Triviality of the Involution on SK_1 for Periodic Groups

by

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Let π be a periodic group or, more generally, a finite group with every p-Sylow subgroup either cyclic, or, for $p = 2$, a dihedral, quaternionic, or semidihedral group. In §1 of this note we show that the involution on $SK_1(\mathbb{Z}\pi)$ is trivial. The proof uses the results of Oliver (3), (4), where he calculates $SK_1(\mathbb{Z}\pi) \cong \mathbb{Z}_2^k$. In §2 we derive some consequences for the groups $H^n(\mathbb{Z}_2; Wh(\pi))$, which have applications in surgery theory (cf. (7, Prop. 4.1)) and the theory of semifree group actions (5, Prop. 3).

By Wall (9) the involution is also trivial on $Wh(\pi)/SK_1(\mathbb{Z}\pi)$. The question remains whether it is trivial on $Wh(\pi)$ for π periodic (see the Remark in §2).

For examples of finite groups with non-trivial involution on SK_1 see (11, proof of Theorem 4.8) and (10, Props. 24 and 25).

Involution on SK_1 for Periodic Groups

§1. The involution on $SK_1(\mathbb{Z}\pi)$.

For any group π the involution on the group ring $\mathbb{Z}\pi$ defined by $x \longmapsto x^{-1}$, $x \in \pi$, induces by conjugate transpose on $GL(\mathbb{Z}\pi)$ an involution on $Wh(\pi)$ and, for finite π, on $SK_1(\mathbb{Z}\pi)$ and $SK_1(\mathbb{Z}\pi)_{(p)}$.

Lemma. *Fix a prime p and let π be a finite group such that the involution on $SK_1(\mathbb{Z}\pi')_{(p)}$ is trivial for every p-hyperelementary subgroup $\pi' \subset \pi$. then the involution is trivial on $SK_1(\mathbb{Z}\pi)_{(p)}$.*

Proof. This follows immediately from the Dress induction isomorphism

$$\varinjlim_{\pi'} SK_1(\mathbb{Z}\pi')_{(p)} \xrightarrow{\cong} SK_1(\mathbb{Z}\pi)_{(p)}$$

of Oliver (4, p. 302), where the limit is taken with respect to inclusion and conjugation (which commute with the involution) among p-$\mathbb{Q}$-elementary (hence p-hyperelementary) subgroups $\pi' \subset \pi$.

Theorem 1. Let π be a finite group whose 2-Sylow subgroup is cyclic, dihedral, quaternionic, or semidihedral. Then the involution on $SK_1(\mathbb{Z}\pi)_{(2)}$ is trivial.

Proof. First reduction. It is not difficult to show that the hypothesis on π is satisfied by every subgroup of π. Hence by the Lemma it suffices to prove the result when π is 2-hyperelementary.

Second reduction. It is also not difficult to show that the 2-Sylow subgroup π_2 has a normal abelian subgroup with cyclic quotient. Hence by (4, Prop. 9 (ii)) we have $SK_1(\hat{\mathbb{Z}}_2\pi) = 0$ in the exact sequence

$$0 \to C\ell_1(\mathbb{Z}\pi)_{(2)} \to SK_1(\mathbb{Z}\pi)_{(2)} \to SK_1(\hat{\mathbb{Z}}_2\pi) \to 0,$$

where $C\ell_1(\mathbb{Z}\pi)$ denotes the kernel of the natural surjection (see (3, p. 184))

$$SK_1(\mathbb{Z}\pi) \to \sum_p SK_1(\hat{\mathbb{Z}}_p\pi).$$

Thus we need only show that the involution is trivial on $C\ell_1(\mathbb{Z}\pi)_{(2)}$.

Third reduction. $SK_1(\mathbb{Z}\pi) = 0$ if π_2 is cyclic by (3, Theorem 2), so finally we are reduced to proving that the involution is trivial on $C\ell_1(\mathbb{Z}\pi)_{(2)}$ if π is 2-hyperelementary and π_2 is dihedral, quaternionic, or semidihedral.

