Equivariant Moore Spaces

by

Justin R. Smith*

Introduction.

This paper studies the following problem, originally proposed by Steenrod in 1960: Given a group π, a right $\mathbb{Z}\pi$-module M and an integer $n>1$, does there exist a topological space X with the properties:

1. $\pi_1(X)=\pi$;
2. $H_i(\tilde{X}) = 0$, $i\neq 0, n$;
3. $H_0(\tilde{X}) = \mathbb{Z}$;
4. $H_n(\tilde{X}) = M$?

where $\tilde{X}$ is the universal covering space of X, equipped with the usual π-action. The space X, if it exists, is called an *equivariant Moore space of type (M, n; π)* or just a *space* of type $(M, n; \pi)$. A triple $(M, n; \pi)$ for which such a space exists will be said to be *topologically realizable.*

Section 1 of the present paper develops an obstruction theory for the existence of equivariant Moore spaces and proves that:

Theorem: Let $(M, n; \pi)$ *be a triple as described above and suppose that the* $n+2$-*dimensional homological k-invariant of the chain complex* $K^+(M, n)\otimes Z_*$ *is nonzero. Then there doesn't exist a topological space,* X, *with the property that* $\pi_1(X) = \pi$, $H_i(X; \mathbb{Z}\pi) = 0$, $0 < i < n$, *or* $i=n+1, n+2$, $H_n(X;\mathbb{Z}\pi) = M$. *In particular, no equivariant Moore space of type* $(M, n; \pi)$ *exists.* □

Remarks: 1. $K^+(M,n)$ is the quotient of the chain complex of a $K(M,n)$ by the 0-dimensional chain module and Z_* is a $\mathbb{Z}\pi$-projective resolution of $\mathbb{Z}$. The tensor product in the hypothesis is over $\mathbb{Z}$ and equipped with the *diagonal* π-action.

2. The statement about the homological k-invariant of $Y=K^+(M,n)\otimes Z_*$ is equivalent to the statement that the evaluation map $H^{n+2}(Y; H_{n+2}(Y)) \rightarrow \mathrm{Hom}_{\mathbb{Z}\pi}(H_{n+2}(Y), H_{n+2}(Y))$ is *not surjective.*

* The author was partially supported by NSF Grant #MCS81-16614.
AMS (MOS) Subject Classification (1980): Primary 55S45; Secondary 18G55.

Sections 2 and 3 of the present paper develop an algorithm for the computation of this homological k-invariant and show that:

Theorem: The hypotheses of the theorem above are satisfied if $\pi=\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *(on generators* s *and* t) *and* M *is the* $\mathbb{Z}\pi$*-module whose underlying abelian group is* $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ *with* s *and* t *acting via right multiplication by the matrices.*

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \text{ respectively.}$$

The first counterexample to the Steenrod conjecture was due to Gunnar Carlsson in [2]. The present counterexample has the advantage that the $\mathbb{Z}$-rank of the module is *minimal* and the fundamental group is the smallest possible -- Peter Kahn (in unpublished work) proved that any module whose underlying abelian group is $\mathbb{Z}\oplus\mathbb{Z}$ is topologically realizable. On the positive side of the Steenrod conjecture we have:

Theorem: Let M *be a* $\mathbb{Z}\pi$*-module of homological dimension* k *and suppose that* $M/p{\bullet}M = M_p = 0$ *for all primes* $p < 1 + k/2$. *Then there exist equivariant Moore spaces of type* $(M, n; \pi)$, *where* n *is any integer* $>k$. □

Here M_p denotes the p-torsion submodule of M.

The Steenrod problem has been studied before by Frank Quinn, James Arnold, Peter Kahn, and Gunnar Carlsson.

Frank Quinn developed an obstruction theory to putting a suitable group action on a pre-existing (non equivariant) Moore space. The main drawback to his theory is that the obstructions (and even the obstruction groups) do not seem to be readily computable -- see [14].

James Arnold, in [19], developed a form of homological algebra based upon *permutation modules* rather than projective modules and used it to prove that all modules over a cyclic group are topologically realizable. Peter Kahn developed an obstruction theory to the existence of equivariant Moore spaces for $\mathbb{Z}$-torsion free modules. When coupled with the results of Kiyoshi Igusa (see [10] and [11]) it implies that the $\mathbb{Z}GL_4(\mathbb{Z})$-module $\mathbb{Z}^4$ (where the group acts by matrix multiplication) is not topologically realizable in any dimension. Unfortunately the results of Igusa don't provide for an easy computation of the obstruction.

The work of Gunnar Carlsson (which resulted in the first counterexample) hinged upon an argument involving cohomology operations that doesn't appear to generalize beyond the examples given in his paper (see [2]). Carlsson approached the problem from the point of view of group actions on CW complexes and the induced actions on homology. The present paper was originally written in 1980 after Carlsson's result.

It was felt that Carlsson's result laid the Steenrod problem to rest but in recent years there has been renewed interest in the approach of the present paper. This is due to connections between the Steenrod problem and the theory of *transformation groups.* The present paper develops an obstruction theory for equivariant Moore spaces that is completely different from all of the theories discussed above and which appears to be much more tractable from a computational point of view. It also turns out that the obstruction theory presented here generalizes to an obstruction theory to topologically realizing *chain complexes* that have nonvanishing homology in more than one dimension. Such an obstruction theory provides a first obstruction to imposing a group-action upon a space (e.g., a manifold). Certainly, if no group-action exists on a CW-complex homotopy-equivalent to the desired space then it can't exist on the desired space either. Also, if one can demonstrate the existence of some desired group-action on a CW-complex homotopy equivalent to a manifold one can explore the (surgery-theoretic) problem of smoothing the action (for instance) to get a similar action on the manifold.

Section 3 of this paper gives an *explicit formula* for the obstruction for all modules whose underlying abelian group is $\mathbb{Z}^3$. This formula can be readily generalized to *all* $\mathbb{Z}$-free modules.

I am indebted to Sylvain Cappell and Andrew Ranicki for their encouragement.

§1 The Obstruction Theory

In this section we will describe the obstructions to the existence of equivariant Moore spaces. Essentially they will turn out to be obstructions to adjoining terms to a partial Postnikov tower in such a way that the first non-vanishing homology module above the one we want to realize, is *annihilated.*

Definition 1.1: Let M be a right $\mathbb{Z}\pi$-module and n be an integer > 1. The equivariant Eilenberg-MacLane space $K_\pi(M, n)$ is defined to be a space homotopy-equivalent to $(K(M,n) \times \tilde{K}(\pi,1))/\pi$ where:

a. the second factor is the universal cover of a $K(\pi,1)$;

b. the cartesian product above is equipped with the *diagonal* π-action. □

Remarks: 1. Note that $K_\pi(M, n)$ has the following properties:

i. $\pi_1(K_\pi(M,n)) = \pi$;

ii. $\pi_i(K_\pi(M,n)) = 0$, $i \neq 1, n$;

iii. $\pi_n(K_\pi(M,n)) = M$, as $\mathbb{Z}\pi$-modules, i.e. the action of the fundamental group on π_n coincides with the action of π on M.

2. In the definition of $K_\pi(M,n)$ above we could have used the cellular bar construction of Milgram for K(M,n) instead of the semi-simplicial complex of Eilenberg and MacLane -- see [13].

Let X be a topological space with fundamental group π and consider a map inducing an isomorphism of fundamental groups:

1.2: $X \xrightarrow{f} K_\pi(M,n)$

The homotopy class of this map defines a cohomology class [f] in $H^n(X;M)$ and it is well-known that the map $f_*: H_n(X;\mathbb{Z}\pi) \rightarrow M$ is the image of [f] under the evaluation map:

$$H^n(X;M) \xrightarrow{e} \mathrm{Hom}_{\mathbb{Z}\pi}(H_n(X;\mathbb{Z}\pi), M)$$

-- i.e. [f] is just the element of $H^n(X;M)$ given by $f^*(\iota)$, where $\iota \in H^n(K(M,n);M)$ is an element whose image under the evaluation map is the *identity map* of M -- see [18, chapter 8]. Furthermore the map f is the classifying map for a fibration over X with fiber a K(M,n-1). If E is the total space of this fibration its homology fits into the Serre exact sequence of a fibration:

$$\ldots \rightarrow H_n(K(M,n-1)) \rightarrow H_n(E;\mathbb{Z}\pi) \rightarrow H_n(X;\mathbb{Z}\pi) \xrightarrow{\alpha} H_{n-1}(K(M,n-1))$$
$$\rightarrow H_{n-1}(E;\mathbb{Z}\pi) \rightarrow H_{n-1}(X;\mathbb{Z}\pi) \rightarrow 0$$

where the map α can be regarded as coinciding with f_* or e[f] since it is essentially the pullback of the transgression homomorphism for the universal fibration over K(M,n) -- which can be regarded as the identity map of M.

Lemma 1.3: Let $H_{n-1}(X;\mathbb{Z}\pi) = 0$. *Then there exists a* K(M,n-1)*-fibration over* X *with total space* E *such that* $H_n(E;\mathbb{Z}\pi) = H_{n-1}(E;\mathbb{Z}\pi) = 0$ *if and only if* $M = H_n(X;\mathbb{Z}\pi)$ *and there exists a map* $f: X \rightarrow K_\pi(M,n)$ *inducing an isomorphism of* π_1, *and an isomorphism in homology in dimension* n. *A map* f *with those properties exists if and only if the evaluation map with local coefficients in* M --

$e:H^n(X;M) \rightarrow \mathrm{Hom}_{\mathbb{Z}\pi}(M,M)$, *where* $M=H_n(X;\mathbb{Z}\pi)$, *is surjective.*

Proof: Most of this follows immediately from the Serre exact sequence above. We need only prove the last statement about the evaluation map. Let Λ denote $\mathrm{Hom}_{\mathbb{Z}\pi}(M,M)$, regarded as a *ring*. From the remarks following 1.1 it is clear that a map $f:X \rightarrow K_\pi(M,n)$ inducing an isomorphism in homology in dimension n represents a cohomology class $[f] \in H_n(X;M)$ whose image under the evaluation map is an *automorphism* of M, and conversely, the existence of such a cohomology class implies the existence of the map. Since the evaluation map $H^n(X;M) \rightarrow \mathrm{Hom}_{\mathbb{Z}\pi}(M,M)$ is *natural* with respect to changes of coefficients it follows that it is a homomorphism of Λ-modules (where Λ acts on the right by *changes of coefficients* in the cohomology, and by *composition* in the Hom-group). Since Λ is generated, as a module over itself, by any automorphism of M it follows that an automorphism is in the image of the evaluation map if and only if that map is *surjective.* □

Lemma 1.4: *A topological realization for the triple* $(M,n;\pi)$ *exists if and only if there exists a sequence of spaces* (X_i) *such that:*

1. X_1 *is a* $K(M,n)$*-fibration over a* $K(\pi,1)$;

2. X_{i+1} *is a* $K(N_i,n+i)$*-fibration over* X_i *with* $N_i = H_{n+i}(X_i;\mathbb{Z}\pi)$ *and whose characteristic class is a cohomology class of* $H^{n+i}(X_i;N_i)$ *whose image under the evaluation map is an automorphism of* N_i.

