Some Remarks on Local Formulae for p_1

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§0. Introduction

A well-known paper [GGL1] of Gabrielov, Gelfand and Losik, which was further explicated by MacPherson [M] and Stone [S_1,S_2], shows how a rational cocycle representing the first rational Pontrjagin class p_1 of a manifold may be computed directly from a combinatorial triangulation of the manifold, provided that certain other data, viz, "configuration" and "hyper-simplicial" data, are given as well. There has been, in addition, a further attempt by Gabrielov [G] to extend these ideas to the higher Pontrjagin classes. But this is only partially successful in that there is a conceptual obstruction to carrying out the suggested procedure which resides in the fact that the topology of certain configuration spaces, in particular the rational homology thereof, is not at all well understood.

The point of the present paper is that much of the apparatus of the original Gabrielov-Gelfand-Losik work is needlessly complicated and obscures what is, at base, a relatively straight-forward geometric concept. The essentials of the methodology can, in fact, be transcribed into a framework that has been in the literature for forty years: the Cairns proof [Ca 1] [Ca 2] of the smoothability of PL 4-manifolds. For that matter it is not too much to say that a prescription for determining local formulae for p_1 is already implicit in Cairns' foundational work on smoothing theory. Recall that this work [which contained some minor lacunae subsequently repaired by J.H.C. Whitehead [W] considered combinatorially triangulated manifolds simplex-wise convex-linearly embedded in general position in a high-dimensional Euclidean space. The main question posed by Cairns was whether a transverse hyperplane field can be found; for the existence of such a field was shown to imply smoothability. (In more modern language, Cairns was essentially demonstrating that a vector-bundle reduction of the PL normal bundle of a PL manifold yields a smoothing.) For our purposes, the most salient fact is Cairns' discovery that there is no obstruction to obtaining a normal field over the 4-skeleton of the dual cell-structure. This fact reduces to a theorem about the path-connectedness of certain configuration spaces which was proved in [Ca 2].

We shall show in what follows that the rather explicit construction of transverse plane fields readily allows the local calculation of a real 4-cocycle representing the real Pontrjagin class p_1. In fact, by slightly amplifying the combinatorial data, we may in fact obtain an <u>integral</u> cocycle representing the integral p_1.

It should be emphasized that the formulae we obtain are quite similar in spirit to those of [GGL1].

Perhaps one should raise, at this point, the question of explicitness. The methods developed below for evaluating a cocycle representing p_1 are formulated in such a way as to involve appeal to an explicit transverse field over the dual 4-skeleton. While it is shown that the value of the cocycle on a typical 4-cell depends only on the restriction of the field on the boundary 3-sphere (where it can easily be constructed explicitly), the actual computation would seem to depend on having a specific field on the 4-cell itself. Cairns' construction is, essentially, an "existence proof" rather than an explicit algorithm. However, we show in the concluding section how this difficulty can be avoided in principle so that the computation can go forward quite constructively in the presence of the appropriate data. Moreover, our derivation is far more transparent than that of [GGL1] and, in particular, avoids the "hypersimplicial" formalism of that paper. The superfluous complexity of the Gabrielov-Gelfand-Losik approach is, as we have intimated, an artifact of its failure to exploit directly the work of Cairns on transverse fields. Finally, this outline, read in conjunction with [G], makes it clear why the attempt to extend this approach to higher Pontrjagin classes runs into difficulty.

§1. Transverse fields

Let M^n be a topological manifold embedded in R^{n+k}. Recall [Ca 1, Wh] that a linear k-plane P in R^{n+k} is said to be transverse to M^n at $x \in M^n$ provided that there exists an open neighborhood U of x in M^n such that for any two distinct points $u_1, u_2 \in U$, $u_1 - u_2 \notin P$. Here, subtraction denotes ordinary vector-subtraction in R^{n+k}. We let $G_{k,n}$ denote the usual Grassmannian of k-planes in R^{n+k}. A continuous assignment $F: M^n \to G_{k,n}$ such that $F(x)$ is transverse to M^n at x is called a <u>transverse field</u>. Clearly, if we view F as the classifying map of a k-dimensional vector bundle ν over M_n, ν is then a vector-bundle reduction of the stable TOP normal bundle of M^n. Cairns and Whitehead showed that the existence of such a transverse field implies that M^n is smoothable and, in fact, that it allows an isotopy of the embedding, pointwise arbitrarily small, to an embedding whose image is a smooth submanifold of R^{n+k}.

We next consider a closed n-form $\omega \varepsilon \Omega(G_{k,n})$ representing a stable real characteristic class for vector bundles (so, implicitly, $n \equiv 0(4)$). Consider, now, a combinatorially-triangulated oriented n-sphere Σ^n together with a distinguished equatorial (n-1)-sphere Σ^{n-1}, assumed to be a subcomplex. We also assume a PL embedding of Σ^n in R^{n+k} and a transverse k-plane field $\phi:\Sigma^n \to G_{k,n}$ (which is piecewise-smooth on the n-simplices of Σ^n). We orient Σ^n and distinguish one of the hemispheres into which Σ^{n-1} divides it as D^n_+, so that $\partial D^n_+ = \Sigma^{n-1}$.

We now define a real number $\eta = \eta(\Sigma^n, \Sigma^{n-1}, \omega, \phi)$ by:

$$\eta = \int_{D^n_+} \phi^* \omega \ .$$

Lemma 1. For a given embedding of Σ^n in R^{n+k}, η depends only on $\phi|D^n_-$, where D^n_- is the hemisphere of Σ^n opposite to D^n_+.

Proof. Since ϕ classifies a stably-trivial vector bundle (the normal bundle of a sphere in Euclidean space), it follows that $\int_{\Sigma^n} \phi^*\omega = 0$. Consequently, if ϕ and ϕ_1 are transverse fields which agree on D^n_-, we have:

$$\int_{D^n_+} \phi^*\omega = -\int_{D^n_-} \phi^*\omega = -\int_{D^n_-} \phi_1^*\omega = \int_{D^n_+} \phi_1^*\omega .$$

Lemma 2. For a given embedding of Σ^n, η depends only on $\phi|\Sigma^{n-1}$.

Proof. By the same reasoning as in Lemma 1 above, η depends only on $\phi|D^n_+$. Let ϕ,ψ now be transverse fields which agree on Σ^{n-1}. Let ϕ_+,ϕ_- denote restriction to D^n_+, D^n_- respectively, and similarly for ψ_+,ψ_-. Let $\eta(\alpha)$ abbreviate $\eta(\Sigma^n, \Sigma^{n-1}, \alpha,\omega)$ for any transverse field α. Then, by the preceding observations:

$$\eta(\phi) = \eta(\phi_+ \cup \psi_-) = \eta(\psi_+ \cup \psi_-) = \eta(\psi).$$

In like manner, we have the following:

Lemma 3. η depends only on the restriction to a neighborhood of Σ^{n-1} of the embedding and the transverse field ϕ.

