A Nonconnective Delooping of Algebraic K-theory

by

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Abstract: Given a ring R, it is known that the topological space $BGl(R)^+$ is an infinite loop space. One way to construct an infinite loop structure is to consider the category $\underline{\underline{F}}$ of free R-modules, or rather its classifying space $B\underline{\underline{F}}$, as food for suitable infinite loop space machines. These machines produce connective spectra whose zeroth space is $(B\underline{\underline{F}})^+ = \mathbb{Z}\times BGl(R)^+$. In this paper we consider categories $\underline{\underline{C}}_o(\underline{\underline{F}}) = \underline{\underline{F}}$, $\underline{\underline{C}}_1(\underline{\underline{F}}),\ldots$ of parametrized free modules and bounded homomorphisms and show that the spaces $(B\underline{\underline{C}}_o)^+ = (B\underline{\underline{F}})^+$, $(B\underline{\underline{C}}_1)^+,\ldots$ are the connected components of a nonconnective Ω-spectrum $B\underline{\underline{C}}(F)$ with $\pi_i B\underline{\underline{C}}(F) = K_i(R)$ even for negative i.

0. Introduction.

Given a ring R, let $\underline{\underline{F}}$ be the category of finitely generated free R-modules and isomorphisms. Form the "group completion" category $\underline{\underline{F}}^{-1}\underline{\underline{F}}$ of $\underline{\underline{F}}$ (see [G]); it is known that its classifying space $B\underline{\underline{F}}^{-1}\underline{\underline{F}}$ is the algebraic K-theory space $BGl(R)^+\times\mathbb{Z}$. The purpose of this paper is to produce a nonconnective delooping of $BGl(R)^+ \times K_o(R)$ by using the parametrized versions $\underline{\underline{C}}_o(\underline{\underline{F}}) = \underline{\underline{F}}$, $\underline{\underline{C}}_1(\underline{\underline{F}}),\ldots$ of $\underline{\underline{F}}$ given in [P]. Our main result is this:

Theorem A. Write B_i for the classifying space of the category $\underline{\underline{C}}^{-1}\underline{\underline{C}}$, except that $B_o = BGl(R)^+$. Then the spaces B_i are connected, and for $i \geq 0$ we have

$$\Omega B_{i+1} = B_i \times K_{-i}(R).$$

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Thus the sequence of spaces $\hat{B}_i = B_i \times K_{-i}(R)$ forms a nonconnective Ω-spectrum $\hat{\underline{\underline{B}}}$ with homotopy groups

$$\pi_i(\hat{\underline{\underline{B}}}) = K_i(R) , \quad i \text{ any integer.}$$

In particular, the negative homotopy groups of $\hat{\underline{\underline{B}}}$ are the negative K-groups of Bass [B].

Actually, we work in the generality of a small additive category $\mathcal{A}$, rather than just with the additive category $\mathcal{F}$ of finitely generated free R-modules. For example, one could take $\mathcal{P}$, the category of finitely generated projective R-modules. The category $\mathcal{P}$ is the idempotent completion of $\mathcal{F}$, and we recover the same spectrum $\hat{\underline{\underline{B}}}$ if we replace $\mathcal{F}$ by $\mathcal{P}$. Note that $B\underline{\underline{P}}^{-1}\underline{\underline{P}}$ is $BGl(R)^+ \times K_o(R)$, where $\underline{\underline{P}}$ is the category of isomorphisms in $\mathcal{P}$.

Given $\mathcal{A}$, we consider the additive categories $\mathcal{C}_i(\mathcal{A})$ of $\mathbb{Z}^i$-graded objects and bounded homomorphisms (see section 1 for details). If $\mathcal{A} = \mathcal{F}$ this definition specializes to the groups $\mathcal{C}_i$ of [P]. Let $\hat{\mathcal{C}}_i$ be the idempotent completion of $\mathcal{C}_i(\mathcal{A})$, and let $\underline{\underline{A}}$, $\underline{\underline{C}}_i$, $\hat{\underline{\underline{C}}}$ be the subcategories of isomorphisms in $\mathcal{A}$, $\mathcal{C}_i$ and $\hat{\mathcal{C}}_i$, respectively. Our second result is this:

<u>Theorem</u> $\underline{B}$. Write $\hat{B}_i$ for the classifying space of the category $\hat{\underline{\underline{C}}}_i^{-1}\hat{\underline{\underline{C}}}_i$ and B_i for the classifying space of $\underline{\underline{C}}_i^{-1}\underline{\underline{C}}_i$. Then

$$\Omega\hat{B}_{i+1} = \hat{B}_i$$

$$\Omega^i \hat{B}_i = \hat{B}_o = \text{"group completion" } (B\underline{\underline{A}})^+ \text{ of } B\underline{\underline{A}} .$$

The connected component of $\hat{B}_i$ is B_i (except for $i=0$), and the sequence of spaces $\hat{B}_o, \hat{B}_1, \ldots$ is a nonconnective Ω-spectrum. In particular, $\hat{B}_i$ is an i-fold delooping of $(B\underline{\underline{A}})^+$.

