EVALUATING THE SWAN FINITENESS OBSTRUCTION FOR PERIODIC GROUPS

by

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In [17] R. Swan introduced the finiteness obstruction $\sigma_n(G)$ for free actions of a periodic group G on finite complexes having the homotopy type of the sphere S^{n-1}. It takes its value in a certain quotient of $\tilde{K}_0(Z(G))$, $\tilde{K}_0(Z(G))/T$, and was one of the main motivations in the development of algebraic K-theory.

More recently Ib Madsen, C. Thomas, and C.T.C. Wall [22], [20], [10] proved a sharpened version of one of Swan's theorems, roughly that if G has period n then G acts freely on a homotopy S^{n-1} or S^{2n-1}, and no examples were known for which 2n-1 was actually necessary. Indeed it was somewhat hesitantly suggested that n-1 is always correct.

In [4], [5], [6], [8], [14], [15], a subgroup $D(G) \subset \tilde{K}_0(Z(G))$ was studied and shown to be computable in terms of determinants or reduced norms and the structure of units in certain algebraic number fields. D(G) contains T and we prove

<u>Theorem</u> 2.B.1: $\sigma_n(G) \in D(G)/T$.

In particular, we relate $\sigma_n(G)$ to the behaviour of these groups $\mathrm{Tor}^i_{Z(G)}(M,Z)$ where M is a maximal order containing Z(G) in $\mathbb{Q}(G)$. In §3 we calculate these Tor groups for hyperelementary groups (this section may have independent interest), and in §4 we study D(G) for a class of periodic groups of period 4, Q(4p,q,1).

The Swan obstructions for these groups are written down in Theorem 4.B.6, and we have

<u>Theorem A</u> (4.C.2, 4.C.5, and 4.C.8): <u>The Swan obstruction</u> $\sigma_4(G) \neq 0$ <u>for</u> G = Q(12,5,1), Q(12,7,1), <u>or</u> Q(12,11,1). (The groups Q(a,b,c) are defined in §3.B, but the notation is standard.)

In fact we have

<u>Theorem B</u>: <u>Among periodic groups of period n and order</u> < 280 <u>only one group of each order</u> 120, 168, 240, <u>and</u> 264 <u>fails to act freely on a</u>

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finite complex having the homotopy type of S^{n-1}.

However, for the next group in the series of theorem A we have

Proposition 4.C.1: $\sigma_3(Q(12,13,1)) = 0$.
Hence there is a 3-complex homotopic to S^3 on which Q(12,13,1) acts freely.

The number theory involved in these questions is subtle since Theorem A shows that Q(12,5,1), Q(12,7,1) and Q(12,11,1) can't even act freely on a homology 3-sphere.

Remark D: The proof that $\sigma_n(G) \in D(G)/T$ came out of several conversations with R. Oliver, and I also profited from a conversation with H. Bass. The initial question which led to this work came up in discussions with I. Hambleton.

This paper was originally written in 1978. For various reasons it has not been previously published but it has been circulated privately. It initiated a vigorous attack on the space form problem for the last and demonstrably most interesting class of groups, the Q(8a,b,c) with a,b,c odd coprime integers.

The initial theorem A and B above were quickly extended in [25] by constructing large numbers of odd index subgroups of the units in the cyclotomic fields $\mathbb{Q}(\lambda_p)$, $\mathbb{Q}(\lambda_{4p})$, $\mathbb{Q}(\lambda_p,\lambda_q)$ studied here. Indeed the proofs of the result in [25] are direct extensions of the proofs here. The only thing preventing their being given in this original paper was the fact that the unit theorems had not yet been proved.

The main result of [25] as slightly extended in [26] is

Theorem C: Let K_n be the maximal 2-abelian extension of $\mathbb{Q}$ contained in the cyclotomic field $Q(\zeta_n)$. If K is K_p, K_{pq} with the quadratic symbol $\left(\frac{p}{q}\right) = -1$, or the maximal 2 extension in $\mathbb{Q}(\zeta_{2^n}+\zeta_{2^n}^{-1},\zeta_p+\zeta_p^{-1})$ $p \not\equiv 1(8)$, then the cyclotomic units have odd index in the units of K.

Next, by the techniques developed here and in [25] we obtain

Theorem D: Let p be a prime and

a) suppose $p \equiv 3(4)$ then $\sigma_4(q(8p,q,1)) = 0$
for q prime if and only if
(i) $q \equiv 1(8)$ or

(ii) $q \equiv 5(8)$ <u>but</u> $p^{\nu} \equiv \pm 1(q)$ <u>for some odd</u> ν.

b) $\sigma_4(Q(8p,q,1)) = 0$ <u>if</u> $p \equiv q \equiv 1(4)$

<u>but</u> $\left[\frac{p}{q}\right] = -1$.

Using this as a starting place a spirited attack by the author and Ib Madsen on the actual surgery problems took place. (Preliminary results had already been indicated in [25]).

Again the results depended on finding sufficient units, hence had to be restricted to the cases covered by theorem C. The results [26] are

<u>Theorem</u> E: <u>Let</u> p,q <u>be distinct odd primes and suppose</u> $p \equiv -1 \pmod 8$; <u>then</u>

a) $Q(8p,q,1)$ <u>acts freely on</u> $\mathbb{R}^{8k+4}-(pt)$ $(k \geq 1)$ <u>if</u> $q \equiv 1 \pmod 4$ <u>and</u> p <u>has odd order</u> (mod q).

b) $Q(8p,q,1)$ <u>acts freely on</u> $\mathbb{R}^{8k+4}-(pt)$ <u>but not on</u> S^{8k+3} $(k \geq 1)$ <u>if</u> $q \equiv 5(8)$ <u>and</u> p <u>has</u> <u>odd order</u> (mod q).

c) $Q(8p,q,1)$ <u>acts freely on</u> S^{8k+3} $(k \geq 1)$ <u>if</u> $q \equiv 1(8)$ <u>and</u> p <u>has</u> odd order (mod q).

For special classes of numbers there are further results [27]. But all these results rest on the idea of identifying the Swan obstruction studied here as the image under ∂ of an element Θ where

$$\Theta \in \coprod_{\nu=2,p,q} \operatorname{im} \left(K_1(\hat{Q}_{\nu}(G)) \xrightarrow{\partial} \tilde{K}_0(Z(G))\right)$$

which is developed here. This class Θ rather than its image is shown to correspond to the surgery obstruction for certain surgery problems over the Poincaré complex constructed here, and using recent results on the calculation of these groups, if (as usual) one has sufficient information about units, Theorem E follows.

I would like to thank A. Ranicki and the other editors of this Proceedings for giving me the opportuniry to finally publish this work.

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§1. $K_0(Z(G))$ and $D(G)$ for G a finite group.

A. Preliminary remarks on $K_0(Z(G))$

Let $M_n(D)$ be a matrix ring over a division algebra D whose center F is a finite extension of the rationals $\mathbb{Q}$ or the complete local field $\hat{\mathbb{Q}}_p$. The reduced norm homomorphism

$$\tilde{N}: GL_n(D) \to F$$

is given by linear embedding $M_n(D) \subset M_n(K)$ for some extension K of F and taking the determinants of the images. Then Wang's theorem [21] identifies

1.A.1: $$K_1(M_n(D)) = K_1(D) \underset{\tilde{N}}{\hookrightarrow} F$$

where im $(K(D))$ is all those elements of F which are positive at all ∞ places at which D is a quaternion algebra.

$$0 \to D(G) \to \tilde{K}_0(Z(G)) \to \tilde{K}_0(M) \to 0.$$

A second group E(G) can be introduced which depends only on reduced norms. Let M_p be a maximal order in $\hat{\mathbb{Q}}_p(G)$ containing $\hat{Z}_p(G)$, and define

1.B.3: $$\mathcal{U}_p \subset K_1(\hat{\mathbb{Q}}_p(G))$$

as the image of $K_1(M_p)$. Writing

$$\hat{\mathbb{Q}}_p(G) = \coprod M_{n_i}(D_i)$$

we have $$M_p = \coprod_i M_{n_i}(N_i)$$

and $K_1(\hat{\mathbb{Q}}_p(G)) = \coprod_i (\dot{F}_i)$ where F_i is the center of D_i. Thus (one could use for example Quillen's localization sequence [12] to show this)

1.B.4: $$\mathcal{U}_p = \coprod_i \mathcal{U}(F_i)$$

the product of the units of the F_i's.

