Two-dimensional flows in slowly deforming domains

Adiabatic invariance and geometric angle

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Motivation: effect of slow, conservative perturbations on perfect fluids

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- 2D, incompressible fluid
Flow in slowly deforming domain

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- Lagrangian fluid-particle positions for $O(1)$ deformations of the domain.
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\[\begin{array}{c}
\text{Amplitude (energy):} \\
\begin{itemize}
\item determined by the \text{adiabatic invariance} of action $I$, $\Delta I = O(\epsilon)$.
\item adiabatic invariance of $I$ stems from invariance of $pdq$.
\item energy depends on $\Lambda$ \text{instantaneously}.
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Flow in slowly deforming domain

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Angle $\langle \theta \rangle$:
- dynamical angle $\int \omega \, dt$
- + Hannay–Berry (geometric) angle
- depends on path in parameter space
Flow in slowly deforming domains

Geometric angle depends only on the curve in parameter space, not on the speed at which this curve is traced.
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**Analogy f-d system/fluid:**

<table>
<thead>
<tr>
<th>f-d system</th>
<th>fluid</th>
</tr>
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<tbody>
<tr>
<td>amplitude</td>
<td>Eulerian flow</td>
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<tr>
<td>angle</td>
<td>particle position</td>
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</table>
Formulation

Domain defined by parameters: $\Lambda = \Lambda(\epsilon t), \quad \epsilon \ll 1.$
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Perfect, incompressible flow in 2D: 
\[ \partial_t u + u \cdot \nabla u = -\nabla p, \quad \text{div} \ u = 0. \]

Vorticity formulation: 
\[ \omega = \text{curl} \ u, \ u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi), \]

\[ \partial_t \omega + [\psi, \omega] = 0, \quad \Delta \psi = \omega, \]

where 
\[ [\psi, \omega] = \partial_{x_1} \psi \partial_{x_2} \omega - \partial_{x_2} \psi \partial_{x_1} \omega. \]
Boundary condition: $\partial D_\Lambda$ is a material curve
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The vorticity is rearranged: $\omega(x, t) = \omega_0(g_t^{-1}x)$, i.e.

$$\omega = \omega_0 \circ g_t^{-1},$$

where $g_t$ is an area-preserving diffeomorphism, with $\dot{g}_t = v$.

Steady Flows: $[\psi, \omega] = 0$

$$\Rightarrow \psi = F_\Lambda(\omega) \quad \text{in } D_\Lambda.$$

Vorticity and streamfunction are functionaly related.
Perturbation expansion

Expand in power series:
\[ \omega = \omega^{(0)} + \epsilon \omega^{(1)} + \cdots, \quad \psi = \psi^{(0)} + \epsilon \psi^{(1)} + \cdots \]

and substitute into 2D Euler.

Assuming all coefficients depend on \( \epsilon t \), we find:
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At leading order,
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\[ \psi^{(0)} = F(\omega^{(0)}). \]

At the next order, \( \partial_{\epsilon t} \omega^{(0)} + [\psi^{(1)}, \omega^{(0)}] + [\psi^{(0)}, \omega^{(1)}] = 0. \)

Rewriting as
\[ \partial_{\epsilon t} \omega^{(0)} + [\phi, \omega^{(0)}] = 0, \quad \text{with} \quad \phi = \psi^{(1)} - F'(\omega^{(0)})\omega^{(1)} \]

shows that \( \omega^{(0)} \) is rearranged by velocity \( \nabla \perp \phi \).
Eulerian flow

The leading-order flow $\omega^{(0)}$ can be found by imposing that it is:

- steady, $\psi^{(0)} = \Delta^{-1} \omega^{(0)} = F(\omega^{(0)})$ for some $F(\cdot)$.
- a rearrangement, $\omega^{(0)}(x, t) = \omega_0(\Lambda^{-1} x)$ for an area-preserving diffeomorphism $\Lambda$. 