The outline of this paper is as follows. In section 1 we give the definitions of the $\mathbb{Z}^i$-graded categori $\mathcal{C}_i(\mathcal{A})$. In section 2, we recall the passage from categories to spectra, and review the main points of Thomason's paper [T] that we need. In section 3, we prove Theorems A and B.

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1. The categories $\mathcal{C}_i$.

In this section we give the definition of the categories $\mathcal{C}_i(\mathcal{A})$ associated to a small additive category $\mathcal{A}$. We also review the notions of filtered additive categories and of the idempotent completion of $\mathcal{A}$ for the convenience of the reader.

Definition 1.1. An additive category $\mathcal{A}$ is said to be filtered if there is an increasing filtration

$$F_o(A,B) \subseteq F_1(A,B) \subseteq \ldots \subseteq F_n(A,B) \subseteq \ldots$$

on Hom(A,B) for every pair of objects A,B of $\mathcal{A}$. Each $F_n(A,B)$ is to be a subgroup of Hom(A,B) and we must have $\bigcup F_n(A,B) = \mathrm{Hom}(A,B)$. We require 0_A and 1_A to be in $F_o(A,B)$, and assume that the composition of morphisms in $F_m(A,B)$ and $F_n(B,C)$ belongs to $F_{m+n}(B,C)$. We also assume that the projections $A\oplus B \to A$, and inclusions $A \to A\oplus B$ and coherence isomorphisms all belongs to F_o. If ϕ is in $F_d(A,B)$ we say that ϕ has filtration degree d.

The reason for concerning ourselves with filtered categories is that the categories $\mathcal{C}_i$ come with a natural filtration. Of course every additive category has a trivial filtration, obtained by setting $F_o(A,B) = \mathrm{Hom}(A,B)$.

Example (1.1.1). Given a $\mathbb{Z}$-graded ring A such as $R[t,t^{-1}]$, let $\mathcal{A}$ be the category of graded A-modules. We can filter $\mathcal{A}$ by legislating that homogeneous maps of degree $\pm d$ have filtration degree d.

We now give our definition of the filtered category $\mathcal{C}_i$. Let the distance between points $J = (j_1,\dots,j_i)$ and $K = (k_1,\dots,k_i)$ in $\mathbb{Z}^i$ be given by

$$\|J-K\| = \max_s |j_s - k_s| \quad .$$

Definition (1.2). Let $\mathcal{A}$ be a (filtered) additive category. We define $\mathcal{C}_i(\mathcal{A})$ to be the category of $\mathbb{Z}^i$-graded objects and bounded homomorphisms. This means that an object A of $\mathcal{C}_i$ is a collection of objects A(J) in $\mathcal{A}$, one for each J in $\mathbb{Z}^i$. A morphism $\phi : A \to B$ in $\mathcal{C}_i$ of filtration degree d is a collection

$$\phi(J,K) : A(J) \to B(K)$$

of $\mathcal{A}$-morphisms, where we require $\phi(J,K) = 0$ unless $\|J-K\| \leqslant d$. If $\mathcal{A}$ is filtered, we also require each $\phi(J,K)$ to have filtration degree $\leqslant d$. Composition of $\phi = A \to B$ with $\psi : B \to C$ is defined by

$$(\psi\circ\phi)(J,L) = \sum_K \psi(K,L)\circ\phi(J,K) \quad .$$

Note that composition is well-defined because only finitely many elements in this sum are different from 0. It is easily seen that $\mathcal{C}_o(\mathcal{A}) = \mathcal{A}$.

Example (1.2.1). If $\mathcal{F}$ is the category of finitely generated free R-modules (with trivial filtration), the category $\mathcal{C}_i(\mathcal{F})$ is the same as the category $\mathcal{C}_i(R)$ constructed in [P]. In that paper it was proven that

$$K_1(\mathcal{C}_{i+1}(R)) = K_{-i}(R) \quad , \quad i \geqslant 0 \ .$$

This indicated that $\mathcal{C}_{i+1}$ might be a delooping of K-theory, and was the original motivation for this paper. That it cannot be exactly the case follows from (1.3.1) below.

Example (1.2.2). Since $\mathcal{C}_i(\mathcal{A})$ is filtered, we can iterate the construction. It is easy to see that

$$\mathcal{C}_i(\mathcal{C}_j(\mathcal{A})) = \mathcal{C}_{i+j}(\mathcal{A}) \ .$$

However, if we forget the filtration on $\mathcal{C}_j(\mathcal{A})$ this is no longer the case.

Remark (1.2.3). If V is any metric space, we can define a category $\mathcal{C}_V(\mathcal{A})$ in a way generalizing the case $V = \mathbb{Z}^i$. An object A of $\mathcal{C}_V$ is a collection of objects A(v), one for each v in V, subject to the following constraint: for every $d > 0$ and v, $A(w) \neq 0$ for only finitely many w of distance less than d from v. Morphisms are defined as for $\mathcal{C}_i$. It is easy to see that if $V = \mathbb{R}^i$ then $\mathcal{C}_V$ is naturally equivalent to its subcategory $\mathcal{C}_i$.This shows that the difference between $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ is the rate of growth in d of the number n(d,J) of points K within a distance of d from J.