Remarks: 1. It is clear from lemma 1.3 that X_{i+1} can't exist unless the evaluation map $H^{n+i}(X_i;N_i) \rightarrow \mathrm{Hom}_{\mathbb{Z}\pi}(N_i,N_i)$ is surjective. Consequently the i^{th} obstruction to the existence of a topological realization of the triple $(M,n;\pi)$ is *defined* if and only if the previous i-1 obstructions *vanish.*

2. Since the evaluation map for *integral* cohomology is well-known to be surjective it is easy to see why all of the obstructions in the theory presented here vanish if π is the *trivial group.*

Proof: The *if* part of the statement follows from 1.3 which implies that

$H_0(X_i;\mathbb{Z}\pi) = \mathbb{Z}$;

$H_j(X_i;\mathbb{Z}\pi) = 0$ if $0 < j < n$;

$H_n(X_i;\mathbb{Z}\pi) = M$;

$H_j(X_i; \mathbb{Z}\pi) = 0$ if $n < j < n+i$.

The X_i form a convergent sequence of fibrations whose limit will be a suitable equivariant Moore space.

The *only if* part of the argument is a consequence of the existence and uniqueness of equivariant Postnikov towers -- see [1]. □

The remainder of this section will be spent developing algebraic criteria for the *surjectivity of the evaluation map* -- since the results above show that the equivariant Moore space can be constructed if this map is surjective. Essentially we will show that it is possible to define *obstructions* to the surjectivity of the evaluation map -- these will turn out to be closely related to the *homological k-invariants* of chain complexes defined by Heller in [8].

Definition 1.5: Let C_* be a projective $\mathbb{Z}\pi$-chain complex with the following properties:

i. $H_i(C_*) = 0$, $1 < n$;

ii. $H_n(C_*) = M$;

iii. $H_i(C_*) = 0$, $n < i < n+k$;

iv. $H_{n+k}(C_*) = N$.

Let P_* be a projective resolution for M. Then there exists a unique chain-homotopy class of chain maps $c: C_* \to \Sigma^n P_*$ inducing an isomorphism in homology in dimension n. Let $\mathfrak{A}(c)$ denote the algebraic mapping cone of c. We have the exact sequence:

$$0 \to \Sigma^n P_* \to \mathfrak{A}(c) \to \Sigma C_* \to 0$$

and $H_i(\mathfrak{A}(c)) = 0$ for $i < n+k+1$ and $H_{n+k+1}(\mathfrak{A}(c)) = N$ so that there exists a map $\mathfrak{A}(c) \to \Sigma^{n+k+1} Q_*$, where Q_* is a projective resolution of N. By composition there is a chain map $\Sigma^n P_* \to \Sigma^{n+k+1} Q_*$ defining a class $x \in \mathrm{Ext}_{\mathbb{Z}\pi}^{k+1}(M,N)$. This will be called the *homological k-invariant of* C_* *in dimension* n+k. □

Remarks: 1. It is not hard to see that this definition agrees with that of Heller in [8]: the pair $(\Sigma^{-n} C_*, \Sigma^{-n}\mathfrak{A}(c))$ can be regarded as a *0-truncated segregated pair* as in §6 of [8] -- see §3 of that paper also; C_* is regarded as a triangular complex such that $T_{*,i}=0$ if $i>0$.

2. It is also clear that if C_* is the (cellular or singular) chain complex of a connected topological space and n=0, the homological k-invariant defined above agrees with the *topological k-invariant* of the topological space.

Definition 1.6: Let $f{:}C_* \to D_*$ be a chain map of chain complexes. Then $\mathfrak{S}(f)$ is defined to be $\Sigma^{-1}\mathfrak{A}(f)$ -- the *desuspension* of the algebraic mapping cone. □

Remark: We have the usual short exact sequence of chain complexes:

$0 \to \Sigma^{-1}D_* \to \mathfrak{S}(f) \to C_* \to 0$. Let C_* be a chain complex such that

$H_0(C_*) = \mathbb{Z}$;
$H_n(C_*) = M$;
$H_i(C_*) = 0,\ n < i < n+k$;
$H_{n+k}(C_*) = H$

where M and H are $\mathbb{Z}\pi$-modules, n and k are positive integers and $n > 1$.

There exists a unique chain-homotopy class of chain maps $f_0{:}C_* \to Z_*$ inducing an isomorphism of H_0, where Z_* is some projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$. Then $H_0(\mathfrak{S}(f)) = 0$ and the canonical projection $\mathfrak{S}(f_0) \to C_*$ induces homology isomorphisms in all higher dimensions. If P_* is a projective resolution of M then there exists a unique chain-homotopy class of chain maps $f_n{:}\mathfrak{S}(f_0) \to \Sigma^n P_*$ inducing an isomorphism of homology in dimension n.

Proposition 1.7: Under the conditions discussed above the following diagram commutes, with all horizontal and vertical sequences exact:

$$\begin{array}{ccccc}
\mathrm{Ext}_{\mathbb{Z}\pi}^{k+1}(M,H) & \xleftarrow{\ \lambda\ } & H^{n+k+1}(V_*;H) & \longleftarrow & H^{n+k+1}(\pi, H) \\
\| & & \uparrow \xi & & \uparrow \\
\mathrm{Ext}_{\mathbb{Z}\pi}^{k+1}(M,H) & \xleftarrow{\ \zeta\ } & \mathrm{Hom}_{\mathbb{Z}\pi}(H,H) & \xleftarrow{\ \delta\ } & H^{n+k}(\mathfrak{S}(f_0);H) \\
 & & \uparrow e & & \uparrow \rho \\
 & & H^{n+k}(C_*; H) & & H^{n+k}(C_*; H)
\end{array}$$

where e *and* δ *are the evaluation maps, respectively, of* C_* *and* $\mathfrak{S}(f_0)$. If $c = \xi(1_H)$, *then the evaluation map of* C_* *is surjective if and only if* $c = 0$. *Furthermore,* $\lambda(c)$ *is the homological k-invariant of* $\mathfrak{S}(f_0)$ *in dimension* n+k. *The complex* V_* *is the algebraic mapping cone of the composite* $\Sigma^{-1}Z_* \subset \mathfrak{S}(f_0) \xrightarrow{\alpha} \Sigma^n P_*$, *where* $\alpha = f_n$ *and* λ *is induced by the canonical inclusion* $\Sigma^n P_* \subset V_*$.

Remarks: 1. The element c defined above is just the homological k-invariant of C_* in dimension n+k -- the simple definition given in 1.5 doesn't apply here because C_* has homology in dimension 0. See [8] for the general definition of homological k-invariants.

2. When C_* is the chain complex of X_k in 1.4 the element c will be the obstruction to the existence of X_{k+1} and the k^{th} obstruction to the existence of the corresponding equivariant Moore space. In order to prove the nonexistence of a given equivariant Moore space it clearly suffices to show that $\lambda(c) \neq 0$ at some stage of the construction. This will form the basis of the proof of the main results stated in the introduction since $\lambda(c)$ turns out to be more readily computable than c itself.

Proof: The diagram in the statement is the result of applying $H^{n+k}(*;H)$ to the *following* commutative exact diagram of chain complexes (where, as usual, Z_* is a projective resolution of $\mathbf{Z}$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{n-1}P_* & \longrightarrow & V' & \longrightarrow & \Sigma^{-1}Z_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{n-1}P_* & \longrightarrow & \mathfrak{C}(f_n) & \longrightarrow & \mathfrak{C}(f_0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & C_* & \longrightarrow & C_* & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where:

1. The middle row and the right column are the canonical exact sequences for $\mathfrak{C}(f_n)$ and $\mathfrak{C}(f_0)$, respectively;
2. $\mathfrak{C}(f_n) \to C_*$ is the composite of the canonical projections $\mathfrak{C}(f_n) \to \mathfrak{C}(f_0)$ and $\mathfrak{C}(f_0) \to C_*$ and V' is defined to be its kernel.

The existence and exactness of the remaining maps in the diagram now follows by a straightforward diagram chase. The fact that $H^{n+k}(\mathfrak{C}(f_n);H) = Hom_{\mathbf{Z}\pi}(H,H)$ follows from the fact that the homology of $\mathfrak{C}(f_n)$ vanishes below dimension n+k. That e and δ are the evaluation maps followsfrom the naturality of evaluation maps with respect to maps of chain complexes. That $\lambda(c)$ is the n+k-dimensional homological k-invariant of $\mathfrak{C}(f_0)$ folllows from the definition. It is also not hard to see that $V' = \mathfrak{C}(g)$, where g is the composite $\sum^{-1}Z_* \subset \mathfrak{C}(f_0) \to \sum^n P_*$ so that $V_* = \sum V'$. □

It was noted above that the elements $\lambda(c)$ are easier to calculate than the obstructions themselves. The $\lambda(c)$ do, however, occur in a natural geometric setting that will be described now.

Definition 1.8: Let $(M, n; \pi)$ be a triple as in 1.1. A ***split topological realization of*** $(M, n; \pi)$ is a space X of type $(M, n; \pi)$ such that the characteristic map $c:X \to K(\pi,1)$ (inducing an isomorphism of fundamental groups) is *split* by a map $s:K(\pi,1) \to X$ (i.e. $c \circ s$ is homotopic to the identity). □

Remarks: 1. Note that in the split case the first k-invariant of the space X must vanish. The condition that X be split is stronger than the vanishing of the first k-invariant, however. It essentially implies that there is *no* interaction between the fundamental group and the homotopy groups above π_2.

2. Since the splitting map $s:K(\pi,1) \to X_k$ must lift to X_{k+1} in each stage of the construction it follows that the classifying element of the fibration used to construct X_{k+1} comes from the *relative* cohomology group $H^{n+k}(X_k, K(\pi,1); H)$ -- since the relative chain complex is essentially $\mathfrak{C}(f_0)$ (in the setting of 1.7) -- it follows that the obstructions of the existence of a *split* equivariant Moore space are the images of the *nonsplit* obstructions under λ.

The geometric significance of the split case is connected with Steenrod's original definition of an equivariant Moore space as a CW-complex acted upon by a group, π, such that its equivariant homology had prescribed properties (so, for instance, the equivariant Moore space was generally *simply connected* and corresponded to the *universal cover* of an equivariant Moore space in *our* sense).