The composite maps

1.B.5:
$$\alpha_p: K_1(\hat{Z}_p(G)) \to K_1(M_p) \to \mathcal{U}_p$$
$$\beta_p: K_1(M) \to K_1(M_p) \to \mathcal{U}_p$$

then give

<u>Definition</u> 1.B.6: <u>The local defect</u> $E_p(G)$ <u>at</u> p <u>is</u> $\mathcal{U}_p/\mathrm{im}(\alpha_p)$.

<u>Remark</u> 1.B.7: If $p \nmid |G|$ then $M_p = \hat{Z}_p(G)$ so $E_p(G) = 1$. In any case $E_p(G)$ is a partial measure of the deviation of $\hat{Z}_p(G)$ from M_p and is finite for every finite group G.

$E_p(G)$ consists of a p primary part which is difficult to analyze and a somewhat easier part of order prime to p. To obtain this second part we can use the map

1.B.8: $$K_1(\hat{Z}_p(G)/J) \to \coprod_i \mathcal{U}(F_i)/(1+\mathfrak{m}_i)$$

where J is the Jacobson radical in $\hat{Z}_p(G)$ and $\mathfrak{m}_i$ is the maximal ideal in $\mathcal{O}(F_i)$.

Returning to 1.B.5 the map,

1.B.9: $$\beta = \coprod_{p \mid |G|} \beta_p : K_1(\mathcal{M}) \to \coprod_{p \mid |G|} E_p(G)$$

defines the quotient

1.B.10: $$E(G) = \coprod_{p \mid |G|} E_p(G)/\mathrm{im}(\beta).$$

Thus E(G) is the product of the local defects factored out by the image of global units coming from $K_1(\mathcal{M})$. Its calculation is difficult but in specific cases presents no insuperable obstacles.

<u>Theorem</u> 1.B.11: E(G) = D(G)
(See for example [5].)

<u>Remark</u> 1.B.12: When the author initiated this work he was unaware of 1.B.11 and so derived his own proof, which we present in outline from here, as it may be easier for some people to reconstruct. Note, to begin, that there is a k so $|G|^k . \mathcal{M} \subset Z(G)$. So set $L_s(G) = \mathrm{im}(Z(G) \subset \mathcal{M}/|G|^{ks}\mathcal{M}$, and we have the pullback diagram

1.B.13: $$\begin{array}{ccc} Z(G) & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \pi_s \\ L_s(G) & \longrightarrow & \mathcal{M}/|G|^{ks}\mathcal{M} \end{array} .$$

Moreover, π_s is onto so Milnor's Mayer-Vietoris sequence can be applied obtaining

1.B.14: $$K_1(Z(G)) \to K_1(L_s(G)) \oplus K_1(\mathcal{M}) \to K_1(\mathcal{M}/|G|^{ks}\mathcal{M}) \xrightarrow{\partial_s} \tilde{K}_0(Z(G)) \to \tilde{K}_0(\mathcal{M}) \to 0$$

so im ∂_s = D(G). Now let s become large and pass to limits. In the limit we need some information about $K_1(\mathcal{M})$, $K_1(\mathcal{M}_p)$. This may be supplied using Quillen's exact sequence of a localization to show that the kernels of the <u>local</u> reduced norm maps are in the image of the elements in $K_1(\mathcal{M})$.

§2. <u>Swan's finiteness obstruction</u>.

A. <u>The definition and basic properties</u>

Let $T(G) \subset D(G) \subset \tilde{K}_0(Z(G))$ be defined as the image of $\coprod_{p \mid |G|} U(\hat{Z}_p^+)$

where $\hat{Z}_p^+ \subset M_p$ corresponds to the trivial representation. Now suppose G is periodic of period n. That is, there is an exact sequence

2.A.1: $$0 \to Z \to C_{n-2} \to \dots \to C_1 \to Z(G) \to Z \overset{\varepsilon}{\to} 0$$

where the C_j are projective Z(G) modules. In [17] Swan defined an invariant $\sigma_n(G) \in \tilde{K}_0(Z(G))/T$ which is zero if and only if G acts freely on an n-1 dimensional complex having the homology type of S^{n-1}. In fact, $\sigma_n(G)$ is the Euler class in $\tilde{K}_0(Z(G))/T$ of 2.A.1.

2.A.2: $$\sigma_n(G) = [C_0] - [C_1] + [C_2] \dots \pm [C_{n-1}].$$

If n is even then sequences 2.A.1 may be spliced either together to give $\sigma_{sn}(G) = s\sigma_N(G)$. Also if $H \subset G$ is a subgroup then by restriction Z(G) projectives become Z(H) projective so $r_H:\tilde{K}_0(Z(G)) \to \tilde{K}_0(Z(H))$ induces $\bar{r}_H:\tilde{K}_0(Z(G))/T \to \tilde{K}_0(Z(H))/T$ and

2.A.3: $$\bar{r}_H(\sigma_n(G)) = \sigma_n(H)$$

Proof: It suffices to consider hyperelementary subgroups since $\coprod r_H|$ in 2.B.2 is injective.

2.B.2:

$$\begin{array}{ccc} T(G) & \longrightarrow & \coprod_H T(H) \\ \downarrow & & \downarrow \\ D(G) & \longrightarrow & \coprod_H D(H) \\ \downarrow & & \downarrow \\ \tilde{K}_o(Z(G)) & \xrightarrow{\coprod r_H} & \coprod_H \tilde{K}_o(Z(H)) \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle \coprod p_H} \\ \tilde{K}_o(\mathcal{M}) & \xrightarrow{\coprod r_H|} & \coprod_H \tilde{K}_o(\mathcal{M}_H) \end{array}$$

Let N be a finitely generated torsion $\mathcal{M}$ module. By [13] N has projective length 1 so there is a short exact sequence

$$0 \to P_1 \to P_0 \to N \to 0$$

with P_o, P_1 finitely generated projective, and

$$\chi(N) = [P_0] - [P_1] \in \tilde{K}_0(\mathcal{M})$$

is well defined. We now have (and this is the heart of the matter)

<u>Lemma</u> 2.B.3: $p_G(\sigma_n(G)) = \sum_{i=0}^{n-1} (-1)^i \chi(\mathrm{Tor}^i_{Z(G)}(\mathcal{M}(G), Z))$.

<u>Proof</u>: Tensor 2.A.1 over $Z(G)$ with $\mathcal{M}$ obtaining

2.B.4: $$0 \to C_{n-1} \otimes \mathcal{M} \to \ldots \to \mathcal{M} \xrightarrow{\varepsilon} Z \to 0 .$$

The homology of 2.B.4 is $\sum_0^{n-1} \widetilde{\mathrm{Tor}}^i_{Z(G)}(\mathcal{M}, Z)$ which differs from the usual Tor group only in the part of involving Z^+ which is, in any case, a P.I.D. Clearly, $P_G(\sigma_n(G)) = \Sigma\ (-1)^j (C_j \otimes M)$, but this is given by the Tor formula in 2.B.3.

By results in [22], $\sigma_n(H)$ is zero except for H a 2-hyperelementary subgroup with 2-Sylow subgroup a generalized quaternion group.

For this group write $M = \coprod_i M_i$ where the M_i correspond to the irreducible representations of M_i so $\mathrm{Tor}^j_{Z(G)}(M,Z) = \coprod_i \mathrm{Tor}^j_{Z(G)}(M_i,Z)$.

In $3.6 we prove

<u>Lemma</u> 2.B.5 a. $\mathrm{Tor}^j(M_i,Z)$ <u>is a</u> q-<u>torsion abelian group with</u> $q \mid |G|$.

b. <u>If</u> q <u>is not prime then</u> $M_i = Z$.

c. <u>If</u> q <u>is an odd prime then</u> $M_i = M_2(\mathcal{O}_{q^j})$ with $\mathcal{O}_{q^j} = Z(\rho_{q^j}+\rho_{q^j}^{-1})$ <u>or</u> $M_i = \hat{Z}_q(\rho_{q^j})$.

d. <u>If</u> q <u>is</u> 2 <u>then</u> $M_i = M_2(\mathcal{O}_{2^j})$ <u>or</u> L <u>where</u> L <u>is the maximal order in the usual quaternion algebra over</u> $\mathbb{Q}(\rho_{2^j}+\rho_{2^j}^{-1})$.

