

Gravity waves generated by sheared three-dimensional potential-vorticity anomalies

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Abstract

The gravity waves (GWs) produced by three-dimensional potential-vorticity (PV) anomalies are examined under the assumption of constant vertical shear, constant stratification, and unbounded domain. As in the two-dimensional case analysed in an earlier paper, the disturbance near the PV anomaly is well modelled by quasi-geostrophic theory. At larger distances the nature of the disturbance changes across the two inertial layers that are located above and below the anomaly, and it takes the form of a vertically propagating GW beyond these.

For an horizontally monochromatic PV anomaly of infinitesimal depth, the disturbance is described analytically using both an exact solution and a WKB approximation; the latter includes an exponentially small term which captures the change of the solution near the PV anomaly induced by the radiation boundary condition in the far field. The analytical results reveal a strong sensitivity of the emission to the Richardson number and to the orientation of the horizontal wavenumber: the absorptive properties of the inertial layers make that the emission is maximised in the northern hemisphere for wavenumbers at negative angles to the shear.

For localised PV anomalies, numerical computations give the temporal evolution of the GWs field. Analytical and numerical results are also used to establish an explicit form for the Eliassen-Palm flux that could be used to parameterize GWs sources in GCMs. The properties of the Eliassen-Palm flux vector imply that in a westerly shear, the GWs exert a drag in a south-west direction in the upper inertial-layer, and exert a drag in a north-west direction at the altitudes where the GWs dissipate aloft.

1 Introduction

Spontaneous adjustment, the mechanism whereby a well balanced flow radiates gravity waves (GWs) in the course of its near-balanced evolution (Ford et al. 2000, Vanneste 2008), is a possible source of atmospheric GWs. It can be distinguished from other mechanisms, including topographic forcing and the classical adjustment originally described in Rossby (1937), by the

fact that it involves no process external to the flow itself. In realistic configurations however, spontaneous adjustment is mixed with other mechanisms. For instance, large mountain GWs produce potential-vorticity (PV) anomalies when they break (Shär and Smith 1993) as well as secondary GWs (Scavuzzo et al. 1998). To measure the relative importance of these two signals, Lott (2003) studied the large-scale response to mountain-wave breaking near critical levels and showed that substantial GWs are re-emitted during the breaking itself, while the long term evolution is dominated by the balanced response. Martin (2008) subsequently found that the PV field associated with the balanced response radiates GWs well after the end of the initial breaking. This emission, although weaker than the initial one, is potentially more persistent since it is tied to the slowly evolving PV. It is plausible, therefore, that it contributes to the low-frequency GWs observed in the wakes of breaking topographic waves (Plougonven et al 2010).

To quantify this emission by PV more precisely, Lott et al. (2010, hereinafter LPV10) examined the GWs emitted by small-amplitudes PV anomalies in a shear. In this scenario, the separation between balanced motion and GWs does not hold: because of the Doppler shift, motions which are balanced in the vicinity of the anomalies become, in the far-field, gravity waves (see also Plougonven et al 2005, Mamatsachvili et al 2010). In the linear approximation, and assuming constant wind shear Λ , and constant Brunt-Väisälä frequency N , LPV10 found that substantial GWs reach the far field when the Richardson number $J = N^2/\Lambda^2$ is not too large (say between 1 and 10). By substantial, we mean that, for PV anomalies representative of those likely to be found when thin layers of stratospheric air enter the troposphere, the Eliassen-Palm (EP, or pseudomomentum) flux associated with the GWs is comparable to that measured in the stratosphere far from mountains (Hertzog et al. 2008).

A practical result of LPV10 is a simple analytical estimate for the EP flux which could be used in General Circulation Models (GCMs) that include the stratosphere. In non-dimensional form, this estimate is given by

$$\mathcal{F} \sim \frac{1}{4} e^{-\pi\sqrt{J}}. \quad (1.1)$$

LPV10 also showed that half of this flux is absorbed in the inertial layers above and below the PV anomaly while the other half is radiated in the far field as GWs. The dimensional EP flux follows from (1.1) by multiplication by the scaling factor

$$F_0 = \frac{\rho_r g^2}{f \theta_r^2 N^3} (\rho_r q_r \sigma_z)^2, \quad (1.2)$$

where g is the gravity constant, f the Coriolis parameter, ρ_r and θ_r background reference values for the density and potential temperature, q_r the amplitude of the PV anomaly and σ_z its depth. In a GCM, these last two quantities could be related to the grid-scale PV value and to the vertical gridspacing.

A limitation of LPV10 is the restriction to two-dimensional PV anomalies, with no structure in the direction transverse to the basic shear. This is a significant limitation since the absorption of GWs at inertial levels strongly depends on the orientation of the horizontal wavevector. This is known from the investigations on GW propagating upward into inertial levels: Grimshaw (1975) and Yamanaka and Tanaka (1984) showed that the absorption at the lowest inertial level is large for $\nu\Lambda < 0$, where $\nu = l/k$ is the ratio between the transverse and parallel horizontal wavenumbers, and much smaller for $\nu\Lambda > 0$. This results in a 'valve effect' which

Yamanaka (1985) interpreted by analyzing the tilt of the phase lines of the GWs (i.e. of particle displacements) relative to the isentropes. The configuration that we analyse is quite different since the GW associated with a PV disturbance is generated between the critical levels and propagates outwards of them. Nevertheless, the argument of Yamanaka (1985) applies and we find strong absorption at the inertial level if $\nu\Lambda > 0$ and much weaker absorption if $\nu\Lambda < 0$.

The motivation of the present paper is to extend the results in LPV10 to three-dimensional PV anomalies. Accordingly, its first aim is to obtain the vertical structure of the 3D singular modes associated with PV anomalies that have the form of a Dirac function in the vertical. The analytic results derived for monochromatic anomalies can then be integrated to obtain the vertical structure associated with anomalies of arbitrary horizontal structure and show, in particular, that an horizontally isotropic PV anomaly produces a very specific anisotropic GW signature beyond the inertial levels. A second aim is to deduce further, by integration over the continuous spectrum, the GW response to a vertically smooth, localised PV distribution. A third aim is to extend the EP flux predictions in (1.1) and (1.2) to the 3D case. In this case, the (vertical component of the) EP flux, which can also be interpreted as a wave stress, is a horizontal vector. For a horizontally isotropic shear in the Northern hemisphere, this vector is shown to make an angle with the shear that decreases with altitude, from zero at the anomaly level to some negative value in the far field. This implies that a PV anomaly in a westerly shear exerts a drag that is oriented to the south-west in the upper inertial region and to the north-west where the associated GW dissipates aloft.

The plan of the paper is as follows. The general formulation of the problem and its transformation to a dimensionless form are given in Section 2. There we discuss both the exact response to a δ -PV distribution in the vertical, and its WKB approximation valid for $J \gg 1$. The WKB analysis extends that of LPV10 by resolving the Stokes phenomenon associated with the existence of an exponentially small (in J) contribution to the solution which grows exponentially between the PV anomaly and the inertial levels. Taking this contribution into account, we obtain a 3D generalisation of the EP flux estimate (1.1). Section 3 presents the vertical structure of the response in some detail. It emphasizes the directional aspects and relates them to the tilt of the solution about isentropes in the meridional plane. Section 4 recasts the results in dimensional form and considers PV distributions that are localised horizontally and that have a finite depth, in which case the GW-response is transient. Section 5 summarizes the results. Appendices A and B provide technical details on the exact and WKB solutions, respectively.

2 Formulation

2.1 Disturbance equations and potential vorticity

In the absence of mechanical and diabatic forcings, the linearized hydrostatic–Boussinesq equations for a three-dimensional disturbance in the shear flow $\bar{\mathbf{u}}_0 = (\Lambda z, 0, 0)$ read

$$(\partial_t + \Lambda z \partial_x) u' + \Lambda w' - f v' = -\frac{1}{\rho_r} \partial_x p', \quad (2.1a)$$

$$(\partial_t + \Lambda z \partial_x) v' + f u' = -\frac{1}{\rho_r} \partial_y p', \quad (2.1b)$$

$$0 = -\frac{1}{\rho_r} \partial_z p' + g \frac{\theta'}{\theta_r}, \quad (2.1c)$$

$$(\partial_t + \Lambda z \partial_x) g \frac{\theta'}{\theta_r} - f \Lambda v' + N^2 w' = 0, \quad (2.1d)$$

$$\partial_x u' + \partial_y v' + \partial_z w' = 0. \quad (2.1e)$$

Here u' , v' , and w' are the three components of the velocity disturbance, p' is the pressure disturbance, θ' is the potential temperature disturbance, and $N^2 = g \bar{\theta}_{0z} / \theta_r$ is the square of the constant Brunt–Väisälä frequency, with $\bar{\theta}_0(y, z)$ the background potential-temperature. Without loss of generality we assume that $\Lambda > 0$.

Equations (2.1a)–(2.1e) imply the conservation equation

$$(\partial_t + \Lambda z \partial_x) q' = 0, \quad (2.2)$$

for the potential-vorticity (PV) perturbation

$$q' = \frac{1}{\rho_r} (\bar{\theta}_{0z} (\partial_x v' - \partial_y u') + \bar{\theta}_{0y} \partial_z u' + \Lambda \theta'_y + f \partial_z \theta'). \quad (2.3)$$

It follows that the PV at any time t is given explicitly in terms of the initial condition $q'_0(x, y, z) = q'(x, y, z, t = 0)$ by

$$q'(x, y, z, t) = q'_0(x - \Lambda z t, y, z). \quad (2.4)$$

2.2 Normal-mode decomposition

To evaluate the disturbance field associated with the PV anomaly (2.4), we express this solution in Fourier space,

$$q'(x, y, z, t) = \iint \hat{q}(k, l, z, t) e^{ikx + ily} dk dl = \iint \hat{q}_0(k, l, z) e^{i(kx + ly - k\Lambda z t)} dk dl, \quad (2.5)$$

where \hat{q}_0 is the horizontal Fourier transform of q'_0 ,

$$\hat{q}_0(k, l, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q'_0(x, y, z) e^{-i(kx + ly)} dx dy. \quad (2.6)$$

Here and henceforth integrations without limits are understood to be over the whole space. We rewrite (2.5) in the form

$$q'(x, y, z, t) = \frac{k\Lambda}{f} \iiint \hat{q}_0(k, l, z') e^{i(kx + ly - k\Lambda z' t)} \delta\left(\frac{k\Lambda}{f}(z - z')\right) dz' dk dl, \quad (2.7)$$

where $\delta(\xi)$ is the Dirac function of the variable

$$\xi = \frac{k\Lambda}{f}(z - z'). \quad (2.8)$$

Note that (2.7) can be interpreted as the expansion of the perturbation PV in the (singular) normal modes of (2.3); these modes form a continuum, parameterised by the phase speed $\Lambda z'$. The scaling used in (2.8) places the inertial levels $z = z' \pm f/(k\Lambda)$ of these modes at $\xi = \pm 1$ (Inverarity and Shutts 2000).

The expansion of the vertical velocity w' corresponding to the expansion (2.7) of the PV can be written as

$$w'(x, y, z, t) = \iiint \hat{w}_0(k, l, z') e^{i(kx+ly-k\Lambda z't)} W\left(\frac{k\Lambda}{f}(z-z')\right) dz' dk dl, \quad (2.9)$$

where $\hat{w}_0(k, l, z')$ is the amplitude of the normal mode, and $W(\xi)$ its vertical structure. Note that this expansion describes the part of w' slaved to the PV: an additional continuum of singular modes, representing free sheared GWs, would need to be added to the expansion to solve an arbitrary initial-value problem.