Proposition: ***A triple*** $(M, n; \pi)$ ***has a split topological realization (in the sense of the present paper) if and only if it is realizable (in Steenrod's sense) by a*** π***-complex that has a fixed point.***

Proof: Suppose $(M, n; \pi)$ has a split realization X, in our sense. Then there exists a π-equivariant map $z:\tilde{K}(\pi,1) \to \tilde{X}$ whose mapping cone is a *pointed* π-complex realizing $(M, n; \pi)$ in Steenrod's sense.

The converse follows by taking a Steenrod realization and taking the topological product with $\tilde{K}(\pi,1)$ and defining the group action diagonally. The group π acts on the resulting space freely so we can take the quotient to get a realization in *our* sense. The existence of the *fixed point* in the original space implies that the final space will be split (it will contain a copy of $K(\pi,1)=\tilde{K}(\pi,1)\times$fixed point$/\pi$). □

Suppose we are at the beginning of the process of constructing an equivariant Moore space of type $(M, n; \pi)$, in the non split case. Then X_1 will be the total space of a $K(M,n)$-fibration over $K(\pi,1)$. The results of V. K. A. M. Gugenheim in [7] imply that the (equivariant) chain-complex of X_1 will be a *twisted tensor product* of the chain complex for a $K(M,n)$ by that of $\tilde{K}(\pi,1)$ (we can use the description of these chain-complexes that appears in [4]). Since in the stable range (all dimensions $<2n$, in this case) a twisted tensor product is the same as an *ordinary* tensor product followed by a *twisted direct sum* it follows that:

Proposition 1.9: Under the hypotheses of 1.4 and 1.7 (with $n>2$, $k = 2$, *and* C_* *the chain complex of* X_1*) the obstruction,* c_3, *to the existence of* X_3 *is an element of* $H^3(V_*;\ M/2M)$ *that maps under* λ *to the* $n+k$*-dimensional homological k-invariant of* $K^+(M,n)\otimes\mathbb{Z}_*$.

Remarks: 1. This result has the interesting consequence that, if the first obstruction c_2 is nonvanishing in the *split* case it doesn't vanish in the *general* case, i.e., one can't "cancel out" the first obstruction by introducing a non-trivial topological k-invariant in the lowest dimension.

Phrased in the terms of Steenrod's original formulation of the problem this says that if the first obstruction to introducing an appropriate π-action to a pointed complex is nonvanishing, then letting the basepoint move freely won't simplify matters.

2. In the non-split case there will turn out to be *three* essentially different sources of obstruction:

A. *The "homological" obstructions* -- coming from the homological k-invariants of the equivariant Eilenberg-MacLane spaces;

B. *The "topological" obstructions* -- defined when the homological obstructions vanish and coming from the rightmost colume of 1.7. They derive from the effects of cohomology operations on the first topological k-invariant and *vanish identically* in the split case.

C. *The "multiplicative" obstructions* -- derived from the fact that fibrations correspond to *twisted tensor products* rather than twisted direct sums. This obstruction is explored in [18] and is shown to be nonvanishing in general.

3. $K^+(M,n)$ denotes the quotient of $K(M,n)$ by the subcomplex of 0-dimensional elements. See 1.7 for definitions of V_* and λ.

Proof: This follows from the fact that, in the stable range (i.e. dimensions n through 2n-1), $\tilde{K}(\pi,1)\otimes_{\xi}K(M,n)=\tilde{K}(\pi,1)\oplus_{\xi'}\tilde{K}(\pi,1)\otimes K^{+}(M,n)$, (where ξ' is the restriction of ξ to the stable range), and by direct computation of $\mathfrak{S}(f_0)$ in this setting (see the discussion preceding 1.7). □

At this point we are in a position to state sufficient conditions for the existence of equivariant Moore spaces. We will make use of the results involving the homology of Eilenberg-MacLane spaces in [5] and [3].

We begin with the following well-known result (which can be proved directly using the Hurewicz homomorphism):

Corollary 1.10: If M *is a* $\mathbb{Z}\pi$*-module of homological dimension* ≤ 2, *there exists a space of type* $(M, n; \pi)$ *for any* $n>1$. □

Remark: Using the obstruction theory described above, this follows from the fact that $H_{n+1}(K(M,n);\mathbb{Z}) = 0$ for all $n>1$ and all abelian groups M -- see [5, §20].

It follows that the first *nontrivial obstruction* is $c_2 \in \mathrm{Ext}_{\mathbb{Z}\pi}{}^3(M,H_{n+2}(K(M,n);\mathbb{Z})$. Since it is proved in [5, §§21, 22] that:

$H_4(K(M,2);\mathbb{Z}) = \Gamma(M)$;

$H_{n+2}(K(M,n);\mathbb{Z}) = M/2M$, if $n > 2$.

(where $\Gamma(M)$ is the Whitehead functor).

Corollary 1.11: Let M *be a* $\mathbb{Z}\pi$*-module of homological dimension* ≤ 3 *and suppose that* $\mathrm{Ext}_{\mathbb{Z}\pi}{}^3(M,\Gamma(M)) = 0$. *Then there exists an equivariant Moore space of type* $(M, 2; \pi)$. □

Theorem 1.12: Let M *be a* $\mathbb{Z}\pi$*-module of homological dimension* k *and suppose* M_p *(the p-torsion submodule)*$= M/p\bullet M = 0$ *for all primes* $p < 1 + k/2$. *Then there exist equivariant Moore spaces of type* $(M, n; \pi)$, *where* n *is any integer* $\geq k$.

Proof: This follows immediately from the results on the homology of Eilenberg-MacLane spaces in the stable range in [3]. Those results imply that the homology of a K(M,n) in dimension n+k is a sum of copies of M_p and $M/p\bullet M$ for primes p such that $2(p-1) \leq k$ where $k < n$. □

§ 2 The First Homological k-Invariant of a Chain-Complex.

In this section we will develope methods for computing the homological k-invariants of a chain complex. We will make extensive use of the perturbation theory of DGA-algebras. This theory was developed by H. Cartan in unpublished work and later elaborated by V.K.A.M. Gugenheim (see [7]).

Definition 2.1: Let $f:C \to D$, $g:D \to C$ be maps of chain-complexes. Then:

1. if f maps each C_i to D_{i+k} then f will be called a ***map of degree*** k;

2. if f is a map of degree k then df is defined to be $d_D \cdot f + (-1)^{k+1} f \cdot d_C$. The map f is defined to be a ***chain map*** if it is of degree 0 and $df=0$.

3. if f and g, above, are ***both chain maps*** and:
   a. $f \cdot g = 1_D$, and $g \cdot f = d\varphi$, where φ is some map of degree +1; and
   b. $f \cdot \varphi = 0$, $\varphi \cdot g = 0$, and $\varphi^2 = 0$;

then ***the triple*** (f, g, φ) is called a ***contraction of*** C ***onto*** D. The map f is called the ***projection*** of the contraction, and g is called the ***injection***. □

Remarks: 1. Since df has the special meaning given above, we will follow Gugenheim in [7] in using $d \cdot f$ to denote the composite.

2. **We will also use the convention that if $f:C_1 \to D_1$, $g:C_2 \to D_2$ are maps, and $a \otimes b \in C_1 \otimes C_2$ (where a is a homogenous element), then** $(f \otimes g)(a \otimes b) = (-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b)$. This convention simplifies some of the common expressions in homological algebra. For instance the differential, $d_{\otimes}$, of the tensor product $C \otimes D$ is just $d_C \otimes 1 + 1 \otimes d_D$.

3. It is not difficult to see that the definition of a chain-map given above coincides with the usual definition.

4. The definition of a contraction of chain complexes given here is slightly stronger than the original definition due to Eilenberg and MacLane in [4], since they don't require the chain-homotopy to be *self-annihilating*. The *present* definition is due to Weishu Shih in [16]. Its use in the present paper is justified by the fact that it enables us to use **the following lemma, which is central to perturbation theory in differential algebra:**

Lemma 2.2* (Perturbation Lemma)**: *Let* $(f, g, \varphi):C \to D$ ***be a contraction of chain complexes with differentials d_C ***and*** d_D, ***respectively. Suppose*** C ***is equipped***

with second differential d' *and an increasing filtration* (F_iC) *such that:*

1. $t=d'-d_C$ *lowers filtration degree by at least* 1;
2. φ *and* d_C *preserve the filtration;*
3. $t(F_0C)=0$.

Then there exists a second differential d" *on* D *and a contraction* $(f',g',\varphi'):(C,d')\rightarrow(D,d")$. *The contraction* (f',g',φ') *is defined by:*

1. $f'=f\cdot(1+t\cdot T_{\infty}\cdot\varphi)$;
2. $g'=T_{\infty}\cdot g$;
3. $\varphi'=T_{\infty}\cdot\varphi$;

where $T_{\infty}=1+\sum_{i=1}^{\infty}(\varphi\cdot t)^i$ *and the differential* d", *on* D, *is given by* $d"=d+f\cdot t\cdot T_{\infty}\cdot g$.
□

Remarks: 1. The summation above has i going from 1 to infinity. Note that this "infinite series" actually reduces to a *finite sum* when evaluated on elements of C because of the conditions on the filtration of C. Throughout the remainder of this section we will use the notation $T_{\infty}=(1-\varphi\cdot t)^{-1}$. This is *more than just a notational convention* -- the condition of the filtration of C implies that $T_{\infty}\cdot(1-\varphi\cdot t)=1_C$.

2. This lemma first appeared in [7], although it was used *implicitly* in [16].

Definition 2.3: If M and N are $\mathbb{Z}\pi$-modules, F is a free $\mathbb{Z}\pi$-module with preferred basis $\{y_i\}$ and $f:M\rightarrow N$ is a homomorphism of abelian groups that doesn't necessarily preserve the action of π then the *F-extension of* f, denoted $\tilde{f}_F:M\otimes_{\mathbb{Z}}F\rightarrow N\otimes_{\mathbb{Z}}F$ (with *diagonal* π-action) is defined to be the $\mathbb{Z}$-linear map for which $\tilde{f}_F(m\otimes(y_i\bullet v))=f(m\bullet v^{-1})\bullet v\otimes(y_i\bullet v)$ for all $m\in M$ and $v\in\pi$. □

Remarks: 1. This construction will be used as a way to convert *maps* into *module homomorphisms* -- it is not difficult to see that $\tilde{f}_F$ is a $\mathbb{Z}\pi$-module homomorphism. The construction was motivated by the *Borel Construction* for making a group action free (i.e. take the product with a space upon which the group acts freely and give the product the *diagonal* action).

2. The F-extension of f clearly depends upon the preferred basis for F that was used in its construction. If f is already a module homomorphism $\tilde{f}_F=f\otimes1$.

3. The definition above can clearly be generalized to the case where M, N, and F are *chain complexes*. In this case bases for the chain modules of F must be defined in each dimension. If f was *originally a chain map* its F-extension will *also* be a chain map if the *differential* on F is *identically zero*.