(Here ρ_ℓ is a primitive ℓ^{th} root of unity, and (a) is well known.)

Now the proof of 2.B.1 follows directly since $\tilde{K}_0(Z) = 0$ and the prime ideal over p in $\mathcal{O}(\rho_{p^j}+\rho_{p^j}^{-1})$ is principal. The only remaining case is 2.B.5 (d), but there the results of [18] identify $K_0(L)$ with $K_0(\mathcal{O}(\rho_{2^j}+\rho_{2^j}^{-1}))$ (using Weber's theorem see e.g. [7]). This completes the proof of 2.B.1.

C. <u>The value of</u> $\sigma_n(G)$ <u>in</u> $D(G)/T(G)$

We suppose the groups $\mathrm{Tor}^j{}_{Z(G)}(M,Z)$ are known for the periodic group G, and local solutions

2.C.1: $$0 \to \hat{Z}_p \to C_{p,n-1} \xrightarrow{\partial p,\ n-1} C_{p,n-2} \to \ldots \to C_{p,1} \xrightarrow{p,1} Z_p(G) \xrightarrow{\xi} \hat{Z}_p \to 0.$$

of 2.A.1, with the $C_{p,j}$ free are also given.

Next, let the complex of free M-modules

2.C.2: $$0 \to Z \to M_{n-1} \to M_{n-2} \to \ldots \to M \to Z \to 0$$

be given with homology the $\mathrm{Tor}^j{}_{Z(G)}(M,Z)$. Localizing we have

<u>Lemma</u> 2.C.3: <u>Let</u> N_p <u>be a maximal order in a simple algebra over</u> $\hat{Z}_p$. <u>Then an isomorphism classification of finite chain complexes of free</u> N_p <u>modules</u> $C = \{C_i \to C_{i-1} \to \ldots \to C_0 \to 0\}$ <u>is given by the groups</u> $H_*(C)$

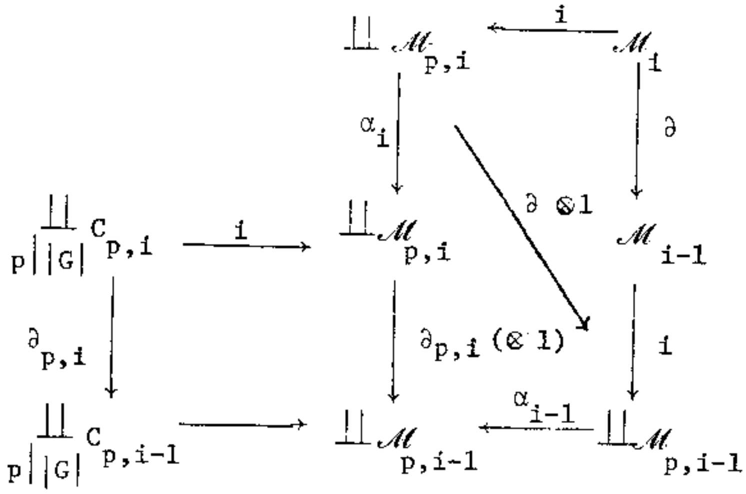
<u>and the ranks</u> $\dim \mathcal{N}_p(C_j)$.

<u>Proof</u>: This follows in the same way as the analogous theorem for free Z-chain complexes proved using the diagonalization theorem for matrices with coefficients in $\mathcal{N}_p$. (See e.g. [13], p. 173, Theorem 17.7)

In particular, we assume 2.C.2 is a direct sum of complexes one for each simple order in $\mathcal{M}$, and rank $_{\mathcal{M}}M_j$ = rank$_{\hat{Z}_p(G)}(C_{p,j}) = k_j$ for all $p \mid |G|$. Then we have

<u>Lemma 2.C.4: There are elements</u> $\alpha_i \varepsilon \coprod_{p \mid |G|} GL_{k_i}(\mathcal{M}_p)$ <u>which make</u> 2.C.5 <u>commute for each</u> i

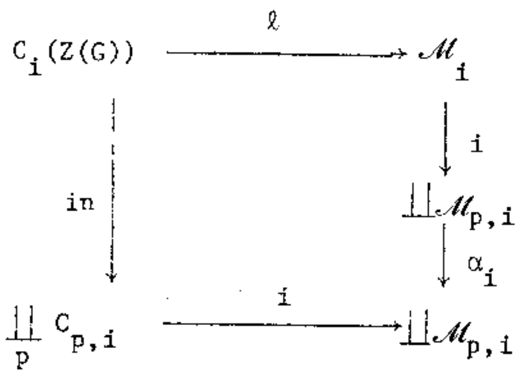
2.C.5:



<u>and</u>

$$\sigma_n(G) = \{\alpha_1 \alpha_2^{-1} \alpha_3 \alpha_4^{-1} \ldots \alpha_{n-1}^{\varepsilon}\}^{-1}$$

<u>Proof</u>: The existence of the α_i is given by 2.C.3. Now let $C_i(Z(G))$ be obtained as the pullback diagram



Then $C_i(Z(G))$ is projective and represented by $\{\alpha_i\}$ in $D(G)$. Moreover the following sequence of complexes is exact

$$0 \to C_*(Z(G)) \xrightarrow{(\ell,\ -in)} M_* \oplus \coprod (C_{p,*}) \xrightarrow{\alpha_{*i}\ \oplus\ \bar{i}} \coprod M_{p,*} \to 0$$

and passing to homology, since α_{*i} induces isomorphisms (as $\mathrm{Tor}^i_{Z(G)}(M,Z)$ is torsion) in positive degrees, and in 0 $H_0(C_*(Z(G)) = Z$ we have $H_j(C_*(Z(G)) = 0$ $j > 0$ and the sequence obtained is a periodic resolution

Remark 2.C.6: We should be more explicit about the classes $\alpha_i \in \coprod \mathrm{Aut}\,(M_{p,i})$. Note that since $M_i = M \oplus \dots \oplus M$ we have an evident map $C_{p,i} = \hat{Z}_p(G) \oplus \dots \oplus \hat{Z}_p(G) \to M_{p,i}$. This identifies $M_{p,i}$ with $C_{p,i} \otimes M_p$. It is with respect to this that α_i is defined.

D. Wall's finiteness obstruction.

In [23, especially pp. 64-68] a finiteness obstruction $f_n(X)$ for complexes in a homotopy type satisfying "$D_n: H_i(\tilde{X}) = 0 \quad i > n,\ H^{n+1}(X,\mathcal{B}) = 0$ for all abelian coefficient bundles $\mathcal{B}(n>2)$", was defined which genralizes Swan's obstruction. It makes its values in $\tilde{K}_0(Z(\pi_1(X))$, and for $\pi_1(X)$ finite we have the result corresponding to 2.B.3.

Lemma 2.D.1: $\quad p_{\pi_1(X)}(f_n(X)) = \pm \sum_{i=0}^{n} (-1)^i \chi(H_i(X,M)).$

(The proof is completely analogous to 2.B.3.)

Similarly, in case $p_{\pi_1(X)}(f_n(X)) = 0$, analysis of the Wall obstruction in terms of local piecing analogous to that in 2.C. may be casried through.

§3. Calculations of $\mathrm{Tor}^i_{Z(H)}(M(H),Z)$ for H a Hyperelementary Periodic Group.

A. Reduction to local cases.

The formal reduction

3.A.1: $$\mathrm{Tor}^i_{\hat{Z}_p(H)}(M \otimes \hat{Z}_p, \hat{Z}_p) = \mathrm{Tor}^i_{Z(H)}(M,Z) \otimes \hat{Z}_p$$

allows us to calculate the Tor^i locally. Thus, to begin we study the structure of the local group rings.

Lemma 3.A.2 Suppose p prime and $(p, \ell) = 1$, then

$$\hat{Z}_p(Z/\ell) = \hat{Z}_p \oplus \coprod_{j|\ell} \coprod_{i=1}^{r_j} \hat{Z}_p(\rho_j), \text{ where}$$

$r_j = \ell - 1/\varphi_j(p)$ and $\varphi_j(p)$ is the least positive integer for which $p^{\varphi_\ell(p)} \equiv 1(j)$.