The velocities u' , v' and the potential temperature θ' have expansions analogous to (2.9), with \hat{w}_0 replaced by \hat{u}_0 , \hat{v}_0 , and $\hat{\theta}_0$, and W replaced by U , V , and Θ , respectively. Introducing these expansions into (2.1a)–(2.1e), and choosing

$$\hat{u}_0 = i\frac{\Lambda}{f}\hat{w}_0, \quad \hat{v}_0 = -\frac{\Lambda}{f}\hat{w}_0, \quad \text{and} \quad \hat{\theta}_0 = i\frac{\theta_r\Lambda^2}{fg}\hat{w}_0 \quad (2.10)$$

gives

$$U = \frac{\xi - i\nu}{\xi(1 + \nu^2)}W_\xi + \frac{\nu^2}{\xi(1 + \nu^2)}W, \quad V = \frac{1 - i\nu\xi}{\xi(1 + \nu^2)}W_\xi + \frac{i\nu}{\xi(1 + \nu^2)}W, \quad (2.11a)$$

$$\text{and} \quad \Theta = \frac{1 - i\nu\xi}{\xi^2(1 + \nu^2)}W_\xi + \left(\frac{i\nu}{\xi^2(1 + \nu^2)} + \frac{J}{\xi}\right)W, \quad (2.11b)$$

where $\nu = l/k$. We now introduce (2.10)–(2.11), into the expressions (2.3) and (2.7) for the PV. Choosing the vertical-velocity amplitude

$$\hat{w}_0(k, l, z') = -i\frac{\rho_r g(1 + \nu^2)}{\theta_r \Lambda^2}\hat{q}_0(k, l, z'), \quad (2.12)$$

then leads to the differential equation

$$\frac{1 - \xi^2}{\xi^2}W_{\xi\xi} - \left(\frac{2}{\xi^3} - \frac{2i\nu}{\xi^2}\right)W_\xi - \left(\frac{(1 + \nu^2)J}{\xi^2} + \frac{2i\nu}{\xi^3}\right)W = \delta(\xi), \quad (2.13)$$

for the structure function $W(\xi)$. Note that W depends on J and ν in addition to ξ , and that we use the notation $W(\xi)$ as a shorthand for the more complete, but more cumbersome $W(J, \nu; \xi)$.

In Appendix A, we follow Yamanaka and Tanaka (1984), Plougonven et al (2005), and LPV10 and solve this equation exactly using a change of variable that transforms the homogeneous part of (2.13) into the hypergeometric equation. The solution satisfies

$$W(\xi) \sim E\xi^{1/2+i\sqrt{J(1+\nu^2)-1}/4} \quad \text{as} \quad \xi \rightarrow +\infty, \quad (2.14)$$

$$\text{and} \quad W(\xi) = W^*(-\xi) \quad \text{for} \quad \xi < 0, \quad (2.15)$$

corresponding to an upward- (downward-) propagating GW as $\xi \rightarrow +\infty$ ($\xi \rightarrow -\infty$). An explicit expression for the amplitude E of this GW is given in (A.13).

2.3 WKB approximation

In the limit $J \gg 1$, it is possible to derive an approximation to $W(\xi)$ using a WKB approach. This approximation, which we now derive, is more transparent than the exact solution in terms of hypergeometric functions and proves remarkably accurate for moderately large J .

We focus on the region $\xi > 0$ since the solution for $\xi < 0$ follows immediately from (2.15). The WKB approximation does not provide a single solution that is valid uniformly in $\xi > 0$; instead, four regions, which we label (i)–(iv), need to be distinguished. The form of the solution in each of these regions, given below, is derived in Appendix B.

In region (i), close to the PV anomaly and specifically for $\xi = O(J^{-1/2}) \ll 1$, the quasi-geostrophic approximation applies, leading to

$$W^{(i)}(\xi) \sim A^{(i)} \left(\sqrt{J(1+\nu^2)}\xi + 1 \right) e^{-\sqrt{J(1+\nu^2)}\xi} + B^{(i)} \left(\sqrt{J(1+\nu^2)}\xi - 1 \right) e^{\sqrt{J(1+\nu^2)}\xi}. \quad (2.16)$$

One solution is exponentially decaying away from the PV anomaly, consistent with the expectation from the QG approximation. The other, exponentially growing solution, is absent in the standard QG approximation but will need to be retained in order to derive the EP flux between the inertial levels, as discussed below.

In region (ii), between the PV anomaly and the inertial level, and more precisely where $\xi = O(1)$ and $\xi < 1$,

$$W^{(ii)}(\xi) \sim \frac{\xi}{(1-\xi)^{1/4-i\nu/2}(1+\xi)^{1/4+i\nu/2}} \times \left(A^{(ii)} e^{-\sqrt{J(1+\nu^2)}\sin^{-1}\xi} + B^{(ii)} e^{\sqrt{J(1+\nu^2)}\sin^{-1}\xi} \right). \quad (2.17)$$

In region (iii), close enough to the inertial level that $|\xi - 1| = O(J^{-1})$, the solution is expressed in terms of the scaled variable $\zeta = J(1+\nu^2)(\xi - 1)$ as

$$W^{(iii)}(\xi) \sim \zeta^{i\nu/2} \left(A^{(iii)} H_{i\nu}^{(1)}(\sqrt{2\zeta}) + B^{(iii)} H_{i\nu}^{(2)}(\sqrt{2\zeta}) \right), \quad (2.18)$$

where $H_{i\nu}^{(1)}$ and $H_{i\nu}^{(2)}$ are Hankel functions (Abramowitz and Stegun 1964).

Finally, in region (iv) above the inertial level where $\xi = O(1)$ and $\xi > 1$, the solution is

$$W^{(iv)}(\xi) \sim \frac{\xi}{(\xi-1)^{1/4-i\nu/2}(\xi+1)^{1/4+i\nu/2}} \times \left(A^{(iv)} e^{i\sqrt{J(1+\nu^2)}\ln(\xi+\sqrt{\xi^2-1})} + B^{(iv)} e^{-i\sqrt{J(1+\nu^2)}\ln(\xi+\sqrt{\xi^2-1})} \right), \quad (2.19)$$

The 8 constants $A^{(i)} \dots B^{(iv)}$ are fixed by imposing a jump conditions at $\xi = 0$ given in (A.1), radiation condition as $\xi \rightarrow \infty$, and continuity of the solution across the 4 regions. Starting with the radiation condition, we obtain from (2.14) and (2.19) that

$$A^{(iv)} = 2^i \sqrt{J(1+\nu^2)} E \quad \text{and} \quad B^{(iv)} = 0. \quad (2.20)$$

Matching between regions (iv) and (iii) then gives

$$A^{(iv)} = 2^{1/2+i\nu/2} \pi^{-1/2} (J(1+\nu^2))^{-1/4+i\nu/2} e^{\nu\pi/2} e^{-i\pi/4} A^{(iii)} \quad \text{and} \quad B^{(iii)} = 0 \quad (2.21)$$

(see Appendix B for details).

Some care needs to be exercised when matching from region (iii) to region (ii). Standard matching as carried out in Lott et al. (2010) gives that $B^{(ii)} = 0$, in agreement with the expectation from quasi-geostrophic theory of a solution that decays exponentially with altitude above $\xi = 0$. This solution is not entirely satisfactory, however, in that it fails to capture the feedback that radiation (as $\xi \rightarrow \infty$) has on the solution in regions (ii) and (i). In particular, a single exponentially decaying solution (in regions (i) and (ii)) has a zero EP flux, inconsistent with the non-zero flux from the exact solution. To resolve this apparent difficulty, we need to recognise that $B^{(ii)} = 0$ is only an approximation. In fact, $B^{(ii)}$ takes an exponentially small, non-zero value, which can be captured using the more sophisticated matching procedure applied in Appendix B. This procedure yields

$$A^{(iii)} = 2^{-1/2 - i\nu/2} \pi^{1/2} (J(1 + \nu^2))^{1/4 - i\nu/2} e^{-\pi/2 \sqrt{J(1 + \nu^2)} - \nu\pi} A^{(ii)}, \quad (2.22)$$

$$B^{(ii)} = -ie^{-\sqrt{J(1 + \nu^2)}\pi} \cosh(\nu\pi) A^{(ii)}. \quad (2.23)$$

Eq. (2.23) implies that the exponentially decaying solution of quasi-geostrophic theory in regions (i) and (ii) is always accompanied by an exponentially growing solution. The amplitude of this solution is exponentially small in region (ii) but becomes comparable to the decaying solution as $\xi \rightarrow 1$. This combination of exponentially growing and decaying solutions is enforced by the radiation condition and is consistent with the consequent non-zero EP flux. By retaining the exponentially small $B^{(ii)}$ (in spite of the neglect of much larger, $O(J^{-1/2})$ terms in the dominant solution), we capture this important part of the physics of the problem. An comparable situation arises for the Schrödinger equation in quantum mechanics, in the semi-classical study of wave propagation through a potential barrier (e.g., Bender and Orszag 1999). In this context, a wave-like solution radiating outside the barrier is associated with a combination of exponentially decaying and exponentially growing solutions inside the barrier. While the solution that decays towards the interior of the barrier (and corresponds to $B^{(ii)}$ in our problem) is usually neglected (Bender and Orszag 1999), it can be retained, e.g. to obtain a direct estimate of the so-called decay width (Shepard 1983).

The WKB solution is completed by matching regions (i) and (ii) to obtain

$$A^{(ii)} = \sqrt{J(1 + \nu^2)} A^{(i)} \quad \text{and} \quad B^{(ii)} = \sqrt{J(1 + \nu^2)} B^{(i)}. \quad (2.24)$$

Taking (2.15) into account and applying the jump conditions (A.1) at $\xi = 0$ yields

$$A^{(i)} = \frac{1}{2(J(1 + \nu^2))^{3/2}}, \quad (2.25)$$

on neglecting an exponentially small term against $O(1)$ terms. It then follows that

$$B^{(i)} = -\frac{i}{2(J(1 + \nu^2))^{3/2}} e^{-\sqrt{J(1 + \nu^2)}\pi} \cosh(\nu\pi), \quad (2.26)$$

which provides the amplitude of the exponentially growing solution in the quasi-geostrophic region.

The amplitude of the GW radiating at $\xi \rightarrow \pm\infty$ is found by combining (2.20), (2.21), (2.22) and (2.25). It is given by

$$|E| \sim \frac{e^{-\nu\pi/2}}{2J(1+\nu^2)} e^{-\pi\sqrt{J(1+\nu^2)}/2}. \quad (2.27)$$

This large- J approximation will be compared with the exact solution in Sec. 3 and found to provide a reasonable estimate for J as small as 1.

2.4 EP flux

An important property of (2.13) is the conservation of the EP flux (Eliassen and Palm 1961), or pseudomomentum flux. Multiplying (2.13) by $J^{3/2}(1+\nu^2)^{3/2}W^*$ and integrating by parts results in a conservation for the non-dimensional EP flux,

$$\mathcal{F} = \frac{J^{3/2}(1+\nu^2)^{3/2}}{2} \operatorname{Re} \left\{ i \frac{1-\xi^2}{\xi^2} W_\xi W^* - \nu \frac{WW^*}{\xi^2} \right\} = \text{const} \quad (2.28)$$

that is valid away from $\xi = 0, \pm 1$. The scaling factor on the left of the real part symbol is introduced so that \mathcal{F} coincides with the conventional definition

$$\rho_r \left(-\overline{u'w'} + f \frac{\overline{v'\theta'}}{\theta_{0z}} \right) \quad (2.29)$$

of the EP flux (e.g. Andrews *et al.* 1987) up to the J -independent dimensional factor (1.2).