Example (1.2.4). If we take $V = \{0,1,2,...\}$ then we will let $\mathcal{C}_+(\mathcal{A})$ denote $\mathcal{C}_V(\mathcal{A})$. This is the full subcategory of $\mathcal{C}_1(\mathcal{A})$ whose objects satisfy $A(j) = 0$ for $j < 0$. Similarly, if we take $V = \{0,-1,-2,...\}$, we will write $\mathcal{C}_-(\mathcal{A})$ for $\mathcal{C}_V(\mathcal{A})$. We can identify $\mathcal{C}_+(\mathcal{A}) \cap \mathcal{C}_-(\mathcal{A})$ with $\mathcal{A}$ in the obvious way.

There is a shift functor $T : \mathcal{C}_1(\mathcal{A}) \to \mathcal{C}_1(\mathcal{A})$ sending A to TA with $TA(j) = A(j-1)$, and T restricts to an endofunctor of $\mathcal{C}_+(\mathcal{A})$. There is an obvious natural isomorphism t from A to TA in both $\mathcal{C}_1$ and $\mathcal{C}_+$. We include the following result here for expositional purposes, and will generalize it in section 3 below.

Lemma (1.3). Every object of $\mathcal{C}_+(\mathcal{A})$ is stably isomorphic to 0. In particular, the Grothendieck group $K_o(\mathcal{C}_+)$ is zero.

Proof. Given A in $\mathcal{C}_+$, let $B = \Sigma T^n A$. That is, $B(j) = A(j)\oplus A(j-1)\oplus ...\oplus A(0)$. It is clear that $A\oplus TB = B$.The result follows from the observation that $t : B \cong TB$ is an isomorphism in $\mathcal{C}_+(\mathcal{A})$.

Corollary (1.3.1). If $i \neq 0$ then every object of $\mathcal{C}_i(\mathcal{A})$ is stably isomorphic to 0. In particular, $K_o(\mathcal{C}_i) = 0$.

Proof. By (1.2.2) we can assume that $i = 1$. But every object of

$\mathcal{C}_1$ can be written $A_+ \oplus A_-$ with A_+ in $\mathcal{C}_+$ and A_- in $\mathcal{C}_-$. Hence $K_o(\mathcal{C}_1)$ is a quotient of $K_o(\mathcal{C}_+) \oplus K_o(\mathcal{C}_-) = 0$.

Here is a quick discussion of idempotent completion, as applied to the $\mathcal{C}_i$ construction.

Definition (1.4) (see, e.g., [F, p.61]). Let $\mathcal{A}$ be an additive category. The idempotent completion $\hat{\mathcal{A}}$ of $\mathcal{A}$ has as objects all morphisms $p : A \to A$ of $\mathcal{A}$ satisfying $p^2 = p$. An $\hat{\mathcal{A}}$-morphism from p_1 to p_2 is an $\mathcal{A}$-morphism ϕ from the domain A_1 of p_1 to the domain A_2 of p_2 satisfying $\phi = p_2 \phi p_1$. It is easily seen that $\hat{\mathcal{A}}$ is an additive category and that $\mathrm{Hom}(p_1, p_2)$ is a subgroup of $\mathrm{Hom}(A_1, A_2)$. Hence $\hat{\mathcal{A}}$ inherits any filtered structure that $\mathcal{A}$ might have. There is a full embedding of $\mathcal{A}$ in $\hat{\mathcal{A}}$ sending A to 1_A; if this is an equivalence ogf categories, we say that $\mathcal{A}$ is idempotent complete.

Example (1.4.1). The idempotent completion of the category $\mathcal{F}$ of free R-modules is equivalent to the category $\mathcal{P}$ of projective R-modules.

Lemma (1.4.2). The categories $\mathcal{A}$ and $\mathcal{C}_i(\mathcal{A})$ are cofinal in their idempotent completions $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}_i(\mathcal{A})$. Moreover, $\mathcal{C}_i(\mathcal{A})$ is cofinal in $\mathcal{C}_i(\hat{\mathcal{A}})$.

Proof. This is an easy computation. For example, if p is an object of $\mathcal{C}_i(\hat{\mathcal{A}})$, define q by $q(J) = 1-p(J)$. Then $p \oplus q$ belongs to $\mathcal{C}_i(\mathcal{A})$.

To compute the K-theory of $\mathcal{A}$, we need to know which sequences are "exact": a different embedding of $\mathcal{A}$ in an ambient abelian category will result in a different family of short exact sequences (see [Q]). In particular, we cannot talk about $K_1\mathcal{C}_i(\mathcal{A})$ unless we know which sequences in $\mathcal{C}_i$ are "exact". It is not clear what the notion of "exact" should be, unless either (a) all exact sequences in $\mathcal{A}$ split (we insist the same is true of $\mathcal{C}_i$), or (b) $\mathcal{A}$ is embedded in an abelian category $\tilde{\mathcal{A}}$ closed under countably infinite direct sum (for then $\mathcal{C}_i$ is embeddable in $\tilde{\mathcal{A}}$). In either case, it follows from

(1.4.2) and Theorem 1.1 of [Gr] that

$$K_n\mathcal{C}_i(\mathcal{A}) = K_n\mathcal{C}_i(\hat{\mathcal{A}}) = K_n\hat{\mathcal{C}}_i(\mathcal{A}) \ , \quad n \geq 1 \ .$$

Note that our proofs of theorem A and B only apply to situation (a).

