SEMIFREE FINITE GROUPS ACTIONS ON COMPACT MANIFOLDS

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INTRODUCTION

One of the classical problems in transformation groups has been to study the properties of the stationary point sets of actions on manifolds, and to characterize them whenever possible. P. A. Smith theory in combination with various other topological considerations provide a number of necessary conditions to be satisfied by the stationary point sets of some restricted classes of actions. In the case of smooth actions of a compact Lie group G on a manifold W, the stationary point set, say F, is a manifold and its normal bundle in W, say ν, is a G-bundle which determines the action in a (tubular) neighborhood of F.

For a complete characterization (of the diffeomorphism type) of F, one needs to show that the above mentioned necessary conditions are sufficient as well, in the following sense. Assuming that the submanifold F of the prescribed manifold W, and the G-bundle ν given, one tries to find an action on W which would restrict to the given action in the tubular neighborhood of F provided by the G-bundle ν. Special cases of such problems have been considered under various circumstances by various authors: [J1], [J2], [A1], [A2], [A3], [A-B 1], [A-B 1], [L], [D-R], [S] to mention a few. In these and other related contexts, a common hypothesis is that W is simply-connected and this assumption is indispensable for the techniques and the arguments to be applicable.

In the following, we consider this and some other relevant questions in the case of non-simply connected compact manifolds on which a finite group G has a "simple semifree action," i.e. where action is free outside of the stationary point set, and a certain localized Borel construction becomes fibre homotopy trivial. Although semifree actions comprise a restricted class, their understanding seems essential in developing general theories with more complicated isotropy group structures. The further restriction of "simplicity" of actions has been imposed to bring the homotopy-theoretic constructions and algebraic

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calculations within reach, as well as to provide a satisfactory answer to the above-mentioned questions in the form of less-complicated necessary and sufficient conditions.

In the presence of the fundamental group of the ambient manifold — on which the desired G-action is to be constructed — much of the methods and results of the simply-connected cases (in their various forms and contexts) are inapplicable. Thus, one is led to construct a new obstruction group and a new invariant (depending on both $\pi_1 W$ and G) whose vanishing is one of the necessary conditions for the existence of such actions. The obstruction group fits in a five-term exact sequence relating various Whitehead groups, and conceivably it can be defined as the fundamental group of the fibre of a transfer map between two Whitehead spaces involved in the problem, although its definition given below is in purely algebraic terms. The above-mentioned invariant is related to a certain Reidemeister torsion-type invariant.

If $\pi = 1$, Wh_1^T becomes simply $\tilde{K}_0$. This functor takes into account the interaction between $\tilde{K}_0$ (the finiteness obstruction in the presence of G-actions) and Wh_1 (the Whitehead torsion involving the fundamental group $\pi_1 = \pi$) in a way which is necessary to study the above mentioned problems. Thus, in the geometric context, Wh_1^T plays the same role in the study of finite group actions on non-simply connected compact manifolds that $\tilde{K}_0$ does in the simply-connected case.

The organization of the paper is as follows. In Section I we introduce Wh_1^T and state some of its algebraic properties which are used subsequently to detect the (combined) finiteness and Whitehead torsion type obstructions as the image of a Reidemeister torsion type invariant. Section II illustrates some computations of Wh_1^T. (The details of the results in these sections will appear elsewhere.) Section III considers semifree simple actions and gives necessary and sufficient conditions for existence of simple actions in this context. The problem of characterization of the stationary point sets of simple semi-free actions on compact bounded manifolds and an extension theorem for free simple actions are reduced to the homotopy theoretic problem of constructing appropriate Poincaré complexes, which are carried out using mixing the localizations of diagrams of spaces involved. Section IV gives an indication of the proofs of the theorems of Section III. Section V gives some useful theorems on constructing free simple actions either by extending a given action on a subspace or by pulling back actions from a given space, thus formalizing and generalizing the constructions needed in Section III. Although these are non-simply connected versions of analogous results in [A2] and [A3] where free actions are constructed from homotopy actions on simply-connected spaces (which are not simple in general), there is little overlap in scope or the methods.

There is somewhat of an overlap between some of the results obtained independently by S. Cappell and S. Weinberger [CW] as well as S. Weinberger [W], P. Löffler [L], P. Löffler and M. Raußen [LR]. The papers of L. Jones [J] and F. Quinn [Qu] also deal with related problems.

