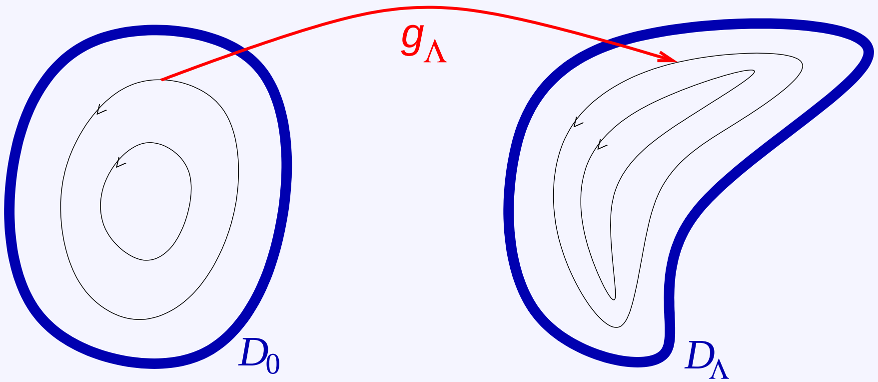

The map g_Λ

- satisfies $\det g_\Lambda = 1$,
- maps ∂D_0 to ∂D_Λ.
- depends on t only through Λ

Eulerian flow (continued)

The existence of g_Λ also answers the question of robustness of steady flows to domain deformation:

given a steady flow in the domain D_0, does it persist when the domain is deformed to D_Λ? (Wirosoetisno & V, 2005)

Eulerian flow (continued)

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The problem is written entirely in terms of g_Λ and F:

- g_Λ satisfies the nonlinear PDE,

$$\omega_0 = \Delta(F \circ \omega_0 \circ g_\Lambda^{-1}) \circ g_\Lambda,$$

- F is determined by a solvability condition.

Eulerian flow (continued)

Use an implicit-function-theorem argument to show:

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For adiabatic deformations, the constraint $T < \infty$ is a requirement of slowness.

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Eulerian flow (continued)

How can we find g_Λ?

- The PDE may be solved numerically, e.g. using an iterative scheme.
- This is simpler if D_0 is a channel or a disc, but convergence appears limited to very small boundary deformations.
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Lie series: write g_Λ as the flow at δ of vector field $\nabla^\perp\varphi(\delta)$, and expand φ to find

$$f(g_\Lambda x) = f(x) + \delta[\varphi_1, f](x) + \frac{\delta^2}{2}([\varphi_1,[\varphi_1,f]](x) + [\varphi_2, f](x)) + \cdot$$

This leads to a sequence of linear problems for the φ_i and $F = F_0 + \delta F_1 + \cdots$.

Eulerian flow (continued)

Example: deformation of an axisymmetric flow in a disc.

Take D_0 to be the unit disc, and $\psi(0) = \psi(r,0) = r^{1/2}$.

The deformed domain D_Λ is defined by

$$r = 1 + \delta \sum_m \Lambda_m \exp(\mathrm{i}m\sigma) + O(\delta^2)$$

We find that:

$$\varphi_1 = \sum_m \frac{\mathrm{i}}{m} r_m^{\beta} \Lambda_m \exp(\mathrm{i}m\sigma), \quad \text{with} \quad \beta_m = \sqrt{m^2 - 3/4} + 3/2 \quad \text{and} \quad F_1 = 0.$$

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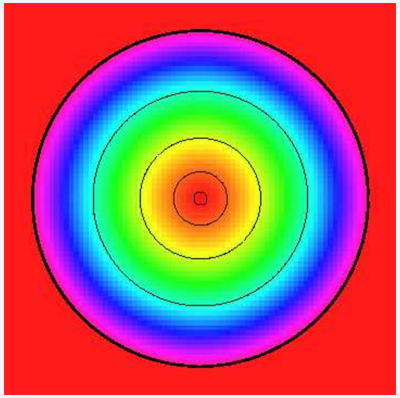