Lemma 2.4 **(The Module Lemma)**: *Let* C *and* D *be chain complexes and let* (f, g, φ):C→D *be a* **Z***-contraction (i.e. the maps involved aren't necessarily module homomorphisms). Let* Z_* *be a free resolution of* **Z** *over* **Z**π *and suppose some preferred basis has been chosen in each dimension. Define:*

1. $\hat{f} = \tilde{f}_Z \circ (1 - (1\otimes d_Z)\circ\hat{\varphi}_Z)^{-1}$;
2. $\hat{g} = (1 - \hat{\varphi}_Z \circ (1\otimes d_Z))^{-1}$;
3. $\tilde{\varphi} = (1 - \hat{\varphi}_Z \circ (1\otimes d_Z))^{-1}$;
4. $d' = (\tilde{d}_D)_Z + \tilde{f}_Z \circ (1\otimes d_Z)\circ(1-\hat{\varphi}_Z \circ (1\otimes d_Z))^{-1} \circ \tilde{g}_Z$;
5. $c' = (\tilde{d}_C)_Z + 1\otimes d_Z$;

Remark: Notice that when the composite $\varphi_Z \circ (1\otimes d_Z)$ is evaluated on $a\otimes b \in C_* \otimes Z_*$ the dimension of the first factor is *lowered* by 1 and that of the second factor is *raised* by 1.

Proof: This is a straightforward application of the Perturbation Lemma to the contraction $(\tilde{f}, \tilde{g}, \hat{\varphi}): C_*\otimes Z_* \to D_*\otimes Z_*$, where the differentials of $C_*\otimes Z_*$ and $D_*\otimes Z_*$ are taken to be $(\tilde{d}_C)_Z$ and $(\tilde{d}_D)_Z$, respectively. The "perturbation", t, is $1\otimes d_Z$, which evaluates to $(-1)^{\dim(a)} a\otimes d_Z(b)$ on $a\otimes b$, by the convention regarding evaluation of maps on tensor products. The filtration degree of such an element is defined to be the dimension of b. □

Let C_* be a chain complex over **Z**π and suppose its lowest dimensional non-vanishing homology module is H_n (in dimension n). Furthermore suppose the next non-vanishing homology module is H_{n+k} (in dimension n+k) with $k \geq 1$. Let D_* be a **Z**π-chain complex with:

1. $D_i = 0, i < n$;
2. $D_n = H_n$ (as a **Z**π-module);
3. $D_i = 0, n < i < n+k$;
4. D_* is **Z**-chain homotopy equivalent to C_*.

(to simplify the discussion somewhat we'll assume that the boundary homomorphisms of D_* commute with the action of π, although this isn't necessary).

The theory of chain-complexes over a PID ($\mathbb{Z}$ in this case) guarantees the existence of such a D_* and a contraction (see theorem 5.1.15 on p. 164 of [9]):

2.5: $(f, g, \varphi): C_* \rightarrow D_*$

over $\mathbb{Z}$. If Z_* is a free $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$ with preferred bases for its chain modules chosen (so $\tilde{f}_Z$, $\tilde{g}_Z$, and $\hat{\varphi}_Z$ can be defined as in 2.3) then 2.4 implies the existence of a contraction over $\mathbb{Z}\pi$ -- $(\hat{f}, \hat{g}, \tilde{\varphi}): C_*\otimes Z_* \rightarrow (D_*\otimes Z_*, d')$ where $\hat{f}$, $\hat{g}$, and $\tilde{\varphi}$ are defined as in 2.4.

Corollary 2.6: Under the conditions in the discussion above, the first non-trivial homological k-invariant of $C_*\otimes Z_*$ *is given by the cocycle:*

$$(-1)^{n+k} p \circ \tilde{f}_Z \circ (1\otimes d_Z) \circ (\hat{\varphi}_Z \circ (1\otimes d_Z))^k \circ \tilde{g}_Z : H_n \otimes Z_{k+1} \rightarrow H_{n+k}$$

where $p: D_{n+k}\otimes Z_0 \rightarrow H_{n+k}$ *is the projection of the cycle module to the homology module.*

Remarks: 1. We may consider this cocycle as being defined in either $\mathrm{Hom}_{\mathbb{Z}\pi}(H_n\otimes Z_*, H_{n+k})$ which defines an element of $\mathrm{Ext}_{\mathbb{Z}\pi}^{k+1}(H_n, H_{n+k})$ or $\mathrm{Hom}_{\mathbb{Z}\pi}(Z_*, \mathrm{Hom}_{\mathbb{Z}}(H_n, H_{n+k}))$, which gives rise to the isomorphic group $H^{k+1}(\pi, \mathrm{Hom}_{\mathbb{Z}}(H_n, H_{n+k}))$ -- see [17].

2. By the remarks following 1.4 and 1.7 it follows that, if $C_* = K(M,n)$ with M a $\mathbb{Z}$-free $\mathbb{Z}\pi$-module then $C_*\otimes Z_*$ is the (equivariant) chain complex of $K_\pi(M,n)$ and the homological k-invariant in question is the first obstruction to the existence of an equivariant Moore space of type $(M,n;\pi)$.

3. Note that this cocycle *vanishes* if any of the $\mathbb{Z}$-homomorphisms f, g, or φ is also *$\mathbb{Z}\pi$-linear*, in the dimension range of the formula.

4. It should be kept in mind that the boundary map in the resolution, $H_n\otimes Z_*$, of H_n *is not* $1\otimes d_Z$ -- it is $(\tilde{f}_Z)_n \circ (1\otimes d_Z) \circ (\tilde{g}_Z)_n$ (it is not difficult to see that $H_n\otimes Z_*$, with this differential, is *still* a resolution of H_n -- at least up to dimension n+k). This twisted differential coincides with $1\otimes d_Z$ if and only if f is $\mathbb{Z}\pi$-linear, i.e. $\tilde{f}_Z = f\otimes 1$ in dimension n.

Proof: Throughout this argument the term characteristic map of a chain complex will refer to the canonical map (up to a chain homotopy) from a chain complex to a projective resolution of its lowest dimensional homology module, i.e. if the lowest dimensional nonvanishing homology module of C_* is in dimension n and has a projective resolution P_* then the characteristic map of C_* is a chain map $C_* \to \sum^n P_*$.

We will prove that the cocycle in the statement of the theorem represents the first homological k-invariant of $D''=(D_* \otimes_{\mathbf{Z}} Z_*, d')$, which is chain homotopy equivalent to $C_* \otimes Z_*$. First note that D'' can be regarded as the direct sum $D''=H_n \otimes Z_* \oplus D'$, where D' has no nonvanishing chain modules below dimension n+k. This is not necessarily a direct sum decomposition of *chain complexes*. In fact, by the description of d' in 2.4 it follows that:

1. $D' \otimes Z_*$ is a chain subcomplex of D'';
2. The boundary of $H_n \otimes Z_*$ may contain components in $D' \otimes Z_*$.

This follows from the existence of a corresponding direct sum decomposition of *chain complexes* when the unperturbed differential is used, and the fact the perturbation terms in d' *lower* the dimension of the Z_*-factor and *raise* that of the D_*-factor. Let d_B denote "the portion of the boundary of $H_n \otimes Z_*$ that lies in D', i.e. the composite $H_n \otimes Z_* \to D'' \to_{d'} D'' \to D'$, where the leftmost map is the inclusion and the rightmost map is the projection.

It is not hard to see that there is a chain homotopy equivalence $h:\mathfrak{A}(c) \to \Sigma D'$ (recall that $\mathfrak{A}(c)$ is the algebraic mapping cone of c), where $c:D'' \to H_n \otimes Z_*$ is the projection, which is also the *characteristic map*. The map h can be described on the various direct summands of $\mathfrak{A}(c)$ as follows (where D'' is regarded as the direct sum $H_n \otimes Z_* \oplus D'$):

1. $h| \Sigma D' = 1: \Sigma D' \to \Sigma D'$;
2. $h| \Sigma H_n \otimes Z_* = 0: \Sigma H_n \otimes Z_* \subset \mathfrak{A}(c) \to \Sigma D'$;
3. $h|H_n \otimes Z_i = (-1)^{i+1} d_B : H_n \otimes Z_* \subset \mathfrak{A}(c) \to \Sigma D'$.

where in the last statement d_B is regarded as a map from $H_n \otimes Z_i$ to $D'_{i-1} = (\Sigma D')_i$.

The conclusion now follows from the fact that the component of d' that maps $H_n \otimes Z_{k+1}$ to $D_{n+k} \otimes Z_0$ is the cocycle given in the statement of this theorem. □

Definition 2.7: Let n and k be integers ≥ 1, let $E=(f,g,\varphi):C_* \to D_*$ be a $\mathbb{Z}$-contraction of $\mathbb{Z}\pi$-chain complexes and let $\mu_1,\dots,\mu_{k+1} \in \pi$. The *c-symbol of the elements* $\mu_1,\dots,\mu_{k+1}$, *for the contraction* E, *and in dimension* n, is *denoted* $\mathfrak{C}_n(E;\mu_1,\dots,\mu_{k+1})$ and is defined to be the element of $\mathrm{Hom}_{\mathbb{Z}}(D_n,D_{n+k})$ given by:

$$\mathfrak{C}_n(E;\mu_1,\dots,\mu_{k+1}) = (-1)^{n+k} f\cdot(\alpha_k\cdot\mu_1^{-1})\cdot\mu_1\cdot\dots\mu_{k+1}$$

where α_k is defined inductively by $\alpha_i = \varphi\cdot(\alpha_{i-1}\cdot\mu_{k+2-i}^{-1})$ and $\alpha_0=g$.□

Remarks: 1. By abuse of notation we have used the *symbol* for μ_i to denote the *homomorphism* of C_* or D_* *induced* by μ_i.

2. Note that, due to the self- and mutual annihilating properties of f, g, and φ, the c-symbol will *vanish* if any of the μ_i is equal to the *identity element.*

3. The definition above will be extended to the case where the μ_i are arbitrary elements of the *group-ring* $\mathbb{Z}\pi$. This is done by simply defining it to be *Z-linear* in each **argument**, μ_i.

Definition 2.8: **Let Z_* be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$ and suppose preferred basis** elements have been chosen in each dimension. If $a \in Z_t$ is a preferred basis element its *boundary tree with respect to* Z_* is defined to be a tree whose nodes are labelled with triples (k,λ,b) as follows:

1. k is an integer, called the *dimension* of the node;
2. $\lambda \in \mathbb{Z}\pi$ is called its *multiplier;*
3. b is a preferred basis element of Z_t, called the *base* of the node.

The boundary tree of a is constructed inductively as follows:

1. There is one node of dimension t labelled with $(t,0,a)$;
2. Given a node, n, of dimension i with label (i,λ,b), suppose the boundary of b is equal to $\sum\lambda_j c_j$, $\lambda_j \neq 0$, where the c_j are *preferred basis elements.* Then there is one *descendant* of n for each *term* of that linear combination and these descendant nodes are labelled $(i-1,\lambda_j,c_j)$, respectively. Each node of dimension $i<t$ is joined to a *unique* node of dimension $i+1$.
3. The process described above terminates in nodes of dimension 0. □

Remarks: This is simply a way of keeping track of all the terms that arise from taking boundaries of elements and then taking boundaries of the individual terms of the linear combinations that arise. This notational device will turn out to be indispensible in performing explicit computations.