Proof: This is standard. Indeed since $(\ell, p) = 1$ $\hat{Z}_p(Z/\ell)$ is a maximal $\hat{Z}_p$ order. Also, the global maximal order is $\coprod_{j|\ell} Z_p(\rho_j)$ and

3.A.3: $$\hat{Z}_p \otimes Z(\rho_j) = \coprod_{i=1}^{r_j} \hat{Z}_p(\rho_j).$$

Compare with [9, p. 39].

More generally

Corollary 3.A.4: Write $n = p^m \ell$ with $(\ell, p) = 1$ then

$$\hat{Z}_p(Z/n) = \hat{Z}_p(Z/p^m) \oplus \coprod_{j|\ell} \coprod^{r_j} \hat{Z}_p(\rho_j)(Z/p^m).$$

and

3.A.8: $$\hat{Z}_p(H) = \hat{Z}_p(Z/_p m)[\mathrm{im}\lambda] \oplus N$$

Factoring still further, we have the restriction map

$$r_p\colon \mathrm{Aut}(Z/n) \longrightarrow \mathrm{Aut}(Z/_p n)$$

and applying the above argument to the first factor in 3.A.8 (Which is $\hat{Z}_p(Z/_p m \times {}_T(\mathrm{im}\lambda))$ we obtain

3.A.9: $$\hat{Z}_p(H) = \hat{Z}_p(Z/_p m \times {}_T r_p(\mathrm{im}\lambda)) \oplus N_1 \oplus N.$$

Of course, when p = q, 3.A.8 takes the form

3.A.10: $$\hat{Z}_q(\mathbb{H}) = \hat{Z}_q(\mathbb{H}_q) \oplus N.$$

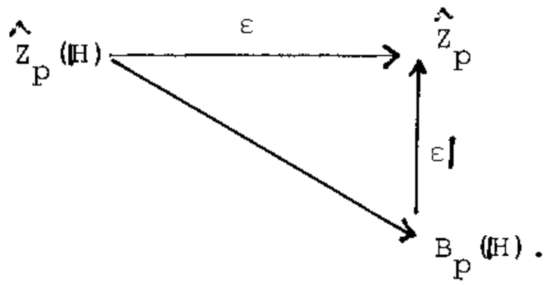
We call the various summands in 3.A.9, respectively 3.A.10, the p-blocks, respectively q-blocks of $\mathbb{H}$. Further, on tensoring with $\hat{\mathbb{Q}}_p$ the p-blocks each become direct sums of simple algebras, and we distinguish the block containing the trivial representation (the left-most block in 3.A.9 or 3.A.10) and write it

$$B_p(\mathbb{H}).$$

<u>Proposition</u> 3.A.10: $\mathrm{Tor}^*_{Z(H)}(M_i, Z)$ <u>contains</u> p-<u>torsion only if</u>

$$M_i \cap B_p(\mathbb{H}) \neq 0$$

<u>Proof</u>: We check locally. Now $B_p(\mathbb{H})$ is a direct summand of $\hat{Z}_p(\mathbb{H})$, hence projective. Moreover, the augmentation factors as



Thus, a suitable resolution of the augmentation is obtained by resolving $\varepsilon|$ over $B_p(\mathbb{H})$. Finally, tensoring this resolution with M_i gives zero identically unless $M_i \cap B_p(\mathbb{H}) \neq 0$.

<u>Corollary</u> 3.A.11: $\mathrm{Tor}^i_{Z(H)}(M_i, Z)_p = \mathrm{Tor}^i_{B_p(H)}(M_i \otimes \hat{Z}_p, \hat{Z}_p)$.

This reduces tor calculations for these $\mathbb{H}$ to those for $Z/p^m \times_T V$ $V \subset \mathrm{Aut}\,(Z/p)$, for which calculations are easy, or in case p = q, for $\mathbb{H}_p$.

B. <u>Calculations for $B_p(\mathbb{H})$ where</u> $\mathbb{H}$ <u>is a 2-hyperelementary group with</u> H_2 <u>generalized quaternion</u>, $p \neq 2$.

In Milnor's notation $\mathbb{H} = Z/m \times \mathbb{Q}(2^i n, k, \ell)$. This means $\mathbb{Q}(2^i n, k, \ell)$ has a presentation $\{x,y,z,w,v \mid x^2 = y^{2^{i-1}},\ y^{2^i} = 1,\ xyx^{-1} = y^{-1},\ z^n = w^k = v^\ell = 1,\ xzvx^{-1} = z^{-1}v^{-1},\ y(wv)y^{-1} = w^{-1}v^{-1},\ yzy^{-1} = z,\ xwx^{-1} = w$ and z, w, v commute$\}$. Moreover, in order that $\mathbb{H}$ be periodic, m, n, k, ℓ must be coprime odd integers.

$$\text{The block } B_p(\mathbb{H}) = \begin{cases} \hat{Z}_p(Z/p^\ell) & \text{if } p/m \\ \hat{Z}_p(Z/p^\ell \times {}_T Z/2) & \text{if } p \nmid m. \end{cases}$$

The first case is well understood so we concentrate on the second case.

$$r_j(g) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda_{p^j} \end{pmatrix}$$

3.B.3
$$r_j(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Similarly
$$r_{\pm} : B_p(\mathbb{H}) \longrightarrow \hat{Z}_p^{\pm}$$

is given by $r_{\pm}(g) = 1$, $r_{\pm}(t) = \pm 1$. These representations define actions of $Z/p^{\ell} \bar{\times}_T Z/2$ on $\bar{} \ M_2(\hat{Z}_p(\lambda_{p^j}))$ or $\hat{Z}_p$, and we have

Lemma 3.B.4: $H_i^{r_j}(Z/_{p^{\ell}}, M_2(\hat{Z}_p(\lambda_{p^j}))) = \begin{cases} (Z/p)^2 & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

Moreover the action of t on the groups above is via $t(x) = (-1)^{i/2}x$.

Proof: Take the usual resolution of $Z(Z/_{p^{\ell}})$ with one copy of $Z(Z/_{p^{\ell}})$ in each degree, $\partial_{2i+1} = g-1$, $\partial_{2i} = 1+g+g^2+ \ldots +g^{p^{\ell}-1} = \Sigma_g$.
Then, tensor with $M_2(\hat{Z}_p(\lambda_{p^j}))$ over r_j and we have

$$\tilde{\partial}_{2i+1} = \begin{pmatrix} -1 & 1 \\ -1 & \lambda-1 \end{pmatrix}, \quad \tilde{\partial}_{2i} = 0$$

This means
$$\partial_{2i+1}(\alpha) = \alpha \cdot \begin{pmatrix} -1 & 1 \\ -1 & \lambda-1 \end{pmatrix} = \begin{pmatrix} a, & -a+b(\lambda-2) \\ c, & -c+d(\lambda-2) \end{pmatrix}$$

and the cokernel is $Z/p \times Z/p$ since $\lambda - 2$ is a uniformizing parameter for $\hat{Z}_p(\lambda)$.

Representing generators can be chosen as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

and $A \to A(t) = A\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ fixes these generators mod $\operatorname{im}(\partial_{2i+1})$.

Next note that the periodicity is given by capping with a generator e of $H^2(Z/_{p^\ell}, \hat{Z}_p) = \hat{Z}/_p$ [1, p. 115]. Then $e \cap H^{(r_j)}_{i+2}(Z/_{p^\ell}, M_2(\hat{Z}_p(\lambda_{p^j})))$

$= H^{(r_j)}_i(Z/_{p^\ell}, M_2(\hat{Z}_p(\lambda_{p^j})))$, $t(\alpha \cap \beta) = t(\alpha) \cap t(\beta)$ and $t(e) = -1(e)$,

so 3.B.4. follows.