Using the fact that $W \sim E\xi^{1/2+i\sqrt{J(1+\nu^2)-1/4}}$ for $\xi \rightarrow +\infty$ and the asymptotics in (A.14a)–(A.14b) for $\xi \ll 1$ gives

$$\mathcal{F} = \frac{J^{3/2}(1+\nu^2)^{3/2}}{2} \begin{cases} 3i(BA^* - B^*A)|E|^2/2 & \text{for } |\xi| < 1, \\ \sqrt{J(1+\nu^2)-1/4}|E|^2 & \text{for } |\xi| > 1, \end{cases} \quad (2.30)$$

where A , B and E are given explicitly in terms of Γ -functions in Appendix A. A more convenient expression is obtained by using the WKB form of W . To compute \mathcal{F} below the inertial level, we introduce (2.16), (2.25) and (2.26) into (2.28); to compute \mathcal{F} above the inertial level we introduce (2.27) into (2.30). The result is the large- J approximation

$$\mathcal{F} \sim e^{-\pi\sqrt{J(1+\nu^2)}} \begin{cases} \cosh(\nu\pi)/4 & \text{for } |\xi| < 1, \\ e^{-\nu\pi}/8 & \text{for } |\xi| > 1. \end{cases} \quad (2.31)$$

Note that the expression for $|\xi| < 1$ relies on our estimate (2.26) of the exponentially small constant $B^{(i)}$.

Eq. (2.31) shows that inertial-level absorption results in a jump in the EP flux such that

$$\frac{\mathcal{F}(1^+)}{\mathcal{F}(1^-)} \sim \frac{1}{1+e^{2\nu\pi}}. \quad (2.32)$$

This formula extends LPV10's result which showed that for $\nu = 0$ half of the EP flux is deposited at the inertial level.

3 Results for $W(\xi)$

In this section we examine the structure of $W(\xi)$ and compare the exact solution with the WKB approximation.

3.1 Vertical structure

The four panels in Fig. 1 show $W(\xi)$ for $J = 4$ and for four different orientations of the horizontal wavenumber $\mathbf{k} = (k, l) = K(\cos \varphi, \sin \varphi)$, with $\varphi = -45^\circ, -15^\circ, 15^\circ, 45^\circ$, that is for $\nu = -1, -0.267 \dots, 0.267 \dots, 1$ and $J = 4$. In all cases, the real part of W is approximated by its geostrophic estimate $W^{(i)}$ (2.16) some distance away from the neighbourhood of $\xi = 0$ where it is strictly valid. The imaginary part of W is substantially smaller than the real part, in particular near and around $\xi \approx 0$ where the quasi-geostrophic approximation predicts a purely real $W^{(i)}$. The real and imaginary parts of W only become comparable near the inertial levels, where balanced approximations do not apply.

Between (and away from) the inertial levels $\xi = \pm 1$, $\text{Im } W$ follows in quadrature $\text{Re } W$ when $\nu < 0$, but precedes $\text{Re } W$ in quadrature when $\nu > 0$. As discussed in the next subsection, this behaviour implies that the solutions always tilt along the isentropes in the (y, z) -plane. Note that this behaviour is well captured by the WKB solution in 2.17 but that can also be captured by correcting the QG solution to higher order as in Plougonven et al. (2005). Beyond $\xi = \pm 1$ the solution almost behaves as a pure gravity wave solution, in agreement with the asymptotic approximation in (2.14). The real part of the latter is shown by the grey dots in Fig. 1.

The most striking feature in Fig. 1 is the strong sensitivity of W to ν . According to the WKB estimates, ν affects the amplitude of W in three ways. First near $\xi = 0$, W decreases with increasing $|\nu|$, according to the approximation $W(0) = W^{(i)}(0) \sim (J(1 + \nu^2))^{-3/2}/2$ obtained from (2.16) and (2.25). Second, the decay rate of W in region (ii), is given by $\sqrt{J(1 + \nu^2)}$ (see (2.17)) and thus increases with $|\nu|$. Third, the amplitude of W in the GW region (iv) depends strongly on ν through the factor $e^{-\nu\pi/2}$ in (2.27). The first two effects explain the decrease in W between $\xi = \pm 1$ from Figs.1a,d to Figs.1b,c. The third effect depends on the sign of ν ; this introduces a meridional asymmetry and explains the changes from Fig. 1a to Fig. 1d and from Fig. 1b to Fig. 1c. We discuss this effect further in the next section.

3.2 Meridional asymmetry and valve effect

A strong meridional asymmetry in absorption was highlighted by Grimshaw (1975) and Yamanaka and Tanaka (1984) in their studies of GWs propagating upward toward a critical level surrounded by two inertial levels. The latter authors showed that there is a very strong absorption at the lowest inertial level if $\nu\Lambda < 0$. If $\nu\Lambda > 0$, the wave crosses the first inertial level with little attenuation, but it is almost entirely reflected downward at a turning point located between the critical level in $\xi = 0$ and the lowest inertial level in $\xi = -1$. The reflected wave is then strongly absorbed as it returns to the lowest inertial level. Even though in both scenarios the initial upward GW is ultimately absorbed at the lowest inertial level, this potential intrusion of the GW signal between the inertial levels is quite remarkable and was referred to as a ‘‘valve’’ effect by these authors. This effect was interpreted heuristically by Yamanaka (1985), who analysed with detail the behaviour of two independent solutions near the lowest inertial level. He pointed out that the phase lines of one of the solutions change direction rapidly around the inertial level, and lie between the horizontal plane and the isentropes in a narrow region. Applying a static-stability method to analyse the stability of the air parcels displaced along phase lines leads to the conclusion that, for this solution, the region is baroclinically unstable (Pedlosky 1987).

A similar heuristic argument can be invoked to explain why the absorption at the inertial

levels is much stronger for $\Lambda\nu > 0$ than $\Lambda\nu < 0$ (recall that we assume $\Lambda > 0$). If we follow Yamanaka and Tanaka (1985) and translate their description of the valve effect in our context, this sensitivity is related to the mathematical fact that around the inertial level in $\xi = 1$, the two independent solutions of (2.13),

$$W_1^{(1)} = (1 + \xi)^{-i\nu} F(a, b; a + b + 1 - c; 1 - \xi^2), \quad (3.1a)$$

$$W_1^{(2)} = (\xi - 1)^{+i\nu} F(c - b, c - a; c - a - b + 1; 1 - \xi^2), \quad (3.1b)$$

behave very differently. The first changes smoothly through $\xi = 1$ whereas the second varies sharply and jumps by a multiplicative factor equal to $e^{\nu\pi}$ in $\xi = 1$ (see the analytical continuation in (A.8)).

Following Plougonven et al. (2005), a good way to assess the significance of these two solutions is to visualize them in the (y, z) - plane (Figs. 2b, 2c, 2f, 2g). Fig. 2b indicates that the smooth solution always tilts in the direction of the isentropes. In contrast, the other solution also tilts in the direction of the isentropes for $\xi < 1$ but tilts in the other direction for $\xi > 1$. The structure of the upward waves above $\xi = 1$, namely $\xi^{1/2+i\mu} e^{i\mu y}$, also tilts in the direction of the isentropes when $\nu < 0$ but in the opposite direction when $\nu > 0$ (Figs. 2a, e). It is therefore not a surprise that the smooth solution plays the greater role to match the PV anomaly and the upward wave when $\nu < 0$, and that the other, rapidly changing solution plays the greater role when $\nu > 0$. Of course this can be checked analytically since the parameter α' and β' in (A.6b) exactly control the role of the rapidly changing and of the smooth solution in the connection through the upper critical level respectively. It turns out that $|\alpha'/\beta'| = e^{\nu\pi}$, consistent with our argument.

Interestingly, the structure of the rapidly changing and smooth solutions are not much different well below $\xi = 1$ (compare for instance the Figs 2b and 2g between $\xi = 0.4$ and $\xi = 0.6$). In fact, the two solutions have the same Taylor expansion near $\xi = 0$ up to $O(\xi^3)$. According to (A.14a) it means that as $\xi \rightarrow 0$ both can equally be used to produce the δ -PV anomaly, which is consistent with the fact that the exact solution around and above $\xi = 0$ are not much sensitive to the sign of ν (see Figs 2d and 2h).

3.3 GW amplitude and EP flux

The combined effect of the two parameters J and ν on the GW emission is shown in Fig. 3 which compares the exact values of the GW amplitude $|E|$ with the WKB approximation (2.27). For a fixed value of ν , $|E|$ decreases with J as in LPV10. For fixed values of J , the cases with $\nu > 0$ and $\nu < 0$ need to be distinguished. For $\nu > 0$ and increasing, $|E|$ decreases monotonically as a result of increasing exponential decay in region (ii) and increasing inertial-level absorption. When $\nu < 0$ those two effects oppose: increasing $|\nu|$ increases the exponential decay but decreases the inertial-level absorption. Accordingly, $|E|$ is maximized for some $\nu(J) < 0$. Importantly, the WKB approximation (2.27) provide a good approximation for $|E|$ for $J \gtrsim 1$, thus well beyond the theoretical range of validity $J \gg 1$ of the asymptotics.

The EP flux within and outside the inertial levels are shown in the Figs. 4a and 4b respectively. The exact and WKB solutions (2.31) are compared. Fig. 4a indicates that the EP flux between the inertial levels is only weakly sensitive to the angle $\varphi = \tan^{-1} \nu$ of the wavevector. It remains almost constant, for instance, for $-45^\circ \lesssim \varphi \lesssim 45^\circ$ when $J \approx 3$, or for $-30^\circ \lesssim \varphi \lesssim +30^\circ$ and $J \approx 10$. For larger values of φ , however, the EP flux decreases rapidly

and vanishes for $\varphi = \pm\pi/2$. An important aspect of Fig. 4a is that the EP flux between the inertial levels is symmetric about the axis $\varphi = 0$.

This symmetry is broken outside the inertial levels, i.e. for $|\xi| > 1$, as a result of the asymmetric absorption at the inertial levels. This is clear from Fig. 4b: when $\varphi \lesssim -30^\circ$ the EP fluxes for $|\xi| > 1$ are almost equal to the fluxes for $|\xi| < 1$, but they are much smaller for $\varphi \gtrsim 30^\circ$. This is well captured by the WKB approximation (2.31) which again provides a good estimate for $J \gtrsim 1$ both for $|\xi| < 1$ and $|\xi| > 1$. In particular, it leads to the prediction

$$\nu_M \approx -\frac{1}{\sqrt{J}}, \quad (3.2)$$

for the value of ν for which \mathcal{F} is maximum for $|\xi| > 1$. The corresponding angle φ_M is shown as a dotted line in Fig. 4b.

4 Application to localised PV distributions

4.1 Horizontally localised δ -PV

To gauge the significance of the directional effects discussed above on the structure of the GWs associated with a 3D PV anomaly, we first consider the case of an infinitely thin PV distribution with Gaussian distribution in the horizontal direction:

$$q'_0(x, y, z) = \sigma_z q_r e^{-(x^2+y^2)/(2\sigma_H^2)} \delta(z), \quad (4.1)$$

where σ_H gives the characteristic horizontal width of the PV anomaly, q_r its characteristic amplitude, and σ_z its characteristic depth. The introduction of the scale σ_z naturally follows from the fact that $\delta(z)$ scales as an inverse length. For such a distribution, the vertical velocity field in (2.9) reads

$$w'(x, y, z) = \sigma_z \int_0^\infty \int_0^{2\pi} \hat{w}_0(K, \varphi) e^{i(kx+ly)} W\left(\varphi; \frac{k\Lambda z}{f}\right) K d\varphi dK = \sigma_z w'_0(x, y, z), \quad (4.2)$$

where for clarity we have made explicit the dependence of W on the angle φ , and where

$$\hat{w}_0(K, \varphi) = -i \frac{\rho_r g (1 + \tan^2 \varphi)}{\theta_r \Lambda^2} \frac{q_r \sigma_H^2}{2\pi} e^{-\frac{K^2 \sigma_H^2}{2}}, \quad (4.3)$$

according to the scaling in (2.12) and introducing the Fourier transform of (4.1). Note that w'_0 is defined in (4.2) to simplify the notation in the full 3D case treated at the end of the section.