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The map $g_\Lambda$

- satisfies $\det g_\Lambda = 1$,
- maps $\partial D_0$ to $\partial D_\Lambda$.
- depends on $t$ only through $\Lambda$
The existence of $g_\Lambda$ also answers the question of robustness of steady flows to domain deformation:

given a steady flow in the domain $D_0$, does it persist when the domain is deformed to $D_\Lambda$? (Wirosoetisno & V, 2005)
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The problem is written entirely in terms of $g_\Lambda$ and $F$:

- $g_\Lambda$ satisfies the nonlinear PDE,

$$\omega_0 = \Delta(F \circ \omega_0 \circ g_\Lambda^{-1}) \circ g_\Lambda,$$

- $F$ is determined by a solvability condition.
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- the solution is unique modulo translation along constant \( \omega^{(0)} \) (gauge freedom).

This leads to unique \( \omega^{(0)} \) and \( \psi^{(0)} \) that depend on \( t \) only through \( \Lambda \).
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For adiabatic deformations, the constraint \( T < \infty \) is a requirement of slowness.
Eulerian flow (continued)

How can we find $g_\Lambda$?

- The PDE may be solved numerically, e.g. using an iterative scheme.
- This is simpler if $D_0$ is a channel or a disc, but convergence appears limited to very small boundary deformations.
- A perturbative scheme can be developed for small boundary deformations $\delta \ll 1$, using Lie series.
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**Lie series**: write $g_\Lambda$ as the flow at $\delta$ of vector field $\nabla^\perp \phi(\delta)$, and expand $\phi$ to find

$$f(g_\Lambda x) = f(x) + \delta [\phi_1, f](x) + \frac{\delta^2}{2} ([\phi_1, [\phi_1, f]](x) + [\phi_2, f](x)) + \cdots.$$ 

This leads to a sequence of linear problems for the $\phi_i$ and

$$F = F_0 + \delta F_1 + \cdots.$$
Eulerian flow (continued)

Example: deformation of an axisymmetric flow in a disc. Take $D_0$ to be the unit disc, and $\psi(0) = \psi(r, 0) = r^{1/2}$. The deformed domain $D_\Delta$ is defined by

$$r = 1 + \delta \sum_m \Lambda_m \exp(im\sigma) + O(\delta^2)$$

We find that:

$$\varphi_1 = \sum_m \frac{i}{m} r^\beta_m \Lambda_m \exp(im\sigma), \quad \text{with} \quad \beta_m = \sqrt{m^2 - 3/4} + 3/2 \quad \text{and} \quad F_1 = 0.$$
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Lagrangian trajectories

For $t = O(\epsilon^{-1})$, position $(x, y)(t)$ of particles is governed by:

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{\partial}{\partial y}(\psi^{(0)} + \epsilon\psi^{(1)}) + O(\epsilon^2), \\
\frac{dy}{dt} &= \frac{\partial}{\partial x}(\psi^{(0)} + \epsilon\psi^{(1)}) + O(\epsilon^2).
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This is a ‘doubly perturbed’ Hamiltonian system with Hamiltonian

\[
H(x, y; \Lambda) = \psi^{(0)}(x, y, \Lambda(\epsilon t)) + \epsilon \psi^{(1)}(x, y, \Lambda(\epsilon t)).
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Determination of \(\psi^{(1)}\): recall

\[
\partial_{\epsilon t} \omega^{(0)} + [\phi, \omega^{(0)}] = 0, \quad \text{with} \quad \phi = \psi^{(1)} - F'(\omega^{(0)})\omega^{(1)}.
\]

Since \(\omega^{(0)}\) is known, \(\phi\) can be determined up to an arbitrary function of \(\omega^{(0)}\):

\[
\nabla^\perp \phi = \frac{d}{d\epsilon t} g_\Lambda x = d_\Lambda g_\Lambda \cdot \dot{\Lambda}.
\]
Hence, $\psi^{(1)} = \Delta^{-1} \omega^{(1)}$ is found by solving

$$\psi^{(1)} - F'(\omega^{(0)})\omega^{(1)} = \phi,$$

The gauge freedom in $\phi$ is fixed by the condition that the total vorticity is rearranged:

$$\int \int_{\omega^{(0)} + \epsilon \omega^{(1)} = \Omega} dx - \int \int_{\omega^{(0)} = \Omega} dx = O(\epsilon^2) \implies \int_{\omega^{(0)} = \Omega} \omega^{(1)} ds = 0,$$

where $ds = dl/|\nabla \omega^{(0)}|$.