Proof. Let $f_1\colon \Sigma^n \subset R^{n+k}$, ϕ_1 and $f_2\colon \Sigma^n \subset R^{n+k}$, ϕ_2 both be pairs consisting of an embedding and a transverse field. Assume first that f_1,ϕ_1 coincides with f_2,ϕ_2 on a neighborhood of D^n_-. By the reasoning of Lemma 1, the respective η's agree. Then, extending the reasoning of Lemma 2, we see that the η's still agree on the weaker assumption that (f_1,ϕ_1), (f_2,ϕ_2) merely agree on a neighborhood of Σ^{n-1} in S^n. Note that for this last part of the calculation, we may have to refer to a field transverse to an immersed, rather than embedded Σ^n. But this makes no difference, practically speaking.

Corollary. η depends only on the restriction of the embedding to a neighborhood of Σ^{n-1} and on $\phi|\Sigma^{n-1}$.

Proof. Given an embedding and two fields ϕ_1,ϕ_2 which agree on Σ^{n-1} it is easy to deform both ϕ_1,ϕ_2 to transverse fields ϕ_1', ϕ_2' such that on a collar neighborhood $\Sigma^{n-1} \times I$, $\phi_i'(x,t) = \phi_i(x)$, $x \in \Sigma^{n-1}$, and so that $\eta(\phi_i') = \eta(\phi_i)$ (for $i = 1,2$). Therefore $\eta(\phi_1) = \eta(\phi_2)$ since, by the lemma $\eta(\phi_1') = \eta(\phi_2')$.

From the discussion above, it follows that an invariant η may be defined by data consisting of:

an embedding i of $\Sigma^{n-1} \times I$;

a transverse field ϕ to the embedded $\Sigma^{n-1} \times I$;

a closed form ω on $G_{k,n}$

(Provided, of course, that it is understood that the embedding i and the transverse field ϕ extend, in some fashion, to an embedding and a transverse field on Σ^n. We also assume, obviously, that $\Sigma^{n-1} \times I$ is oriented.)

Thus, we shall revise our notation and speak of $\eta(i,\phi,\omega)$.

We may then note the following, understanding that in so doing, we identify Σ^{n-1} with $\Sigma^{n-1} \times \{\frac{1}{2}\} \subset \Sigma^{n-1} \times I$.

Let M^n be a compact oriented n-manifold with boundary $\partial M^n = \Sigma^{n-1}$; let $f: M^n \to R^{n+k}$ be an embedding (or even an immersion) of M^n which agrees with i on $\Sigma^{n-1} \times \{0,\frac{1}{2}\}$ (thought of as a collar of Σ^{n-1} in M^n. Let ψ be a transverse field to the embedded M^n which agrees with ϕ on $\Sigma^{n-1} \times \{0,\frac{1}{2}\}$. Let γ denote the real characteristic class represented by ω. Finally, let M_+^n denote the closed manifold $M^n \; c\Sigma^{n-1}$.

Lemma 4. The characteristic number $\gamma[M_+^n]$ is given by the formula $\gamma[M_+^n] = \int_{M_+^n} \psi^* \omega - \eta(i,\phi,\omega)$.

This is almost self-evident. The reader should note the formal analogy to the η-invariant of oriented Riemannian

(4j-1)-manifolds defined by Atiyah, Patodi, and Singer [APS]. The idea is that Riemannian data has now been replaced by geometric data consisting of the embedding i and the transverse field.

Finally, we develop a slight extension of these ideas. Rather than an (n-1)-sphere Σ^{n-1} and an embedding $\Sigma^{n-1} \times I \subset R^{n+k}$, consider a (j-1)-sphere Σ^{j-1} and an embedding $i: \Sigma^{j-1} \times I \times D^{n-j} \subset R^{n+k}$. ϕ will now be a transverse field to this embedding, and ω becomes a j-form on $G_{k,n}$. It is understood that i and ϕ extend, in some fashion, to $\Sigma^{j} \times D^{n-j}$.

Then $\eta(i,\phi,\omega)$ is defined as $\int_{D^j_+} \psi^*\omega$, where D^j_+ is identified with $D^j_+ \times \{0\}$ in $\Sigma^j \times D^{n-j}$. The statements analogous to Lemmas 1,2, and 3 are then easily proved.

§2. Real cocycles representing p_1

We now consider explicitly the case of 4-dimensional characteristic classes, which reduces, in essence, to a discussion of p_1.

Let M^n be a PL triangulated manifold embedded in R^{n+k} so that each simplex is convex-linearly embedded. Suppose a k-plane field ϕ is defined on $M^{(4)}_*$ = 4-skeleton of the cell complex Poincaré dual to the given triangulation. We assume that for every $x \in M^{(4)}_*$, $\phi(x)$ is transverse to M^n at x. Let ω be a closed differential 4-form on the Grassmannian $G_{k,n}$ representing some real characteristic class γ (e.g. p_1 or p_1 of the complementary bundle). Define an oriented real 4-cochain c on M_* by stipulating for any oriented 4-cell e of M_*

$$c(e) = \int_e \phi^*\omega$$

Lemma 5. c is a cocycle representing $\gamma(M) \in H^4(M^n;R)$.

Proof. Let ν denote the PL normal (block) bundle of M^n. Quite clearly, the vector bundle over $M_*^{(4)}$ defined by ϕ constitutes a vector-bundle reduction of $\nu|M_*^{(4)}$ so that regarded as a cochain, and hence a cocycle, of $M_*^{(4)}$, c certainly represents $\gamma(\nu|M_*^{(4)})$. Since $H^4(M^n;R) \to H^4(M_*^{(4)};R)$ is monic it follows that we need merely show that c is a cocycle of M_* itself, for then its cohomology class must coincide with $\gamma(M)$. To see this, we need merely consider an arbitrary oriented 5-cell d of M_*. Since $\phi|\partial d$ is easily seen to represent the stable normal bundle of $\partial d \cong S^4$, it follows that $\int_{\partial d}\phi^*\omega = 0$. But $\int_{\partial d}\phi^*\omega = \sum_{e^4 \subset \partial d} \int_{e^4}\phi^*\omega = \sum_{e^4 \subset \partial d} c(e^4)$. Hence, $\delta c = 0$ and we are done.

We now recall the work of Cairns [Ca 1] and Whitehead [W] to remind the reader of how a transverse k-plane field ϕ on $M_*^{(4)}$ can always be constructed under the very weak assumptions

(1) The embedding of M^n in R^{n+k} is in general position, i.e., the images of the vertices of any star form a linearly independent set of $(n+k)$-vectors.