SECTION 1. Let Λ be a ring and $P(\Lambda)$ denote the category of finitely generated projective Λ-modules. In the sequel, G will denote a finite group, and π a discrete group which denotes as well the subgroup $\pi \times \{1\} \subset \pi \times G$ for simplicity of notation. Consider the set $A = \{(P, B) \mid P \in P(\mathbb{Z}(\pi \times G)),\ B = \mathbb{Z}\pi\text{-basis for } P\}$. The operation of direct sum of modules and disjoint union of $\mathbb{Z}\pi$-bases in the given order gives A the structure of a monoid with neutral element $(0, \emptyset)$. We introduce the equivalence relation $(P, B) \sim (P', B')$ among the elements of A if there exists a $\mathbb{Z}(\pi \times G)$-linear isomorphism $\alpha : P \xrightarrow{\cong} P'$ such that $\tau_\pi(\alpha) = 0$ with respect to B and B', where $\tau_\pi(\alpha) \in Wh_1(\pi)$ is the Whitehead torsion. The set of equivalence classes $A' = A/\sim$ inherits the monoid structure of A, and contains the submonoid "of trivial elements"; namely, (P, B) represents a trivial element in A' if P is $\mathbb{Z}(\pi \times G)$-free, and B is induced by a $\mathbb{Z}(\pi \times G)$-basis. The quotient monoid A' modulo the submonoid of trivial elements is seen to be an abelian group and is denoted by $Wh_1^T(\pi \subset \pi \times G)$. We have an obvious homomorphism $\alpha : Wh_1^T(\pi \subset \pi \times G) \longrightarrow \tilde{K}_0(\mathbb{Z}(\pi \times G))$ induced by the forgetful map $(P, B) \longrightarrow P \in \tilde{K}_0(\mathbb{Z}(\pi \times G))$. There is a further homomorphism $\beta : Wh_1(\pi) \longrightarrow Wh_1^T(\pi \subset \pi \times G)$ which is induced by the operation of "twisting the standard basis;" namely, let $x \in Wh_1(\pi)$ be represented by $\phi : (\mathbb{Z}\pi)^n \longrightarrow (\mathbb{Z}\pi)^n$. After stabilization, we have a π-linear homomorphism $\phi \oplus id : (\mathbb{Z}(\pi \times G))^m \longrightarrow (\mathbb{Z}(\pi \times G))^m$. Let B be the image of the standard basis of $(\mathbb{Z}(\pi \times G))^m$ under the $\mathbb{Z}\pi$- linear map $\phi \oplus id$. Then B is a $\mathbb{Z}\pi$-basis for $(\mathbb{Z}(\pi \times G))^m$ and $((\mathbb{Z}(\pi \times G))^m, B)$ represents $\beta(x) \in Wh_1^T(\pi \subset \pi \times G)$.

1.1 Theorem. There is an exact sequence

$$Wh_1(\pi \times G) \xrightarrow{Tr} Wh_1(\pi) \xrightarrow{\beta} Wh_1^T(\pi \subset \pi \times G) \xrightarrow{\alpha} Wh_0(\pi \times G) \xrightarrow{tr} Wh_0(\pi)$$

in which Tr and tr are transfer homomorphisms and $Wh_0 \cong \tilde{K}_0$.

The homomorphism $\mathbb{Z}\pi \to \mathbb{Z}_q\pi$ induces a homomorphism $Wh_1(\pi) \longrightarrow Wh_1(\pi;\mathbb{Z}_q) \overset{def}{\equiv} K_1(\mathbb{Z}_q\pi)/\{\pm\pi\}$ where $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. One has a further map $\gamma : Wh_1(\pi;\mathbb{Z}_q) \longrightarrow Wh_1^T(\pi \subset \pi \times G)$ defined as follows. Let $GL_n'(\mathbb{Z}\pi)$ be the monoid of $(n\times n)$-matrices which have an inverse in $GL_n(\mathbb{Z}_q\pi)$. Given $\phi \in GL_n'(\mathbb{Z}\pi)$, one has an exact sequence

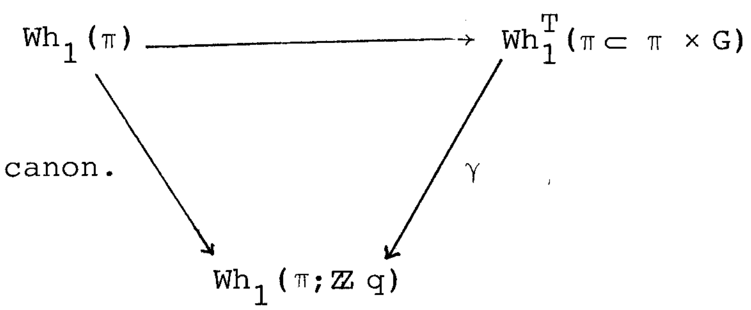
$$(C_*) : 0 \longrightarrow (\mathbb{Z}\pi)^n \xrightarrow{\phi} (\mathbb{Z}\pi)^n \longrightarrow M \to 0$$

Thus $M_q = M \otimes \mathbb{Z}_q = 0$. It follows that proj $\dim_{\mathbb{Z}(\pi\times G)} M \le 1$, and we may take a short projective resolution over $\mathbb{Z}(\Pi \times G)$ for M, where order $(G) = q$:

$$(C_*') : 0 \longrightarrow C_1' \longrightarrow C_0' \longrightarrow M \longrightarrow 0$$

such that C_1' is free and C_0' is projective over $\pi \times G$. There is a $\mathbb{Z}\pi$-linear chain homotopy equivalence $\zeta : C_* \to C_*'$. Since the finiteness obstruction of C_* over $\mathbb{Z}\pi$ vanishes, C_0' is stably trivial over $\mathbb{Z}\pi$ also. After stabliization, we choose $Z\pi$-basis for C_1' and C_0', say $\mathcal{B}_1'$ and $\mathcal{B}_2'$. If we choose the "standard bases $\mathcal{B}_1$ and $\mathcal{B}_0$ in the resolution (C_*) above for $C_1 \cong (\mathbb{Z}\pi)^n$ and $C_0 \cong (\mathbb{Z}\pi)^n$, then it is possible to arrange for the choices of $\mathcal{B}_1'$ and $\mathcal{B}_0'$ so that becomes a simple homotopy equivalence over $\mathbb{Z}\pi$. Let $\gamma(\phi) = [(C_1',\mathcal{B}_1')] - [(C_1',\mathcal{B}_1')]$ in $Wh_1^T(\pi \subset \pi \times G)$. In general, for $\phi \in GL_n(\mathbb{Z}_q\pi)$, we take $\phi = \frac{1}{s}\psi$, where $(s,q) = 1$. Then $s(Id) \in GL_n'(\mathbb{Z}\pi)$ and $\psi \in GL_n'(\mathbb{Z}\pi)$.
Let $\gamma(\phi) = \gamma(s(Id))$.