To evaluate the double integral in (4.2) we next proceed numerically by tabulating in the vertical direction the structure function $W(\varphi; \xi)$ for 180 discrete values of the angle φ . This yields an angular resolution $\Delta\varphi = 2^\circ$. We also consider 50 discrete values for the horizontal wavenumber K , with a resolution $\Delta K = \pi/(10\sigma_H)$. For the physical grid we take for both horizontal directions the resolution $\Delta x = \Delta y = 0.2\sigma_H$.

In the following, we express our results in dimensional form. We consider a $\sigma_z = 1$ km-thick layer of stratospheric air entering in the troposphere. We therefore take a PV amplitude of $\rho_r q_r = 1$ PVU, and assume a horizontal width $\sigma_H = 55$ km. Assuming that this air enters the troposphere at midlatitudes, we take $\rho_r = 1$ kg m $^{-3}$, $N = 0.01$ s $^{-1}$, $\theta_r = 300$ K, $f = 10^{-4}$ s $^{-1}$ and $J = 4$.

4.2 Vertical velocity field

The vertical velocity calculated from (4.2) is shown in Fig. 5 for 6 different altitudes. Near the PV anomaly, that is for $z = 0$ km (Fig. 5a), the vertical velocity is positive to the east of the positive PV anomaly and negative to the west. This is of course consistent with the balanced picture that the meridional geostrophic winds are toward the North on the Eastern flank of a positive PV anomaly, and to the South on the Western flank (not shown). As the advective terms are very small in the thermodynamic Eq. (2.1d) at this altitude, the vertical velocity balances the meridional advection of the background potential temperature ($f\Lambda v' \approx N^2 w'$). At the higher altitude $z = 1$ km (Fig 5b), the signal in vertical velocity decays in magnitude and spreads in horizontal scale, consistent with the QG predictions that all wavelengths decay exponentially with altitude, with the long wavelengths decaying less rapidly than the short ones. Note, however, the two large-scale lobes of opposite sign of the vertical velocity which have moved slightly to the North, which is a first sign that the QG prediction starts to break down (the QG prediction is insensitive to the sign of ν , see Appendix B). At $z = 2$ km (Fig. 5c), the signal in vertical velocity has decayed further in magnitude and spread further horizontally (note the contour-interval decrease between Fig. 5b and Fig. 5c), again somehow in agreement with the QG prediction. Nevertheless, the two large scale lobes of vertical velocity start to be modulated by a smaller scale oscillatory signal, clearly apparent aloft the PV disturbance. Higher up in altitude this oscillatory signal entirely dominates the response, its lines of constant phase make a positive angle with the longitude axis, because the waves with $\nu < 0$ are less absorbed at the inertial levels than those with $\nu > 0$. Note also that the amplitude between $z = 3$ km and $z = 10$ km increases in agreement with the $z^{1/2}$ dependence predicted in (2.27). Because of the superposition of wavenumbers, the transition between decaying and wave-like perturbation does not occur sharply at a single inertial level but rather smoothly across an inertial-layer region. The altitude of the centre of this region is given by the estimate $\sigma_H f / \Lambda \approx 1.1$ km, consistent with Fig. 5.

4.3 EP-flux vector

The EP flux in (2.29) is significant because its vertical derivative gives the x -component of the force exerted by the GWs on the (transformed Eulerian) mean flow (Andrews *et al.* 1987). Because our model (2.1) is both x - and y -independent and the GWs are plane waves in both directions, the two horizontal components of the force can in fact be obtained from the EP-flux *vector* (or, up to a sign, vertical pseudomomentum-flux vector)

$$- \left(\overline{u'w'} - f \frac{\overline{v'\theta'}}{\theta_{0z}}, \overline{v'w'} + f \frac{\overline{u'\theta'}}{\theta_{0z}} \right) \quad (4.4)$$

(see Bühler 2009, section 8.2, and the discussion on the angular momentum flux in Jones 1967). The two components of this EP-flux vector are in proportion to k and l , since these are the proportion of the x - and y -components of the corresponding pseudomomentum density (Bühler 2009). The non-dimensional EP-flux vector can therefore be written as $(1, l/k)\mathcal{F}$.

This can be made transparent using Bretherton's (1969) interpretation of the EP flux as the wave stress exerted by pressure force on undulating isentropes:

$$\mathbf{F} = \frac{1}{2\sigma_H^2} \iint p' \nabla \eta' \, dx dy = \frac{2\pi^2}{\sigma_H^2} \iint -i \mathbf{k} \hat{p} \hat{\eta}^* \, dk dl, \quad (4.5)$$

where η' denotes the vertical displacement satisfying $D_t \eta' = w'$, and the factor $1/(2\sigma_H^2)$ is introduced in the definition of the average so that \mathbf{F} has the dimension of a pressure. Using that $ik\Lambda z \hat{\eta} = \hat{w}$, and the disturbance equations (2.1), we obtain that

$$-i\mathbf{k}\hat{p}\hat{\eta}^* = -\rho_r \left(\hat{w}^* - f \frac{\hat{v}\hat{\theta}^*}{\theta_{0z}}, \hat{v}\hat{w}^* + f \frac{\hat{u}\hat{\theta}^*}{\theta_{0z}}, \right) \quad (4.6)$$

in agreement with (4.4). Using the scaling in (2.10) and the structure function (2.11a) this relation leads to the EP-flux vector of a single plane wave,

$$-i\mathbf{k}\hat{p}\hat{\eta}^* = \frac{\mathbf{k}}{k} \frac{\rho_r \Lambda \sigma_z^2 |\hat{w}_0|^2}{f(1+\nu^2)} \left(i \frac{1-\xi^2}{\xi^2} W_\xi W^* - \nu \frac{WW^*}{\xi^2} \right) = \frac{\rho_r \Lambda^4 \sigma_z^2 |\hat{w}_0|^2}{N^3 f (1+\nu^2)^2} \frac{\mathbf{k}}{K} \mathcal{F} \quad (4.7)$$

where \mathcal{F} is the normalized EP flux (2.28). For the localised PV distribution (4.3) the EP-flux vector then becomes

$$\mathbf{F} = F_0 \int_0^\infty \int_0^{2\pi} \sigma_H^2 \mathbf{k} e^{-K^2 \sigma_H^2} \mathcal{F}(\varphi, \frac{k\Lambda z}{f}) d\varphi dK. \quad (4.8)$$

The scaling factor F_0 is given in (1.2) and is exactly the same as in LPV10. For the parameters chosen, it is about

$$F_0 = 10 \text{ Pa} \quad (4.9)$$

and directly gives the amplitude of the EP-flux vector since the double integral in (4.8) is non-dimensional.

The exact results for the EP-flux vector \mathbf{F} in (4.8) is shown in Fig. 6 and for two different values of the Richardson number J . For $J = 4$ (Fig. 6a), \mathbf{F} at $z = 0$ is purely zonal, with a magnitude near 5 mPa. The zonal orientation follows from the symmetry of the PV distribution about the x -axis. At higher altitudes, \mathbf{F} decreases in amplitude and changes direction. These two effects result from the absorption of an increasingly large portion of the wave spectrum at inertial levels, and from the fact that waves with $\nu < 0$ are much less absorbed than those with $\nu > 0$. When $J = 4$ \mathbf{F} as $z \rightarrow \infty$ makes an angle close to $\varphi \approx -30^\circ$ with the x -axis, almost the angle for which the normalized EP flux has a maximum according to (3.2) (see also Fig. 4b). For $J = 10$ (Fig. 6b), \mathbf{F} in the far field has an amplitude that is about half that at $z = 0$, and makes an angle with the x -axis that is close to $\varphi \approx -15^\circ$, a value again consistent with (3.2).

For practical purposes it is useful to estimate the EP-flux vector near the PV anomaly and in the far-field using the WKB form for \mathcal{F} in (2.31). Introducing (2.31), (4.8) becomes

$$\mathbf{F}(0^+) \approx \frac{F_0}{4} \int_{-\pi/2}^{\pi/2} \cos \varphi \hat{\mathbf{x}} e^{-\pi \sqrt{J(1+\nu^2)} - \nu\pi} d\varphi \quad (4.10a)$$

$$\mathbf{F}(\infty) \approx \frac{F_0}{8} \int_{-\pi/2}^{\pi/2} (\cos \varphi \hat{\mathbf{x}} + \sin \varphi \hat{\mathbf{y}}) e^{-\pi \sqrt{J(1+\nu^2)} - \nu\pi} d\varphi, \quad (4.10b)$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the zonal and meridional unit vectors. Since the WKB approximation assumes $J \gg 1$ is large, these expressions can be further simplified using Laplace's method to obtain

$$\mathbf{F}(0^+) \approx \frac{F_0}{2\sqrt{2\sqrt{J}}} e^{-\pi\sqrt{J}} \hat{\mathbf{x}}, \quad \mathbf{F}(\infty) \approx \frac{F_0}{4\sqrt{2\sqrt{J}}} e^{-\pi\sqrt{J}} (\hat{\mathbf{x}} - J^{-1/2}\hat{\mathbf{y}}). \quad (4.11)$$

These formula give very good approximations for the EP-flux vector for $z \ll 1$ and $z \gg 1$ when $J \gg 1$ as in Fig. 6b, but underestimate it by a factor of almost 2 when $J \approx 1$.

Comparing (4.11) to the 2D results in LPV10 (see also (1.1)–(1.2)) shows that the orders of magnitude of \mathbf{F} are comparable in 2D and 3D (the F_0 term), and that in both cases about half of the EP flux in the direction of the shear is deposited in the inertial layer. The most remarkable difference is that the EP-flux vector rotates with altitude. Consider for example a westerly shear in the Northern Hemisphere: the EP-flux vector tends, for large z , to an angle close to $\varphi_M = -1/\sqrt{J}$ to the right of the shear. As a consequence, the disturbance produced by the PV anomaly exerts a south-westward drag on the large-scale flow in the upper inertial layer and, assuming dissipation at high altitude, a north-westward drag in the far field aloft. Both drags are almost equal in the direction of the shear (as in the 2D case) and opposite in the transverse direction. More generally, the transverse component of the drag in the inertial layers are to the right of the wind (e.g. northward at the lower inertial layer in the above example), and to the left in the Southern Hemisphere.