(2) Every star of an $(n-4)$-simplex is a Brouwer star, i.e., $st(\sigma^{n-4})$ embeds in R^n with the embedding convex linear on each simplex.

(It is well known that any combinatorially triangulated manifold admits a subdivision wherein all stars are Brouwer stars.)

If we consider an abstract Brouwer star of the form $\Delta^p * \Sigma^{n-p-1}$, where Σ^{n-p-1} denotes a triangulated $(n-p-1)$-sphere, we note that the property of being a Brouwer star is equivalent to the fact that the complex $c\,\Sigma^{n-p-1}$ embeds in

R^{n-p} so that the embedding is convex-linear on each simplex. We define the <u>configuration space</u> $CF(\Sigma^{n-p-1})$ to be the space (with the obvious topology) of all such embeddings (normalized so that the cone-point $* \to 0$) modulo the action of $GL(n-p,R)$. The role of configuration spaces in analyzing transverse fields is revealed by the following:

<u>Lemma 6 (see [W])</u>. Let the Brouwer star $\Delta^p * \Sigma^{n-p-1}$ be embedded in R^{n+k} in general position convex-linearly on simplices. Let N $G_{k,n}$ denote the set of k-planes P such that P is transverse to $\Delta^p * \Sigma^{n-p-1}$ at b = barycenter of Δ^p. Then N is homeomorphic to $CF(\Sigma^{n-p-1}) \times R^i$. where $i = nk - q(n-p)$ and q = number of vertices of Σ^{n-p-1}.

(By convention, if $p = n$, i.e., $\Sigma^{n-p-1} = \emptyset$ then $CF(\Sigma^{n-p-1}) = *$.)

Let us review Cairns' proof of the smoothability of 4-manifolds, assuming the crucial result that the existence of a transverse field implies smoothability. We need only analyze, in a rather straightforward way, the obstructions to obtaining a field transverse to a simplex-wise convex-linear, general position embedding of M^4_* in R^{4+k}.

First, to each simplex σ assign, in arbitrary fashion, a k-plane P_σ transverse to M^4 at the barycenter b_σ. Next, we try to extend this to a transverse field defined everywhere. An obvious fact is that if $x \in M^4$ and $\sigma(x)$ denotes the unique simplex such that $x \in \text{int}\,\sigma$, and if we let $N(x)$ denote the set of k-planes transverse to M^4 at x, then

$N(x) \cong N(b_{\sigma(x)}) \sim CF(\ell k \sigma)$.

Now, since $\ell k \sigma$ is of dimension ≤ 3, there are only a few cases we need analyze.

Lemma 7. If $\dim \ell k\sigma < 2$, i.e $\dim \sigma = 2,3$ or 4, then $CF(\ell k\sigma)$ is contractible; if $\dim(\ell k\sigma) = 2$, i.e., $\dim \sigma = 1$, then $CF(\ell k\sigma)$ is path connected.

The first part of the lemma is a triviality. The second part, however, is far from trivial; it represents the substance of a separate paper of Cairns [Ca 2].

Remark: As of this writing, it remains an open question whether $CF(\Sigma^2)$ is, in fact, contractible for a triangulated 2-sphere Σ^2. We may reformulate this question slightly by characterizing $CF(\Sigma^2)$ as the space of geodesic triangulations of the standard S^2 realizing the simplicial complex Σ^2 such that each simplex is contained in an open hemisphere and such that one particular 2-simplex, σ^2, is realized in a fixed way. Block, Conolly and Henderson [BCH] have proved the following: Let K^2 be a subdivision of the standard 2-simplex such that K triangulates the boundary S^1 in the standard way. Let C denote the set of simplex-wise convex linear homeomorphisms $K^2 \to \Delta^2$ which are the identity on the boundary. Then C is contractible. This can be read as strong evidence for a positive answer to the stated open question.

Returning to Cairns' proof of smoothability for 4-manifolds, we exploit Lemma 7 in the following way. Consider the first barycentric subdivision K of the given triangulation of M^4.

We wish to construct a transverse field ϕ on M such that $\phi(b_\sigma) = P_\sigma$ for simplices σ of the original triangulation. The assignment $b_\sigma \longmapsto P_\sigma$ defines ϕ on the 0-skeleton of K. Now consider a 1-simplex τ of K; extending ϕ to

τ is tantamount to finding a path between two points in $N(b_\sigma) \sim CF(\ell k\sigma)$, where σ is the smallest simplex of the original triangulation such that $\tau \subset \sigma$. Lemma 7 guarantees that we can do this, the most difficult case occurring when $\dim \sigma = 1$. Proceeding to the s-skeleton we must, for any 2-simplex τ of K, find a way of extending ϕ, now defined on $\dot{\tau}$ to all of τ. Again, with σ the smallest original simplex containing τ, this is a question of contracting a loop in a space homotopy equivalent to $CF(\ell k\sigma)$; but since $\dim \sigma \geq 2$, Lemma 7 tells us that $CF(\ell k\sigma)$ is contractible. We continue in like manner to define ϕ on the 3-skeleton and then the 4-skeleton of K, which is to say, all of M. Hence as asserted, a transverse field does exist.

Transposing this argument to the more general context of triangulated n-manifolds (for arbitrary n), we see that exactly the same procedure works to construct a transverse field over the union of all simplices τ of the first subdivision such that the smallest original simplex σ containing τ satisfies $\dim \sigma \geq n-4$. In other words, the method works to construct a transverse k-plane field ϕ over the 4-skeleton $M_*^{(4)}$ of the cell-complex M_* Poincaré dual to the original triangulation.

Given now a closed 4-form $\omega \in \Omega(G_{k,n})$ whose deRham class is the real characteristic class γ, it follows from Lemma 5 that the real 4-cochain C defined on oriented 4-cells e of M_* by:

$$c(e) = \int_e \phi^* \omega$$

is a cocycle representing $\gamma(M)$.

We may now, without loss of generality, make the following assumptions about the plane-field ϕ. Let e be a 4-cell of M_* with boundary ∂e. Then, typically, e is Poincaré dual to an (n-4)-simplex σ and is identified with a particular subspace of $b_\sigma * \ell k(\sigma) \subset st(\sigma)$. Thus, e naturally has the structure of a cone , viz., $c\partial e$ where b_σ corresponds to the cone point. Let C_e^+ denote a collar neighborhood of ∂e in $c\partial e = e$. We can assume that, for $x \varepsilon C_e^+$, $\phi(x) = \phi(px)$, where p denotes projection of C_e^+ upon ∂e.