Definition 2.9: 1. A *track* through a boundary tree is defined to be a path that starts at the node of highest dimension and proceeds to a node of dimension 0 without ever covering a given edge more than *once.*

2. A track will be called *essential* if the number $1 \in \mathbf{Z}\pi$ never occurs as a multiplier. Otherwise it will be called *inessential.*

3. A boundary tree that has no inessential tracks will be called *reduced.* □

Remarks: 1. Since a track isn't allowed to double back on itself, dimension is clearly a monotone decreasing function of distance along a track.

2. Given an arbitrary boundary tree it is clearly possible to *reduce* it, i.e. to find a subtree containing all of the essential tracks of the original tree and not containing any inessential tracks. Simply delete from the original tree any subtree whose root has a multiplier of 1.

3. Given a track, T, its *multiplier sequence* is defined to be the sequence of multipliers encountered in traversing the track from the root to the end. This sequence is assumed to begin at one dimension below the top dimension. If T is a track in the boundary tree of $a \in Z_t$, as in 2.8, then the multiplier sequence is denoted $\{T_{t-1},\dots,T_0\}$.

Our main result is the following:

Theorem 2.10: Let $E=(f,g,\varphi):C_* \to D_*$ *be a* $\mathbf{Z}$*-contraction of* $\mathbf{Z}\pi$*-chain complexes such that:*

1. The lowest-dimensional nonvanishing homology module of C_* *is* H_n *in dimension* n, *and it is* $\mathbf{Z}$*-torsion free;*

2. The next nonvanishing homology module of C_* *is* H_{n+k} *in dimension* $n+k$;

3. $D_i=0,\ i < n,\ D_n=H_n,\ D_i=0,\ n < i < n+k.$

Let $\mathbf{Z}_*$ *be a free resolution of* $\mathbf{Z}$ *over* $\mathbf{Z}\pi$ *with preferred basis elements chosen in each dimension and with the property that the augmentation* $\mathbf{Z}_0 \to \mathbf{Z}$ *maps preferred basis elements to* 1. *Then the first homological k-invariant of* $C_* \otimes \mathbf{Z}_*$ *is an element of* $H^{k+1}(\pi,\mathrm{Hom}_{\mathbf{Z}}(H_n,H_{n+k})) = \mathrm{Ext}_{\mathbf{Z}\pi}^{k+1}(H_n,H_{n+k})$

represented by a cocycle that maps a preferred basis element $b \in \mathcal{Z}_{k+1}$ ***to***

$$\sum_T \mathfrak{C}_n(E;T_0,\ldots,T_k)$$

where the sum is taken over all essential tracks in a boundary tree of b ***with respect to*** $\mathcal{Z}_*$.

Remarks: 1. The condition involving the augmentation homomorphism of $\mathcal{Z}_*$ won't prove to be very restrictive.

2. Let $\mathcal{Z}_*$ be the ***right bar resolution*** of $\mathbb{Z}$ with the symbols $[\mu_1|\ldots|\mu_i]$ as preferred basis elements (where i is any positive integer and the μ's run over all elements of π). Then it isn't hard to see that a ***reduced boundary tree*** for $[\mu_1|\ldots|\mu_{k+1}]$ is:

$$(k+1,0,[\mu_1|\ldots|\mu_{k+1}])\rightarrow(k,\mu_{k+1},[\mu_1|\ldots|\mu_k])\rightarrow\ldots\rightarrow(0,\mu_1,[])$$

so that the first homological k-invariant is a cocycle whose value on $[\mu_1|\ldots|\mu_{k+1}]$ is $p\cdot\mathfrak{C}_n(E;\mu_1,\ldots,\mu_{k+1})\in \mathrm{Hom}_{\mathbb{Z}}(D_n,H_{n+k})$ where $D_n=H_n$ and $p:D_{n+k}\rightarrow D_{n+k}/\partial D_{n+k+1} = H_{n+k}$, is the projection.

Proof: First note that the defintions of the two quantities equated in the statement of the theorem ***never*** make use of the self-annihilating property of the boundary homomorphism $d_{\mathcal{Z}}$. Thus, in principle, it is possible to define the terms in the theorem with $d_{\mathcal{Z}}$ an ***arbitrary sequence of homomorphisms,*** $d_i:\mathcal{Z}_i\rightarrow\mathcal{Z}_{i-1}$. We will, consequently, separate the proof of the theorem into two cases:

Case I: We assume that the boundary homomorphisms $d_{\mathcal{Z}}$ have the property that $d_{\mathcal{Z}}(b)=\sum m_j b_j \mu_j$ if b_j is a preferred basis element, where $\mu_j\in\pi$, $m_j\in\mathbb{Z}$. (In case II the coefficients of the b_j will be arbitrary elements of $\mathbb{Z}\pi$).

Define:

1. B_i to be the subtree of the boundary tree of b spanned by all the nodes of dimension $\geq i$;

2. The ***end*** of a track, T, to be the base of its lowest-dimensional node, denoted e(T). See 2.8 for a definition of the base of a node in a boundary tree.

3. $A_i(T_k,\ldots,T_i)$ (where each $T_j=m_j v_j$ for some $m_j\in\mathbb{Z}$, $v_j\in\pi$) to be $\alpha_{k+1-i}\cdot T_i\cdot\ldots\cdot T_k$, where α_i is defined as in 2.7 using $\mu_j=v_{j-1}$ and T is some track in B_i;

4. V_i to be $(\hat{\varphi}_Z \cdot (1 \otimes d_Z))^{k+1-i} \cdot \tilde{g}_Z$, as in 2.6;

Remarks: 1. Note that $\tilde{g}_Z(x \otimes b)$, where $x \in D_n$, is equal to $g(x) \otimes b$ if b is a preferred basis element -- see 2.3.

2. We will *actually* give an inductive proof of the *following* statement:

Claim: Under the hypotheses of the theorem

$$\sum A_i(T_k,\ldots,T_i)(x) \otimes e(T) \bullet \hat{T}_i \ldots \hat{T}_k = V_i(x \otimes b)$$

(where b is a preferred basis element and the sum is over all tracks in B_i)

for all $x \in D_n$ *and all* $1 \leq i < k+1$, *where* $\hat{T}_j$ *denotes the element* $v \in \pi$ *whenever* T_j *is of the form* mv *and* $m \in \mathbb{Z}$.

Remarks: 1. Since the c-symbol vanishes *identically* on tracks that *aren't essential* (and this doesn't depend upon d_Z being self-annihilating) the sum above can be regarded as a sum over *essential* tracks.

2. Proving the claim above proves the theorem in Case I because, when i=1 we simply take the boundary d_Z one more time and take $\tilde{f}_Z$ and apply the *augmentation*, which maps all preferred basis elements to 1 (by hypothesis).

Proof of claim: First we will verify the claim in the case where i=k. Suppose $d_Z(b) = \sum m_j b_j \mu_j$, where $m_j \in \mathbb{Z}$, b_j are preferred basis elements of Z_* and $\mu_j \in \pi$. Then $V_k(x \otimes b) = \hat{\varphi}_Z \cdot (1 \otimes d_Z)(g(x) \otimes b)$ (see remark 1 preceding the claim) $= \hat{\varphi}_Z(\sum m_j g(x) \otimes b_j \mu_j) = \sum m_j(\varphi(g(x)\mu_j^{-1})\mu_j \otimes b_j \mu_j)$ (see 2.3) $= \sum A_k(T_k) \otimes e(T)\hat{T}_k$ (where the sum is over all $T \in B_k$). The last equality is a consequence of the definition of a boundary tree (2.8) which implies that the ends of tracks in B_k are in a 1-1 correspondence with the b_j.

Now we will assume the inductive hypothesis and assume the claim is true in dimensions $\geq$ i. Note that $V_{i-1}(x \otimes b) = \hat{\varphi}_Z \cdot (1 \otimes d_Z) V_i(x \otimes b)$. By hypothesis $V_i(x \otimes b) = \sum A_i(T_k,\ldots,T_i) \otimes e(T)\hat{T}_i \ldots \hat{T}_k$. The inductive definition of a boundary tree implies that $(1 \otimes d_Z) \sum A_i(T_k,\ldots,T_i) \otimes e(T)\hat{T}_i \ldots \hat{T}_k$ (summed over all $T \in B_i$) $= \sum A_i(T_k,\ldots,T_i) \otimes e(T)T_{i-1}\hat{T}_i \ldots \hat{T}_k$ (summed over all $T \in B_{i-1}$--see 2.8) and evaluating $\hat{\varphi}_Z$ on *this* gives

$$(1) \quad \sum \varphi(A_i(T_k,\ldots,T_i)\hat{T}_k^{-1} \ldots \hat{T}_{i-1}^{-1})\, \hat{T}_{i-1}\, \hat{T}_i \ldots \hat{T}_k \otimes e(T)T_{i-1}\, \hat{T}_i \ldots \hat{T}_k$$

(summed over all $T \in B_{i-1}$)

Let $T_{i-1} = m\hat{T}_{i-1}$. Then $\varphi(A_i(T_k,\ldots,T_i)\hat{T}_k^{-1}\ldots\hat{T}_{i-1}^{-1})\hat{T}_{i-1}\hat{T}_i\ldots\hat{T}_k$ = $\varphi(\alpha_{k+1-i}T_i\ldots T_k\hat{T}_k^{-1}\ldots\hat{T}_{i-1}^{-1})\hat{T}_{i-1}\hat{T}_i\ldots\hat{T}_k = \varphi(\alpha_{k+1-i}\hat{T}_{i-1}^{-1})\hat{T}_{i-1}T_i\ldots T_k = \alpha_{k+2-i}\hat{T}_{i-1}T_i\ldots T_k =$ $A_{i-1}(T_k,\ldots,T_{i-1})/m$. **Substituting this into formula (1) gives**

$$(1)\quad \sum A_{i-1}(T_k,\ldots,T_{i-1})\otimes e(T)\hat{T}_{i-1}\hat{T}_i\ldots\hat{T}_k$$

(summed over all $T\in B_{i-1}$)

which proves the induction step and, by the remarks following the claims, also proves the theorem in case I. Case II follows from case I by noting that each of the terms in the formula

$$(2)\quad \sum_T \mathfrak{C}_n(E;T_0,\ldots,T_k)(x) = (-1)^{n+k}p\circ\tilde{f}_Z\circ(1\otimes d_Z)\circ(\hat{\varphi}_Z\circ(1\otimes d_Z))^k\circ\tilde{g}_Z(x\otimes b)$$

(summed over all $T\in B$)

is *linear* in the boundary maps in the following sense:

1. Suppose d, d', d" are sequences of homomorphisms $Z_i\rightarrow Z_{i-1}$ that are identical except that, in dimension j $d_j=d'_j+d''_j$. Then the value of the right-hand side of equation (2) calculated using $d_Z=d$ (in all dimensions) will be the *sum* of the values obatined using d' and d".