Example (1.5). Let p_- be the idempotent natural transformation in $\mathcal{C}_1(\mathcal{A})$ given by

$$(p_-)_A : A \to A \ , \quad p_-(j,k) = \begin{cases} 1 & \text{if } j = k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Given an object A of $\mathcal{A}$. let A_- denote the image of p_- on the constant object $A(j) = A$ of $\mathcal{C}_1(\mathcal{A})$. Thus $A_-(j) = 0$ if $j > 0$ and $A_-(j) = A$ if $j \leq 0$. The map t is an endomorphism of the constant object $A \cong TA$; write s for the restriction of p_-t to A_-. Then $1-s : A_- \to A_-$ is both a monomorphism and an epimorphism in $\mathcal{C}_1(\mathcal{A})$, but not an isomorphism. This is because its "inverse" Σs^n is not bounded. In particular, $\mathcal{C}_1(\mathcal{A})$ can never be an abelian category, even if $\mathcal{A}$ is.

We conclude this section with the following result, which provides motivation for our Theorem B. It is also a consequence of Theorem B. Since we will not use this result, we merely sketch the proof.

Proposition (1.6). If all short exact sequences in $\mathcal{A}$ split, then $K_1(\mathcal{C}_{i+1}(\mathcal{A})) = K_0(\hat{\mathcal{C}}_i(\mathcal{A}))$. In particular, $K_1\mathcal{C}_1(\mathcal{A}) = K_o(\hat{\mathcal{A}})$.

Sketch of proof. This is proven in section 1 of [P], modulo terminology.

First of all, we can assume that $\mathcal{A}$ is idempotent complete and that $i = 0$ by (1.4.2) and (1.2.2). The map from $K_o(\mathcal{A})$ to $K_1\mathcal{C}_1(\mathcal{A})$ sends the object A of $\mathcal{A}$ to the shift automorphismt of the constant object $A(j) = A$ of $\mathcal{C}_1(\mathcal{A})$. The map $\phi : K_1(\mathcal{C}_1) \to K_o(\mathcal{A})$ is defined by sending the class of $\alpha \in \mathrm{Aut}(A)$ to the difference (for $d \gg 0$) in $K_0(\mathcal{A})$:

$$\phi(\alpha) = [(\alpha p_-\alpha^{-1})(\bigoplus_{j=-2d}^{2d} A(j))] - [p_-(\bigoplus_{j=-2d}^{2d} A(j))] \ .$$

If α has filtration degree less than d, one shows as in [P,(1.11)] that this map ϕ is well-defined and independant of d. Clearly the composition is the identity on $K_o(\mathcal{A})$. The proof of [P, (1.20)] applies to show that ϕ is monic, which proves the proposition.

Example (1.6.1). Again, let $\mathcal{F}$ be the category of finitely generated free R-modules. Then for $i \geq 1$ we have
$K_o\mathcal{C}_i(R) = 0$ but $K_o\hat{\mathcal{C}}_i(R) = K_1\mathcal{C}_{i+1}(R) = K_{-i}(R)$.

Note: Example (1.6.1) follows from [P], not from (1.6).

2. The pasage to topology.

In this section we recall various results on the passage from the categories $\mathcal{A}$, $\mathcal{C}_i$ etc. to infinite loop spaces and spectra. We also recall Thomason's simplified double mapping cylinder from section 5 of [T]. We urge the reader to consult [T] for more details.

A symmetric monoidal category $\underline{\underline{S}}$ is a category together with a functor $\oplus : \underline{\underline{S}}\times\underline{\underline{S}} \to \underline{\underline{S}}$ and natural isomorphisms

$$\alpha : (A\oplus B)\oplus C \cong A\oplus(B\oplus C)$$

$$\gamma : A\oplus B \cong B\oplus A \ .$$

These natural isomorphisms are subject to coherence conditions that certain diagrams commute. We refer the reader to [Mac] for a more detailed definition, contending ourselves with:

Example (2.1). If $\mathcal{A}$ is an additive category then $\mathcal{A}$ is a symmetric monoidal category under $\oplus$ = direct sum. The subcategory $\underline{\underline{A}}$ of the isomorphisms in $\mathcal{A}$ is also symmetric monoidal under $\oplus$ = direct sum. It follows that $\mathcal{C}_i(\mathcal{A})$ and its category $\underline{\underline{C}}_i(\mathcal{A})$ of isomorphisms are also symmetric monoidal.

There is a functor Spt from the category of small symmetric monoidal categories to the category of connective Ω-spectra (i.e., sequences of spaces X_n with X_n being $(n-1)$-connected and with $X_n = \Omega X_{n+1}$). This functor satisfies

(a) A functor $\underline{\underline{A}} \to \underline{\underline{B}}$ preserving $\oplus$ up to coherent natural transformation , a "lax" functor, induces a map $\mathrm{Spt}(\underline{\underline{A}}) \longrightarrow \mathrm{Spt}(\underline{\underline{B}})$ of infinite loop spectra.

(b) The zeroth space $\mathrm{Spt}_o(\underline{\underline{A}})$ is the "group completion" of $B\underline{\underline{A}}$, the classifying space of the category $\underline{\underline{A}}$.