<u>I.2. Theorem</u>. γ induces a well-defined homomorphism such that the following diagram commutes



Suppose C_* is a chain complex over $\mathbb{Z}\pi$ such that $H_*(C_*\otimes\mathbb{Z}q) = 0$. Then the Reidemeister torsion of C_* is a well-defined element of $Wh_1(\pi;\mathbb{Z}q)$ and is denoted by $\tau(C_*)$. The main algebraic result of this section is the following:

<u>1.3. Theorem.</u> Let A_*' be a finite $\mathbb{Z}\pi$-based chain complex, and A_* be a finite $\mathbb{Z}(\pi\times G)$-based chain complex. Suppose there exists a $\mathbb{Z}\pi$-linear map $f:A_*' \longrightarrow A_*$ which is a $\mathbb{Z}\pi$-chain homotopy equivalence. Further, suppose $H_*(A\otimes\mathbb{Z}_q) = 0$ and that G acts trivially on $H_*(A)$, where order $(G) = q$. Then there is a finite $\mathbb{Z}(\pi\times G)$-based complex B_* and a $\mathbb{Z}(\pi\times G)$-chain homotopy equivalence $h:A_*\to B_*$ such that $hf:A_*\to B_*$ is π-simple if and only if $\gamma(\tau(A_*)) = 0$.

The above algebraic theory has the following application which is crucial in the construction of surgery problems of the next sections.

<u>1.4 Theorem.</u> Suppose we have a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \hookrightarrow & \tilde{X} \\ \beta\downarrow & & \downarrow\alpha \\ Y & \hookrightarrow & X \end{array}$$

with the following properties:

i) $\tilde{X},\tilde{Y}$ and Y are finite connected CW complexes, and X is a connected CW-complex.

ii) $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) = \pi, \pi_1(X) = \pi_1(Y) = \pi\times G$.

iii) $\tilde{Y}$ is a covering space of Y and α induces a homotopy equivalence from $\tilde{X}$ to the covering space of X with the fundamental group π.

iv) $H_*(\tilde{X},\tilde{Y};\mathbb{Z}_q[\pi]) = 0$ and the Reidemeister torsion of $(\tilde{X},\tilde{Y})$ is $\tau(\tilde{X},\tilde{Y})$ in $Wh_1(\pi;\mathbb{Z}_q)$.

v) G acts trivially on $H_*(\tilde{X},\tilde{Y});\mathbb{Z}[\pi]) = H_*(X,Y;\mathbb{Z}[\pi\times G])$.

Then there exists a homotopy equivalence from X to a finite complex Z such that the composite map $\tilde{X}\xrightarrow{\alpha}X\longrightarrow Z$ induces a simple homotopy equivalence from $\tilde{X}$ to a covering space of Z, if and only if $\gamma(\tau(\tilde{X},\tilde{Y})) = 0$.

<u>Indication of Proof</u>: Let us denote by $C_*(-;M)$ the cellular chain complex with (twisted coefficients M. We have a π-linear homotopy equivalence $f : C_*(\tilde{X},\tilde{Y};\mathbb{Z}\pi)\longrightarrow(C_*(X,Y;\mathbb{Z}[\pi\times G])$. If there exists such a Z, then we have a π-simple homotopy equivalence

$$C_*(\tilde{X},\tilde{Y};\mathbb{Z}\,\pi) \longrightarrow C_*(Z,Y;\mathbb{Z}\,[\pi \times G])$$

from a finite π-based complex to a finite $\pi \times G$-based complex. Hence by Theorem 1.3, $\gamma(\tau(\tilde{X},\tilde{Y})) = 0$.

Conversely, suppose that $\gamma(\tau(\tilde{X},\tilde{Y}))$ vanishes. Then there exists a finite $\pi \times G$-based chain complex B_* and a $\pi \times G$-homotopy equivalence g from $C_*(X,Y;\mathbb{Z}\,[\pi \times G])$ to B_* such that $g \circ f$ is π-simple. This implies that the finiteness obstruction of X vanishes and there exists a homotopy equivalence from X to a finite complex $\tilde{Z}_1$. Moreover, we can add 2-cells and 3-cells to $\tilde{Z}_1$ in order to modify the simple type of Z_1 to obtain a finite complex Z such that the composite map

$B_* \xrightarrow{g^{-1}} C_*(X,Y;\mathbb{Z}\,[\pi \times G]) \longrightarrow C_*(Z,Y;\mathbb{Z}\,[\pi \times G])$ is a $\pi \times G$-simple homotopy equivalence. It is easy to see that the composite map $\tilde{X} \to X \to Z$ induces a simple homotopy equivalence from $\tilde{X}$ to the covering space of Z with fundamental group π.