2. The same is true of the *left* hand side if we define the boundary trees of d, d', d" to have the same underlying tree structure (with the possibility of many of the multipliers begin 0).

Since, in each dimension, we can decompose the boundary homomorphism $(d_Z)_i$ into linear combinations of homomorphisms that satisfy the conditions of case I, it follows that the theorem is true in all cases. □

We will now give a second example of how to calculate and use boundary trees (recall that the first example used the bar resolution of $\mathbb{Z}$ and appeared immediately after the *statement* of the theorem). This second example will be more important than the first since it will be used extensively in the next section.

We will consider the case where Z_* is the *Gruenberg Resolution* of $\mathbb{Z}$ over $\mathbb{Z}\pi$. Suppose the group π has the presentation $\langle x_1,\ldots,x_s; r_1,\ldots,r_t\rangle$ (we are only assuming that the presentation is finite to simplify the discussion). Then theorem 10.9 on p.271 of [15] implies the existence of a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$ with chain modules generated by the symbols:

$(R_{i1}\ldots R_{ij})$, in dimension 2j, and $(R_{i1}\ldots R_{ij}X_{ij+1})$, in dimension 2j+1 where the R-symbols are in a 1-1 correspondence with a set of *free generators for the relation subgroup* -- this is the ***normal closure*** of the relations $\{r_j\}$ in the **free group generated by the x_i. They may be obtained by computing a Reidemeister-Schreier system of generators.** The X-symbols are in a 1-1 correspondence with the generators $\{x_i\}$.

In order to define the boundary of an element it is necessary to recall the notion of a *Fox Derivative* or *free derivative:*

These are symbols $\{\partial/\partial x_i\}$ that operate on the words in the $\{x_i\}$ via the following rules:

1. $\partial x_i/\partial x_j = \delta_{ij}$;
2. $\partial(w_1 w_2)/\partial x_i = \partial w_1/\partial x_i + w_1 \bullet \partial w_2/\partial x_i$

where w_1 and w_2 are arbitrary words in the x_i -- see [6,§2] as a general reference. The formula

$$w-1 = \sum \partial w/\partial x_i \bullet (x_i - 1)$$

was proved in [6, p.551]. We will need another version of this formula though. Let $^{-}$ be the anti-involution on the group ring of the free group that maps all group elements to their inverses. If w is a word in the free group then

$w^{-}-1 = \sum \partial w^{-}/\partial x_i \bullet (x_i - 1)$, and taking $^{-}$ of both sides gives

$w-1 = \sum (x_i^{-} - 1) \bullet (\partial w^{-}/\partial x_i)^{-}$, which implies

2.11: $w - 1 = \sum (x_i - 1) \bullet \bar{\partial}_i w$, *where we have written* $\bar{\partial}_i w = -x_i^{-1}(\partial w^{-}/\partial x_i)^{-}$.

The $\bar{\partial}_i$ are similar to the Fox derivatives -- they satisfy the following relations (which are sufficient to *define* them):

2.12: $\bar{\partial}_j x_i = \delta_{ij}$;

$\bar{\partial}_j(w_1 w_2) = (\bar{\partial}_j w_1) w_2 + \bar{\partial}_j w_2$.

Statement 2.11 above and the definition of the boundary maps for the Gruenberg resolution on p.271 of [15] imply that the boundary maps are given by:

1. $d(R_{i1} \ldots R_{ij} X_{ij+1}) = R_{i1} \ldots R_{ij}[x_{ij+1} - 1]_{\pi}$;

2. $d(R_{i1}...R_{ij}) = \sum R_{i1}...R_{ij-1}X_k[\bar{\partial}_k(r_{ij})]_\pi$;

where $[*]_\pi$ denotes the image in $\mathbb{Z}\pi$ under the homomorphism $\mathbb{Z}F_s \rightarrow \mathbb{Z}\pi$ defined by the presentation for π given above, where F_s is the free group on the symbols $\{x_i\}$. Theorem 2.10, coupled with the descriptions of the boundary maps in the Gruenberg Resolution immediately implies:

Corollary 2.13: Under the hypotheses of theorem 2.10, if π *has a presentation* $\langle x_1,...x_s;\ r_1,...,r_t\rangle$ *then the first homological k-invariant of* $C_*\otimes\mathbb{Z}_*$ *is an element of* $H^{k+1}(\pi, \mathrm{Hom}_{\mathbb{Z}}(H_n, H_{n+k}))$ *represented by a cochain on the Gruenberg Resolution of* $\mathbb{Z}$ *corresponding to the presentation above, as follows:*

1. *if* $k=2m$ *then the value of the cocycle on* $R_{i1}...R_{im}X_{im+1}$ *is* $p\cdot\sum \mathfrak{C}_n(E;[x_{im+1}]_\pi, [\bar{\partial}_{j1}(r_{im})]_\pi, ..., [\bar{\partial}_{jm}(r_{i1})]_\pi, [x_{im+1}]_\pi)$;
2. *if* $k=2m-1$ *then the value of the cocycle on* $R_{i1}...R_{im}$ *is* $p\cdot\sum \mathfrak{C}_n(E;[\bar{\partial}_{j1}(r_{im})]_\pi, ..., [\bar{\partial}_{jm}(r_{i1})]_\pi, [x_{jm}]_\pi)$;

where $p: D_{n+k} \rightarrow D_{n+k}/d(D_{n+k+1}) = H_{n+k}$ *is the projection.* □

Remarks: 1. In the summations above all of the j_i vary *independantly* so that they are *m-fold* summations.

2. Note that we have replaced $[x_{ji}-1]_\pi$ by $[x_{ji}]_\pi$. This is permissible because the c-symbols in question are multilinear and they vanish if any of their arguments is equal to 1.

We will conclude this section with an example. Suppose π is the group $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ presented by $\langle s,t;\ s^2, t^2, (ts)^2\rangle$. Then the Reidemeister-Schreier theory implies that the relation subgroup of the free group on s and t is generated by the following words:

$r_1 = s^2$, $r_2 = t^2$, $r_3 = tsts$, $r_4 = tsts^{-1}$, $r_5 = ts^2t^{-1}$, $r_6 = sts^{-1}t^{-1}$

Corollary 2.13 implies that the first homological k-invariant of $C_*\otimes\mathbb{Z}_*$ in the case where $k=2$ is a cochain whose value on R_1T is $p\cdot\mathfrak{C}_n(E;t,[\bar{\partial}_s s^2]_\pi, s)$ (since $\bar{\partial}_t s^2$ is clearly 0). The $\bar{\partial}$-symbol, is easliy calculated using 2.12 -- for instance $\bar{\partial}_s s^2 = 1 + s$, and we may drop the 1-term.

§3 The first homological k-invariant of an Equivariant Eilenberg-MacLane Space.

In this section we will use the results of the preceding section to compute the first homological k-invariant of an equivariant Eilenberg-MacLane space. This computation, coupled with the results of section 1 will imply the existence of a triple $(M, n; \pi)$ where $M=\mathbb{Z}^3$ and $\pi=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, for which the corresponding equivariant Moore space doesn't exist.

The methods of this section will be generally applicable to any triple $(\mathbb{Z}^3, n; \pi)$, where π is any group acting on $\mathbb{Z}^3$, although the final calculation at the end of the section will be performed with $\pi=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ using the presentation given at the end of section 2 with s and t acting via right multiplication by:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \textit{ respectively.}$$

We begin by constructing a contraction of DGA-algebras $(a,b,G):A(\mathbb{Z}^3,2)\rightarrow P(x,y,z)$, where $P(x,y,z)$ is the *divided polynomial algebra* -- the $\mathbb{Z}$-subalgebra of $\mathbb{Q}[x,y,z]$ generated by the elements $\{\gamma_i(x)=x^i/i!,\ \gamma_j(y)=y^j/j!,\ \gamma_k(z)=z^k/k!\}$ for all values of i, j, and k, and $A(\mathbb{Z}^3,n)$ is the n-fold bar construction $\overline{B}^n(\mathbb{Z}[\mathbb{Z}^3])$ -- see [4, §14].

The DGA-algebra $P(x,y,z)$ is the chain-complex that will be used for D_* in the application of 2.11. The π-action on $P(x,y,z)$ can be regarded as being induced by that on $\mathbb{Q}[x,y,z]$ (where x, y, and z are regarded as generating the module $\mathbb{Z}^3$ and the action on the powers of these elements is defined so that the group π acts via *algebra homomorphisms*).

We begin with the following application of the Perturbation Lemma in the preceding section:

Corollary 3.1: Let $(f,g,\varphi):C\rightarrow D$ *be a contraction of DGA-algebras. Then* $(\overline{B}(f),\ \overline{B}(g),\tilde{\varphi})$: $\overline{B}(C)\rightarrow\ \overline{B}(D)$ *is a contraction of DGA-Hopf algebras, where* $\tilde{\varphi}$ =

$(1-\hat{\varphi}\cdot d_s)^{-1}\cdot\hat{\varphi}$ *and* $\hat{\varphi}$ *is defined by* $\hat{\varphi}([a|u]) = -[\varphi(a)|u] + (-1)^{\dim(a)}[g^\circ f(a)|\hat{\varphi}(u)]$. □

Remarks: This is a straightforward consequence of the Perturbation Lemma, where $\hat{\varphi}$ is the homotopy for $\overline{B}(C)$ defined using only the *tensor boundary* and the *simplicial boundary* is regarded as a *perturbation* -- see [4].

The construction of the contraction (a,b,G) will involve several steps. We will initially construct a contraction from $A(\mathbf{Z}^3,1)$ onto $\Lambda(x,y,z)$ (the *exterior algebra*).

We begin with the contraction

(p,q,θ): $A(\mathbf{Z},1) \rightarrow \Lambda(x)$

where $\Lambda(x)$ is the exterior algebra over $\mathbf{Z}$ on one generator x. The maps are defined by:

1. $p([n_1|...|n_k]) = 0$ is $k>1$;
   $p([n]) = nx$;
2. $q(x) = [1]$;
3. $\theta([n_1|...|n_k]) = 0$ if $n_1 = 1$;
   $\theta([n_1|...|n_k]) = \sum[1|j|n_2|...|n_k]$ if $n_1>1$, where the summation has j going from 1 to $n_1 - 1$.
   $\theta([n_1|...|n_k]) = -\sum[1|-j|n_2|...|n_k]$ if $n_1 < 0$, where the summation has j going from 1 to $|n_1|$.