The construction of Spt is basically due to May and to Segal, and Spt is unique up to homotopy equivalence. See [A]. One description of Spt may be found in the Appendix of [T].

Lemma (2.2). Suppose that $\underline{\underline{A}} \to \underline{\underline{B}}$ is a lax functor of small symmetric monoidal categories, and that $B\underline{\underline{A}} \to B\underline{\underline{B}}$ is a homotopy equivalence of topological spaces. Then $\mathrm{Spt}_o(\underline{\underline{A}}) \to \mathrm{Spt}_o(\underline{\underline{B}})$ is a homotopy equivalence.

Proof. See (2.3) of [T].

Lemma (2.3). Suppose that $\underline{\underline{A}}$ is a full, cofinal subcategory of the small symmetric monoidal category $\underline{\underline{B}}$. Then the connected components of $\mathrm{Spt}_o(\underline{\underline{A}})$ and $\mathrm{Spt}(\underline{\underline{B}})$ are homotopy equivalent.

Proof. This is wellknown. The point is that

$$\begin{aligned} H_*[\mathrm{Spt}_o(\underline{\underline{A}})_o] &= \varinjlim_{A\in\underline{\underline{A}}} H_*\, B\, \mathrm{Aut}(A) \\ &= \varinjlim_{B\in\underline{\underline{B}}} H_*\, B\, \mathrm{Aut}(B) \\ &= H_*[\mathrm{Spt}_o(\underline{\underline{B}})_o] \;. \end{aligned}$$

Lemma (2.4) (Quillen). Let $\underline{\underline{S}}$ be a small symmetric monoidal category in which all morphisms are isomorphisms, and assume that all translations $S\oplus : \underline{\underline{S}} \to \underline{\underline{S}}$ are faithful. Then there is a category $\underline{\underline{S}}^{-1}\underline{\underline{S}}$

whose objects are pairs (S_1, S_2) of objects in $\underline{\underline{S}}$, such that $B\underline{\underline{S}}^{-1}\underline{\underline{S}}$ is homotopy equivalent to $Spt_o(\underline{\underline{S}})$.

Proof. See [G, p.221] or p. 1657 of [T].

Corollary (2.4.1). If $\mathcal{A}$ is a small additive category, let $\underline{\underline{A}}$ denote the category of isomorphisms in $\mathcal{A}$. Then $B\underline{\underline{A}}^{-1}\underline{\underline{A}}$ is homotopy equivalent to $Spt_o(\underline{\underline{A}})$.

Example (2.4.2). Let R be a ring for which $R^m \cong R^n$ implies that $m = n$, and let $\underline{\underline{F}}$ be the category of finitely generated free R-modules and isomorphisms. The basepoint component of $\underline{\underline{F}}^{-1}\underline{\underline{F}}$ has objects $R^m = (R^m, R^m)$ and

$$Hom(R^m, R^{m+n}) = Gl_{m+n}(R) \underset{Gl_n(R)}{\times} Gl_{m+n}(R).$$

In particular, $Hom(0, R^m)$ is $Gl_m(R)$. The family of the $Hom(0, R^m)$ gives a map from $BGl(R)$ to the basepoint component $BGl^+(R)$ of $B\underline{\underline{F}}^{-1}\underline{\underline{F}}$

The main ingredient in the proof of Theorem B is the simplified double mapping cylinder construction of R.W. Thomason, described in (5.1) of [T]. Let $\underline{\underline{A}}$ be a symmetric monoidal category with all morphisms isomorphisms and $u : \underline{\underline{A}} \to \underline{\underline{B}}$, $v : \underline{\underline{A}} \to \underline{\underline{C}}$ strong functors of symmetric monoidal categories (i.e. functors preserving direct sum up to natural isomorphism). Define $\underline{\underline{P}} = \underline{\underline{P}}(\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}, u, v)$ to be the category with objects triples (B,A,C) with A an object of $\underline{\underline{A}}$, B of $\underline{\underline{B}}$, and C of $\underline{\underline{C}}$. A morphism $(B,A,C) \to (B',A',C')$ is a 5-tuple $(\psi, \psi_1, \psi_2, U, V)$ where U, V are objects of $\underline{\underline{A}}$, $\psi : A \cong U \oplus A' \oplus V$, $\psi_1 : B \oplus uU \to B'$ and $\psi_2 : C \oplus vV \to C'$. U and V may be varied up to isomorphism. Composition of $(\psi, \psi_1, \psi_2, U, V) : (B,A,C) \to (B',A',C')$ with $(\bar\psi, \bar\psi_1, \bar\psi_2, \bar U, \bar V) : (B',A',C') \to (B'',A'',C'')$ is given by

$$A \cong U \oplus A' \oplus V \cong (U \oplus \bar U) \oplus A'' \oplus (\bar V \oplus V)$$

$$B \oplus u(U \oplus \bar U) \cong (B \oplus uU) \oplus u\bar U) \to B' \oplus u\bar U \to B''$$

$$v(\bar V \oplus V) \oplus C \cong v\bar V \oplus vV \oplus C \to v\bar V \oplus C' \to C''$$

and direct sum in $\underline{\underline{P}}$ is induced by direct sum in $\underline{\underline{A}}$, $\underline{\underline{B}}$ and $\underline{\underline{C}}$. We then have

Theorem 2.5 (R.W. Thomason [T,(5.2)] . Up to homotopy the diagram

$$\begin{array}{ccc} \mathrm{Spt}_0\underline{\underline{A}} & \longrightarrow & \mathrm{Spt}_0\underline{\underline{B}} \\ \downarrow & & \downarrow \\ \mathrm{Spt}_0\underline{\underline{C}} & \longrightarrow & \mathrm{Spt}_0\underline{\underline{P}} \end{array}$$

is a pullback diagram.