To begin the proof we have $\pi \cong \mathbb{Z}_n \rtimes \pi_2$ with n odd. Write $\mathbb{Z}_n = \{1, x, \ldots, x^{n-1}\}$ and $\mathrm{Aut}(\mathbb{Z}_n) = \{a \mid 1 \leq a < n, (a,n) = 1\}$. The action of π_2 on $\mathbb{Z}_n$ is given by a homomorphism $t: \pi_2 \to \mathrm{Aut}(\mathbb{Z}_n)$ with $gxg^{-1} = x^{t(g)}$ for $g \in \pi_2$. Thus $\mathbb{Z}(\mathbb{Z}_n \rtimes \pi_2) = \mathbb{Z}(\mathbb{Z}_n)\{\pi_2\}^t$ is a twisted group ring with involution defined by

$$xg \mapsto \overline{xg} = x^{t(g^{-1})} g^{-1}.$$

Now fix $d \mid n$. Let $\zeta_d = e^{2\pi i/d}$ and let $\mathbb{Z}\zeta_d$ denote the ring of integers in the extension of $\mathbb{Q}$ by the d^{th} roots of unity. Since n is odd and π_2 is a 2-group, we have $(d, t(g)) = 1$ for $g \in \pi_2$. Hence $\zeta_d \mapsto \zeta_d^{t(g)}$ defines an automorphism t_g of $\mathbb{Z}\zeta_d$, and $g \mapsto t_g$ defines an action of π_2 on $\mathbb{Z}\zeta_d$.

Let $\mathbb{Z}\zeta_d(\pi_2)^t$ denote the corresponding twisted group ring, with multiplication given by $\alpha g \cdot \alpha_1 g_1 = \alpha t_g(\alpha_1) g g_1$ for $\alpha, \alpha_1 \in \mathbb{Z}\zeta_d$ and $g, g_1 \in \pi_2$. The involution on $\mathbb{Z}\zeta_d(\pi_2)^t$ is defined by

$$\alpha g \mapsto \overline{\alpha g} = t_{g^{-1}}(\overline{\alpha}) g^{-1} = \overline{t_{g^{-1}}(\alpha)}\, g^{-1},$$

where $\overline{\alpha}$ is the ordinary complex conjugate of $\alpha \in \mathbb{Z}\zeta_d \subset \mathbb{C}$.

Setting $pr_d(x^m g) = \zeta_d^m g$ for $x^m \in \mathbb{Z}_n$ defines the natural projection

$$pr_d : \mathbb{Z}(\mathbb{Z}_n \rtimes \pi_2) \to \mathbb{Z}\zeta_d(\pi_2)^t.$$

One checks easily that pr_d commutes with the involution. Hence the induced homomorphism

$$pr_{d*(2)} : C\ell_1(\mathbb{Z}(\mathbb{Z}_n \rtimes \pi_2))_{(2)} \to C\ell_1(\mathbb{Z}\zeta_d(\pi_2)^t)_{(2)}$$

commutes with the involution on these groups. Therefore

$$\sum_{d|n} pr_{d*(2)} : C\ell_1(\mathbb{Z}(\mathbb{Z}_n \rtimes \pi_2))_{(2)} \to \sum_{d|n} C\ell_1(\mathbb{Z}\zeta_d(\pi_2)^t)_{(2)}$$

commutes with the involution, which leaves each summand on the right invariant. But according to (4, Prop. 11) if π_2 is dihedral, quaternionic, or semidihedral, then $C\ell_1(\mathbb{Z}\zeta_d(\pi_2)^t)_{(2)}$ is either 0 or $\mathbb{Z}_2$, which have only the trivial involution. Hence the involution on the direct sum is trivial. Since $\sum_{d|n} pr_{d*(2)}$ is injective (indeed, bijective) by (4, p.328), it follows that the involution on $C\ell_1(\mathbb{Z}(\mathbb{Z}_n \rtimes \pi_2))_{(2)} \cong C\ell_1(\mathbb{Z}\pi)_{(2)}$ is trivial. This completes the proof of Theorem 1.

<u>Remark</u>. A. Bak has an unpublished proof that the involution is trivial on $C\ell_1(\mathbb{Z}\pi)$ for <u>any</u> finite group π, a general result he conjectured and proved a special case of in (0).