A consequence is that

$$\int_{\omega^{(0)} = \Omega} \psi^{(1)} ds = \int_{\omega^{(0)} = \Omega} \phi ds.$$
To find particle trajectories, we use action–angle coordinates:

- \( I = A(\omega^0) \), area inside contour, is an adiabatic invariant,
- \( \theta \), conjugate to \( I \), gives position along \( \omega \)-contours.

Note: \( 2\pi ds = A'(\omega^0) d\theta \).
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Use a generating function: \( x_2 = \partial_{x_1} S(x_1, I), \ \theta = \partial_I S(x_1, I) \), with the new Hamiltonian

\[
\tilde{H}(I, \theta, \Lambda) = \psi^{(0)}(I, \theta, \Lambda) + \epsilon \psi^{(1)}(I, \theta, \Lambda) + \partial_t S
\]
Lagrangian trajectories (continued)

With $\partial_t S = \partial_t \bar{S} - \bar{x}_1 \partial_t \bar{x}_2$, we find the evolution equation for the angle:

$$\dot{\theta} = \nu + \epsilon \partial_I \left[ \partial_t \bar{S} - \bar{x}_1 \partial_t \bar{x}_2 + \bar{\psi}^{(1)} \right],$$

where $\nu = \partial_I \bar{\psi}^{(0)}$ is the frequency.
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$$

where $\nu = \partial_I \bar{\psi}^{(0)}$ is the frequency.

Consider cyclic deformation of the domain, and use averaging:

$$
\Delta \theta \sim \frac{d}{d I} \int_0^t \left[ \bar{\psi}^{(0)} - \langle \bar{x}_1 \partial_t \bar{x}_2 + \bar{\psi}^{(1)} \rangle \right] dt', \quad \text{where} \quad \langle \cdot \rangle = \frac{1}{2\pi} \int \cdot \, d\theta.
$$
Now, use:

- $\int \psi^{(1)} \, d\theta = \int \phi \, d\theta = \int \Psi \cdot \dot{\Lambda} \, d\theta$, where $\Psi = \sum_m \Psi_m \, d\Lambda_m$ is a function-value one-form (connection), with $\nabla \perp \Psi = d_\Lambda g_\Lambda$,

- $\bar{x}(I, \theta, \Lambda) = g_\Lambda x(I, \theta, 0)$,

- Stokes’ theorem.

to find

$$\Delta \theta(t) \sim \frac{d}{dI} \int_0^t \psi^{(0)}(\Lambda(s)) \, ds + \frac{d}{dI} \int_{\mathcal{D}_\Lambda} d_\Lambda \Psi - \frac{1}{2} [\Psi, \Psi]$$

with $[\Psi, \Psi] = \sum_{m,n} [\Psi_m, \Psi_n] \, d\Lambda_m \wedge d\Lambda_n$. 
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given dynamical angle geometric angle
Lagrangian trajectories (continued)

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- determined completely from the connection $\Psi$ defined by $g_A$ with suitable gauge,
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- curvature of the connection form $\Psi$

Small deformations: $f(g_\Lambda x) = f(x) + \delta[\varphi_1, f](x) + \cdots$, leads to

$$\Psi = \delta d_\Lambda \varphi_1 + \frac{\delta^2}{2} (d_\Lambda \varphi_2 + [d_\Lambda \varphi_1, \varphi_1]) + \cdots$$
Example: deformed axisymmetric flow:

\[ r = 1 + \delta \sum_m \Lambda_m \exp(i m \sigma) + O(\delta^2) \] gives

\[ \langle \theta \rangle_{\text{geom}} = \delta^2 \sum_{m > 0} f_m(r) A_m + O(\delta^3), \]

where \( A_m \) is the area enclosed by path of \( \Lambda_m \) in the complex plane.

For \( \psi(0) = r^{1/2} \), \( f_m(r) \) is a sum of 2 powers of \( r \).
Conclusions

- Procedure for computing flows in slowly deforming domains
- Eulerian flow is quasi-steady
- Eulerian flow depends only on the initial and final domain shapes
- Particle positions defined by their angle along vorticity contours
- Angle depends on the history of the domain, in a geometric manner

Extensions:
- rotation period $T$ unbounded (non-parallel critical levels)
- 3D flows