Corollary: 3.B.5: $\mathrm{Tor}^i_{B_p(\mathbb{H})}(M_2(Z_p(\lambda_{p^\ell})), \hat{Z}_p) = \begin{cases} (Z/p)^2 & i \equiv 0(4) \\ 0 & \text{otherwise.} \end{cases}$

Proof: In the Hochschild-Serre spectral sequence

$$E^2_{i,k} = H_i(Z/2, H^{(r_j)}_k(Z/p^j; M_2(\hat{Z}_p(\lambda_{p^j}))))$$

which collapses to $E^2_{0\,*}$ since $p \neq 2$. But

$$E^2_{0,k} = H_k(Z/p^j, M_2(\hat{Z}_p(\lambda_{p^j})))/\mathrm{im}\ (t-1)$$

and $\mathrm{im}(t-1) = 0$, $k \equiv 0(4)$ but $\mathrm{im}(t-1) = \mathrm{im}(-2)$ if $k \equiv 2(4)$ so 3.B.5. follows.

Similarly, for $r_\pm$ we have

Corollary 3.B.6: (a) $\mathrm{Tor}^i_{B_p(\mathbb{H})}(\hat{Z}^+_p, \hat{Z}_p{}^+) = \begin{cases} \hat{Z}_p & i = 0 \\ Z/p^\ell & i \equiv 3(4) \\ 0 & \text{otherwise.} \end{cases}$

(b) $\mathrm{Tor}^i_{B_p(\mathbb{H})}(\hat{Z}^-_p, \hat{Z}^+_p) = \begin{cases} Z/p^\ell & i \equiv 1(4) \\ 0 & \text{otherwise.} \end{cases}$

Remark 3.B.7: Exactly the same chain of ideas works to calculate the Tor's in the case of $Z/p^\ell \times_T Z/_{q^s}$ where $q^s \mid (p-1)$. Here

3.B.8: $$M_p = \coprod_{1\leq j\leq \ell} M_{q^s}(\hat{Z}_p(\rho_{p^j})^{(\mathrm{inv})}) \oplus \coprod_1^{q^s} \hat{Z}_p$$

the latter terms corresponding to embeddings $Z/q^s \hookrightarrow (\hat{Z}_p)$; taking the generator to the varions q^s'th roots of unity.

C. The situation at 2 and the completion of the proof of 2.B.5

$$B_2(\mathbb{H}) = \hat{Z}_2(H_2) = \hat{Z}_2(\{x,y \mid x^2 = y^{2^i},\ xyx^{-1} = y^{-1}\})$$

and we have

Lemma 3.C.1: The maximal order of $B_2(\mathbb{H})$ is

(a) $$\coprod_{i \geq j \geq 2} M_2(Z_2(\rho_{2^j} + \rho_{2^j}^{-1})) \oplus \hat{Z}_2^{(4)}$$

if $i > 2$

(b) $\mathfrak{Q} \oplus Z_2^{(4)}$, where $\mathfrak{Q}$ is the maximal order $\hat{Z}_2(\rho_3) \oplus \hat{Z}_2(\rho_3)\pi$, $\pi^2 = -2$, and $\pi\lambda = \Psi(\lambda)\pi$ for $\lambda \in \hat{Z}_2(\rho_3)$ where Ψ is the Galois automorphism of $\hat{Z}_2(\rho_3)$.

(This follows from the results of [11]. However, for $i > 2$ the explicit representations are given easily. In particular the faithful representation ($i=j$) is given by

3.C.2: $$g \to \begin{pmatrix} 0 & 1 \\ -1 & \lambda_{2^i} \end{pmatrix}, \qquad t \to \begin{pmatrix} 1 & C-\lambda \\ C & -1 \end{pmatrix}$$

where C is a root of

3.C.3: $$C^2 - \lambda_{2^i} C + 2 = 0.$$

Indeed 3.C.3 splits inside $\hat{\mathbb{Q}}_2(\lambda_{2^i})$ for $i > 2$ since its discriminant

$$d = \lambda^2 - 8 \equiv \lambda^2 (4\pi)$$

and is thus a square. The remaining representations are given by the same formulae as 3.B.3.)

This completes the proof of 2.B.5 since the only representations which restrict non-trivially at both 2 and at least one other prime are the 4 copies of Z.

D. Calculations for $B_2(\mathbb{H})$

A minimal resolution of $\mathbb{H}$ can be found on page 253 of [2]. It is periodic and has the form

$$\overset{\partial}{\to} Z(H_2)e \xrightarrow{\partial_3} Z(\mathbb{H}_2)c \oplus Z(\mathbb{H}_2)c' \xrightarrow{\partial_2} Z(\mathbb{H}_2)b \oplus Z(\mathbb{H}_2)b' \xrightarrow{\partial_1} Z(H_2)a \overset{\varepsilon}{\to} Z$$

where

$$\partial_3(e) = (y-1)c - (yx-1)c'$$

$$\partial_2(c') = (yx+1)b + (y-1)b'$$

3.D.1:

$$\partial_2(c) = (1+y+y^2+\ldots+y^2+\ldots+y^{2^{i-1}-1})b - (x+1)b'$$

$$\partial_1(b') = (x-1)a$$

$$\partial_1(b) = (y-1)a.$$

We can use 3.D.1 to calculate Tor groups explicitly in this case. The only ones we need are for the group of order 8 where the results are given by the table.

3.D.2:

	Tor^0	Tor^1	Tor^2	Tor^3
Z^{++}	$\hat{Z}_2$	$Z/2 + Z/2$	0	$Z/8$
Z^{+-}	$Z/2$	$Z/2$	$Z/2$	0
Z^{-+}	$Z/2$	$Z/2$	$Z/2$	0
Z^{--}	$Z/2$	$Z/2$	$Z/2$	0
$\mathcal{Q}$	F_4	F_4	F_4	0

§4. Evaluating the Swan obstruction for the groups $Q(4p,q,1)$, p,q odd primes.

In this section we calculate the Swan obstruction for some of the groups Q(4p,q,1). We begin in 4.A. with partial calculations of the local defects. The method is to do it separately for each p-block and the maximal orders associated with it. As the methods and the results are rather technical they are summarized in Table 4.A.11. Using 4.A.11 the reader can skip to 4.B. where the Swan obstruction is calculated for these groups. Finally, in 4.C we give examples of specific calculations chosen mainly to illustrate the types of complexities encountered. In particular, these examples imply all the results mentioned in the introduction.

A. The local defects.

The p-blocks have the form

4.A.1: (a) $C(Z/p) \times_T Z/2 \times Z/2$

(b) $L(Z/p) \times_T Z/2$

where $C = \hat{Z}_p$, $\hat{Z}_p(\rho_q)$ or $(\hat{Z}_p(\rho_q))^2$ and $t^2 = -1$, while $L = \hat{Z}_p$, $M_2(\hat{Z}_p(\rho_q+\rho_q^{-1})$ or $M_2(\hat{Z}(\rho_q))$.

The second case occurs in C and L if $p^\nu \equiv -1(q)$ for some ν, and the third case occurs otherwise.

Lemma 4.A.2: The non p-primary part of the local defect of a block of type 4.A.1. is

$$\dot{F}_p(\rho_q+\rho_q^{-1})$$

in cases 2 and 3 in C or L and $\dot{F}_p$ in the first case.

Proof: Let R be the maximal order for 4.A.1 in $\hat{\mathbb{Q}}_p \otimes (\ )$. Then R contains $N = C \otimes_T Z/2 \times_T Z/2$ or $L(Z/2)$ as a direct summand, and

$$R = N \oplus M$$

where M is the maximal order in $\hat{\mathbb{Q}}_p \otimes C(\rho_p) \times_T Z/2 \times Z/2$ or $\hat{\mathbb{Q}}_p \otimes L(\rho_p) \times_T Z/2$.

Let J be the Jacobson radical of the block B, then

$$B|J \cong N/J$$

and hence the local defect C away from (p) for B is obtained from the diagram

4.A.3:

$$\begin{array}{ccccc} \mathcal{N}/J' & \xrightarrow{\ i\ } & B/J & \xrightarrow{\ \ell\ } & (\mathcal{N} \oplus \mathcal{M})/J'' \\ & & \downarrow \cong & & \downarrow p_2 \\ & & \mathcal{N}/J' & \xrightarrow{\ p_2 \circ \ell \circ i\ } & \mathcal{M}/J''' \end{array}$$

Thus, factoring out by imB/J simply identifies N/J with its image under $p_2 \circ \ell \circ i$ in M/J. Finally, observe that any element in one of the Jacobson radicals is p-adic so contributes only p-torsion to the local defect.

At 2 the situation is slightly more involved.