<u>Theorem</u> 2. <u>Let</u> π <u>be a finite group whose</u> p-<u>Sylow subgroups are cyclic or, for</u> p = 2, <u>dihedral, quaternionic, or semidihedral (e.g.</u> π <u>periodic).</u> <u>Then the involution on</u> $SK_1(\mathbb{Z}\pi)$ <u>is trivial.</u>

<u>Proof</u>. $SK_1(\mathbb{Z}\pi)_{(p)} = 0$ for $p > 2$ by (3, Theorem 2). So $SK_1(\mathbb{Z}\pi) = SK_1(\mathbb{Z}\pi)_{(2)}$ and the result follows by Theorem 1.

§2. $H^n(Wh(\pi))$

Wall has shown [9, p. 617] (see also [2, Corol. 6.10]) that for a finite group π the involution $\tau \longmapsto \bar{\tau}$ is trivial on $Wh'(\pi) \equiv Wh(\pi)/SK_1(\mathbb{Z}\pi)$. Hence setting $h(\tau) = \tau - \bar{\tau}$ for $\tau \in Wh(\pi)$ defines a homomorphism

$$h: Wh(\pi) \longrightarrow SK_1(\mathbb{Z}\pi).$$

From now on assume π satisfies the hypothesis of Theorem 2.

Remark. Although the involution will then be trivial on both $Wh(\pi)/SK_1(\mathbb{Z}\pi)$ and $SK_1(\mathbb{Z}\pi)$, it does not follow that it is trivial on $Wh(\pi)$. (For example, it might take (1,0) to (1,1) in $\mathbb{Z} \times \mathbb{Z}_2$.) Question: Is it, if π is periodic?

By Theorem 2 we have $h\, SK_1(\mathbb{Z}\pi) = 0$, so there is a unique homomorphism $h': Wh'(\pi) \to SK_1(\mathbb{Z}\pi)$ factoring $h = h' \circ \nu$, where $\nu: Wh(\pi) \to Wh'(\pi)$ is the natural projection.

It follows from Oliver [3, Theorem 2] and [4, Theorem 6] that $SK_1(\mathbb{Z}\pi) \cong \mathbb{Z}_2^k$, where k is the number of conjugacy classes of odd cyclic subgroups $C \subset \pi$ such that (i) the 2-Sylow subgroup of the centralizer of C is nonabelian and (ii) there is no $g \in \pi$ with $gxg^{-1} = x^{-1}$ for all $x \in C$. Combining this with Wall's result [9, Theorems 1.2 and 6.1](see also [8, §§6 and 7]) that $SK_1(\mathbb{Z}\pi) = tor(Wh(\pi))$, and Bass' theorem [1] on the rank of $Wh(\pi)$, yields $Wh(\pi) \cong \mathbb{Z}_2^k \times \mathbb{Z}^{r-q}$, where (using [6, Chap. 13]) q is the number of conjugacy classes of cyclic subgroups of π, and r is the number of conjugacy classes of unordered pairs $\{x, x^{-1}\}$, $x \in \pi$.

Let m be the $\mathbb{Z}_2$ rank of $im(h)$ = image of h in $SK_1(\mathbb{Z}\pi) \cong \mathbb{Z}_2^k$. Note $m \leq \min(k, r-q)$ since $im(h) = im(h')$ and we may interpret $h': \mathbb{Z}^{r-q} \to \mathbb{Z}_2^k$.

For an abelian group G with involution $g \longmapsto \bar{g}$ set

$$H^n(G) = \frac{\{g \in G \mid g = (-1)^n \bar{g}\}}{\{g + (-1)^n \bar{g} \mid g \in G\}} = H^n(\mathbb{Z}_2; G).$$

These are elementary 2-groups.

Theorem 3. *If π is a finite group whose p-Sylow subgroups are cyclic or, for p = 2, dihedral, quaternionic, or semidihedral, then*

(i) $H^n(SK_1(\mathbb{Z}\pi)) \cong \mathbb{Z}_2^k$ for all n

(ii) $$H^n(Wh(\pi)) \cong \begin{cases} \mathbb{Z}_2^{k-m} & \text{for } n \text{ odd} \\ \mathbb{Z}_2^{k-m+r-q} & \text{for } n \text{ even.} \end{cases}$$

Proof. (i) This is immediate since $\sigma = \bar{\sigma} = -\bar{\sigma}$ for $\sigma \in SK_1(\mathbb{Z}\pi) \cong \mathbb{Z}_2^k$.