<u>SECTION II</u>. Let $A = \mathbb{Z}\,\pi$ and $\omega = \sum_{g \in G} g$ be the norm of G. For simplicity of notation, let $A[G]/\omega A[G] \equiv A[G]/\omega$, $A/qA \equiv A_q$, and $\mathbb{Z}/2\mathbb{Z} \times M \cong \{+1,-1\} \times M \equiv \pm M$ for any group M. Consider the cartesian diagram:

$$\begin{array}{ccc} A[G] & \xrightarrow{\quad h \quad} & A[G]/\omega \\ \big\downarrow f & & \big\downarrow \\ A & \longrightarrow & A_q \end{array} \qquad \text{(C)}$$

where f is the augmentation and all other homomorphisms are cannonically defined quotient morphisms. The associated Mayer-Vietories sequence is:

$$K_1(A[G]) \longrightarrow K_1(A) \oplus K_1(A[G]/\omega) \longrightarrow K_1(A_q) \longrightarrow K_0(A[G]) \longrightarrow$$
$$K_0(A) \oplus K_0(A[G]/\omega) \longrightarrow K_0(A_q) \qquad \text{(MV)}$$

Corresponding to (MV), one has the following exact sequence if $G \neq \mathbb{Z}_2$

(U) $0 \longrightarrow \pm H_1(\pi) \times H_1(G) \longrightarrow \pm H_1(\pi) \oplus \pm H_1(\pi) \times H_1(G) \longrightarrow \pm H_1(\pi) \xrightarrow{0} \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$

and if $G = \mathbb{Z}_2$, the sequence reads:

$0 \longrightarrow \pm H_1(\pi) \times H_1(G) \longrightarrow \pm H_1(\pi) \oplus H_1(\pi) \times H_1(G) \longrightarrow H_1(\pi) \xrightarrow{0} \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$

The sequences (U) and the corresponding homomorphisms are also obtained from the diagram (C). The sequence (U) admits an injective homomorphism into the sequence (MV) and the quotient sequence is the exact sequence of the Whitehead groups below:

$$Wh_1(\pi \times G) \longrightarrow Wh_1(\pi) \oplus K_1(A[G]/\omega)/\pm H_1(\pi) \times H_1(G) \longrightarrow Wh_1(\pi;\mathbb{Z}_q) \xrightarrow{\partial}$$

$$\tilde{K}_0(A[G]) \longrightarrow \tilde{K}_0(A) \oplus \tilde{K}_0(A[G]/\omega) \longrightarrow \tilde{K}_0(A_q)$$

For simplicity of notation we write this sequence in terms of Whitehead groups (by a slight abuse of notation)

$$Wh_1(A[G]) \longrightarrow Wh_1(A) \oplus Wh_1(A[G]/\omega) \longrightarrow Wh_1(A_q) \xrightarrow{\partial} Wh_0(A[G])$$

$$Wh_0(A) \oplus Wh_0(A[G]/\omega) \longrightarrow Wh_0(A_q) \tag{W}$$

The boundary map ∂ in the sequence is related to a generalization of the Swan homomorphism $(\mathbb{Z}_q)^{\times} \longrightarrow \tilde{K}_0(\mathbb{Z}G)$ in the case of $\pi = 1$, (cf. [Sw] or [M]). We continue to call ∂ the Swan homomorphism.

Let α and γ be as in Theorem I.1. Then the Swan homomorphism is $-\alpha\circ\gamma$. To see this, let $x \in Wh_1(A_q)$ correspond to the isomorphism $\phi : (A_q)^n \longrightarrow (A_q)^n$ induced by the (injective) homomorphism $\phi : A^n \longrightarrow A^n$. As in Section I, it follows that in the exact sequence

$$0 \longrightarrow A^n \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

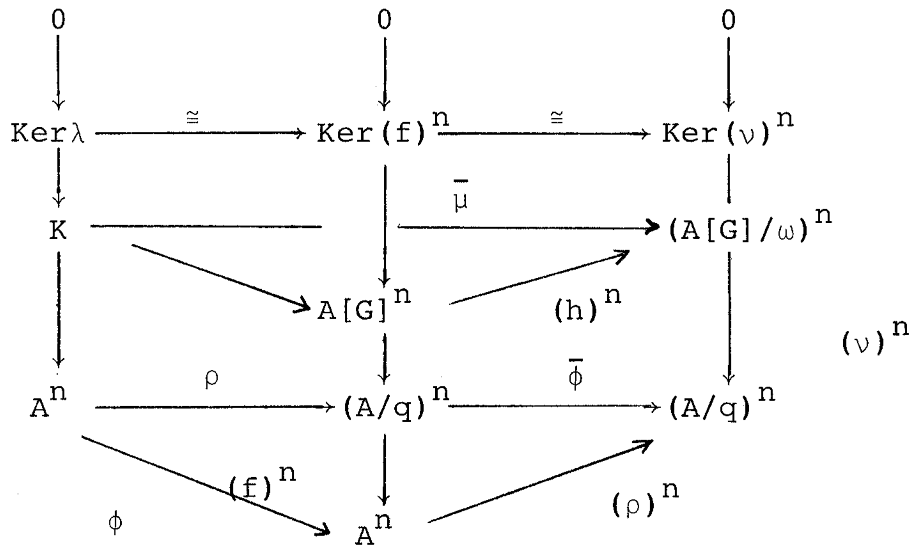
one has $\text{proj dim}_{AG}(M) \leq 1$. Thus one has the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^n & \xrightarrow{\phi} & A^n & \longrightarrow & M & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle (f)^n} & & \uparrow & & \uparrow{\scriptstyle 1} & & \uparrow \\
0 & \longrightarrow & K & \xrightarrow{\mu} & A[G]^n & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where $(f)^n$ is induced by the augmentation f. Thus $\alpha\gamma([\phi]) = -[K]$ and the problem is reduced to show that the following diagram is cartesian:

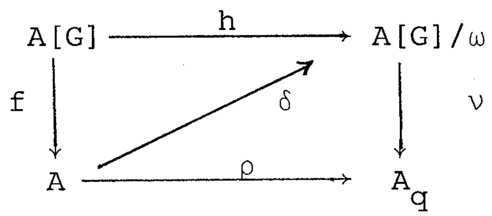
$$\begin{array}{ccccc}
K & & \xrightarrow{\bar{\mu}} & & (A[G]/\omega)^n \\
\downarrow{\scriptstyle \lambda} & & & & \downarrow{\scriptstyle (\nu)^n} \\
A^n & \xrightarrow{(\rho)^n} & (A/q)^n & \xrightarrow{\bar{\phi}} & (A/q)^n
\end{array}$$

(Recall the definition of ∂ in the Mayer-Vietories sequence; cf. [M] e.g.). Since $\mathrm{Ker}\lambda \cong \mathrm{Ker}(f)^n \cong \mathrm{Ker}(\nu)^n$ and $(f)^n \circ \mu = \phi \circ \lambda$, one has the diagram:

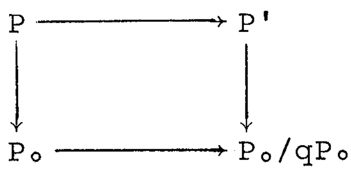


obtained from diagrams (B) and (C) above, and in which $\bar{\phi} \circ (\rho)^n \circ \lambda = (\rho)^n \circ \phi \circ \lambda = (\rho)^n \circ (f)^n \circ \mu = (\nu)^n \circ (h)^n \circ \mu = (\nu)^n \circ \bar{\mu}$. Thus (B) is cartesian.

Next, we identify the transfer $T_i : Wh_1(A[G]) \longrightarrow Wh_i(A)$, $i = 0,1$ in the 5-term exact sequence of Theorem I. Consider the diagram:



where δ is the composite $A \xrightarrow{\cong} A \times \{1\} \longrightarrow AG \longrightarrow AG/\omega$ so that $\nu \circ \delta = \rho$. Let $p \in P(AG)$ be given and tensor P over AG by the diagram (C) to obtain the cartesian diagram:



Thus one obtains four functors from $P(AG)$ to the categories $P(A[G])$, $P(A[G]/\omega)$, $P(A)$, and $P(A_q)$. The above cartesian diagram yields the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & P_o \oplus P' & \longrightarrow & P_o/qP_o & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow {\scriptstyle 1+0} & & \uparrow {\scriptstyle 1} & & \\
0 & \longrightarrow & P_o & \xrightarrow{(\cdot)xq} & P_o & \longrightarrow & P_o/qP_o & \longrightarrow & 0
\end{array}$$

which yields the exact sequence:

$$0 \longrightarrow P_o \longrightarrow P \oplus P_o \longrightarrow P_o \oplus P' \longrightarrow 0$$

The above sequence defines, in fact, a short exact sequence of the corresponding functors due to the functoriality of all the above constructions. It follows from Quillen's theorem on the additivity of functors (cf.[Q]) that the functor $P \longrightarrow P \oplus P_o$ is the sum of functors $P \longrightarrow P_o$ and $P \longrightarrow P_o \oplus P'$, which in turn implies that induced homomorphisms on K-theory satisfy $Tr = f_* \oplus tr\, h_*$, $Tr : K_*(A[G]) \longrightarrow K_*(A)$ and $tr : K_*(AG/\omega) \longrightarrow K_*(A)$ are transfer homomorphisms. Thus on the level of Whitehead groups, one has the following:

<u>II.1 Lemma</u>. Let $T_i : Wh_i(A[G]) \longrightarrow Wh_i(A)$ and $t_i : Wh_i(A[G]/\omega) \longrightarrow Wh_i(A)$ be transfer homomorphisms. Then $T_i = f_* \oplus t_i h_*$, where f_* and h_* are induced from diagram (C) above.

Let $\rho_i \equiv \rho_* : Wh_i(A) \longrightarrow Wh_i(A) \longrightarrow Wh_i(A_q)$. Specializing to the case $G = \mathbb{Z}_2$, the above calculation is continued to show

<u>II.2 Lemma</u>: The sequence $Wh_1(\pi) \xrightarrow{2\rho_1} Wh_1(\pi;\mathbb{Z}_2) \xrightarrow{\gamma} Wh_1^T(\pi \subset \pi \times \mathbb{Z}_2) \longrightarrow$

$\longrightarrow Ker\rho_o \longrightarrow 0$ is exact. In particular $Ker\gamma = Im 2\rho$.