(See [5, p.95])

We now take the bar construction of this and use 3.1 to get the contraction:

3.3: (p,q,Θ):$A(\mathbf{Z},2) \rightarrow P(x) = \overline{B}(\Lambda(x))$,

where, by abuse of notation we are denoting $\overline{B}(p)$ and $\overline{B}(q)$ by p and q, respectively (we won't be using the original definitions of p and q any longer). The chain-homotopy Θ is defined by $\Theta=(1-\Theta'\cdot d_s)^{-1}\cdot\Theta'$ (by 2.3) where Θ' is defined by $\Theta'[a|_2u]=-[\theta(a)|_2u] + (-1)^{\dim(a)}[q\cdot p(a)|_2\Theta'(u)]$.

Remark: The perturbation term $(1-\Theta'\cdot d_s)^{-1}$ will not be significant here because we will only apply Θ' to elements of dimension ≤ 3. In fact we can just assume that $\Theta([a]) =$

$-[\theta(a)]$ because the elements we will work with won't even have a $|_2$.

In the case of $\mathbf{Z}^3$ we have

$$(\Theta'', \hat{p}, \hat{q}): \otimes_1{}^3 A(\mathbf{Z}_i, 2) \to P(x,y,z)$$

(i runs from 1 to 3 in the tensor product), where $P(x,y,z) = P(x) \otimes P(y) \otimes P(z)$ and we have numbered the copies of $\mathbf{Z}$ for the sake of definiteness.

The maps are defined by: $\hat{p} = p_1 \otimes p_2 \otimes p_3$ and $\hat{q} = q_1 \otimes q_2 \otimes q_3$, where $(p_i, q_i, \Theta_i): A(\mathbf{Z}_i, 2) \to P(*)$, with $* = x$ if $i = 1$, y if $i = 2$, and z if $i = 3$, and the contractions are as defined in the statements following 3.3.

3.4: $\Theta'': \otimes_1{}^3 A(\mathbf{Z}_i, 2) \to \otimes_1{}^3 A(\mathbf{Z}_i, 2)$ is defined by

$$\Theta''(U \otimes V \otimes W) = \Theta_1(U) \otimes V \otimes W - (-1)^{\dim(U)} q_1 \cdot p_1(U) \otimes \Theta_2(V) \otimes W -$$
$$(-1)^{\dim(U)+\dim(V)} q_1 \cdot p_1(U) \otimes q_2 \cdot p_2(V) \otimes \Theta_3(W).$$

Now we will develop a contraction

$$(\hat{f}, \hat{g}, \hat{\psi}): \bar{B}(\otimes_1{}^3 A(\mathbf{Z}_i, 1)) \to \otimes_1{}^3 A(\mathbf{Z}_i, 2)$$

This will be done in two stages using the results of chapter I of [5]. First we will construct a contraction

$$(f_1, g_1, \psi_1): \bar{B}(\otimes_1{}^3 A(\mathbf{Z}_i, 1)) \to A(\mathbf{Z}_1, 2) \otimes \bar{B}(A(\mathbf{Z}_2, 1) \otimes A(\mathbf{Z}_3, 1))$$

The maps involved will only be discussed in the dimension range of interest (i.e. dimensions ≤ 3):

$f_1([A \otimes B]) = 0$ unless A or B is 1;

$f_1([1 \otimes B]) = 1 \otimes [B]$, $f_1([A \otimes 1]) = [A] \otimes 1$;

$g_1(1 \otimes [B]) = [1 \otimes B]$, $g_1([A] \otimes 1) = [A \otimes 1]$;

$\psi_1([A \otimes B]) = 0$ if either A or B are 1;

$\psi_1([A \otimes B]) = [1 \otimes B | A \otimes 1]$, otherwise;

where $A \in A(\mathbf{Z}_1, 1)$, $B \in A(\mathbf{Z}_2, 1) \otimes A(\mathbf{Z}_3, 1)$.

Remarks: The statement about ψ_1 follows directly from the formula given at the bottom of p. 53 of [5] for φ and the definitions of the face and degeneracy operators on the bar construction of [4]. Recall that the formula for φ in [5] is only sensitive to the *simplicial dimension* of an element of the bar construction -- and $A\otimes B$ has simplicial dimension 1 (*whatever* the dimensions of A and B might be).

Now we define

$$(f_2,g_2,\psi_2)\colon \bar{\mathcal{B}}(\otimes_2{}^3 A(\mathbf{Z}_i,1)) \to A(\mathbf{Z}_2,2)\otimes A(\mathbf{Z}_3,2)$$

in *exactly the same way* (let $A\in A(\mathbf{Z}_2,1)$, $B\in A(\mathbf{Z}_3,1)$ in the formula above). The two contractions are combined to give $(\hat{f},\hat{g},\hat{\psi})$ where $\hat{f}=(1\otimes f_2)\cdot f_1$, $\hat{g}=g_1\cdot(1\otimes g_2)$, and $\hat{\psi}=\psi_1+g_1\cdot(1\otimes\psi_2)\circ f_1$:

3.5: 1. $\hat{f}([A_1\otimes A_2\otimes A_3]) = 0$ *unless two out of the three terms are* 1 *in which case* $\hat{f}([\ldots\otimes A_i\otimes\ldots])=[A_i]$, i= 1, 2, or 3;

2. $\hat{g}([A_i]) = [A_i]$, i= 1, 2, *or* 3;

3. $\hat{\psi}([A_1\otimes A_2\otimes A_3])= 0$ *if two out of the three terms are 1; **otherwise*** $\hat{\psi}([A_1\otimes A_2\otimes A_3])= [1\otimes A_2\otimes A_3|\, A_1\otimes 1\otimes 1]$ *if all three terms are* $\neq 1$;

$\hat{\psi}([1\otimes A_2\otimes A_3])= [1\otimes 1\otimes A_3|\, 1\otimes A_2\otimes 1]$, ***where*** $A_i\in A(\mathbf{Z}_i,1)$, i= 1, 2, *or* 3. □

In the last step we will define a contraction $(\hat{R},\hat{S},\hat{\Xi})\colon A(\mathbf{Z}^3,2)\to\bar{\mathcal{B}}(\otimes_2{}^3A(\mathbf{Z}_i,1))$, and we will compose the three contractions to get (a,b,G). The contraction $(\hat{R},\hat{S},\hat{\Xi})$ will be defined by applying 3.1 to the contraction $(-,\hat{s},\hat{\xi})\colon A(\mathbf{Z}^3,1)\to\otimes_1{}^3A(\mathbf{Z}_i,1)$. As with the contractions above, *this* contraction will be built in two steps:

$$(r_1,s_1,\xi_1)\colon A(\mathbf{Z}_1\oplus\mathbf{Z}_2\oplus\mathbf{Z}_3,1)\to A(\mathbf{Z}_1,1)\otimes A(\mathbf{Z}_2\oplus\mathbf{Z}_3,1)$$
$$(r_2,s_2,\xi_2)\colon A(\mathbf{Z}_2\oplus\mathbf{Z}_3,1)\to A(\mathbf{Z}_2,1)\otimes A(\mathbf{Z}_3,1)$$

We will use triples (u,v,w) to denote elements of $\mathbf{Z}^3$ and, abusing the notation a little, triples with the first term equal to 0 will denote elements of $\mathbf{Z}^2$.

Definition 3.6: A $\le$ 4-dimensional element of $A(\mathbf{Z}^3,2)$ that is a linear combination of basis elements, each of which contains at least *two adjacent* $|_1$-symbols, will be called *special*.

Remarks: For instance, $[(1,0,0)|_1(1,2,3)|_1(0,1,1)]$ is special and $[(1,0,0)|_2(1,2,3)]$ isn't. In fact it isn't hard to see that the only non-special 4-dimensional canonical basis elements of $A(\mathbb{Z}^3,2)$ are of the form $[u|_2v]$, with $u, v \in \mathbb{Z}^3$.

The following result will enable us to eliminate some terms in the final formula:

Proposition 3.7: ***The map,*** a, ***in the contraction*** $(a,b,G):A(\mathbb{Z}^3,2)\rightarrow P(x,y,z)$ ***maps all special elements to*** 0.

Proof: The map a is the composite of:

$\hat{R}:A(\mathbb{Z}^3,2)\rightarrow\overline{B}(\otimes_1{}^3A(\mathbb{Z}_i,1))$

$\hat{f}:\overline{B}(\otimes_1{}^3A(\mathbb{Z}_i,1))\rightarrow\otimes_1{}^3A(\mathbb{Z}_i,2)$

$\hat{p}:\otimes_1{}^3A(\mathbb{Z}_i,2)\rightarrow P(x,y,z)$

The composite $\hat{f}\cdot\hat{R}$ is already described in theorem 6.1 of [5]. Using the formula presented in the statement of that theorem we get:

$\hat{f}\cdot\hat{R}([(u_1,v_1,w_1)|_1(u_2,v_2,w_2)]) = [(u_1,0,0)|_1(u_2,0,0)]\otimes1\otimes1 + 1\otimes[(0,v_1,0)|_1(0,v_2,0)]\otimes1 + 1\otimes1\otimes[(0,0,w_1)|_1(0,0,w_2)]$

This is mapped to 0 by $\hat{p}$ since p_i maps all terms of $A(\mathbb{Z}_i,2)$ of the form $[x|_1y]$ to zero, with $x,y\in\mathbb{Z}_i$ (the point is that such elements are suspensions of elements of $A(\mathbb{Z}_i,1)$ of dimension >1). A similar argument is used in the higher dimensional cases. □

Remarks: Special elements may be ignored in the formula for a chain homotopy, G, since they will have at *least one* $|_1$-term in them even *after* the boundary is taken(in a bar construction).

Recall that triples (u,v,w) with u=0 denote elements of $\mathbb{Z}_2\oplus\mathbb{Z}_3$. The results of the first chapter of [5] imply that:

3.8: 1. $\hat{-}([(u,v,w)]) = [(u,0,0)]\otimes1\otimes1+ 1\otimes[(0,v,0)]\otimes1+1\otimes1\otimes[(0,0,w)]$;

$\hat{-}([(u_1,v_1,w_1)|(u_2,v_2,w_2)]) = [(u_1,0,0)|(u_2,0,0)]\otimes1\otimes1 + [(u_1,0,0)]\otimes[(0,v_2,0)]\otimes1 + [(u_1,0,0)]\otimes1\otimes[(0,0,w_2)] + 1\otimes[(0,v_1,0)|(0,v_2,0)]\otimes1 + 1\otimes[(0,v_1,0)]\otimes[(0,0,w_2)] + 1\otimes1\otimes[(0,0,w_1)|(0,0,w_2)]$ *(see theorem 6.1 in [5]);*

2. $\hat{s}([(u,0,0)]\otimes1\otimes1) = [(u,0,0)]$;

$\hat{s}(1\otimes[(0,v,0)]\otimes1) = [(0,v,0)]$;

$\hat{s}(1\otimes 1\otimes[(0,0,w)]) = [(0,0,w)]$;

$\hat{s}([(u,0,0)]\otimes[(0,v,0)]\otimes[(0,0,w)])=$

$\hat{s}([(u,0,0)]\otimes 1\otimes 1)*\hat{s}(1\otimes[(0,v,0)]\otimes 1)*\hat{s}(1\otimes 1\otimes[(0,0,w)])$, *where the * denotes the shuffel product in* $A(\mathbf{Z}^3,1)$ *(see [4, p.74]);*

3. $\hat{\xi} = \xi_1 + s_1\cdot(1\otimes\xi_2)\cdot r_1$. *Note that since* s_1 *involves shuffel products it will map special elements to special elements.* □

Since ξ_1 always increases the number of bars in a canonical basis element of $A(\mathbf{Z}^3,1)$ it follows that ξ_1 can be *disregarded* in dimension 2 (since this gives rise to $\hat{\Xi}$ in dimension 3 and that is going to be plugged into a, which annihilates special elements).