(ii) n odd. Recall that $\tau = \bar{\tau} + h(\tau)$ with $h(\tau) \in SK_1(\mathbb{Z}\pi)$ for $\tau \in Wh(\pi)$. Hence $\tau = -\bar{\tau}$ implies $2\tau = h(\tau) \in SK_1(\mathbb{Z}\pi) = tor(wh(\pi))$, whence $\tau \in SK_1(\mathbb{Z}\pi)$. Conversely, $\tau \in SK_1(\mathbb{Z}\pi)$ implies $\tau = -\bar{\tau}$ as above, so $H^n(Wh(\pi)) = \{\tau = -\bar{\tau}\} / \{\tau - \bar{\tau}\} = SK_1(\mathbb{Z}\pi)/im(h) \cong \mathbb{Z}_2^k/\mathbb{Z}_2^m$.

(ii) n even. Choose a basis $\sigma_1,\dots,\sigma_m,\dots,\sigma_k$ for $SK_1(\mathbb{Z}\pi)$ as a vector space over $\mathbb{Z}_2$ such that $\sigma_1,\dots\sigma_m$ is a basis for the subspace $im(h')$. Then by induction on r-q find a basis $\tau'_1,\dots,\tau'_m,\dots,\tau'_{r-q}$ for $Wh'(\pi)$ as a free $\mathbb{Z}$ module such that $h'(\tau'_i) = \sigma_i$ for $i = 1,\dots,m$ and $h'(\tau'_{m+1}) = \dots = h'(\tau_{r-q}) = 0$. Finally, pick $\tau_i \in \nu^{-1}(\tau'_i) \subset Wh(\pi)$, so that $h(\tau_i) = h'(\tau'_i)$, $i = 1,\dots,r-q$. Let $\langle x_1,\dots,x_s\rangle$ denote the subgroup generated by $x_1,\dots,x_s$. Then, since $\tau = \bar{\tau} + h(\tau)$ for $\tau \in Wh(\pi)$ and $h(\tau) = 0$ if $\tau \in SK_1(\mathbb{Z}\pi)$, we get

$$H^n(wh(\pi)) = \frac{\{\tau = \bar{\tau}\}}{\{\tau + \bar{\tau}\}}$$

$$= \frac{\langle\sigma_1,\dots,\sigma_k, 2{}_1,\dots 2{}_m, {}_{m+1},\dots, {}_{r-q}}{\langle 2\tau_1+\sigma_1,\dots,2\tau_m+\sigma_m, 2\tau_{m+1},\dots,2\tau_{r-q}\rangle}$$

$$\cong \frac{\langle\sigma_1,\dots,\sigma_k, 2\tau_1+\sigma_1,\dots,2\tau_m+\sigma_m\rangle}{\langle 2\tau_1+\sigma_1,\dots,2\tau_m+\sigma_m\rangle} \times \frac{\langle\tau_{m+1},\dots,\tau_{r-q}\rangle}{2\langle\tau_{m+1},\dots,\tau_{r-q}\rangle}$$

$$\cong \mathbb{Z}_2^k \times \mathbb{Z}_2^{r-q-m}.$$

From Theorem 3(ii) we derive some consequences which do not depend on the unknown m, but only on the reasily computed r, q, and k.

Corollary. If π is a periodic group, then

(i) <u>the Herbrand quotient</u> $\frac{|H^0(Wh(\pi))|}{|H^1(Wh(\pi))|} = 2^{r-q}$

(ii) <u>for</u> n <u>odd</u>, $H^n(Wh(\pi)) = 0$ <u>only if</u> $k \leqq r-q$

(iii) <u>for</u> n <u>even</u>, $H^n(Wh(\pi)) = 0$ <u>if and only if</u> $Wh(\pi) = 0$

<u>Proof</u>: (i) follows trivially, (ii) follows from $m \leqq r-q$, and (iii) because $m \leqq k$ also, so $k-m+r-q = 0 \Leftrightarrow k = r-q = 0 \Leftrightarrow Wh(\pi) = 0$.

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