3. The proof of Theorem A and B.

In this section we prove Theorems A and B. We make the standing assumption that $\mathcal{A}$ is a small filtered additive category and that $\underline{\underline{A}}$ is the (symmetric monoidal) category of isomorphisms of $\mathcal{A}$. Similarly we write $\underline{\underline{C}}_i$, $\underline{\underline{C}}_+$ and $\underline{\underline{C}}_-$ for the categories of isomorphisms of $\mathcal{C}_i(\mathcal{A})$, $\mathcal{C}_+(\mathcal{A})$ and $\mathcal{C}_-(\mathcal{A})$. The idea is to show that the diagram

$$\begin{array}{ccc} \underline{\underline{A}} & \longrightarrow & \underline{\underline{C}}_+ \\ \downarrow & & \downarrow \\ \underline{\underline{C}}_- & \longrightarrow & \underline{\underline{C}}_1 \end{array}$$

induces a pullback diagram of spectra, and to use the following result:

Proposition (3.1). $\mathrm{Spt}_o(\underline{\underline{C}}_+)$ and $\mathrm{Spt}_o(\underline{\underline{C}}_-)$ are contractible.

Proof. By symmetry it is enough to consider $\underline{\underline{C}}_+$. Recall from the discussion before (1.3) that there is a shift functor $T : \underline{\underline{C}}_+ \to \underline{\underline{C}}_+$ and a natural transformation t from A to TA. The category $\underline{\underline{C}}_+$ has an endofunctor $\sum_{n=0}^{\infty} T^n$ with

$$\left(\sum_{n=0}^{\infty} T^n\right)A(j) = \bigoplus_{n=0}^{j} A(j-n) .$$

(Recall that $A(j) = 0$ for $j < 0$.)We can define $\sum_{n=1}^{\infty} T^n$ similarly. The natural isomorphism t induces a natural isomorphism t from $\sum_{n=0}^{\infty} T^n A$ to $\sum_{n=1}^{\infty} T^n A$. But as endofunctors of $\underline{\underline{C}}_+$ we have $1 \oplus \sum_{n=1}^{\infty} T^n \cong \sum_{n=0}^{\infty} T^n$. Hence as self-maps of the H-space $B\underline{\underline{C}}_+$ we have

$$1 \sim (\sum_{n=0}^{\infty} T^n) - (\sum_{n=1}^{\infty} T^n) \overset{t}{\simeq} 0 .$$

This shows that B is contractible. But then $Spt_o(\underline{\underline{C}}_+)$ is contractible by Lemma (2.2).

<u>Proof</u> <u>that</u> <u>Theorem</u> <u>B</u> <u>implies</u> <u>Theorem</u> <u>A</u>. Write $\hat{B}_i$ for $Spt_o(\hat{\underline{\underline{C}}}_i)$. Since we have $\pi_o(\hat{B}_i) = K_{-i}(R)$ by (1.6.1) and since translations are faithful in $\hat{\underline{\underline{C}}}_i$, it follows that $\hat{B}_i$ is homotopy equivalent to $B_i \times K_{-i}(R)$. Since $\Omega B_i = \Omega \hat{B}_i$, the result is now immediate.

We now begin the proof of theorem B by making a series of reductions. Since

$$\pi_o(B_i) = \pi_o Spt_o(\underline{\underline{A}}_i) = K_o(\underline{\underline{A}}_i) ,$$

connectedness of the B_i for $i \neq 0$ follows from (1.3.1). Now $\underline{\underline{C}}_i$ is full and cofinal in $\hat{\underline{\underline{C}}}_i$ by (1.4.2), so by (2.3) the connected space $B_i = Spt_o(\hat{\underline{\underline{C}}}_i)$. By construction (or by (2.4.1)), $\hat{B}_o = Spt_o(\hat{\underline{\underline{A}}})$ is the group completion of $B\hat{\underline{\underline{A}}}$. Thus the proof of Theorem B is reduced to showing that $\Omega\hat{B}_{i+1} = \hat{B}_i$ for $i \geq 0$.

Next, observe that $\hat{\mathcal{C}}_{i+1}(\mathcal{A}) = \hat{\mathcal{C}}_1\hat{\mathcal{C}}_i(\mathcal{A})$, so that $\hat{B}_{i+1} = Spt_o(\hat{\underline{\underline{C}}}_1(\hat{\mathcal{C}}_i(\mathcal{A}))$ and $\hat{B}_i = Spt_o(\hat{\mathcal{C}}_i(\mathcal{A}))$.