4.4 Horizontally localised, finite depth PV

The PV distributions used so far were infinitely thin and thus neglected the effect of the vertical shear on the PV distribution itself and consequent time evolution of the wave field. As this aspect has been detailed in the 2D case by LPV10, we only describe it briefly here. As in LPV10, we consider a PV distribution at $t = 0$ that is separable in the horizontal and the vertical directions and that has the same vertical integral as (4.1):

$$q'_0(x, y, z) = q_r e^{-(x^2+y^2)/(2\sigma_H^2)} \begin{cases} \cos^2(\pi z/(2\sigma_z)) & \text{for } |z| < \sigma_z \\ 0 & |z| > \sigma_z \end{cases} \quad (4.12)$$

To compute the vertical integral in the response at a low numerical cost, we take full advantage of the preceding calculations and discretize q'_0 as

$$q'_0 \approx q_r e^{-(x^2+y^2)/(2\sigma_H^2)} \sum_{-M+1}^{M-1} \cos^2(\pi z_m/(2\sigma_z)) \Delta z \delta(z - z_m) \quad (4.13)$$

where $z_m = m\Delta z$ and $\Delta z = \sigma_z/M$. In this case, the vertically discretized equivalent of the vertical velocity in (2.9) reduces to

$$w'(x, y, z, t) \approx \sum_{-M+1}^{M-1} \cos^2\left(\frac{\pi z_m}{2\sigma_z} \Delta z\right) w'_0(x - \Lambda z_m t, y, z - z_m) \quad (4.14)$$

where w'_0 is the function introduced in (4.2). Because x and t enter (4.14) only in the combination $x - \Lambda z_m t$ the computation of the sum over the indices m involves straightforward vertical and horizontal translations of w'_0 .

Fig. 7 shows the evolution of the integral of the disturbance PV, $\int_{-\infty}^{+\infty} q'(x, y, z, t) dz$, and of the vertical velocity at the altitude $z = 10$ km, for $\sigma_z = 1$ km, and $J = 4$. All the other parameters are as in the previous sections. The solution is only shown for negative values of t :

for positive t it is almost symmetric to that at negative t . The background velocity shears the PV whose horizontal extent therefore decreases with time until $t = 0$ before increasing again. When it is more spread out horizontally (that is at large negative or positive time) its vertical integral is also relatively small compared to its value at $t = 0$. As a result, the vertical velocity increases as t increases towards 0.

Comparing the four panels in Fig. 7 to the time-independent disturbance produced by the δ -PV of Fig. 5 indicates that the amplitude of the GW patterns are comparable at $t = 0$ h and $t = \pm 6$ h but substantially smaller at $t = \pm 12$ h and $t = \pm 18$ h. Accordingly, it is only in time intervals of half a day or so that the values for GWs emission and for the associated EP flux given in the previous sections apply.

5 Conclusion

The linear motion associated with 3D localized potential vorticity (PV) anomalies in the presence of an unbounded vertical shear Λ has been analysed in the linear approximation. Exact and approximate solutions were obtained analytically for PV anomalies that are monochromatic in x and y , and vary as a Dirac delta-function, $\delta(z)$, in the vertical. Combinations of these yield solutions for more general PV anomalies.

A PV anomaly of horizontal scale σ_H , at $z=0$, induces two inertial critical layers at $z = \pm \sigma_H f / \Lambda$. Through these levels, the intrinsic frequency of the disturbance increases from subinertial to superinertial. Correspondingly, there is a transition from balanced near $z = 0$ (where the solutions can be described as quasi-geostrophic) to sheared GW for $|z| > \sigma_H f / \lambda$. The amplitude of the GW is approximately

$$\exp(-\sqrt{J(1+\nu^2)}\pi/2 - \nu\pi)/J/(1+\nu^2),$$

where $J = N^2/\Lambda^2$ is the Richardson number, and $\nu = l/k$ the ratio of the y - and x -components of the wave vector. As previously noted (Lott et al 2010), these waves can be substantial for moderate Richardson numbers, say J between 1 and 10. The present analysis reveals a new, remarkable result: the emitted waves have a strong meridional asymmetry, with larger amplitudes for waves with $\nu < 0$. For example, in a westerly shear in the northern hemisphere, waves aloft having their wave vector pointing southeast will be larger than waves with wave vector pointing northeast (see Figs. 5 and 7). Using the exact analytical solutions we show how this asymmetry, in a symmetric background flow, is related to the meridional slope of the isentropes (see Fig. 2). This asymmetry has been identified previously in studies of gravity waves propagating toward critical levels in a constant shear (the ‘‘valve effect’’, Yamanaka and Tanaka 1984).

One implication consists in a strong sensitivity to orientation (i.e. to ν) of the absorption of the Eliassen–Palm flux through the inertial levels: almost no jump when ν is large and negative, in contrast to nearly complete absorption when ν is large and positive (see Eqs. (2.31)–(2.32) and Fig. 4). Hence the drag due to the waves absorbed within the upper inertial layer has a substantial component oriented to the right of the shear in the Northern Hemisphere (southeast in the above example). The WKB solutions provide simple expressions for the fluxes, the angle maximizing them and the drag, in very good agreement with the exact analytical solutions (see Fig. 4).

The relevance of this emission in real flows remains to be assessed. Nonetheless, two points are worth noting: first, it has been noted from satellite observations (Wu and Eckermann, 2008) and from high-resolution NWP (Numerical Weather Prediction) models (Shutts and Vosper, 2011) that gravity waves in the mid-latitudes have a favored orientation: phaselines with a North-East to South-West tilt in the Northern Hemisphere, and with a North-West to South-East tilt in the Southern Hemisphere. Waves with these orientations are conspicuous in the stratospheric polar night jets of both hemispheres, i.e. in regions with strong, positive vertical shear. The reasons for this favored orientation are not clear¹. It is noteworthy that this orientation is consistent with that expected in the case of emission from sheared PV anomalies. Whether this emission is occurring or this is only a coincidence due to a more fundamental property of GW in shear remains to be investigated. Second, at smaller scales, we can expect this mechanism to play a role where the breaking of intense orographic gravity waves produces small-scale PV anomalies (Plougonven et al 2010).

As discussed in LPV10, our results could be used for parameterizations in GCMs of the GWs emission by fronts at the tropopause (Charron and Manzini 2002, Richter et al. 2010), where substantial intrusion of stratospheric air occurs and where strong shears are common. In this context the predictor given in LPV10 seems adapted, providing we add the transverse component of the EP flux as in (4.11). The factor 1/4 for the flux emitted by 2D PV disturbance in (1.1) should more accurately be $1/(2\sqrt{2\sqrt{J}})$ (see (4.11)), but they should probably be replaced by a tuning factor of order 1. In all cases the along-shear component of the EP-flux vector should decrease by a factor 2 at the inertial levels and the transverse component in the far field should be oriented to the right of the shear with magnitude $1/\sqrt{J}$ times the along-shear component.

To summarize, the present paper has shown that the formula given in LPV10 and that are recalled here in (1.1)–(1.2) applies quite well in the 3D case. To take directional effects into account one should use (4.11) rather than (1.1) keeping unchanged the dimensional factor (1.2).

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A Exact solution for $W(\xi)$

To find a solution to (2.13), we first derive its homogeneous solutions for $\xi > 0$, and impose a radiation condition for $\xi \gg 1$ to obtain a solution which represents an upward-propagating GW. We deduce from this a solution valid for $\xi < 0$ which represents a downward-propagating GW for $\xi \ll -1$. The amplitudes of these two solutions are then chosen to satisfy the jump conditions

$$[W]_{0^-}^{0^+} = 0, \quad \text{and} \quad \left[\frac{W_\xi}{\xi^2} \right]_{0^-}^{0^+} = 1. \quad (\text{A.1})$$

¹Shutts and Vosper (2011) suggested that this tilt could be tied to the orientation of surface cold fronts, but gravity waves generated in idealized baroclinic lifecycles (O’Sullivan and Dunkerton 1995, Plougonven and Snyder 2007) show the opposite tilt.

A.1 Homogeneous solution for $\xi > 0$

The changes of variables $W = (1 + \xi)^{-i\nu} Y$ and $\eta = \xi^2$, transform (2.13) into the canonical form of the hypergeometric equation (Eq. 15.5.1 in Abramowitz and Stegun 1964, hereafter AS):

$$\eta(1 - \eta)Y_{\eta\eta} + (c - (a + b + 1)\eta)Y_{\eta} - abY = 0, \quad (\text{A.2a})$$

$$\text{where } a = -\frac{1}{4} - \frac{i}{2}\nu + \frac{i}{2}\mu, \quad b = -\frac{1}{4} - \frac{i}{2}\nu - \frac{i}{2}\mu, \quad c = -\frac{1}{2} \quad (\text{A.2b})$$

$$\text{and } \mu = \sqrt{J(1 + \nu^2) - 1/4}. \quad (\text{A.2c})$$

For $\xi > 1$ the two solutions of the hypergeometric equation (A.2a) are given by (15.5.7) and (15.5.8) in AS. We retain the second solution

$$W^{(u)}(\xi) = (1 + \xi)^{-i\nu} \xi^{-2b} F(a', b'; c'; \xi^{-2}), \quad (\text{A.3})$$

where F is the hypergeometric function and $a' = b$, $b' = b - c + 1$, $c' = b - a + 1$, because its asymptotic form

$$W^{(u)}(\xi) \sim \xi^{1/2+i\mu} \quad \text{as } \xi \rightarrow \infty, \quad (\text{A.4})$$

corresponds to an upward propagating GW (Booker and Bretherton, 1967).

For $0 < \xi < 1$ the solution to (A.2a) is best written as a linear combination of the two independent solutions (15.5.3)–(15.5.4) in AS,

$$W^{(u)}(\xi) = (1 + \xi)^{-i\nu} [A F(a, b; c; \xi^2) + B \xi^3 F(a'', b''; c''; \xi^2)] \quad (\text{A.5})$$

where $a'' = a - c + 1$, $b'' = b - c + 1$, $c'' = 2 - c$, and A and B are two complex constants.

To connect (A.5) to (A.3), we use the transformation formula for F (15.3.6 in AS) and obtain the asymptotic approximations

$$W^{(u)}(\xi) \sim \alpha'(\xi - 1)^{i\nu} + \beta' \quad \text{as } \xi \rightarrow 1^+, \quad (\text{A.6a})$$

$$W^{(u)}(\xi) \sim (\alpha A + \alpha'' B)(1 - \xi)^{i\nu} + \beta A + \beta'' B \quad \text{as } \xi \rightarrow 1^-. \quad (\text{A.6b})$$

In these expressions,

$$\alpha = 2^{i\nu} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \quad \text{and} \quad \beta = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (\text{A.7})$$

where Γ is the gamma function (see AS, chapter 6). The other coefficients (α', β') and (α'', β'') are defined by the same formulas with (a, b) replaced by (a', b') and (a'', b'') respectively.

To continue the solution (A.6a) below the inertial level at $\xi = 1$, we follow Booker and Bretherton (1967) and introduce a infinitely small linear damping which shifts the real ξ -axis into the lower half of the complex plane so that

$$\xi - 1 = (1 - \xi)e^{-i\pi} \quad \text{for } \xi < 1. \quad (\text{A.8})$$

Thus, (A.6a) matches (A.6b) provided that

$$\alpha A + \alpha'' B = \alpha' e^{\nu\pi} \quad \text{and} \quad \beta A + \beta'' B = \beta'. \quad (\text{A.9})$$

This determines A and B and completes the evaluation of $W^{(u)}(\xi)$.

A.2 Solution over the entire domain

The solution for $\xi < 0$ can be deduced from $W^{(u)}(\xi)$ by noting that (2.13) applies to W^* when ξ is changed in $-\xi$. A possible solution is simply

$$W^{(d)}(\xi) = W^{(u)}(-\xi)^*. \quad (\text{A.10})$$

This satisfies the radiation condition for $\xi \rightarrow -\infty$ since

$$W^{(d)}(\xi) \sim |\xi|^{1/2-i\mu}, \quad (\text{A.11})$$

which represents a downward-propagating GW.