Now, let C_e^- denote a collar neighborhood of ∂e in $(b_\sigma * \ell k\sigma)$ - int e. It is then easily seen that ϕ may be extended to a field ϕ_1 defined on $e \cup C_e^-$, again by letting $\phi(x) = \phi(px)$ for $x \varepsilon C_e^-$. Now it is quite obvious that the embedding of the 4-ball $e \cup C_e^-$ into R^{n+k} extends to an embedding of a sphere $\Sigma^4 = e \; C_e^- \cup c\partial_1$ (where ∂_1 denotes the copy of ∂e bounding $e \cup C_e^-$, at least if k is large enough. If we consider further the product neighborhood of $e \cup C_e^-$ in M^n, it is clear that this embedding of a 4-ball $\times D^{n-4}$ extends to an embedding $i: \Sigma^4 \times D^{n-4} \subset R^{n+k}$. As for ϕ_1, this extends to a k-plane field $\overline{\phi}$ on Σ^4 transverse to the embedded $\Sigma^4 \times D^{n-4}$. Thus, we are in the situation alluded to at the end of §1, and, reverting to the notation of that section, we have $c(e) = n(i, \overline{\phi}, \omega)$. It follows that $c(e)$ depends only the embedding of M in a neighborhood of ∂e^4 and on $\phi | \partial e$.

Our main objective, be it recalled, is to characterize a local formula for γ. Thus, taking ω, for the moment, as a given, we need to look a bit more closely at the actual construction of ϕ on ∂e. Taking e, as usual, to be the cell dual to a simplex σ, we see that $\phi | \partial e$ has been specified in the following way according to the procedure devised by Cairns:

(1) For each simplex ρ with $\rho > \sigma$ we have picked a point in $CF(\ell k\rho)$ (which yields a particular k-plane P transverse to M^n at b_ρ lying within the $(n+k-\dim \rho)$-plane comprised of all vectors perpendicular to τ.)

(2) For $\dim \rho = n, n-1, n-2$ P can, in fact, be chosen canonically. That is, for ρ of dimension n, the obvious choice is to make P the k-plane perpendicular to ρ; for $\dim \rho = n-1$ we take P to be the k-plane determined by the "obvious" configuration of the cone on the 0-sphere in R^1, viz., the two points of S^0 at $-1,1$ respectively with the cone point at 0. Finally, for $\dim \rho = n-2$, the choice of configuration of $c\ell k\rho$ is almost equally obvious; we embed $c\ell k\rho$ in R^2 as a regular polygon inscribed in the unit 2-disk.

(3) Note, in contrast to the foregoing, that, for $\dim \rho = n-3$, the choice of P, i.e. of an element in $CF(\ell k\rho)$, is not, in any obvious way, canonical. We must therefore rest content with an arbitrary choice.

(4) To complete the construction of ϕ over ∂e we must now choose contractions of the configuration spaces $CF(\ell k\rho)$ to the aforementioned canonical points for ρ of dimension n, n-1, n-2. Fortunately, these choices are also canonical, or very nearly so. To remove lingering ambiguities, it is helpful to make use of the notation of local ordering (introduced in [L-R]).

Definition. A local ordering on a locally-finite simplicial complex K is a partial ordering on the vertices of K such that the vertices of $st(\sigma)$ are linearly ordered for any simplex σ.

We will assume, henceforth, that the triangulations we work with are locally ordered in this sense. I.e., the linear order on each star will be assumed as part of the local data, in addition to the underlying combinatorial data, per se.

With the assumption that $st(\rho)$ is linearly ordered, it is easy to construct canonical contractions of $CF(\ell k\rho)$ for $\dim \rho = n, n-1, n-2$. This may be done trivially in the first two cases. For the case $\dim \rho = n-2$, we may view $CF(\ell k\rho)$ in the following way: Think of $CF(\ell k\rho)$ as the space of all simplex-wise linear embeddings of the triangulated disc $c\ell k\rho$ in R^2 which put the cone point at 0 and which take the "earliest" simplex of $\ell k\rho$ to an edge of the standard regular j-gon (j = # of vertices of $\ell k\rho$). Specifically, "earliest" means with respect to the induced lexicographic ordering on pairs of vertices. We choose the "standard" regular j-gon to have one edge lying in the right half plane and with the ususal x-axis as perpendicular bisector. If (v_0,v_1) is the earliest edge of $\ell k\rho$ ($v_0 < v_1$, in the given ordering), v_0 is assigned to the endpoint of the standard edge lying below the x-axis, and v_1 to the endpoint above the x-axis. The point is [see W] that any element $a \in CF(\ell k\rho)$ is represented by one and only one embedding with this property; thus the space of all embeddings with this property is, in fact, identical to $CF(\ell k\sigma)$. Thus, the "canonical" element of $CF(\ell k\rho)$ is the one which, subject to this condition, embeds $\ell k\rho$ as the standard regular j-gon. Now, if f is some arbitrary embedding satisfying the given constraint, then we form a one-parameter family of embeddings connecting f and the canonical embedding

Clearly we can slide $f(v)$ to $s(v)$ by first sliding it along a circle of radius $|f(v)|$ in the proper angular direction until it is radially in line with $s(v)$ and then sliding radially until it coincides. Do the "angular slide at uniform angular velocity for $0 \leq t \leq \frac{1}{2}$ and the "radial" slide at uniform linear velocity for $\frac{1}{2} \leq t \leq 1$. Doing this simultaneously for all v deforms f through configurations to the standard configuration.

This is jointly continuous in f and t and thus yields the desired contraction of $CF(\ell k\rho)$.

We may conclude, on the basis of (1)-(4) above that the only data needed to specify a cocycle c representing γ are

(1) The local ordering of M

(2) The form ω

(3) The choice of an element of $CF(\rho)$ for $(n-3)$-dimensional simplices ρ.

Clearly, ω being assumed, $c(e)$ will only depend on the data on $st(\sigma)$, σ dual to e. That is, we need only know the linear order on $st(\sigma)$, the embedding of $st(\sigma)$ in R^{n+k}, and the choice of an element in $CF(\ell k\rho)$ for $\rho^{n-3} \subset st(\sigma)$.

In order to obtain a purely local formula, i.e. one depending on the structure of $st(\sigma)$ as a simplicial complex, and on that alone, there are a number of simplifications available. First, we can take the form ω to be invariant under the action of $O(n+k)$ on $G_{k,n}$. Moreover, we might as well assume that we have chosen $\omega \in \Omega^4(G_{k,n})$ consistently for all n,k. This means that in the natural double sequence

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \\
\cdots & \dashrightarrow & G_{k,n} & \dashrightarrow & G_{k+1,k} & \dashrightarrow & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \dashrightarrow & G_{k,n} & \dashrightarrow & G_{k+1,n+1} & \dashrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

the choice of ω is consistent with pullback on Ω^4.