There are 3 types of blocks.

(a) $C_{p,q} \times_T \mathbb{H}_2$

(b) $C_p \times_T \mathbb{H}_2$

(c) $\hat{Z}_2(\mathbb{H}_2)$

Here $C_{p,q}$ is 2 or 4 copies of $\hat{Z}_2(\rho_p, \rho_q)$ and C_p is one or two copies of $\hat{Z}_p(\rho_p)$.

<u>Lemma</u> 4.A.5: <u>The local defect for the block</u> $\hat{Z}_2(\mathbb{H}_2)$ <u>is</u> $Z/2$ <u>and the generator corresponds to</u> $\langle 3 \rangle$ <u>at the trivial representation (hence is in the image of</u> T).

(This may be directly obtained from the calculations of [6], [8].)

<u>Lemma</u> 4.A.6: <u>The</u> 2 <u>primary part of the local defect for a block of type</u> 4.A.4 (a) <u>is zero.</u>

<u>Proof:</u> A block of type 4.A.4 (a) has a maximal order $M_4(\sigma) \oplus M_4(\sigma)$ where σ is one of $\hat{Z}_2(\rho_p, \rho_q)$, $\hat{Z}_2(\rho_p + \rho_p{}^{-1}, \rho_q)$, $\hat{Z}_2(\rho_p, \rho_q + \rho_q^{-1})$ or $\hat{Z}_2(\rho_{pq} + \rho_{pq}{}^{-1})$. The two idempotents for these representations are

$\frac{1}{2}(y^2-1) = e_-$, $\frac{1}{2}(y^2+1) = e_+$. Suppose $C_{p,q} = (\hat{Z}_2(\rho_p,\rho_q))$, and take $\alpha = (a,0)$, then

$$2e_-\alpha + 1 \longmapsto \left(\begin{pmatrix} 1+2a & 0 & 0 & 0 \\ 0 & 1+2\bar{a} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad I\right)$$

which has the image $(N(1 + 2a),1)$. But since the 4 possible extensions of $\hat{Z}_2$ which form the centers are all unramified, the units of the center are all obtained as norms of units in $\hat{Z}_2(\rho_p,\rho_q)$, and 4.A.6 follows. A similar argument holds in case $C = (\hat{Z}_2(\rho_p,\rho_q))^4$.

Finally we evaluate the local defect for 4.A.4(b).

<u>Lemma</u> 4.A.7: <u>The</u> 2 <u>primary part of the defect for</u> 4.A.4(b) <u>is</u> $(\hat{Z}_2(\rho_p+\rho_p^{-1})/m)^+$. <u>The generators come from the representation</u> $C_p \times_T Z/2$ $(t^2 = 1)$, <u>and are represented by units of the form</u> $1 + 2\nu$. <u>The relation with the maximal representation in the block is, in case</u> $C = \hat{Z}_2(\rho_p)$

$$1 + (\alpha + \bar{\alpha})(\iota - 1) \longleftrightarrow 1 + 2(\alpha + \bar{\alpha})$$

<u>and in case</u> $C = \hat{Z}_2(\rho_p)^2$,

$$1 + (\alpha + i\beta)(\iota - 1) \longleftrightarrow 1 + 2(\alpha + \iota).$$

<u>Remark</u> 4.A.8: The non-triviality of the local defect in this case is crucial to the calculations, as without it it would be routine to evaluate the Swan obstruction. Thus the reader is advised to check the proof of 4.A.7 most carefully.

<u>Proof</u>: The maximal order for 4.A.4(b) is

4.A.9: $$M_2(\mathfrak{O})^+ \oplus M_2(\mathfrak{O})^- \oplus M_2(\mathfrak{O})$$

when, in case $C_p = \hat{Z}_2(\rho_p)$, $\vartheta = \hat{Z}_2(\rho_p + \rho_p^{-1})$

$$\vartheta' = \hat{Z}_2(\rho_{4p} + \rho_{4p}^{-1})$$

while if $\quad c_p = \hat{Z}_2(\rho_p)^2 \qquad \sigma = \hat{Z}_2(\rho_p)$.

$$\sigma = \hat{Z}_2(\rho_{4p}).$$

Also

$$C_p \times_T (\mathbb{H}_2)/J = M_2(Z/2(\rho_p + \rho_p^{-1})) \text{ or } M_2(Z/2(\rho_p)).$$

Thus

$$C_p \times_T \mathbb{H}_2 = M_2(F) \oplus M_2(F)[(y-1),2] \oplus M_2(F)[(g-1)^2,4,\ 2(y-1)] \oplus \ldots$$

writing it in filtered form.

The 2 primary component of $SL_2(F) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = W$ and since we are only interested in 2 torsion it suffices to analyze the image of

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}[1 + \theta(y-1) + \theta'_2 + \theta''(y-1)^2 + \theta'''2(y-1) + \theta^{(iv)}4 + \ldots].$$

Next, using the idempotents $\frac{1}{2}(y^2-1)$, $\frac{1}{2}(y^2-1)$ we find that

4.A.10 $\qquad (\dot{\sigma}/_{m^2} \times \dot{\sigma}/_{m^2} \times \dot{\sigma}/_{m^2})$

surjects only the 2 primary defect. Then we use in turn each of the 5 remaining generators $(y-1),2$, etc., to obtain allthe relations in 4.A.10 and complete the proof of **4.A.7**.

We summarize the results of §4.A in the following table

4.A.11: TABLE

Block	Defect	Representations
$\hat{Z}_p(Z/p \times_T Z/2)^+$	$\dot{Z}/p$	Z^{++}, Z^{-+}, $M_2(Z(\rho_p+(\rho_p^{-1}))^+$
$\hat{Z}_p(Z/p \times_T Z/2)^-$	$\dot{Z}/p$	Z^{+-}, Z^{--}, $M_2(Z(\rho_p+\rho_p^{-1}))^-$
$\hat{Z}_g(Z/q \times_T Z/2)^+$	$\dot{Z}/q$	Z^{++}, Z^{-+}, $M_2(Z(\rho_q+\rho_q^{-1}))^+$
$\hat{Z}_g(Z/q \times_T Z/2)^-$	$\dot{Z}/q$	Z^{-+}, Z^{--}, $M_2(Z(\rho_q+\rho_q^{+}))^-$
$M_2(\hat{Z}_p(\rho_q+\rho_q^{-1}))(Z/p \times_T Z/2)$	$\dot{F}_p(\rho_q+\rho_q^{-1})$	$M_2(Z(\rho_q+\rho_q^{-1}))^+$ $M_2(Z(\rho_q+\rho_q^{-1}))^-$ $M_4(Z(\rho_p+\rho_p^{-1},\rho_q+\rho_q^{-1}))$
$M_2(\hat{Z}_q(\rho_p+\rho_p^{-1}))(Z/q \times_T Z/2)$	$\dot{F}_q(\rho_p+\rho_p^{-1})$,	$M_2(Z(\rho_p+\rho_p^{-1}))^+$ $M_2(Z(\rho_p+\rho_p^{-1}))^-$ $M_4(Z(\rho_p+\rho_p^{-1}, \rho_q+\rho_q^{-1}))$
$\hat{Z}_2(\rho_p) \times_T H_8$	$\dot{F}_2(\rho_p+\rho_p^{-1}) \times F_2(\rho_p+\rho_p^{-1})^+$,	$M_2(Z(\rho_p+\rho_p^{-1}))^+$ $M_2(Z(\rho_p+\rho_p^{-1}))^-$ W_p
$\hat{Z}_2(\rho_q) \times_T H_8$	$\dot{F}_2(\rho_q+\rho_q^{-1}) \times F_2(\rho_q+\rho_q^{-1})^+$	$M_2(Z(\rho_q+\rho_q^{-1}))^+$ $M_2(Z(\rho_q+\rho_q^{-1}))^-$ W_q
$\hat{Z}_2(H_8)$	$Z/2$	Z^{++}, Z^{+-}, Z^{-+}, Z^{--}, $\mathcal{Q}$
$\hat{Z}_p(\rho_q)(Z/p) \times_T \mathbb{H}_{8_{r^2=-1}}$	$\dot{F}_p(\rho_q+\rho_q^{-1})$	W_q $W_{p,q}$
$\hat{Z}_q(\rho_p)(Z/q) \times_T H_{8_{t^2=-1}}$	$\dot{F}_q(\rho_p+\rho_p^{-1})$	W_p $W_{p,q}$
$\hat{Z}_2(\rho_p,\rho q) \times_T \mathbb{H}_8$	0	$M_4(Z(\rho_p+\rho_p^{-1}, \rho_q+\rho_q^{-1}))$ $W_{p,q}$

Remark 4.A.12: W_r is the maximal order in

$$\mathbb{Q}(\rho_r) \times_T \mathbb{H}_{8}{}_{t^2=-1}$$

and $W_{p,q}$ is the maximal order in

$$\mathbb{Q}(\rho_{pq}) \times_T \mathbb{H}_{8}{}_{t^2=-1} .$$

B The Swan Obstruction.