The two solutions $W^{(u)}$ and $W^{(d)}$ can be combined to obtain a solution valid over the entire domain which satisfies the jump condition (A.1). This is given by

$$W(\xi) = \begin{cases} EW^{(u)}(\xi) & \text{for } \xi > 0 \\ E^*W^{(d)}(\xi) & \text{for } \xi < 0 \end{cases}, \quad (\text{A.12})$$

where the constant E is found by imposing the jump condition (A.1) and given by

$$E = \frac{A^*}{3(AB^* + A^*B)}. \quad (\text{A.13})$$

To verify this, we note that when $|\xi| \ll 1$, the upper and lower solutions in (A.12) have the asymptotic expansions

$$EA \left(1 - i\nu\xi - \left(\frac{\mu^2}{2} + \frac{1}{8} \right) \xi^2 - i\nu \left(\frac{\nu^2}{3} - \frac{\mu^2}{2} + \frac{5}{24} \right) \xi^3 \right) + EB\xi^3, \quad (\text{A.14a})$$

$$E^*A^* \left(1 - i\nu\xi - \left(\frac{\mu^2}{2} + \frac{1}{8} \right) \xi^2 - i\nu \left(\frac{\nu^2}{3} - \frac{\mu^2}{2} + \frac{5}{24} \right) \xi^3 \right) - E^*B^*\xi^3 \quad (\text{A.14b})$$

for $\xi > 0$ and $\xi < 0$ respectively. For the value of E in (A.13), EA is real which implies that the first terms on the left-hand sides of (A.14a)–(A.14b) are identical, they ensure that $[W]_{0^-}^{0^+} = 0$ and they do not contribute to the jump $[W_\xi/\xi^2]_{0^-}^{0^+} = 1$ in (A.1); the second terms combine so that W_ξ/ξ^2 jumps by 1 at $\xi = 0$ as required. Note also that near $|\xi| = 0$, $W(\xi)$ approaches the value

$$W(0) = \frac{A^*A}{3(AB^* + A^*B)}. \quad (\text{A.15})$$

B WKB approximation

In this Appendix, we derive the WKB approximations (2.16)–(2.19) for W in regions (i)–(iv) and provide details of the matching procedure.

For region (i), we introduce $\zeta = \sqrt{J(1+\nu^2)}\xi = O(1)$ into (2.13) to obtain at leading order the geostrophic approximation

$$\frac{W_{\zeta\zeta}}{\zeta^2} - 2\frac{W_\zeta}{\zeta^3} - \frac{W}{\zeta^2} = \frac{\delta(\zeta)}{(J(1+\nu^2))^{3/2}}. \quad (\text{B.1})$$

The solution is readily found and given in terms of $\xi > 0$ by (2.16).

For regions (ii) and (iv), the standard WKB expansion

$$W(\xi) = (W_0 + J^{-1/2}W_1 + \dots) e^{\sqrt{J(1+\nu^2)} \int^\xi \phi(\xi') d\xi'} \quad (\text{B.2})$$

gives at $O(J)$ and $O(J^{1/2})$

$$\phi = \frac{\pm 1}{\sqrt{1-\xi^2}} \quad \text{and} \quad W_0 = \frac{\xi}{(\xi-1)^{1/4-i\nu/2}(\xi+1)^{1/4+i\nu/2}}, \quad (\text{B.3})$$

respectively. The forms (2.17) and (2.19) of the solution follow immediately.

For the region (iii), finally, we introduce the variable $\zeta = J(1+\nu^2)(\xi-1) = O(1)$. To leading order (2.13) then reduces to

$$2\zeta W_{\zeta\zeta} + 2(1-i\nu)W_\zeta + W = 0 \quad (\text{B.4})$$

whose solution is given in terms of Hankel functions in (2.19). The following asymptotic formulas (9.2.3–4 in AS) are needed to match this solution to the solutions in regions (ii) and (iv):

$$H_{i\nu}^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{\nu\pi/2} e^{i(x-\pi/4)} \quad \text{as } |x| \rightarrow \infty \quad \text{for } -\pi < \arg x < 2\pi, \quad (\text{B.5})$$

$$\text{and } H_{i\nu}^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{-\nu\pi/2} e^{-i(x-\pi/4)} \quad \text{as } |x| \rightarrow \infty \quad \text{for } -2\pi < \arg x < \pi. \quad (\text{B.6})$$

Using these, we find from (2.18) that

$$W \sim \frac{2^{1/4}}{\sqrt{\pi}\zeta^{1/4-i\nu/2}} \left[A^{(\text{iii})} e^{\nu\pi/2} e^{i(\sqrt{2\zeta}-\pi/4)} + B^{(\text{iii})} e^{-\nu\pi/2} e^{-i(\sqrt{2\zeta}-\pi/4)} \right] \quad (\text{B.7})$$

as $\zeta \rightarrow \infty$. Matching with the limiting behaviour of (2.16) as $\xi \rightarrow 1$ gives (2.21).

To match the solutions between regions (ii) and (iii), we need to consider the limit of the Hankel functions for $\zeta = |\zeta|e^{-i\pi}$ with $|\zeta| \rightarrow \infty$, in accordance with the analytic continuation (A.8). Proceeding in similar fashion as above using (B.5) yields relation (2.22) between $A^{(\text{ii})}$ and $A^{(\text{iii})}$, but $B^{(\text{ii})} = 0$. As mentioned, $B^{(\text{ii})} = 0$ is inconsistent with the non-zero EP flux expected because of the wave radiation as $\xi \rightarrow \infty$. To resolve this difficulty, we need to employ a more sophisticated matching which recognises that $B^{(\text{ii})}$ takes in fact a non-zero exponentially small value and provides an estimate for this value.

The non-zero value of $B^{(\text{ii})}$ arises as a result of a Stokes phenomenon (e.g. Ablowitz and Fokas 1997): the line $\zeta < 0$ ($\arg \zeta = -\pi$) is a Stokes line, where one solution (here that multiplied by $A^{(\text{ii})}$) is maximally dominant over the other, recessive solution (that multiplied by $B^{(\text{ii})}$). Across this Stokes line, the dominant solution switches on the recessive solution with an amplitude given by an exponentially small Stokes multiplier. Thus, below the Stokes line the amplitude $B^{(\text{ii})} = 0$, and above it $B^{(\text{ii})} \neq 0$ is given by the Stokes multiplier; on the Stokes line itself, $B^{(\text{ii})}$ is half the Stokes multiplier (Berry 1989). To obtain the Stokes multiplier, we need a large- $|\zeta|$ formula for (2.18) that is valid for $-3\pi < \arg \zeta < -\pi$ so that it holds immediately above the Stokes line and also on the anti-Stokes line $\arg \zeta = -2\pi$ where the two solutions have

the same order and hence can be identified unambiguously. Such a formula is obtained using the connection formula (9.1.37 in AS) to obtain

$$H_{i\nu}^{(1)}(\sqrt{2\zeta}) = 2 \cosh(\nu\pi) H_{i\nu}^{(1)}(e^{i\pi}\sqrt{2\zeta}) + e^{\nu\pi} H_{i\nu}^{(2)}(e^{i\pi}\sqrt{2\zeta}). \quad (\text{B.8})$$

For $-3\pi < \arg \zeta < -\pi$, $-\pi/2 < \arg(e^{i\pi}\sqrt{2\zeta}) < \pi/2$, and the formulas (B.5)–(B.6) can be applied to obtain the large- $|\zeta|$ asymptotics

$$H_{i\nu}^{(1)}(\sqrt{2\zeta}) \sim \frac{2^{1/4}}{\sqrt{\pi}\zeta^{1/4}} e^{\nu\pi/2} \left[2 \cosh(\nu\pi) e^{-3i\pi/4} e^{-i\sqrt{2\zeta}} + e^{-i\pi/4} e^{i\sqrt{2\zeta}} \right]. \quad (\text{B.9})$$

Introducing this into (2.18) with $B^{(\text{iii})} = 0$ and using that $\zeta = J(1 + \nu^2)(1 - \xi)e^{-i\pi}$ leads to

$$W \sim \frac{2^{1/4} e^{\nu\pi} (J(1 + \nu^2))^{-1/4 + i\nu/2}}{\sqrt{\pi} (1 - \xi)^{1/4 - i\nu/2}} A^{(\text{iii})} \left[e^{\sqrt{J(1+\nu^2)}\sqrt{2(1-\xi)}} - 2i \cosh(\nu\pi) e^{-\sqrt{J(1+\nu^2)}\sqrt{2(1-\xi)}} \right]. \quad (\text{B.10})$$

Matching with the limit of (2.17) as $\xi \rightarrow 1$ gives (2.22), and (2.23) on taking into account that Stokes multiplier on the Stokes line is half its value away from it.

The matching between regions (i) and (ii) yielding (2.25)–(2.26) is straightforward.

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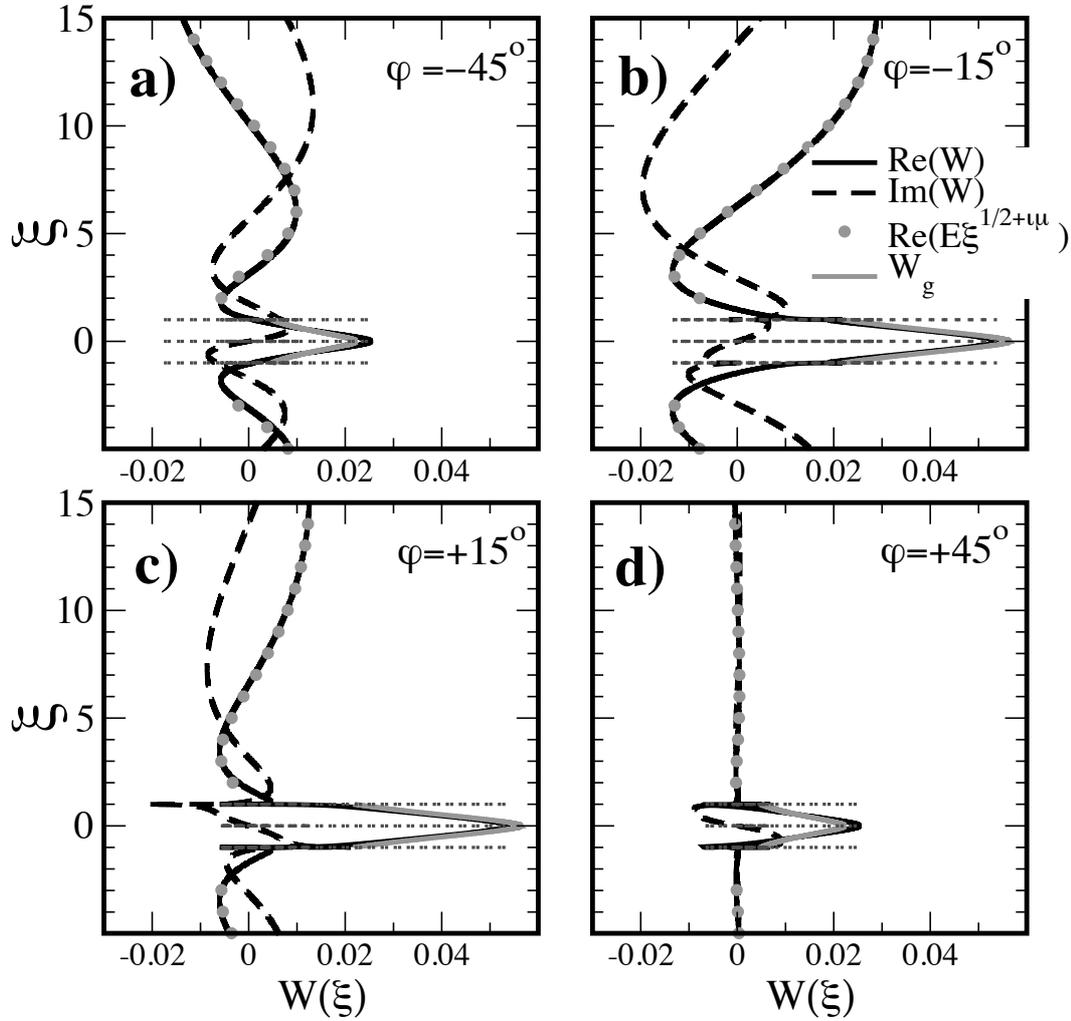


Figure 1: Structure function $W(\xi)$ associated with a monochromatic PV distribution proportional to $\delta(\xi)$ for a Richardson number $J = 4$, and for different values of the wavenumber angle $\varphi = \tan^{-1}(\nu)$, (a) -45° , (b) -15° , (c) 15° , and (d) 45° . The thick black curves and thick dashed curves show the real and imaginary parts of $W(\xi)$, respectively. The grey dotted curves show the real part of the far-field gravity-wave approximation $E\xi^{1/2+i\mu}$, and the thick grey curves show the quasi-geostrophic approximation $W^{(i)}$. The location of $\xi = 0$ and of the inertia levels $\xi = \pm 1$ is also indicated.