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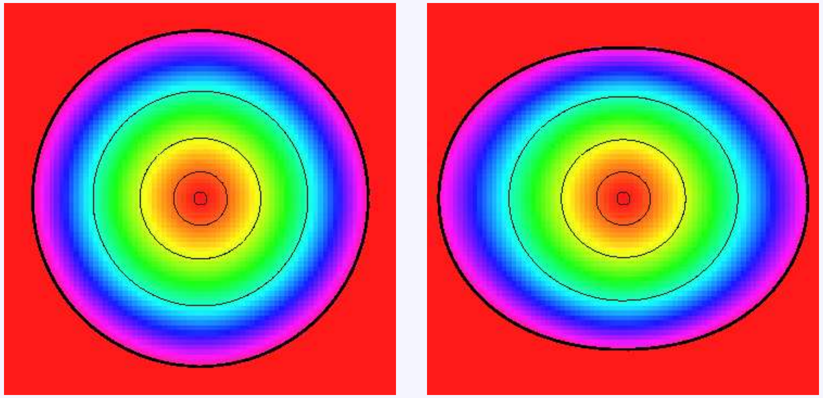

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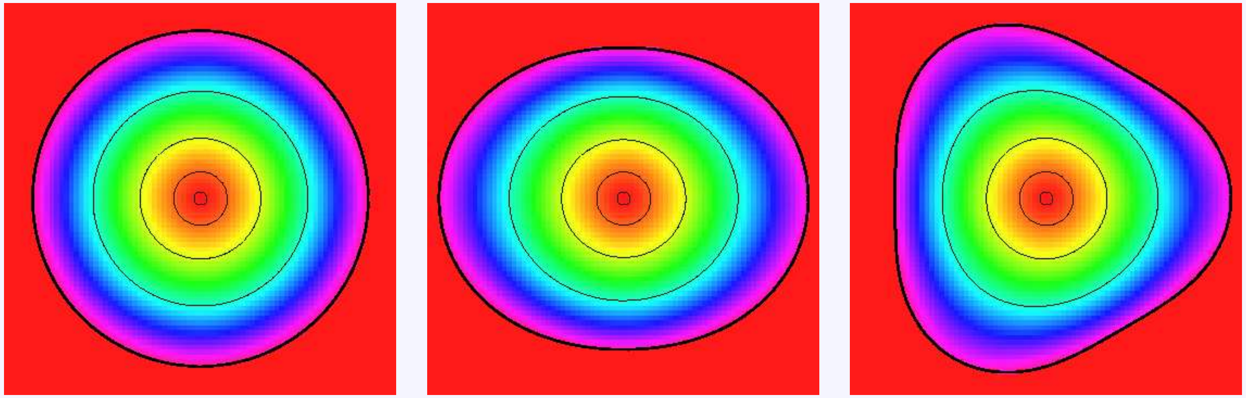

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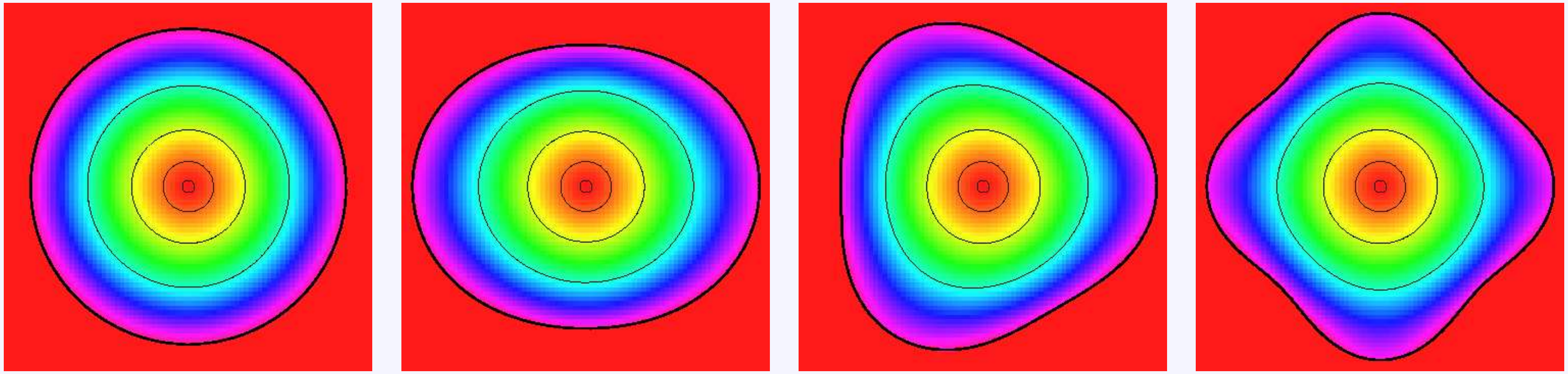