This characterizes completely the obstructions which are discussed in Section I in terms of the $Wh_1(\pi)$ and the mod 2 reduction $Wh_1(\pi) \longrightarrow Wh_1(\pi;\mathbb{Z}_2)$. To obtain examples of nontrivial obstructions, let $\pi = \mathbb{Z}_8$. Then computations show

$$Wh_1(\mathbb{Z}_8;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$$

<u>II.3 Corollary</u>: $Wh_1^T(\mathbb{Z}_8 \subset \mathbb{Z}_8 \times \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $ker\gamma$ consists of the 2-divisible elements of $Wh_1(\mathbb{Z}_8;\mathbb{Z}_2)$.

Although $Wh_1(\mathbb{Z}_5) \cong \mathbb{Z}$, one can show that in the case $\pi = \mathbb{Z}_5$, $G = \mathbb{Z}_2$, all the obstructions vanish.

<u>II.4 Corollary</u>: $Wh_1(\mathbb{Z}_5;\mathbb{Z}_2) \cong \mathbb{Z}_3$ and $Im(\gamma) \equiv 0$.

<u>Remark</u>: In [Kw] Kwun has shown that the transfer $Wh_1(\mathbb{Z}_2 \times \mathbb{Z}_r) \longrightarrow Wh_1(\mathbb{Z}_r)$ is onto if and only if r = odd or $r = 2,4,6$. We thank the referee from bringing Kwun's result to our attention.

<u>SECTION III</u>. Let X be a finite dimensional CW complex with $\pi_1(X) = \pi$, and let G be a finite group of order q acting semifreely on X - i.e. the action is free outside of the stationary point set. In general, there is no explicit relationship between $H_*(X)$ and $H_*(X^G)$. The rather implicit information obtained using the localization theorems of Atiyah-Borel-Quillen-Segal type does not seem sufficient to yield a satisfactory characterization of the stationary point set X^G under general hypotheses. In the sequel, we will consider a class of actions which are encountered often in the geometric considerations, and to which it is possible to apply the present techniques of algebraic topology to obtain rather precise information and characterizations of X^G.

Given a connected space X and a subring of rational numbers Λ or $\Lambda = \mathbb{Z}_q$ we denote by X_Λ the localization of X which preserves $\pi_1 X$ and $\pi_i(X_\Lambda) \cong \pi_i(X) \otimes \Lambda$ for $i > 1$. For instance Bousfield-Kan's localization [B-K] applied to the universal covering space $\tilde{X}$ yields $\tilde{X}_\Lambda$ on which $\pi_1(X)$ operates freely and $\tilde{X} \longrightarrow \tilde{X}_\Lambda$ is equivariant. Then X_Λ can be defined as $\tilde{X}_\Lambda/\pi_1(X)$. For $\Lambda = \mathbb{Z}_q$, $\Lambda = \mathbb{Z}_{(q)}$ and $\Lambda = \mathbb{Z}[\frac{1}{q}]$ we can use the notations X_q, $X_{(q)}$ and $X(\frac{1}{q})$ respectively.

The key observation to reconstruct a space (respectively a diagram of spaces) from its localizations (respectively its diagrams of localizations) is the following:

<u>III.1. Lemma</u>. For any connected space X the following diagram is cartesian:

$$\begin{array}{ccc} X & \longrightarrow & X(\frac{1}{q}) \\ f \big\downarrow & & \big\downarrow f' \\ X_q & \longrightarrow & X_q(\frac{1}{q}) \end{array}$$

<u>Proof</u>: Since $H_*(X_q, X; \mathbb{Z}/q[\pi]) = 0$ it follows that $H_*(X_q, X; \mathbb{Z}\pi)$ is

$\mathbb{Z}[\frac{1}{q}]$-local. Hence the (homotopy) fibre of f is $\mathbb{Z}[\frac{1}{q}]$-local (Cf. [S]). Since the (homotopy) fibre of f' is also $\mathbb{Z}[\frac{1}{q}]$-local, f and f' has the same fibre (up to homotopy).

Definition. Let X be a connected G-CW complex, where G is a finite group of order q. X is called a simple G-space (and the action is called simple) if $(E_G \times_G X)_q$ is fibre homotopy equivalent to $(BG \times X)_q$.

For instance, if X has trivial mod q homology, then any G-action on X will be simple, or if X has the mod q homology of a sphere and $X^G \neq \emptyset$, the X-{point} has a simple action if we take out a point from X^G.

Proposition. Suppose G is a finite group of order q which has a simple semifree action on the finite dimensional complex X with $\pi_1 X = \pi$. Then $H_*(X, X^G; \mathbb{Z}_q\pi) = 0$, where the homology has local coefficients.