Next, given M^n merely as a triangulated manifold, we may pick an embedding $M^n \subset R^{n+k}$ by embedding M^n as a subcomplex of the standard simplex $\Delta^{n+k-1} \subset R^{n+k}$. This amounts to assigning vertices v of M^n to standard basis vectors of R^{n+k} in arbitrary fashion and then extending convex-linearly to a map on M^n.

Having done this, we obtain a transverse k-plane field on $M_*^{(4)}$ upon choosing a local order for M and choosing, for each σ^{n-3}, an element in $CF(\ell k\sigma)$.

If we have proceeded as above it is clear that we obtain a 4-cocycle c such that, for a typical dual 4-cell e, $c(e)$ depends only on the combinatorial structure of $\ell k\sigma$ (σ dual to e), on the linear ordering of $st\ \sigma$, and on the choice of an element in $CF(\ell k\tau)$ for all τ^{n-3} with $\sigma < \tau$. Thus we may write $c(e)$ as a function of such data without reference to M^n per se.

Now we may write down a formula:

$$\overline{c}(e) = E(c(e))$$

where E denotes expected value over all choices of a linear order on $st\ \sigma$ and all choices of configurations $\tau^{n-3} \dashrightarrow \alpha \in CF(\ell k\tau)$. Here we are implicitly assuming that the manifold $CF(\ell k\tau)$ has the natural structure of a measure space; but this is not at all hard to justify.

The following result is immediate:

Theorem 1. Let M^n be a combinatorially-triangulated manifold. Then the assignment $e \to \overline{c}(e)n$ is well-defined on oriented 4-cells and the resulting oriented cochcain with real coefficients, $\overline{c}(M)$, is a cocycle representing the characteristic class $\gamma(M) \in H^4(M;R)$.

So, in particular, if γ designates the first (tangential or normal) Pontrjagin class, we see that Theorem 1 gives us a local formula, one which is, the author trusts, less obscure than that in [GGL1].

§3. A local-ordered formula for the integral p_1.

In the foregoing section, we constructed a reasonably explicit local formula for the real first Pontrjagin class. On the other hand, the work of the author and C. Rourke [L-R] gives theoretical grounds for asserting the existence of a local formula for the integral p_1, provided that the local data is now understood to encompass a linear order of stars. I.e., we should expect to find, for any triangulated locally-ordered manifold M^n, an oriented integral 4-cocycle g representing the integral (tangential or normal) p_1 such that, given an an oriented dual 4-cell e, $g(e) \in Z$ depends only on the combinatorial type of the complex $st(\sigma^{n-4})$ (σ^{n-4} dual to e) and on the linear order on $st(\sigma)$.

The claim is that the ideas of the previous sections can be somewhat extended in order to create just such a formula, which we call a local-ordered formula. It is slightly unfortunate that in order to create it, we must restore some of the arbitrary choices that were "averaged away" at the end of the previous section. That is, given an oriented dual 4-cell e, in order to know the value of the formula on e one would have to know not only the a linear order on $st(\sigma)$ but, as well, a specific choice of configuration $\alpha(\tau) \in CF(\ell k\tau)$ for every $\tau^{n-3} < \sigma$. Therefore, in a strict sense, we only have a local ordered formula for p_1 once we have made, a priori, a choice of $\alpha(\Sigma^2) \in CF(\Sigma^2)$ for every possible triangulated 2-sphere. Thus we create not one but rather a multiplicity of local-ordered formulae, one for each such assignment.

The basic idea, once more, is to call upon Cairns' work on construction of transverse fields over $M_*^{(4)}$. But rather than using such a field to pull back a 4-form from $G_{k,n}$ to be integrated over 4-cells, we use it to define certain intersection numbers which are, perforce, integers.

As a preliminary step, we recall the work of Thom [T] on dual characteristic cycles in Grassmannians. Consider once more the Grassmannian $G_{k,n}$ of linear k-planes in R^{n+k}, $k > n$. Let Q be an arbitrarily chosen linear n-plane in R^{n+k}. We then have defined a certain subset $V(Q) \subset G_{k,n}$ by

$$V(Q) = \{P \in G_{k,n} | \dim(P \cap Q) \geq 2\}.$$

$V(Q)$ is a manifold of dimension $nk-4$ (i.e., of codimension 4) away from certain low-dimensional singularities. Moreover $V(Q)$ is a naturally-co-oriented cycle. That is to say, the normal bundle in $G_{k,n}$ of the non-singular part of $V(Q)$ has a natural 4-dimensional integral Thom class p. For a given k-plane bundle ξ over an arbitrary C•W complex K classified by $f:K \to G_{k,n}$, $p_1(\xi)$ may be computed as follows: assume f is in general position, so, in particular, $f(K^{(3)}) \cap V(Q) = \emptyset$. Then, for any oriented 4-cell e of K, the intersection number $d(e) = f(e) \cdot V(Q)$ is well-defined, using the natural co-orientation of $V(Q)$ as well as the orientation of e. The assignment $e \longmapsto d(e)$ is an oriented cocycle of M^n, and $p_1(\xi)$ is its integral-cohomology class.

We now update some ideas from §1. Let $\Sigma^4 \times D^{n-4}$ be embedded in R^{n+k}; Σ^3 is taken to be the equatorial sphere; i.e., $\Sigma^3 = D^4_- \cap D^4_+$, $\Sigma^4 = D^4_- \cup D^4_+$. As before, we assume that $\Sigma^4 \times D^{n-4}$ admits a transverse k-plane field ϕ. We make a choice of reference n-plane $Q \subset R^{n+k}$ and assume ϕ is in general position, viz, $\phi(\Sigma^3)$ $V(Q) = 0$. We then obtain an integer

$$\eta = \phi(D^4_+) \cdot V(Q).$$

In analogy to the work of §1 where integrals of forms, rather than intersection numbers, were used to define η, we have:

<u>Lemma 8</u>. η depends only on the embedding and the field ϕ on a neighborhood of Σ^3.

In view of Thom's result, we may easily obtain the following consequence.

Let M^n be a triangulated manifold embedded simplex-wise convex-linearly in R^{n+k} in general position. For each simplex σ of codimension ≤ 4, pick an element $\alpha(\sigma) \in CF(\ell k\sigma)$. Assuming a local ordering of M^n, follow the procedure of §2 to obtain, over $M^{(4)}_*$, a transverse field ϕ to M^n. Now let Q be a generic n-plane in R^{n+k}, i.e., one chosen such that $\phi(\partial e) \cap V(Q) = \emptyset$, for any dual 4-cell e. We then define

$$d(e) = \phi(e) \cdot V(Q).$$

Essentially, $d(e)$ is the value of η determined by the embedding and the field ϕ in a neighborhood of ∂e, as well as by the reference plane Q.