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Lagrangian trajectories

For $t = O(\epsilon^{-1})$, position $(x, y)(t)$ of particles is governed by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial}{\partial y}\left(\psi^{(0)} + \epsilon\psi^{(1)}\right) + O(\epsilon^2), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial}{\partial x}\left(\psi^{(0)} + \epsilon\psi^{(1)}\right) + O(\epsilon^2).$$

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This is a ‘doubly perturbed’ Hamiltonian system with Hamiltonian

$$H(x, y; \Lambda) = \psi^{(0)}(x, y, \Lambda(\epsilon t)) + \epsilon\psi^{(1)}(x, y, \Lambda(\epsilon t)) \qquad .$$

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Determination of $\psi^{(1)}$: recall

$$\partial_{\epsilon t}\omega^{(0)} + [\phi, \omega^{(0)}] = 0, \quad \text{with} \;\; \phi = \psi^{(1)} - F'(\omega^{(0)})\omega^{(1)}.$$

Since $\omega^{(0)}$ is known, ϕ can be determined up to an arbitrary function of $\omega^{(0)}$:

$$\nabla^{\perp}\phi = \frac{\mathrm{d}}{\mathrm{d}\epsilon t} g_{\Lambda} x = d_{\Lambda} g_{\Lambda} \cdot \dot{\Lambda}.$$

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Lagrangian trajectories (continued)

Hence, $\psi^{(1)} = \Delta^{-1}\omega^{(1)}$ is found by solving

$$\psi^{(1)} - F'(\omega^{(0)})\omega^{(1)} = \phi,$$

The gauge freedom in ϕ is fixed by the condition that the total vorticity is rearranged:

$$\iint_{\omega^{(0)}+\epsilon\omega^{(1)}=\Omega} \mathrm{d}x - \iint_{\omega^{(0)}=\Omega} \mathrm{d}x = O(\epsilon^2) \implies \oint_{\omega^{(0)}=\Omega} \omega^{(1)}\mathrm{d}s = 0,$$

where $\mathrm{d}s = \mathrm{d}l/|\nabla\omega^{(0)}|$.
A consequence is that

$$\oint_{\omega^{(0)}=\Omega} \psi^{(1)}\mathrm{d}s = \oint_{\omega^{(0)}=\Omega} \phi\mathrm{d}s.$$

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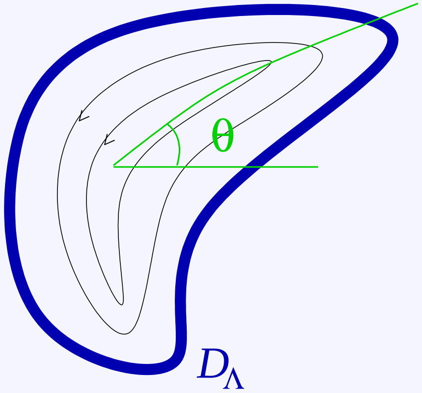
Lagrangian trajectories (continued)

To find particle trajectories, we use action–angle coordinates:

- $I = A(\omega^{(0)})$, area inside contour, is an adiabatic invariant,
- θ, conjugate to I, gives position along ω-contours.