Let us consider $\mathbb{Q}(12,5,1)$. The local defects may be presented in an array as follows

4.B.1:

$$\begin{array}{ccc}
\dot{Z}/3 = E(B_3(\mathbb{H})) & \dot{Z}/2 = E(B_2(\mathbb{H})) & \dot{Z}/5 = E(B_5(\mathbb{H})) \\
\dot{Z}/3 = E(B_3(\mathbb{H})^-) & & \dot{Z}/5 = E(B_5(\mathbb{H})^-) \\
 & Z/2 = E(B_2(\hat{Z}_2(\rho_3)(\mathbb{H}_2)) & \\
 & \dot{F}_4 \times (Z/2)^2 = E(\hat{Z}_2(\rho_5)(\mathbb{H}_2)) & \\
\dot{F}_9 & & \dot{Z}/5 \\
\dot{Z}/3(y^2 = -1) & & \dot{Z}/5(t^2 = -1) \\
\dot{F}_9(y^2 = -1) & & \dot{Z}/5(t^2 = -1)
\end{array}$$

The last 2 rows correspond to the blocks of type 4.A.4(b) and 4.A.4(a) respectively, which at odd primes are obtained from the blocks of type 4.A.1(b) with $Y^2 = -1$ or 4.A.1(a) respectively.

Remark 4.B.2: Lemmas 4.A.2 and 4.A.5 together imply that im T is precisely the first row of 4.B.1, and we have

Lemma 4.B.3 : The units in row 2 of 4.B.1 may be identified with the corresponding units in row 1 on factoring out by global units.

Proof: Take the global representation

$M_2(Z(\rho_5 + \rho_5^{-1}))^-$. Its units go to the generators of $\dot{Z}/5$ in row 2, to some elements in $\dot{F}_4 \times (Z/2)^2$ and to $\dot{F}_9$. But the corresponding units of $M_2(Z(\rho_5 + \rho_5^{-1}))^+$ hit F5 on row one and are identified with the images of $M_2(Z(\rho_5 + \rho_5^{-1}))^-$ in rows 3 and 4. The same argument works for $M_2(Z(\rho_3 + \rho_3^{-1}))$.

Thus, since the Swan obstruction is in $D(\mathbb{H})/T$, we may ignore the first two rows in 4.B.1. But these rows contain all the information from the 4 copies of Z. Now, by 2.B.5, the only remaining M_i which give torsion give it only for 2,3, or 5 separately.

We now describe the Swan obstruction. Choose periodic resolutions of $B_3(\mathbb{H})$, $B_5(\mathbb{H})$, $B_2(\mathbb{H})$, which can all be chosen of the form $\hat{Z}_p \to \Lambda \to \Lambda^2 \to \Lambda^2 \to \Lambda \xrightarrow{\varepsilon} \hat{Z}_p$ and otherwise, on the higher blocks the resolution locally are

$$\begin{array}{ccccccc} B & \xrightarrow{\text{id}} & B & & B & \xrightarrow{\text{id}} & B \\ & & \oplus & \text{id} & \oplus & & \\ & & B & \longrightarrow & B & & \end{array} .$$

Now, the complex of maximal order is

(a)
$$\begin{array}{ccccccc} \mathcal{Q} & \xrightarrow{(i-1)} & \mathcal{Q} & & \mathcal{Q} & \xrightarrow{(i-1)} & \mathcal{Q} \\ & & \oplus & (i-1) & \oplus & & \\ & & \mathcal{Q} & \longrightarrow & \mathcal{Q} & & \end{array}$$

4.B.4

(b) $$M_2(Z(\rho_p + \rho_p^{-1})) = \Lambda\ ,\ \theta = \begin{pmatrix} 2-(\rho_p + \rho_p^{-1}) & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{array}{ccccccc} \Lambda & \xrightarrow{\text{id}} & \Lambda & & \Lambda & \xrightarrow{\theta} & \Lambda \\ & & \oplus & & \oplus & & \\ & & \Lambda & \xrightarrow{\text{id}} & \Lambda & & \end{array} .$$

Then, using §2.C. we have $\sigma_4(Q(12,5,1))$ given by

4.B.5
$$\begin{array}{ccc} * & * & * \\ * & & * \\ 2-\rho_5-\rho_5^{-1} & & 2-\rho_3-\rho_3^{-1} \\ & 2-\rho_3-\rho_3^{-1} & \\ & 2-\rho_5-\rho_5^{-1} & \\ 2 & & 2 \\ 1 & & 1 \end{array}$$

Additionally, $\rho_3+\rho_3^{-1} = -1$ and on F_3, $2 = -1$ so $2-\rho_5-\rho_5^{-1} = -\rho_5^{-1}(\rho_5^3-1)/(\rho_5-1)$.

Also, at level 4, $\rho_5^2 + \iota\rho_5^{-2}$ at $Z(\rho_{20}) \times_T Z/2 | t^2 = -1$ has norm

$$2 + \iota\rho_5 - \iota\rho_5^{-1}$$

which is congruent to 2 mod the maximal ideal over 5.

At level 5 note that $\rho_5^2 + \rho_5^{-2}$ is the image of $z^2 + yz^{-2}$ (in the notation of 3.B) which, at the maximal representation goes to

$$\begin{pmatrix} \rho_5^2 & 0 & \rho_5^{-2} & 0 \\ 0 & \rho_5^{-2} & 0 & \rho_5^2 \\ -\rho_5^{-2} & 0 & \rho_5^2 & 0 \\ 0 & -\rho_5^2 & 0 & \rho_5^{-2} \end{pmatrix}$$

and the determinant of this is $(\rho_5^2 + \rho_5^{-2})^2 = 2 + \rho_5 + \rho_5^{-1}$.

At level 3 and in the 2 local part

$$z^2 + yz^{-2} \to \left(\begin{pmatrix} \rho_5^2 + \rho_5^{-2} & 0 \\ 0 & \rho_5^{-2} + \rho_5^2 \end{pmatrix}, \begin{pmatrix} \rho_5^2 - \rho_5^{-2} & 0 \\ 0 & \rho_5^{-2} - \rho_5^2 \end{pmatrix}, * \right)$$

and so using $(\rho_5^2 + \iota\rho_5^{-2})$ we change the obstruction to

$$\begin{matrix} * & * & * \\ * & & * \\ -2 + \rho_5 + \rho_5^{-1} & & -3 \\ & 1 & \\ & 1 & \\ 1 & & 1 \\ (\rho_5 + \rho_5^{-1})^2 & & 1 \end{matrix} .$$

Actually, the procedure outlined above is general and we have

<u>Theorem</u> 4.B.6: <u>In</u> $E_p \oplus E_q \oplus E_2$ <u>the lift of the Swan obstruction</u> (σ_4) <u>for</u> $\mathbb{Q}(4q,p,1)$ <u>can be chosen to be</u>

$$\begin{array}{ccc}
* & * & * \\
* & & * \\
\mathrm{im}(-2+\rho_q+\rho_q^{-1}) & & \mathrm{im}(-2+\rho_p+\rho_p^{-1}) \\
 & \begin{array}{c}1\\1\end{array} & \\
1 & & 1 \\
(\rho_q+\rho_q^{-1})^2 & & (\rho_p+\rho_p^{-1})^2
\end{array} \quad .$$

<u>C. Examples.</u>

<u>Proposition</u> 4.C.1: <u>The Swan obstruction</u> $\sigma_4(\mathbb{H})$ <u>for</u> $\mathbb{H} = \mathbb{Q}(12,13,1)$ <u>is zero.</u>