Lemma 9. d is an integral 4-cocycle whose cohomology class is the first integral Pontrjagin class p_1 of the stable normal bundle of M^n.

The proof proceeds much as in the case of Lemma 5.

Remark: At this point, we could, effectively, claim to have obtained another local formula for the real first Pontrjagin class. Again, the formula would emerge through an averaging procedure, i.e., by choosing a "standard" embedding of M^n and by averaging over the set of reference planes Q as well as the ordering and configuration data which lead to the explicit construction of ϕ. Details are omitted.

Comparison with [GGL2] suggests that this formula might be replacable by one which involves only a finite averaging procedure, rather than one expected value over a measure space. This we leave as a conjecture, observing only that the key point seems to be this: Pick a reference plane Q and configuration data at the simplices $\tau > \sigma$, (σ^{n-4} dual to e^4). We see that $d(e)$, provided that it is well defined (i.e., provided that the field $\phi|\partial e$ determined by configuration data has image disjoint from $V(Q)$) remains constant under small perturbations of Q and of configuration data. This would seem to suggest that averages (perhaps weighted) should be taken over connected components of the set of all possible choices for Q and for configuration data, rather than an expected value whose determination involves an integral. Whether this would, in practice, represent an actual computational improvement is not clear.

It is now our purpose to sharpen Lemma 9 to the extent of obtaining a local formula for the integral (normal) first Pontrjagin class which disregards all data save the local ordering, the combinatorics of links of $(n-4)$-simplices and the choice of configuration for links of $(n-3)$-simplices.

Our first observation is directed towards eliminating the role of the arbitrarily-chosen reference plane Q from our existing formula.

Lemma 10. Let ϕ be a transverse k-plane field to M^n R^{n+k}. Let $q: M^n \to G_{n,k}$ be a homotopically trivial map. Let $V(q) \subset M^n \times G_{k,n}$ denote the set $\{(x,P) \mid \dim(P \cap q(x)) \geq 2\}$. Let Φ be the section of $M^n \times G_{k,n} \xrightarrow{\text{proj.}} M^n$ induced by ϕ, i.e., $\Phi(x) = (x, \phi(x))$. Finally, assume Φ is in general position with respect to $V(q)$. Then $p_1(\nu M^n)$ is the primary obstruction in $H^4(M^n;Z)$ to deforming Φ off $V(q)$.

The proof is trivial. For the case of q = the constant map $q(x) = Q \in G_{n,k}$, the lemma is merely a slightly roundabout statement of Thom's result. But if q_1, q_2 are homotopic maps then $V(q_1)$ and $V(q_2)$ are homologous cycles in $M^n \times G_{k,n}$ from which it readily follows that the primary obstructions to deforming Φ off $V(q_1)$ and $V(q_2)$ respectively must coincide. The lemma is then immediate.

We may paraphrase Lemma 10 as follows: Assume that Φ is in general position with respect to $V(q)$ in the sense that $\Phi(M_*^{(3)})$ $V(q) = \emptyset$ and $\Phi|e^4$ is transverse to $V(q)$, for any oriented dual 4-cell e^4. Then the intersection number $d(e) = \phi(e) \cdot V(q)$ is well-defined if e be oriented. (In fact, strictly speaking, we need not assume $\Phi|e^4$ is transverse to $V(q)$ to have this intersection number well defined.) The assignment $e \longrightarrow d(e)$ is then an oriented integral cocycle representing the first Pontrjagin class $p_1(\nu M^n)$.

We may even go further: the observation above remains valid provided merely that Φ arises from a transverse field ϕ defined over $M_*^{(4)}$.

We now construct a locally-determined map $q: M_n \to G_{n,k}$, using only data provided by the local ordering, so that q will turn out to be globally trivial. In fact, we define a frame-field over M^n. First consider the barycentric subdivision of M^n, and a typical vertex b_σ (= barycenter of σ) where σ is a simplex of the initial triangulation. We let $q(b_\sigma)$ be defined as the n-frame $(v_1(\sigma)\cdots v_n(\sigma))$ where $v_1(\sigma)\ldots v_n(\sigma)$ represent the earliest vertices of $st(\sigma)$ in the presumed ordering. To define q more generally, recall that the generic point x of M^n may be uniquely denoted

$$x = \alpha_1 b_{\sigma_1} + \cdots + \alpha_s b_{\sigma_s}$$

where $\alpha_i > 0$, $\Sigma\alpha_i = 1$. (I.e., $b_{\sigma 1} \ldots b_{\sigma s}$ are the vertices of the unique simplex of the barycentric subdivision of M^n which contains x as an interior point; thus $\sigma_1 < \sigma_2 \cdots < \sigma_s$).

Let $v_j(x) = \Sigma\alpha_i v_j(\sigma_i) \in R^{n+k}$ for $j = 1,2 \ldots n$.

<u>Lemma 11</u>. For any x, $\{v_1(x),\cdots,v_n(x)\}$ is a linearly independent set.

<u>Proof</u>. Suppose there is a dependence relation:

(i) $$c_1 \cdot v_1(x) + \ldots + c_n \cdot v_n(x) = 0$$

for some x and some set of coefficients $c_1 \cdots c_n$ not all 0,

Let us remember that $v_i(x)$ is a linear combination of vertices of $st(\sigma_i)$. Since $st(\sigma_1) \supset st(\sigma_2) \cdots \supset st(\sigma_s)$ it follows that the left side of (i) may be rewritten as a linear combination of the vertices of $st(\sigma_1)$. But since this set, by the general position assumption concerning the embedding $M^n \subset R^{n+k}$, is linearly independent, we see that in this reformulation all coefficients must be 0.

Consider the earliest coefficient in (i) which is nonzero; call it c_j. Consider now the vertex $v_j(\sigma_1)$. In general, for any p, $v_p(\sigma_i) > v_p(\sigma_1)$ in the order on $st(\sigma_1)$. We use this fact to examine the coefficient of $v_j(\sigma_1)$ in the reformulation of the right-hand side of (i) in terms of the vertices of $st(\sigma_1)$.