In dimension 1

$\xi_1([(u,v,w)])= [(0,v,w)|_1(u,0,0)]$

$\xi_2([(0,v,w)])= [(0,0,w)|_1(0,v,0)]$

and, using the expression $r_1([(u,v,w)])= [(u,0,0)]\otimes 1+1\otimes[(0,v,w)]$, we get $\hat{\xi}([(u,v,w)])= [(0,v,w)|_1(u,0,0)] + [(0,0,w)|_1(0,v,0)]$ in dimension 1. All of this implies that $(\hat{R},\hat{S},\hat{\Xi})$ is defined by:

3.9:

1. $\hat{R}$ *is given by 3.8 (except that the terms in the right-hand side are enclosed in brackets);*
2. $\hat{S}$ *is as given in 3.8;*
3. $\hat{\Xi}$ *in dimension 2 maps* $[(u,v,w)]$ *to* $[(0,v,w)|_1(u,0,0)]$ - $[(0,0,w)|_1(0,v,0)]$;
4. *In dimension 3,* $\hat{\Xi}$ *is special.* □

Now we are in a position to combine 3.9, 3.5, and 3.3 to get a formula for (a,b,G), where $a=\hat{p}\cdot\hat{f}\cdot\hat{R}$, $b=\hat{S}\cdot\hat{g}\cdot\hat{q}$, and $G=\hat{\Xi}+\hat{S}\cdot\hat{\psi}\cdot\hat{R}+\hat{S}\cdot\hat{g}\cdot\Theta''\cdot\hat{f}\cdot\hat{R}$:

3.10: 1. $a([(u,v,w)]) = u\bullet x + v\bullet y + w\bullet z \in P(x,y,z)$;

$a([(u_1,v_1,w_1)|_1(u_2,v_2,w_2)]) = 0$;

$a([(u_1,v_1,w_1)|_1(u_2,v_2,w_2)|_1(u_3,v_3,w_3)]) = 0$;

$a([(u_1,v_1,w_1)|_2(u_2,v_2,w_2)]) = u_1u_2\bullet\gamma_2(x) + v_1v_2\bullet\gamma_2(y) + w_1w_2\bullet\gamma_2(z) +$

$u_1v_2 \bullet xy + u_1w_2 \bullet xz + v_1w_2 \bullet yz$;

2. $b(u \bullet x + v \bullet y + w \bullet z) = u \bullet [(1,0,0)] + v \bullet [(0,1,0)] + w \bullet [(0,0,1)]$;
3. $G([(u,v,w)]) = -[(0,v,w)|_1(u,0,0)] - [(0,0,w)|_1(0,v,0)] + \Theta_1([(u,0,0)]) + \Theta_2([(0,v,0)]) + \Theta_3([(0,0,w)])$;
4. $G([(u_1,v_1,w_1)|_1(u_2,v_2,w_2)]) = [(0,v_2,0)|_2(u_1,0,0)] + [(0,0,w_2)|_2(u_1,0,0)] + [(0,0,w_2)|_2(0,v_1,0)]$ + *special terms*. □

Remarks: Note that a is $\mathbb{Z}\pi$-linear in dimension 2 so that the differential on the resolution of $M \otimes Z_*$ is *untwisted*-- see remark 4 folowing 2.6. Since a is *not* $\mathbb{Z}\pi$-linear *in dimension* 4, the obstruction isn't *trivially* 0.

We will conclude this section by performing a concrete calculation in the case where $\pi = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ using the presentation $\langle s, t; s^2, t^2, (ts)^2 \rangle$ given at the end of section 2 with s and t identified with:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \text{ respectively.}$$

The example at the end of section 2 implies that the first homological k-invariant of $A(\mathbb{Z}^3,3) \otimes Z_*$, where Z_* is the Gruenberg resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$ corresponding to the presentation given above, is a cocycle whose value on the preferred basis element R_1T (where $r_1 = s^2$) is $\mathfrak{C}_3(E;t,s,s) \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, \mathbb{Z}^3/2\mathbb{Z}^3)$, where $E = \bar{\mathcal{B}}(a,b,G): A(\mathbb{Z}^3,3) \to \bar{\mathcal{B}}(P(x,y,z))$. Since we will be in the stable range (i.e. the top dimension is 5, which is < 2×the bottom dimension of 3), we can assume $\bar{\mathcal{B}}(G) = -G$ and we get:

3.11: $c(R_1T) = p^\circ(G(G(b(x) \bullet s) \bullet s) \bullet t) \bullet t$. □

This lends itself to a straightforward computation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -\bullet s \rightarrow \quad \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \quad -G \rightarrow$$

$$\begin{bmatrix} \Theta_1([[(-1,0,0)]]) = [(1,0,0)|(-1,0,0)], \\ -[(0,0,-1)|(-1,0,0)] + \Theta_1([[(-1,0,0)]]) + \Theta_3([[(0,0,-1)]]), \\ -[(0,-1,0)|(1,0,0)] + \Theta_2([[(0,-1,0)]]) \end{bmatrix}$$

Remark: We have deleted all terms containing (0,0,0) and written the main terms in a form suggestive of matrix notation. The i^{th} row of the formula is derived from the i^{th} row of the identity matrix and will give rise to the i^{th} row of the result (i=1,2,3). Continuing, we get:

$$-s\rightarrow \begin{bmatrix} [(-1,0,0)(1,0,0)], \\ -[(-1,1,0)|(1,0,0)]+[(-1,0,0)|(1,0,0)]+[(1,-1,0)|(-1,1,0)], \\ -[(1,0,1)|(-1,0,0)]+[(-1,0,-1)|(1,0,1)] \end{bmatrix}$$

$$-G\rightarrow \begin{bmatrix} 0, & & 0, \\ [(0,1,0)|_2(1,0,0)] & -t\rightarrow & [(1,0,1)|_2(0,1,1)] \\ [(0,0,1)|_2(-1,0,0)] & & [(0,0,-1)|_2(0,-1,-1)] \end{bmatrix}$$

$$-a\rightarrow$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} -t\rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} -p\rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Remarks: 1. Note that the first application of G makes use of the formula in line 3 of 3.10 and the second makes use of the formula in line 4.

2. The map p is just reduction mod 2.

Our computations show that:

$$c(R_1T)= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in Hom_{\mathbf{Z}}(\mathbf{Z}^3, \mathbf{Z}^3/2\mathbf{Z}^3)$$

A straightforward calculation shows that this is *not* a coboundary, i.e.:

1. the boundary of R_1T is $R_1(t-1)$, so the value of any coboundary on R_1T is $(t-1)\bullet$some cochain on $R_1 \in Z_2$.

2. If a cochain takes the value

$$M=\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix}$$

on R_1 (here all the letters are 0 or 1 and we are working mod 2), then the coboundary is $t\bullet M\bullet t-M$ (recall that t acts upon the Hom-group by *conjugation* and $t^{-1}=t$). This is

$$\begin{vmatrix} A+E+H & A+B+D & A+B+D+E+F \\ B+D+H & A+E+G & A+B+C+G+H+J+F \\ G+H & G+H & G+H \end{vmatrix}$$

so that all entries on the third row must be the *same*.

References

1. H. Baues, *Obstruction Theory*, Springer-Verlag Lecture Notes in Mathematics 628 (1977).

2. G. Carlsson, "A counterexample to a conjecture of Steenrod," *Invent. Math.*, vol. 64 (1981), 171-174.

3. H. Cartan, "Algebras D'Eilenberg-MacLane et Homotopie," *Seminaire Henri Cartan 1954/55*, ENS, Paris.

4. S. Eilenberg and S. MacLane, "On the groups H(π,n). I," *Ann. of Math.*, vol.58 (1954), 55-106.

5. S. Eilenberg and S. MacLane, "On the groups H(π,n). II," *Ann. of Math.*, vol.60 (1954), 49-139.

6. R. Fox, "Free differential calculus," *Ann. of Math.*, vol.57 (1954), 547-560.

7. V.K.A.M. Gugenheim, "On the chain-complex of a fibration," *Illinois J. of Math.*, vol. 16 (1972), 398-414.

8. A. Heller, "Homological resolutions of complexes with operators," *Ann. of Math.*, vol.60 (1954), 283-303.

9. P. Hilton and S. Wylie, *Homology theory*, Cambridge University Press, 1965.

10. K. Igusa, "The generalized Grassman invariant $K_3(\mathbb{Z}[\pi]) \rightarrow H\bullet(\pi;\mathbb{Z}_2[\pi])$," to appear in *Springer-Verlag Lecture Notes in Mathematics.*

11. ______, "On the algebraic theory of A_∞-ring spaces," Springer-Verlag Lecture Notes in Mathematics, no. 967, 146-194.

12. P. Kahn, "Steenrod's problem and k-invariants of certain classifying spaces," Springer-Verlag Lecture Notes in Mathematics, no. 967, 195-214

13. J. Milgram, "The bar construction and abelian H-spaces," *Illinois J. of Math.*, vol. 11 (1967), 234-241.

14. F. Quinn, "Finite abelian group-actions on finite complexes," *Springer-Verlag Lecture Notes in Mathematics 658*.

15 J. Rotman, *An Introduction to Homological Algebra*, Academic Press, 1979.

16. Shih Weishu, "Homologie des Espaces Fibrès," *I. H. E. S Publ. Math.*, vol. 13 (1962), 93-176.

17. J. Smith, "Group cohomology and equivariant Moore spaces," *J. of Pure and Appl. Alg.*, vol. 24 (1982), 73-77.

18______, "Equivariant Moore spaces. II -- The low-dimensional case," *to appear in the J. of Pure and Appl. Alg.*

19. J. Arnold, "Homological algebra based upon permutation modules," *J. of Algebra*, vol. 70 (1981), 250-260.

Department of Mathematics and Computer Science
Drexel University
Philadelphia, PA 19104