<u>Proof</u>: In $Z(\rho_{13})$ the ideal (3) splits as $(3) = P_1\, P_2\, P_3\, P_4$ and these are interchanged in pairs by conjugation. Hence the image of $-2+\rho_{13}+\rho_{13}^{-1}$ at the 2 primes over (3) in $Z(\rho_{13}+\rho_{13}^{-1})$ are respectively

$$-2+\rho_{13}+\rho_{13}^{-1} \quad \text{and} \quad -2+\rho_{13}^{2}+\rho_{13}^{-2}$$

but $Z(\rho_{13}+\rho_{13}^{-1})/P_1 = F_{27}$ and we have

$$N(1+\rho_{13}+\rho_{13}^{-1}) = \rho_{13}\left(\frac{\rho_{13}^{3}-1}{\rho_{13}-1}\right)\left(\frac{\rho_{13}^{9}-1}{\rho_{13}^{3}-1}\right)\left(\frac{\rho_{13}-1}{\rho_{13}^{9}-1}\right) = 1 \ ,$$

hence $1+\rho_{13}+\rho_{13}^{-1}$ is a square in F_{27} and similarly for $1+\rho_{13}^{2}+\rho_{13}^{-2}$.

Again $3^3 \equiv -3(13)$ so -3 is a square but not a 4^{th} power mod (13). Note in particular that 2 is a primitive generator mod (13) and

$(\rho_{13} + \rho_{13}^{-1})^{10} \equiv -3(\rho_{13}-1)$. Now use the unit $(\rho_{13} + \rho_{13}^{-1})^2$ at $M_2(Z(\rho_{13} + \rho_{13}^{-1}))$. Since it is a square it leaves $(1 + \rho_{13}^{\varepsilon} + \rho_{13}^{-\varepsilon})$ equal to a square and sets $-3 \equiv 1$. The only remaining obstruction is $(\rho_{13}+\rho_{13}^{-1})^{-2}$ at level 5. But this is a square in each F_{27}.

Hence, an odd multiple of $\sigma_3(\mathbb{H})$ is trivial. But there is an orthogonal, free representation of $\mathbb{Q}(2^i a,b,c)$ in $O(8)$ (see e.g. [22]), hence $2\sigma_3(\mathbb{H}) = 0$ and 4.C.1 follows.

In marked contrast to 4.C.1 we have

Proposition 4.C.2: *The Swan obstruction $\sigma_3(\mathbb{H})$ for the group $\mathbb{Q}(12,5,1)$ is non-zero.*

Proof: We begin by noting that all global representations at levels 4 and 5are quaternionic at ∞. This is evident for $\mathbb{Q}(\rho_{4p}) \times_T Z/2, t^2 = -1$, and proved in [11] for the faithful representation. Hence, the global units which occur for them must be positive at all infinite places.

The centers in question are

$$K_1 = \mathbb{Q}(\rho_{12} + \rho_{12}^{-1}),\ K_2 = \mathbb{Q}(\rho_{20} + \rho_{20}^{-1}),\ K_3 = \mathbb{Q}(\rho_5 + \rho_5^{-1})$$

$$U(K_1) = Z \times Z/2,\ U(K_2) = Z^3 \times Z/2,\ U(K_3) = Z \times Z/2.$$

Lemma 4.C.3: a. *The generating unit of $U(K_1)$ is positive at all infinite places, and $N(\rho_3 + i\rho_3^{-1})$ is an odd power of this generator.*

b. The positive elements of $U(K_2)$ are generated by $U(K_2)^2$, $N(\rho_5 + i\rho_5^{-1})$.

c. *The positive elements of $U(K_3)$ are $U(K_3)^2$.*

Proof: Consider the units of $\mathbb{Q}(\rho_n)$ as compared with those in the maximal real subfield $\mathbb{Q}(\rho_n + \rho_n^{-1})$. By the Dirichlet unit theorem the ranks of the torsion free parts are equal, and H. Hasse [7] has proved index (torsion free part $U(\mathbb{Q}(\rho_n + \rho_n^{-1}))$) in (torsion free part $U(\mathbb{Q}(\rho_n))$) is 2 if n is composite, and one if n is a prime power. In particular, for n = 4p the extra unit ν has the property $\bar{\nu} = i\nu$. Hence, $\rho_p + i\rho_p^{-1}$ represents this extra unit. Clearly, its norm is positive, and evidently, a non-square. This proves (a).

To show (b) we check the signs of the quotients $\rho_p + \iota\rho_p^{-1}/\rho_p^j + \iota\rho_p^{-j} = \lambda_j$. These are invariant under conjugation, so contained in the real subfield, and we easily check that the signs are independent for a suitable subclass of them. Thus, the λ_j and $N(\nu)$ generate $\mathbf{U}(K_2)$ up to odd index, so (b) follows.

(c) is similar, but easier. $\rho_5 + \rho_5^{-1} = \frac{-1+\sqrt{5}}{2}$ has norm -1. Hence, the signs of its infinite embeddings are (+,-).

We now return to the proof of 4.C.2. Look at the level 4 and 5 part of 4.B.6. An element λ_j^2 changes

$$(\rho_5 + \rho_5^{-1})^{-2} \quad \text{to} \quad (\rho_5 + \rho_5^{-1})^{-2}(\tilde{\lambda})^4.$$

But in $\dot{F}_9$ $(\rho_5 + \rho_5^{-1})$ has order 8 since taking norms gives

4.C.4: $$N(\rho_5 + \rho_5^{-1}) = N\left(\rho_5^{-1}\left(\frac{\rho_5 - 1}{\rho_5 - 1}\right)\right) = -1$$

Thus, the only possible element for removing $(\rho_5 + \rho_5^{-1})^{-2}$ is $(\rho_5 + \rho_5^{-1})^2$ in $\mathbf{U}(K_3)$. But at 5 this is $4 \equiv -1(5)$ and these are no remaining global units to convert this -1 to a 1. 4.C.2. follows.

For $\mathbb{Q}(12,5,1)$ the Swan obstruction was non-zero in level 5. Our next example shows the obstruction can also be non-trivial in level 3.

<u>Proposition</u> 4.C.5: <u>The Swan obstruction</u> $\sigma_3(\mathbf{H})$ <u>for</u> $\mathbb{Q}(12,7,1)$ <u>is non-zero.</u>

<u>Proof:</u> Referring to 4.B.6, and noting that $\hat{\mathbb{Q}}_3(\rho_7)$ has degree 6 over $\hat{\mathbb{Q}}_3$, we apply a calculation analogous to 4.C.4 to show that an odd multiple of $\sigma_3(\mathbf{H})$ is represented by

4.C.6: $$\begin{bmatrix} * & * & * \\ * & & * \\ -1 & & 1 \\ & 1 & \\ & 1 & \\ 1 & & 1 \\ 1 & & 1 \end{bmatrix}$$

We must, as before, study units, this time for $K_4 = \mathbb{Q}(\rho_7 + \rho_7^{-1})$ and $K_5 = \mathbb{Q}(\rho_{28} + \rho_{28}^{-1})$. We have

<u>Lemma</u> 4.C.7: a. <u>The units of</u> K_4 <u>are generated up to odd order by</u> $\rho_7 + \rho_7^{-1}$, $\rho_7^3 + \rho_7^{-3}$, -1.

b. <u>The positive units in</u> K_5 <u>are generated by</u> $(\mathcal{U}(K_5)^2)$, $N(\rho_7 + \iota\rho_7^{-1})$)

(a) is well known, see e.g. [3], and (b) follows as in 4.C.3.

Now, we cannot use -1 to cancel the -1 in 4.C.6, as we easily check. Moreover, the effect of $N(\rho_7 + \iota\rho_7^{-1})$ at E_2 has already been noted, and the remaining positive units of K_5, being squares, have no effect. Hence, the only remaining candidates are $\rho_7 + \rho_7^{-1}$, $\rho_7^3 + \rho_7^{-3}$. However, a calculation analogous to 4.C.4 shows

$$N(\rho_7 + \rho_7^{-1}) = N(\rho_7^3 + \rho_7^{-3}) = 1$$

from F_{27} to F_3 and 4.C.5 follows.

but these appear to be very difficult problems.

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