Since we can replace $\mathcal{A}$ by $\hat{\mathcal{C}}_i(\mathcal{A})$, it is enough to prove that $\Omega \hat{B}_1 = \hat{B}_o = Spt_o(\hat{\underline{\underline{A}}})$.There is also no loss in generality in assuming that $\mathcal{A}$ is idempotent complete, since

$$\Omega \hat{B}_1 = \Omega Spt_o(\hat{\underline{\underline{C}}}_1(\mathcal{A})) = \Omega Spt_o(\hat{\underline{\underline{C}}}_1(\hat{\mathcal{A}}))$$

by (2.3). In fact, by (2.3) we also have

$$\Omega Spt_o(\hat{\underline{\underline{C}}}_1) = \Omega Spt_o(\underline{\underline{C}}_1) .$$

Therefore, Theorem B will follow from:

Theorem (3.2). Let $\mathcal{A}$ be a small, filtered additive category which is idempotent complete. Then $\Omega\ \mathrm{Spt}_o(\underline{\underline{C}}_1)$ is homotopy equivalent to $\mathrm{Spt}_o(\underline{\underline{A}})$.

Lemma (3.3). Let $\mathcal{A}$ be a small filtered additive category. Recall that $\underline{\underline{C}}_+$ and $\underline{\underline{C}}_-$ are subcategories of $\underline{\underline{C}}_1$ whose intersection is $\underline{\underline{A}}$. Let $\underline{\underline{P}}$ be the simplified double mapping cylinder construction applied to $\underline{\underline{A}} \to \underline{\underline{C}}_-$ and $\underline{\underline{A}} \to \underline{\underline{C}}_+$. Then $\Omega\ \mathrm{Spt}_o(\underline{\underline{P}})$ is homotopy equivalent to $\mathrm{Spt}_o(\underline{\underline{A}})$.

Proof. This is immediate from Thomason's Theorem (2.5), since by (3.1) the spaces $\mathrm{Spt}_o(\underline{\underline{C}}_+)$ and $\mathrm{Spt}_o(\underline{\underline{C}}_-)$ are contractible.

By the universal mapping property of $\underline{\underline{P}}$ (see p. 1648 of [T]), there is a strong symmetric monoidal functor $\Sigma : \underline{\underline{P}} \to \underline{\underline{C}}_1$. This functor is defined on objects by

$$\Sigma(A^-, A, A^+) = A_- \oplus A \oplus A_+$$

where A^-, A, A^+ are objects of $\underline{\underline{C}}_+$, $\underline{\underline{A}}$ and $\underline{\underline{C}}_-$, respectively. A morphism $(\psi^-, \psi, \psi^+, U^-, U^+)$ in $\underline{\underline{P}}$ from (A^-, A, A^+) to (B^-, B, B^+) is sent by Σ to the composite

$$A^- \oplus A \oplus A^+ \xrightarrow{1\oplus\psi\oplus 1} A^- \oplus U^- \oplus A \oplus U^+ \oplus A^+ \xrightarrow{\psi^- \oplus 1 \oplus \psi^+} B^- \oplus B \oplus B^+ .$$

Theorem (3.4). Let $\mathcal{A}$ be idempotent complete, and let $\underline{\underline{P}}$ be the double mapping cylinder of Lemma (3.3). Then the functor $\Sigma : \underline{\underline{P}} \to \underline{\underline{C}}_1$ induces a homotopy equivalence between the classifying spaces $B\underline{\underline{P}}$ and $B\underline{\underline{C}}_1$.

Note that Theorem (3.4) immediately implies Theorem (3.2) by (3.3) and (2.2). Thus we have reduced the proof of Theorem B to the proof of Theorem (3.4).

Proof. We will show that this functor satisfies the conditions of Quillen's Theorem A from [Q]. Fix an object Y of $\underline{\underline{C}}_1$; we need to show that $Y{\downarrow}\Sigma$ is a contractible category. To do this, we use the bound d for $\mathcal{C}_1(\mathcal{A})$ to filter $Y\downarrow\Sigma$ as the increasing union of subcategories Fil_d, and show that each Fil_d has an initial object $*_d$. Therefore Fil_d is contractible; their union $Y\downarrow\Sigma$ must also be contractible by standard topology.

The category Fil_d is the full subcategory of all $\alpha : Y \to \Sigma(A^-,A,A^+)$ where both α and α^{-1} are bounded by d. Define Y_d, Y_d^- and Y_d^+ in $\underline{\underline{A}}$, $\underline{\underline{C}}_-$ and $\underline{\underline{C}}_+$ respectively by setting

$$Y_d \quad = Y(-d)\oplus\ldots\oplus Y(d) \text{ in } \underline{\underline{A}}$$

$$Y_d^- = Y(j) \quad \text{if } j < -d \text{ , and } = 0 \text{ otherwise}$$

$$Y_d^+ = Y(j) \quad \text{if } j > -d \text{ , and } = 0 \text{ otherwise.}$$

The obvious isomorphism $\sigma : Y \cong Y_d^-\oplus Y_d\oplus Y_d^+$ in $\underline{\underline{C}}_1$ is bounded by d, and forms the object $*_d : Y \to \Sigma(Y_d^-,Y_d,Y_d^+)$ of Fil_d. We will show that $*_d$ is an initial object of Fil_d.