First of all,

$$c_j v_j(x) = c_j \cdot \alpha_1 v_j(\sigma_1) + c_j \alpha_2 v_j(\sigma_2) \cdots$$

Since $c_j \neq 0$, $\alpha_i > 0$, it follows that $c_j \cdot v_j(x) = A \cdot v_j(\sigma_1) +$ {other terms} where the additional terms do not involve $v_j(\sigma_1)$ and $A \neq 0$. On the other hand, consider the term $c_p \cdot v_p(x) = c_p \cdot \Sigma \alpha_i \cdot v_p(\sigma_i)$ for $p > j$. Now $v_p(\sigma_i) \geq v_p(\sigma_1) > v_j(\sigma_1)$. So $c_p \cdot v_p(x)$, rewritten as a linear combination of vertices of $st(\sigma_1)$, does not involve $v_j(\sigma_1)$. Thus $c_1 v_1(x) \cdots c_n \cdot v_n(x) = A \cdot v_j(\sigma_1) + B$ where B does not involve $v_j(\sigma_1)$, $A \neq 0$. But this contradicts the presumed linear independence of the vertices of $st(\sigma_1)$. Hence the lemma follows.

Thus we may define $q(x)$ for any x as the n-frame $(v_1(x) \cdots v_n(x))$. By slight abuse of notation, we also use q to denote the map $M^n \to G_{n,k}$ defined by $x \longmapsto \mathrm{span}(v_1(x), \cdots, v_n(x))$. Clearly, q represents a trivialized bundle and is thus null-homotopic in $G_{n,k}$.

Consider once more the k-plane field ϕ transverse to M^n defined over $M_*^{(4)}$ by data consisting of a local order on M^n and configuration data for simplices of dim $\geq$ n-4. We have, simultaneously, the n-frame field q. Thus, by the remarks following Lemma 10, we shall have a well-defined cocycle d(e) (depending on the embedding as well as the afore-mentioned data) simply provided that $\Phi(M_*^{(3)}) \cap V(q) = \emptyset$. We claim that this last condition will hold for a generic choice of configuration data, though the proof will be omitted.

As to dependency on the embedding, this may be eliminated simply by picking any "standard" embedding of M^n as a sub-complex of Δ^{n+k-1}, i.e., by assigning each vertex of M to an element of the standard basis for R^{n+k} and then extending convex linearly. It turns out that d(e) is independent of the particular embedding, and so d(e) is now seen to be intrinsic, depending only on the local ordering and the configuration data.

We may emphasize the intrinsic nature of the formula we have by freeing it entirely from the notion of embeddings and transverse fields.

Consider the typical dual 4-cell e^4 dual to σ^{n-4}. Decompose it into sub-cells $\{d_\tau\}_{\sigma \leq \tau}$, defined by $d_\tau = st(b_\tau, M'') \cap e^4$ where st(,M") refers to the combinatorial star in the second barycentric subdivision of M.

Omitting details, we note that we may slightly modify the definition of the frame-field q so that for $x \in d_\tau$, $v_i(x)$ will lie in the vector subspace spanned by the vertices of st(τ). We also assume: $\phi(x)$ is transverse to M at b_τ if $x \in d_\tau$. These modifications in the definitions of ϕ and q are relatively simple to make and leave d(e) unaffected.

Concentrating on $d_\tau \cap e$, we let $\overline{\phi}(x)$ denote the $CF(\ell k\tau)$-coordinate of $\phi(x)$. For convenience, we shall think of $\overline{\phi}(x)$ as an embedding $\overline{\phi}(x)$: $st\tau, b_\tau \subset R^n, 0$ where the embedding is convex linear on simplices and is the "standard embedding" on δ^n where δ^n denotes the n-simplex of $st\ \tau$ smallest with respect to the lexicographic ordering induced by the linear ordering of the vertices of $st\ \tau$. In fact, rather than using the standard R^n, we think of the n-plane P parallel to δ^n passing through the origin. So therefore, if we consider the direct-sum decomposition of R^{n+k} as $P \oplus \phi(x)$, the configuration of $st\ \tau$ in P comes merely by projection π_P onto the P-factor of $t-b_\tau$ for all vertices t.

Having oriented e and therefore d_τ we may define an integer $g(e,\tau)$ as the algebraic number of points $x \in d\tau$ such that

(ii) $\dim \text{span } (\pi_P v_1(x) \ldots \pi_P v_n(x)) < n-2.$

The point is that given the configuration of $st\ \tau$ in P we can determine π_P in purely algebraic terms (without reference to the embedding or the normal field). For these purposes, we may, without loss of generality assume that R^{n+k} is spanned by the vertices $\{t\}$ of $st\ \tau$. Now, from δ^n pick the earliest vertices among the t's (label them $u_1,\ldots,u_n$) so that $\{y_i\} = \{u_i - b_\tau\}$ spans P. Label the remaining vertices $w_1,\ldots,w_k$. Consider $z_i = w_i - \overline{w}_i - b_\tau$ where $\overline{w}_i$, a linear combination of y's, is the image of w_i under the configuration embedding $\overline{\phi}(x)$. Let B be the matrix $(y_1,\ldots,y_n, z_1\ldots z_k)$ (writing vectors as columns with respect to the ordered basis $t_1,\ldots,t_{n+k}$). Let $Q = B^{-1}$, and set = first n rows of Q. Write $R_1 = \binom{R}{0}$ an $n \times n$ matrix so R_1

is a projection expressed in y,z coordinates and $BR_1 = \Pi$ is projection π_p expressed in t-coordinates. Π, of course, depends on x. So $g(e,\tau)$ is the algebraic number of points x such that

(iii) $\quad$ rank $\Pi(x)(v_1(x),\ldots,v_n(x)) \leq n-2$.

Now define $\eta(e)$ by

$$\eta(e) = \sum_{\tau} g(e,\tau)$$

Theorem. The assignment $e \mapsto \eta(e)$ is an integral oriented cocycle representing the first integral normal Pontrjagin class $p_1(\nu M)$.

We omit an explicit proof, but we do note that the formula for $\eta(e)$ represents, essentially, a computation of what we previously denoted as $d(e)$. This computation is a rather routine working out of consequences of the characterizations given in [Ca 1] and [W] of the space of k-planes transverse to M at a point in terms of configuration spaces.

This formula should be compared with the formula of [GGL].

There remains the slight problem of deriving a formula for the integral "tangential" first Pontrjagin class. Suffice it to say that the same general approach will work. I.e., in constructing a "normal" field over $M_*^{(4)}$--the transverse k-plane field ϕ--the Cairns procedure simultaneously constructs a "tangent" plane field $\phi^{\perp}$. By the same token, q must be replaced by a locally determined trivial k-plane bundle.