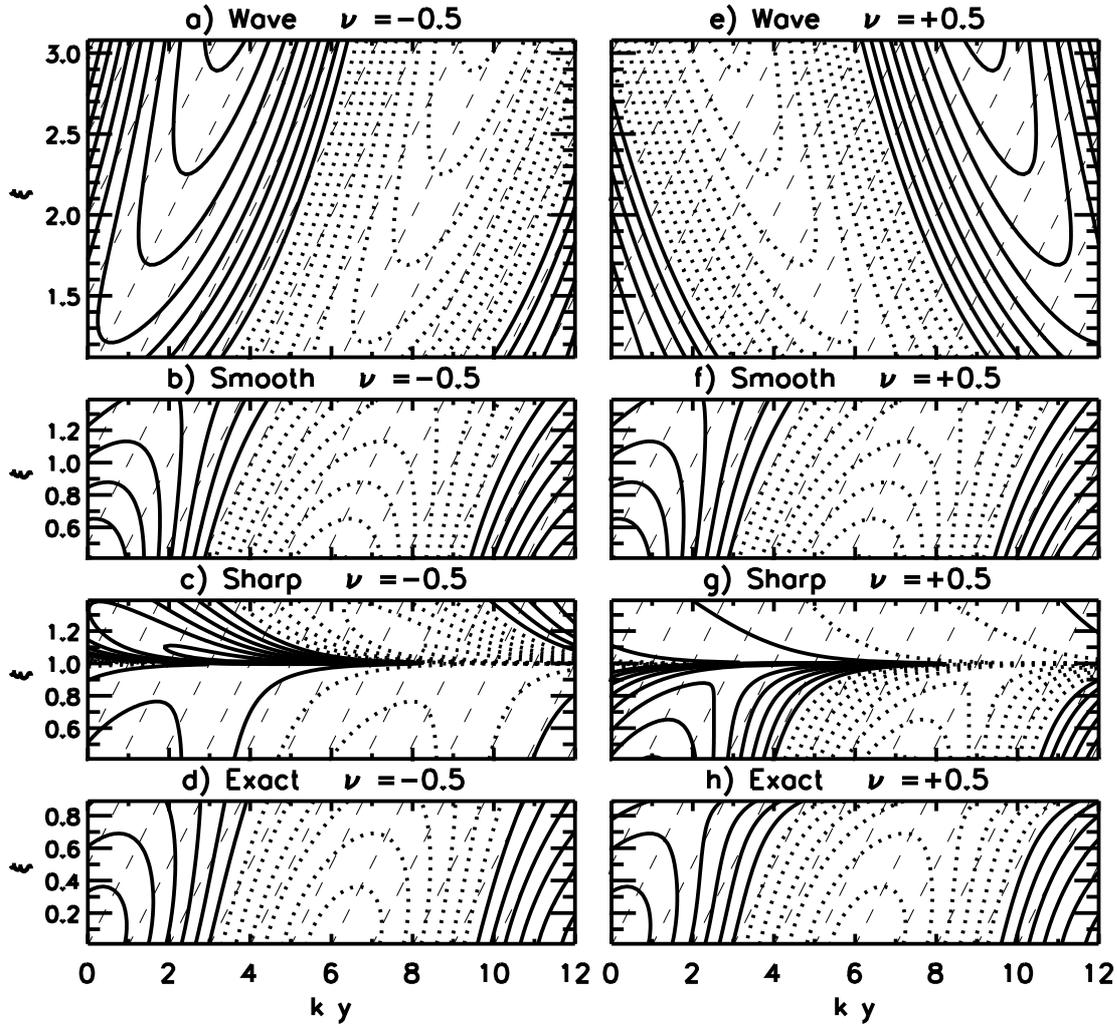


Figure 2: Meridional structures of various solutions and approximations used to analyse $W(\xi)$ at the transition through the inertial level, for $J = 2$, and $\nu = \pm 0.5$. (a), (e): GW asymptotics $\xi^{1/2+i\mu}e^{i\ell y}$; (b), (f): smooth solution (3.1a) $W_1^{(1)}(\xi)e^{i\ell y}$; (c), (g): sharp solution (3.1b) $W_1^{(2)}(\xi)e^{i\ell y}$; (d), (h): exact solutions (A.12). The contour intervals are arbitrary but are identical between the panels (d) and (h). In all panels, the slope of the isentropes is indicated by the gray dashed lines.

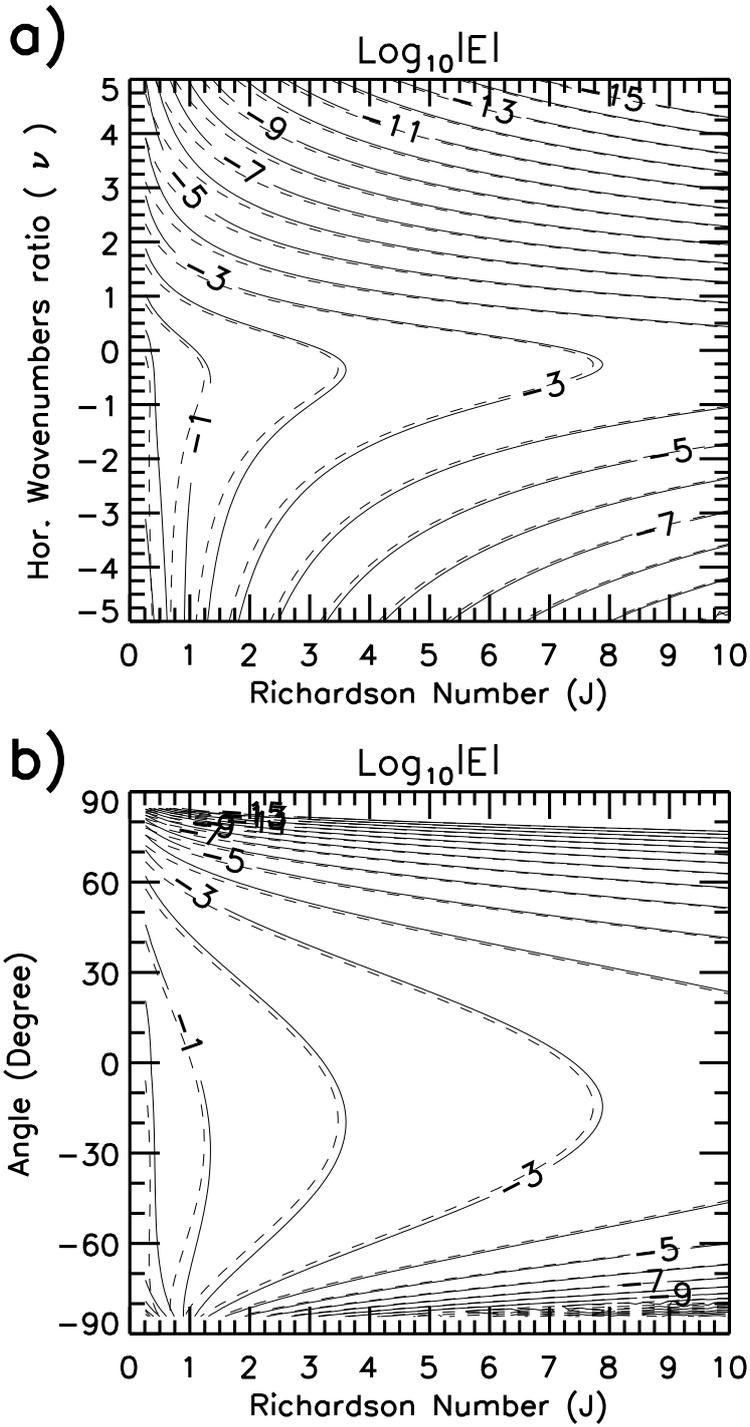


Figure 3: Amplitude $|E|$ of the GW associated with a monochromatic Dirac PV anomaly as a function of J and (a) $\nu = l/k$, or (b) $\varphi = \tan^{-1} \nu$. The exact solution (solid contours) is compared with the WKB approximation (dashed contours).

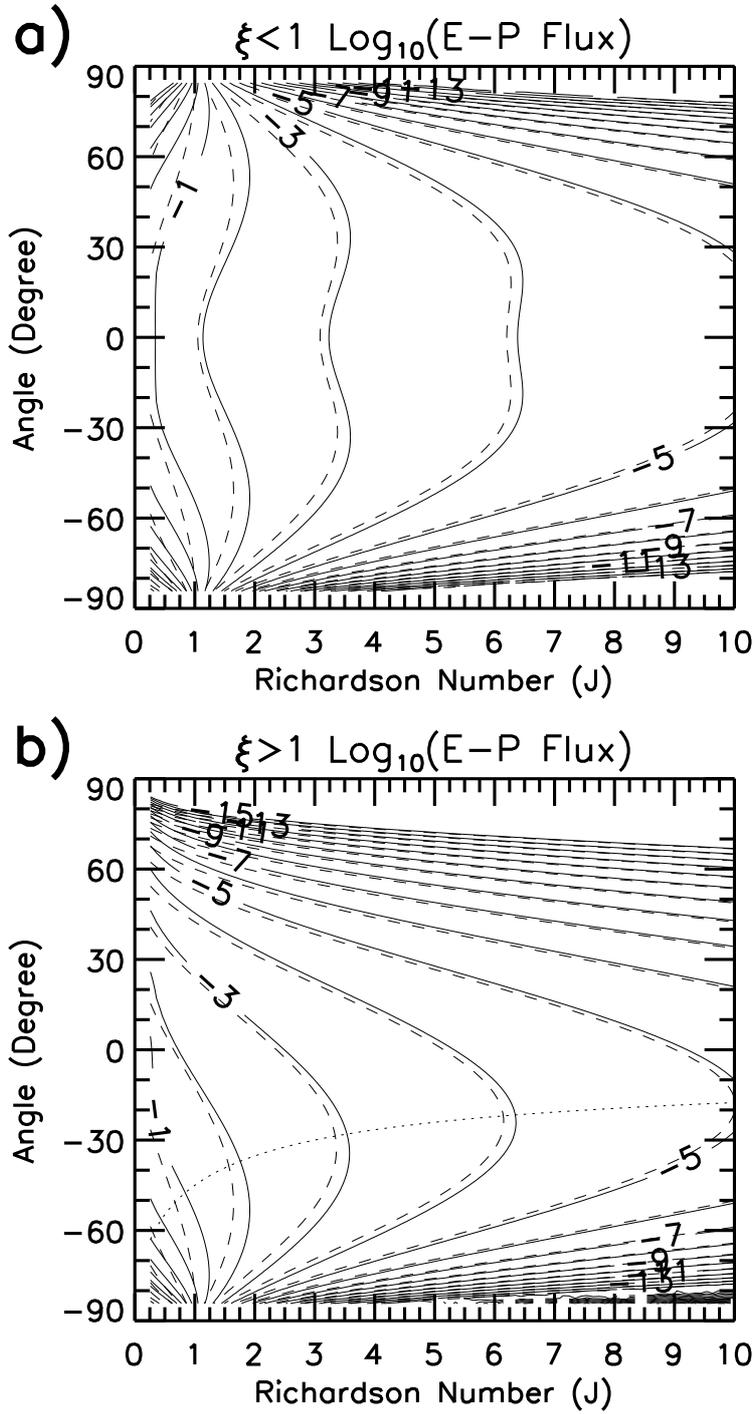


Figure 4: EP flux associated with a monochromatic Dirac PV anomaly for (a) $|\xi| < 1$ and (b) $|\xi| > 1$. The exact solution (solid contours) is compared with the WKB approximation (dashed contours). The dotted line in (b) shows the WKB prediction of the angle maximising \mathcal{F} .