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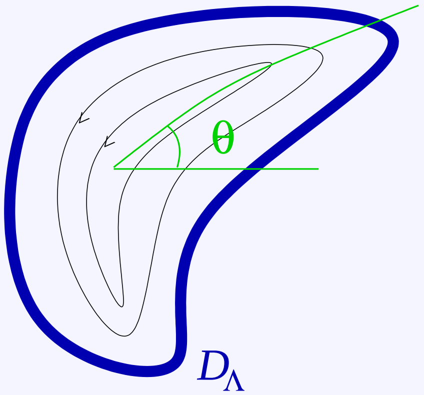

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Use a generating function: $x_2 = \partial_{x_1} S(x_1, I), \;\; \theta = \partial_I S(x_1, I)$, with the new Hamiltonian

$$\bar{H}(I, \theta, \Lambda) = \bar{\psi}^{(0)}(I, \theta, \Lambda) + \epsilon \bar{\psi}^{(1)}(I, \theta, \Lambda) + \partial_t S$$

Lagrangian trajectories (continued)

With $\partial_t S = \partial_t \bar{S} - \bar{x}_1 \partial_t \bar{x}_2$, we find the evolution equation for the angle:

$$\dot{\theta} = \nu + \epsilon \partial_I \left[\partial_t \bar{S} - \bar{x}_1 \partial_t \bar{x}_2 + \bar{\psi}^{(1)} \right],$$

where $\nu = \partial_I \bar{\psi}^{(0)}$ is the frequency.

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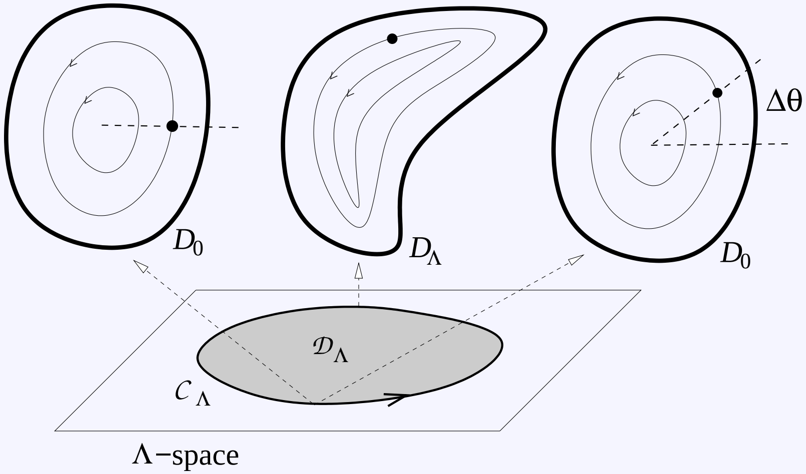

Consider cyclic deformation of the domain, and use averaging:

$$\Delta\theta \sim \frac{\mathrm{d}}{\mathrm{d}I} \int_0^t \left[\psi^{(0)} - \langle \bar{x}_1 \partial_t \bar{x}_2 + \bar{\psi}^{(1)} \rangle \right] \mathrm{d}t', \quad \text{where} \quad \langle \cdot \rangle = \frac{1}{2\pi} \int \cdot \, \mathrm{d}\theta.$$

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Lagrangian trajectories (continued)

Now, use:

- $\int \psi^{(1)}\,\mathrm{d}\theta = \int \phi\,\mathrm{d}\theta = \int \Psi \cdot \dot{\Lambda}\,\mathrm{d}\theta$, where $\Psi = \sum_m \Psi_m \mathrm{d}\Lambda_m$ is a function-value one-form (connection), with $\nabla^\perp \Psi = d_\Lambda g_\Lambda$,
- $\bar{x}(I,\theta,\Lambda) = g_\Lambda x(I,\theta,0)$,
- Stokes' theorem.

