THE KAKEYA PROBLEM

PARTIAL DIFFERENTIAL EQUATIONS REDUCED TO PUZZLES

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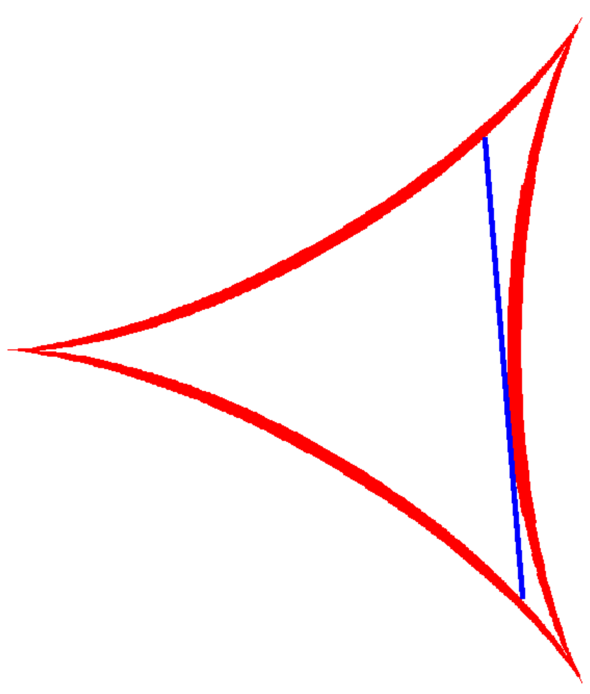
An economical "three-point-turn"

Q: How large a space do you need in order to do a three-point-turn?

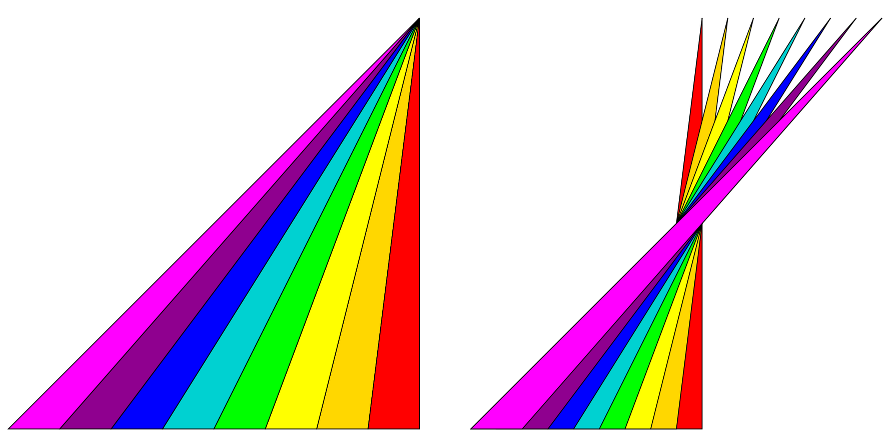
A: An arbitrarily small area, provided that you have an infinitesimally thin car!

- "Kakeya needle problem" (1917): What's the minimal plane area in which a line of length 1 can be turned around?



  - Spinning on the spot gives area $\pi/4$;
  - Three-point turn (pictured) gives $\pi/8$;

    BUT...
  - Besicovitch (1928): NO minimum—arbitrarily small!
- Proof uses overlapping triangles.
  - Sideways shifting is easy: just slide very far away lengthwise, rotate by a tiny angle, and slide back.
  - Hard part is turning the needle through 180°. Use this overlapping triangle construction.

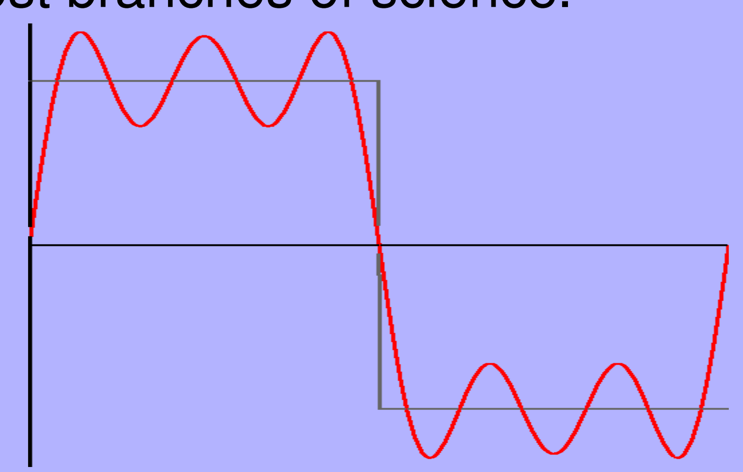


  Needle starts vertical, in the red triangle on the right. Turn it within this triangle until it lies along the sloping side. Then, by one of the faraway-sliding shifts, get it into the corresponding side of the orange triangle.
  Continue like this to turn the needle right round.
- The hard part of the construction is in showing that the overlapped-triangle arrangement can be made to cover an arbitrarily small area, if only we cut into a very large number of very thin triangles.
- This construction was used by Fefferman in a seminal paper on Fourier inversion—see box.