§4. Computational considerations

In the various local formulae we have exhibited above, there remains the problem of explicitness. That is, we have shown that given, say, an embedding of M^n in R^{n+k}, together with a local ordering a choice of configuration data at simplices of dimension $\geq n-4$, we may construct a field transverse to M^n over $M_*^{(4)}$, from which a cocycle representing p_1 (with real or integral coefficients) may be computed. Note, however, these contrasting facts: We have shown that the value of the cocycle on a dual 4-cell e depends only on the given data on ∂e. (Strictly speaking, it depends on data for simplices τ of dimension $\geq n-3$ to which σ, the dual simplex to τ, is incident.) Yet, in point of actual computation our formulas refer to data defined over all of e. Thus, by way of concrete example, if we adopt the approach of §2 and make use of a differential 4-form ω on $G_{k,n}$ we see that the value of the associated Pontrjagin cocycle on a typical e^4 is given as $\int_{e^4}\phi^*\omega$, even though this number is an invariant of ϕ restricted to ∂e^4. So, to compute this number, we should have to know ϕ explicitly on e^4, i.e., we should have to carry out the Cairns construction in detail using the detailed methods, as well as the actual results, of [Ca 2]. In some ways, the situation is analogous to what one finds for the Atiyah-Patodi-Singer η-invariant of an oriented Riemannian $4j-1$ manifold M. $\eta(M)$ is, of course, dependent only upon M but is in general quite difficult to compute merely in terms of M. However, if M is known to be the boundary of the Riemannian $4j$-manifold W (with the product metric near the boundary) then the value of $\eta(M)$ may be computed with relative ease.

In what follows, we indicate a computational alternative to specifying the transverse field ϕ explicitly on the interior of e^4. Again, for simplicity's sake, we work in the context of real coefficients and differential forms.

The general idea is that one wants to avoid the difficulties of making Cairns' construction of the field ϕ over the 4-cells explicit. What is proposed, instead, is to use configuration data at (n-4)-simplices, together with the explicitly-constructible transverse field ϕ over the dual 3-skeleton to construct a certain field F over 4 cells $\bar{e}$ (one for each dual 4-cell e) so that integration can be done over $\bar{e}$ in place of $\int_e \phi^*\omega$ with essentially the same result, albeit F is not to be understood as a transverse field to e. Rather, F arises from constructing a 1-parameter family of embedded 3-spheres bounding a one-parameter family of 4-cells so that, for each 3-sphere in the family, we have an explicit field extending, in principle, to a field over the corresponding 4-cell. The process begins with ϕ and ends with a constant field so we can, for computations sake, regard it as a field over a 4-cell. The integral we get therefore represents the difference between the invariant $\int_e \phi^*\omega$ defined by ϕ itself and the invariant one gets from a constant field over a "flat" 3-sphere, i.e. one which bounds a "flat" 4-cell. Since the latter is obviously 0, we will have computed the former. The point is that F is, in principle, directly constructible without appeal to Cairns' theorem on connectivity of configuration spaces of cones on α-spheres.

Once more, assume that the combinatorially triangulated manifold M^n is simplex-wise linearly embedded in R^{n+k} in general position. We assume further data consisting of a local ordering on the triangulation and a choice, for every simplex σ of dimension $\geq$ n-4, of an element in $CF(\ell k \sigma)$.

With this data in hand, as we have noted continually, we obtain, by a perfectly straightforward construction, a k-plane field ϕ transverse to M^n over $M_*^{(3)}$. This, of course, makes no use of the configuration data at (n-4)-simplices. Rather than concerning ourselves with extending the transverse field to $M_*^{(4)}$, however, we now make use of the remaining data in a different way.

Given an (n-4)-simplex σ, the choice of a configuration of $\ell k\sigma$ is essentially a choice of a k-plane P transverse to M^n at b_σ. Choose an n-simplex ρ of st σ, e.g., the smallest in the lexicographic order induced by the linear order on the vertices of st σ. It follows, then, that every vertex v of st σ may be represented uniquely (in the conventional (n+k)-vector notation) as: $v = b_\sigma + r_v + n_v$ where r_v is a vector in the linear n-plane Q parallel to ρ and $n_v \in P$. Since P is, as assumed, transverse to M^n at b_σ, it follows that the projection map $v \longmapsto b_\sigma + r_v$ is not only 1-1 but extends, in fact, by convex-linear extension, to an embedding st $\sigma \subset b_\sigma + Q$. Furthermore, if we let u_t be defined as the convex-linear extension of the map $v \longmapsto b_\sigma + r_v + (1-t)\cdot n_v$ for $0 \leq t \leq 1$, we obtain a simplex-wise linear embedding st $\sigma \subset R^{n+k}$. If $t < 1$, u_t is in general position; therefore by Lemma 6 for such t we may use the configuration data on simplices of dimension $\geq$ n-3 to construct a k-plane field ϕ_t on $u_t(\partial e^4) - u_t(\text{st } \sigma)$, where e^4 is the 4-cell dual to σ, such that $\phi_t(x)$ is transverse to $u_t(\text{st } \sigma)$ at $u_t(x)$. This process is continuous on $\partial e^4 \times [0,1)$. Moreover, $\lim_{t \to 1} \phi_t(x) = \phi_1(x)$ exists for $x \in \partial e^4$ and we obtain thereby a vector bundle on $\partial e^4 \times I$, in fact a map $\Phi: \partial e^4 \times I \to G_{k,n}$.

Finally, $\phi_1(x)$ is transverse to the n-plane $b_\sigma + Q$.

Now we may deform ϕ_1 further (regarding it as a map $\partial e^4 \to G_{k,n}$) to a trivial map, i.e., the constant map $x \longmapsto Q^\perp$. We think of this deformation (which may, of course, be constructed explicitly in terms of ϕ_1 without difficulty) as defining a map $\Psi: c\partial e^4 \to G_{k,n}$ with $\psi|\partial e^4 = \phi_1$. Thus concatenating with Φ, i.e., taking $F = \Phi \cup \Psi: \partial e^4 \times I \cup c\partial e^4 \cong c\partial e^4 \to G_{k,n}$, we get a map from a topological 4-cell to $G_{k,n}$. Call this 4-cell $\overline{e}$. Now let $\omega \in \Omega^4(G_{k,n})$ be, as in §2, a closed form representing the real Pontrjagin class p_1 in DeRham cohomology. We then have the following.

Theorem: Let $\overline{\phi}$ be any extension of ϕ to e^4 with $\overline{\phi}$ transverse to M. Then

$$\int_{e^4} \overline{\phi}^* \omega = \int_{\overline{e}'} F^* \omega .$$

We omit a proof. Suffice it to remark that what is being exploited here is a principle cognate to, though much simpler than, the well-known fact that, although the Atiyah-Patodi-Singer η-invariant of a $4j-3$ dimensional Riemannian manifold is not the integral of a locally-determined form, nonetheless the difference between the η-invariants of two Riemannian structures which differ by deformation can be represented as such an integral over the manifold $\times$ I.

As a final note, we observe that this method for computing a cocycle representing p_1 may easily be adapted to the integer coefficient case studied in §3 above, where intersection numbers are used in place of integrals.

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