Given the object $\alpha : Y \to \Sigma(A^-,A,A^+)$, we have to show that there is a unique morphism

$$\eta = (\psi,\psi^-,\psi^+,e_-(Y_d),e_+((Y_d)) : (Y_d^-,Y_d,Y_d^+) \to (A^-,A,A^+)$$

in $\underline{\underline{P}}$ so that $\Sigma(\eta) = \alpha\sigma^{-1}$ in $\underline{\underline{C}}_1$. Let pr_-, pr, pr_+ be the projections of $\Sigma(A^-,A,A^+)$ onto A^-, A and A^+, respectively. Since α^{-1} is bounded by d, $\alpha^{-1}(A)$ is contained in Y_d, or rather in the image $\sigma^{-1}(Y_d)$ of Y_d in Y. Hence it makes sense to let e be $\sigma\alpha^{-1}(pr)\alpha\sigma^{-1}$ restricted to Y_d, and it is clear that e is an idempotent of Y_d. Similarly, $\sigma\alpha^{-1}(A^-)$ is contained in $Y_d^-\oplus Y_d$, and $\alpha^{-1}(A^+)$ is contained in $Y_d\oplus Y_d^+$. Let e_- and e_+ be $\sigma\alpha^{-1}(pr_-)\alpha\sigma^{-1}$ and $\sigma\alpha^{-1}(pr_+)\alpha\sigma^{-1}$ restricted to Y_d. These maps are also idempotents of Y_d, and it is easy to see that $e_- + e + e_+ = 1$. Since $\mathcal{A}$ is idempotent complete, the composition

$$Y_d \cong e_-(Y_d) \oplus e(Y_d) \oplus e_+(Y_d)$$

makes sense in $\mathcal{A}$. Define ψ to be the composite

$$Y_d \cong e_-(Y_d) \oplus e(Y_d) \oplus e_+(Y_d) \xrightarrow{1\oplus\alpha\oplus 1} e_-(Y_d)\oplus A\oplus e_+(Y_d)$$

Similarly, define maps

$$\psi^- : Y_d^- \oplus e_-(Y_d) \xrightarrow{\alpha\sigma^{-1}} A^- \text{ in } \underline{\underline{C}}_-$$

$$\psi^+ : e_+(Y_d) \oplus Y_d^+ \xrightarrow{\alpha\sigma^{-1}} A^+ \text{ in } \underline{\underline{C}}_+.$$

This completes the definition of the map $\eta : (Y_d^-, Y_d, Y_d^+) \to (A^-, A, A^+)$ in $\underline{\underline{P}}$. By definition of Σ we have $\Sigma(\eta) = \alpha\sigma^{-1}$. Because all maps in $\underline{\underline{A}}$, $\underline{\underline{C}}_-$ and $\underline{\underline{C}}_+$ are isomorphisms, it is an easy task to verify that η is the unique map with $\Sigma(\eta) = \alpha\sigma^{-1}$. It follows that $*_d$ is an initial object of Fil_d. Q.E.D.

4. An overview.

To place our construction in perspective, it is appropriate to review a little history. The definition of the functors $K_{-i}(R)$ was given by Bass [B] in 1966 during an attempt to formalize his decomposition of $K_1(R[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}])$. In 1967, Karoubi [K-1] gave another definition of $K_{-i}(R)$ by defining $K_{-i}(\mathcal{A})$ for any abelian category. A third and fourth definition of $K_{-i}(R)$ were given independantly by Karoubi Villamayor [K-V] using the ring $S(R)$ and by Wagoner [W-1] using the subring $\mu(R)$ of $S(R)$. Happily all these definitions were shown to agree by Karoubi's axiomatic treatment in [K-136].

In 1971, Gersten [Ger] constructed a nonconnective delooping of $K_o(R) \times BGl^+(R)$ using the fact that $\Omega BGl^+(S(R)) = K_o(R) \times BGl^+(R)$. Wagoner [W-2] then constructed the Ω-spectrum $K_o(\mu^i(R)) \times BGl^+(\mu^i(R))$ and showed that the inclusions $\mu(R) \to S(R)$ induced an equivalence of spectra. To our knowledge, nonconnective deloopings of the K-theory of other additive categories besides $\mathcal{F}$ has not been studied until now.

The construction in [P] is very much in the spirit of the early definitions of the $K_{-i}(R)$, but works for any additive category. Needless to say, an open questiom in our work is whether or not the $\Omega BQ\mathcal{C}_n(\mathcal{A})\hat{}$ yield a nonconnective delooping of any (idempotent complete) additive category with exact sequences. A major difference between the categories $\mathcal{C}_i(\mathcal{A})$ and Karoubi's categories $S^i\mathcal{A}$ is that $S\mathcal{A}$ is defined as a quotient of the flasque category $C\mathcal{A}$ (see [K-136]) while $\mathcal{C}_1(\mathcal{A})$ may be viewed as an enlargement of the flasque category

$\mathcal{C}_+(\mathcal{A})$. It would be interesting to see if the natural inclusion of $C\mathcal{A}$ in $\mathcal{C}_+(\mathcal{A})$ could be made to induce an isomorphism between K-groups.

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