Kakeya Sets of Curves, Geometric and Functional Analysis 15 (2005) 1319-1362

Kakeya and the Fourier Transform

- The Kakeya Needle problem is more than just a puzzle...
- *Kakeya Conjecture:* A set in n-dimensional space containing lines in all directions must have fractal dimension n.
  - Unsolved for $n \geq 3$; much research and recent progress.
  - Surprising links with applicable branches of mathematics.
- Fourier Inversion:
  - The Fourier transform expresses a complicated signal in terms of simple sine waves. It is an essential tool in most branches of science.



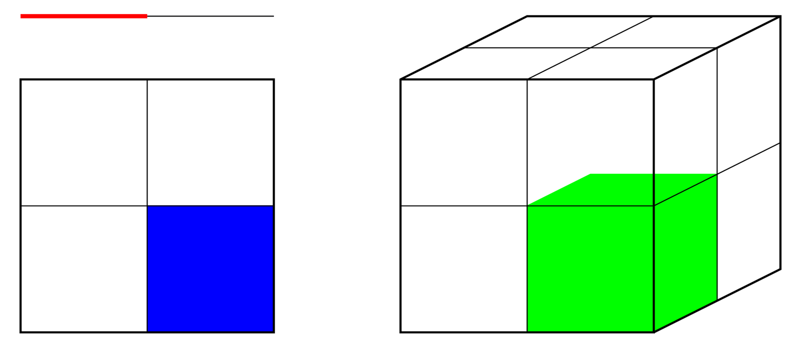
  - Need to compute the signal from its Fourier transform. Try *partial sums*: build up a signal from its lowest frequencies, hoping that higher ones are negligible.
  - BUT: Fefferman showed that (for many senses of convergence) this does not happen, and his proof used the overlapping-triangle construction shown on the left.
  - Instead, try *smooth summability methods*. If the convergence is true for these, then so is the Kakeya conjecture.
- Partial Differential Equations (PDEs):
  - *Local smoothing* for the wave equation implies the Kakeya conjecture.
  - Solutions of PDEs (e.g. Schrödinger's equation) use *oscillatory integrals*, of which the Fourier transform is just the prototype.
  - These give Kakeya-like sets consisting of curves instead of straight lines.
  - I have found hyperbolas for which it is false—dimension only $\frac{n+1}{2}$—and related parabolas for which the dimension is at least the best known values for the straight line case.

What is fractal dimension?

- Ordinary sets have a *dimension* that is a whole number:

Line	1-dimensional
Plane	2-dimensional
Solid cube	3-dimensional

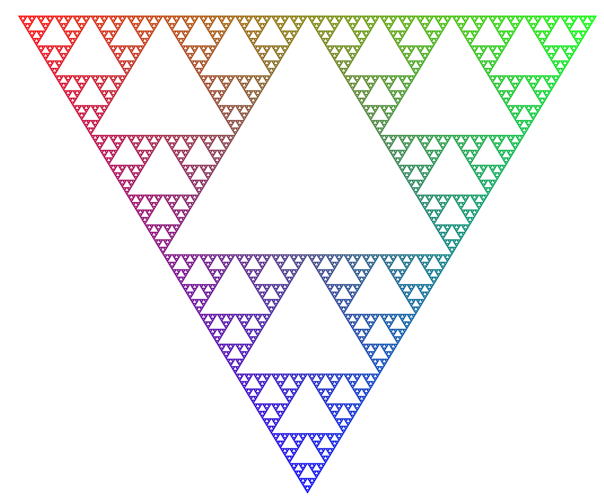
- Fractals are complicated shapes with a very fine structure.
- Whole number concepts don't measure their size very well—the "length", "area" or "volume" of a fractal set is usually either 0 or ∞.



- Fractal dimension is best understood through the idea of scaling.
- Scale everything by a factor of 1/2:

Line	becomes	1/2 of itself
Square	becomes	1/4 of itself
Cube	becomes	1/8 of itself

  So in general, it seems that the dimension is that power d to which 1/2 has been raised in each case.
- Here is a set—the *Sierpinski gasket*—whose dimension is not a whole number.



  - Scaling it by a factor of 1/2 gives an exact copy of 1/3 of the original set. So the dimension, being the number d such that $(1/2)^d = 1/3$, must be $\frac{\log 3}{\log 2} \approx 1.585$.
  - The Sierpinski gasket has zero area—the area decreases by 0.75 at each step of the construction
  - Its boundary has infinite length—the length is doubled at each step
  - So ordinary non-fractal measures of size tell us nothing at all.
- Not all fractals have fractional dimension, however:
  - A Kakeya set is conjectured to have dimension n (that of the whole space).
  - Another example is the famous Mandelbrot set, which has dimension 2.