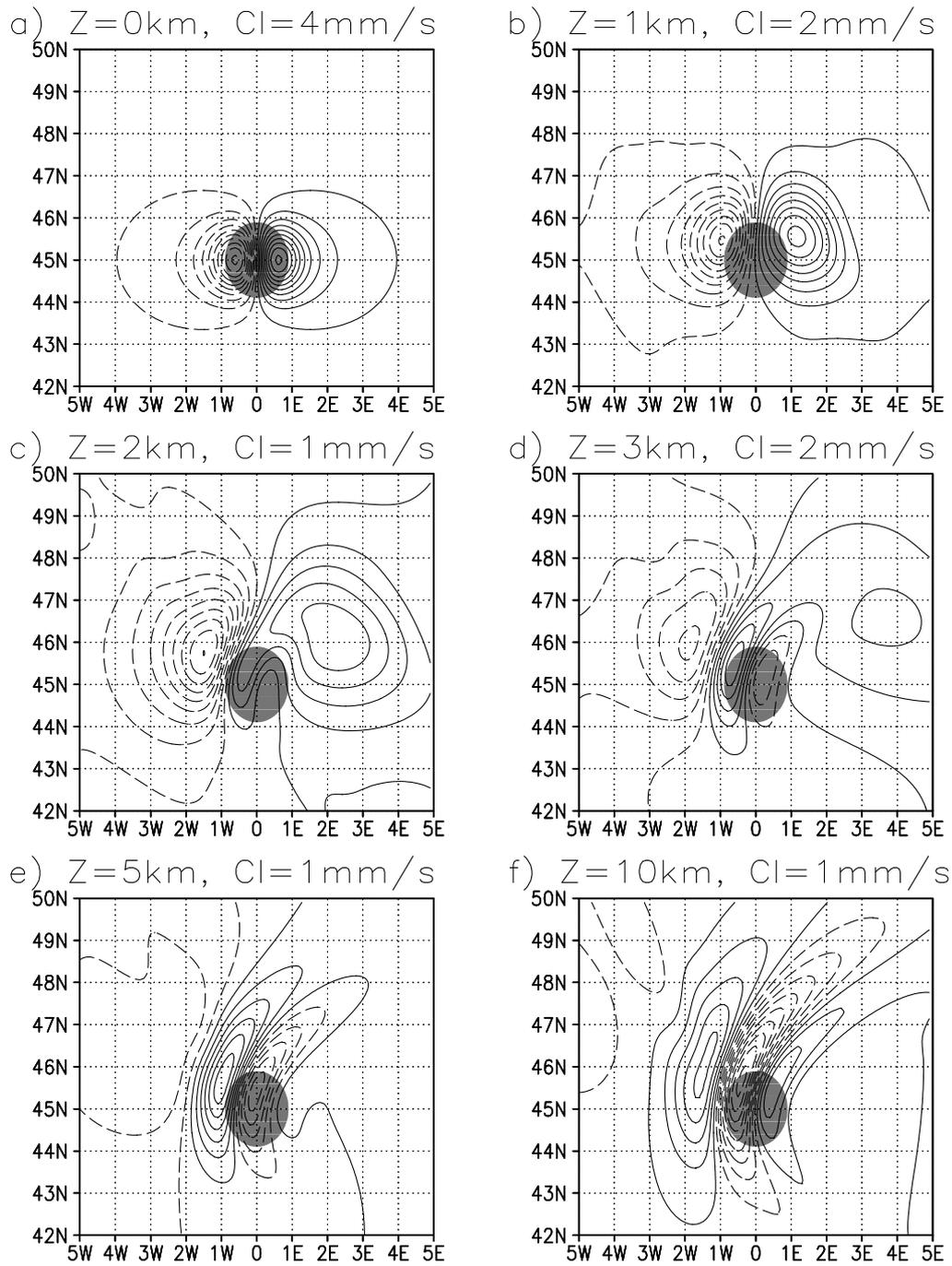


Figure 5: Vertical velocity at various altitudes above a PV anomaly with a Gaussian distribution in the horizontal and Dirac distribution in the vertical. Solid (dashed) contours correspond to positive (negative) values. The grey shading indicates that the disturbance PV $\rho_r q'_0(x, y) > 0.2$ PVU.

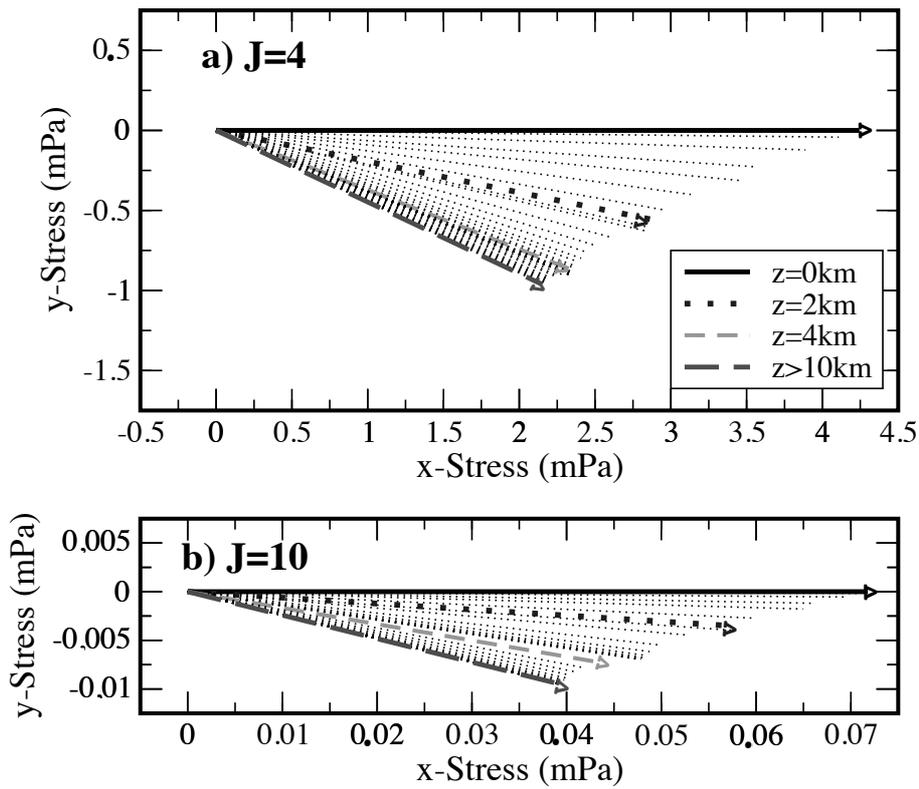


Figure 6: EP-flux vector as a function of altitude for the PV distribution used in Fig. 5 for (a) $J = 4$ and (b) $J = 10$. The dashed lines represent the vector every 200m typically; the bold vectors correspond to the altitudes indicated in panel (a).

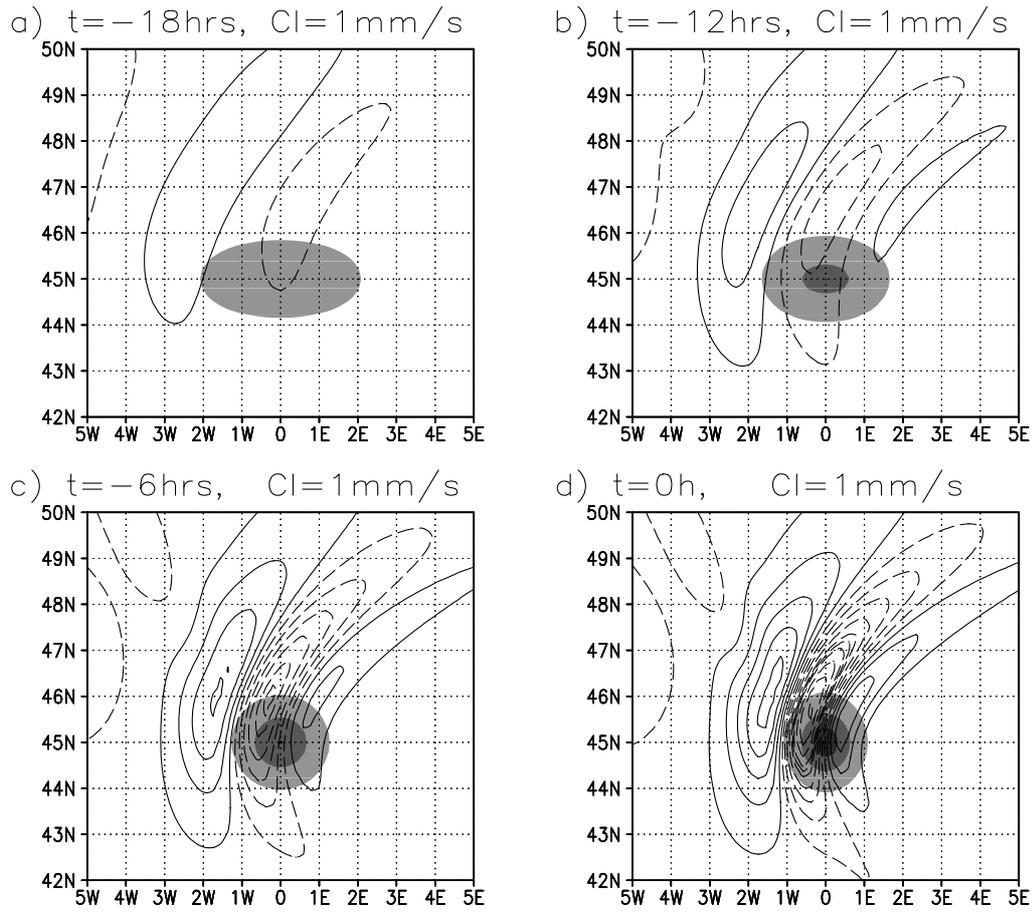


Figure 7: Vertical velocity at $z = 10 \text{ km}$ above a PV anomaly with a Gaussian distribution in the horizontal and a finite depth in the vertical. Solid (dashed) contours correspond to positive (negative) values. The 3 shades of grey indicates vertical integrals of the PV disturbance greater than 0.1, 0.45 and 0.9 PVU km.