In the case of semifree simple actions on compact manifolds, one obtains further restrictions imposed on X^G. For simplicity, let us consider the case of a smooth semifree G-action on a compact manifold W^n with $\pi_1 W = \pi$. Then the stationary point set $W^G = F^k$ is a submanifold with normal bundle ν which is a G-bundle with a free G-representation at each fibre. Assume that $n-k > 2$. We identify the total space of the disk bundle $D(\nu)$ with a closed G-invariant tubular neighborhood of F. Let C^n = W-interior $D(\nu)$. One can choose an appropriate CW structure for W so that W, C, and $D(\nu)$ become G-CW complexes, and various cellular chain complexes have preferred bases. If the action is simple, then $H_*(W,F;\mathbb{Z}_q\pi) = 0$, and G acts trivially on $H_*(W,F;\mathbb{Z}\pi)$, as well as on $H_*(S(\nu);\mathbb{Z}[\frac{1}{q}](\pi)) \cong H_*(S(\nu)/G;\mathbb{Z}[\frac{1}{q}](\pi \times G))$. One further observation is that the geometry provides us with the dotted arrow in the following diagram in which $\pi = \pi_1(W)$:

$$\begin{array}{ccc} \pi_1(S(\nu)/G & \dashrightarrow & \pi \\ \uparrow & \nearrow & \\ \pi_1(S(\nu)) & & \end{array}$$

For a pair (W,F) as above, we define an element $\omega(W,F) \in Wh^T(\pi \subset \pi \times G)$ as follows. Given a free finite $\mathbb{Z}\pi$-based chain complex (A'_*, A') and a free $\mathbb{Z}G$-resolution R_* of $\mathbb{Z}$, we form the $\mathbb{Z}(\pi \times G)$-complex $A_* = A'_* \otimes R_*$ which is $\mathbb{Z}\pi$-chain homotopy equivalent to A'_*. Suppose $H_*(A'_* \otimes \mathbb{Z}_q) = 0$.

Then by theorem I.3 there is a finite $\mathbb{Z}(\pi \times G)$-projective complex B_* with a π-basis $\mathcal{B}'$ such that $(B_*, \mathcal{B}')$ is π-simple homotopy equivalent to $(A'_*, \mathcal{A}')$. Define $\omega(A'_*, \mathcal{A}') = \sum(-1)^i [B_i, \mathcal{B}_i] \in Wh^T(\pi \subset \pi \times G)$ which is seen to be well-defined. Now let A'_* be the $\mathbb{Z}\pi$-chain complex of cellular chains of (W,F) with local $\mathbb{Z}\pi$-coefficients and let $\mathcal{R}'$ be the natural preferred bases provided by the cells. Then $\omega(W,F) = \omega(A'_*, \mathcal{R}')$ is well-defined. From section I, one can compute that $\omega(A'_*, \mathcal{R}') = \gamma\tau(A'_*)$.

<u>III.2. Theorem</u>. Let $\phi : G \times W^n \longrightarrow W^n$ be a smooth simple semifree action with $F^k = W^G$, $n-k > 2$, and ν = normal bundle of F in W, $\pi = \pi_1(W)$. Then:

1) $H_*(W,F;\mathbb{Z}_q\pi) = 0$,

2) G acts trivially on $H_*(S(\nu)/G;\mathbb{Z}[\frac{1}{q}](\pi \times G))$,

3) there is a homomorphism ι making the following diagram commute:

$$\begin{array}{ccc} \pi_1(S(\nu)/G & \xrightarrow{\ \iota\ } & \pi \\ \uparrow & \nearrow & \\ \pi_1(S(\nu)) & & \end{array}$$

4) $\omega(W,F) \in Wh_1^T(\pi \subset \pi \times G)$ vanishes.

Since $C_*(C^n, S(\nu);\mathbb{Z}\pi)$ is $\mathbb{Z}\pi$-chain homotopy equivalent to $C_*(W,F;\mathbb{Z}\pi)$, one verifies that $\omega(W,F)$ is defined under the following more general situation: $F^k \subset W^n$ is a submanifold with normal bundle ν, $n-k > 2$, and ν has G-bundle structure with a free representation on each fibre, and conditions (1) and (3) of Theorem II.2 are satisfied for (W,F).

The main results of this section are the following two theorems.

<u>III.3. Theorem (Characterization of stationary-point sets of simple actions)</u>.

Let W^n be a compact manifold with connected boundary such that $\pi_1(\partial W) \cong \pi_1(W) = \pi$, and let $(F^k, \partial F^k) \subset (W, \partial W)$ be a smooth submanifold with normal bundle ν, $n-k > 2$, $n \geq 6$. Then there is a smooth simple semifree G-action on W^n with $(W^n)^G = F$ if and only if F:

1) ν admits a G-bundle structure over F with a free representation on each fibre.

2) $H_*(W,F;\mathbb{Z}_q\pi) = 0$,

3) $\gamma\tau(W,F) \in Wh_1^T(\pi \subset \pi \times G)$ vanishes.

Remark: Condition (3) is equivalent to $\omega(W,F) = 0$. $\tau(W,F)$ is the Reidemeister torsion, and γ is the homomorphism of Theorem I.2.

The above theorem follows from the following extension theorem and III.2.

III.4. The Extension Theorem. Suppose C^n is a compact smooth manifold with $\partial C = \partial_+ C \underset{\partial}{\cup} \partial_- C$ where $\pi_1(\partial_- C) \cong \pi_1(C) = \pi$, $n \geq 6$.