to find

$$\Delta\theta(t) \sim \frac{\mathrm{d}}{\mathrm{d}I}\int_0^t \psi^{(0)}(\Lambda(s))\,\mathrm{d}s + \frac{\mathrm{d}}{\mathrm{d}I}\int_{\mathcal{D}_\Lambda} d_\Lambda\Psi - \frac{1}{2}[\Psi,\Psi]$$

with $[\Psi,\Psi] = \sum_{m,n}[\Psi_m,\Psi_n] d\Lambda_m \wedge d\Lambda_n$

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$$\Delta\theta(t) \sim \underbrace{\frac{\mathrm{d}}{\mathrm{d}I}\int_0^t \psi^{(0)}(\Lambda(s))\,\mathrm{d}s}_{\text{dynamical angle}} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}I}\int_{\mathcal{D}_\Lambda} d_\Lambda\Psi - \frac{1}{2}[\Psi,\Psi]}_{\text{geometric angle}}$$

Lagrangian trajectories (continued)

Geometric angle:

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- curvature of the connection form Ψ

Lagrangian trajectories (continued)

Geometric angle:

- determined completely from the connection Ψ defined by g_Λ with suitable gauge,
- has 2 parts:
  - $d_\Lambda \Psi$, due to $\psi^{(1)}$
  - $-[\Psi, \Psi]/2$, Hannay–Berry angle, due to ϵt-dependence of $\psi^{(0)}$
- curvature of the connection form Ψ

Small deformations: $f(g_\Lambda x) = f(x) + \delta[\varphi_1, f](x) + \cdots$, leads to

$$\Psi = \delta d_\Lambda \varphi_1 + \frac{\delta^2}{2}(d_\Lambda \varphi_2 + [d_\Lambda \varphi_1, \varphi_1]) + \cdots$$

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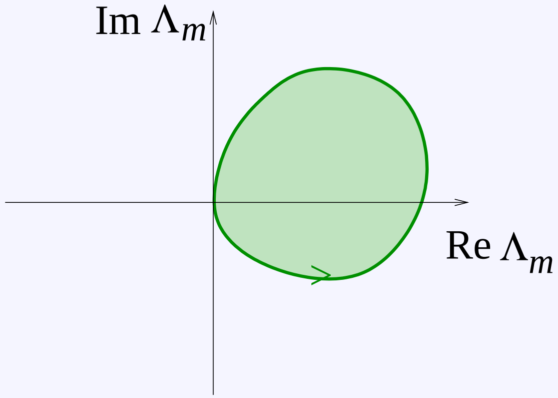
Lagrangian trajectories (continued)

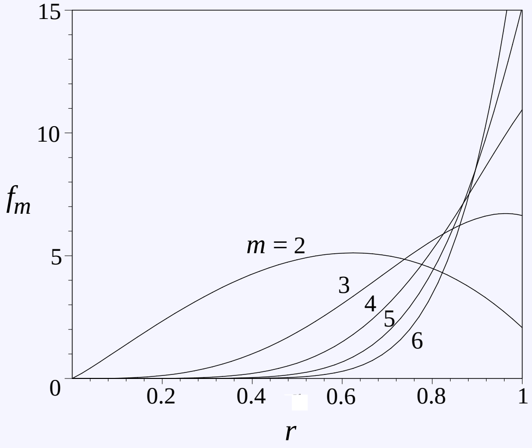
Example: deformed axisymmetric flow:

$r = 1 + \delta \sum_m \Lambda_m \exp(im\sigma) + O(\delta^2)$ gives

$$\langle\theta\rangle_{\text{geom}} = \delta^2 \sum_{m>0} f_m(r)\mathcal{A}_m + O(\delta^3),$$

where $\mathcal{A}_m$ is the area enclosed by path of Λ_m in the complex plane.





For $\psi(0) = r^{1/2}$, $f_m(r)$ is a sum of 2 powers of r.

Conclusions

- Procedure for computing flows in slowly deforming domains
- Eulerian flow is quasi-steady
- Eulerian flow depends only on the initial and final domain shapes
- Particle positions defined by their angle along vorticity contours
- Angle depends on the history of the domain, in a geometric manner

Extensions:

- rotation period T unbounded (non-parallel critical levels)
- 3D flows