Suppose that G is a finite group of order q and $\phi_+ : G \times \partial_+ C \longrightarrow \partial_+ C$ is a free smooth action such that:

1) $H_*(C, \partial_+ C; \mathbb{Z}_q \pi) = 0$,

2) there is a commutative diagram

$$\begin{array}{ccc} \pi_1(\partial_+ C/G) & \dashrightarrow & \pi \\ \uparrow & \nearrow & \\ \pi_1(\partial_+ C) & & \end{array}$$

3) G acts trivially on $H_*(\partial_+ C/G; \mathbb{Z}[\frac{1}{q}](\pi \times G))$.

Then there is a free G-action $\phi : G \times C \to C$ extending ϕ_+ with G acting trivially on $H_*(C/G; \mathbb{Z}[\frac{1}{q}](\pi \times G))$ if and only if $\gamma\tau(C, \partial_+ C) \in \acute{Wh}_1^T(\pi \subset \pi \times G)$ vanishes. Moreover, this action is unique up to concordance.

SECTION IV. We indicate an outline of proofs of the main theorems of Section III. Complete proofs and further applications will appear later.

Outline of the proof of Theorem III.3. The necessity of condition (2) follows from an application of the Atiyah-Segal-Quillen localization theorem for each prime order cyclic subgroup of G to the covering G-action on the universal covering space of W. Condition (3) is necessary due to Theorem I.3.

Given (W,F) satisfying (1) - (3) of III.3, we can apply The Extension Theorem III.4 to W-intD(ν) $\equiv$ C and the induced action of (1) to $S(\nu) \equiv \partial_+ C$ in order to construct a smooth simple semifree G-action on W with $W^G = F$.

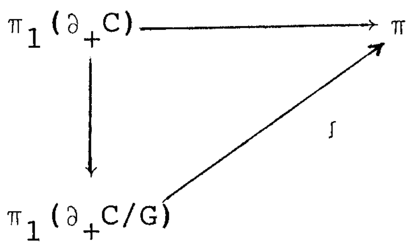
An outline of the proof of III.4 is as follows. Theorems IV.1 and IV.2 allow us to construct an appropriate Poincare pair (X,Y) such that surgery problem provided by (X,Y) would yield the candidate for the

the orbit space $(C/G, C/G)$.

<u>IV.1. Theorem</u>. Let C^n be a compact manifold with $\pi_1(C) = \pi_1(\partial_- C) = \pi$, $\partial C^n = \partial_+ C^n \cup \partial_- C^n$, $\partial_+ C^n \cap \partial_- C^n = \partial_0 C = \partial(\partial_+ C) = \partial(\partial_- C)$. Suppose that $\phi : G \times \partial_+ C \to \partial_+ C$ is a free G-action such that:

1) $H_*(C, \partial_+ C; \mathbb{Z}_q[\pi]) = 0$

2) $\exists$ homomorphism ρ such that



commutes.

3) G acts trivially on $H_*(\partial_+ C/G; \mathbb{Z}[\pi \times G])$, where $\mathbb{Z}[\pi \times G]$ is the local system for $\partial_+ C/G$ via ρ.

Then there exists a Poincaré complex (X,Y) such that $Y = (\partial_+ C/G) \cup (\partial_- X)$, $(\partial_- X) \cap (\partial_+ C/G) = \partial_0 C/G$, and $(\tilde{X}, \tilde{Y})$ is homotopy equivalent to $(C, \partial C)$ rel $\partial_+ C$, where $(\tilde{X}, \tilde{Y})$ is the covering space with the covering transformation group G and fundamental group π.

<u>IV.2 Theorem</u>. Keep the notation and the hypotheses of IV.1. Then (X,Y) can be taken to be a finite Poincaré pair π-simple homotopy equivalent (rel $\partial_+ C$) to $(C, \partial C)$ if and only if $\gamma\tau(C, \partial_+ C) = 0$.

Assuming the proofs of IV.1 and IV.2, the proof of III.4 proceeds as follows. By IV.1, we have a Poincaré pair (X,Y) whose covering pair $(\tilde{X}, \tilde{Y})$ with fundamental group π is homotopy equivalent to $(C, \partial C)$ rel $\partial_+ C$. By virtue of IV.2 and condition (3) of the hypotheses, we can choose (X,Y) so that $(\tilde{X}, \tilde{Y})$ is simple-homotopy equivalent to $(C, \partial C)$. Next we show that the Spivak normal fibre space of (X,Y) has a linear structure which in turn shows that the set of normal invariants of (X,Y) is non-empty. Moreover, we can choose a normal invariant such that the corresponding normal map $(V, \partial V) \xrightarrow{f} (X,Y)$ is relative to $\partial_+ C/G$, i.e. $\partial_+ C/G \subset \partial V$ and $f|\partial_+ C/G : \partial_+ C/G \to \partial_+ C/G \subset Y$ is the inclusion (as a normal map). To see this, let $\lambda : X \to BG$ be the classifying map into Stasheff's classifying space for (stable) spherical fibrations. $\lambda|\partial_+ C/G$ has a life to BO induced by the given smooth structure of a $\partial_+ C/G$. Since G/O is an infinite loop space, the obstruction to extending this lift to X is an element $\alpha \in h^*(X, \partial_+ C/G)$ where h^* is the generalized cohomology theory associated with G/O. We need the following lemma to show that this obstruction vanishes.

IV.3. Lemma. For any generalized cohomology theory $h^*(X, \partial_+ C/G)$ is $(\frac{1}{q})$-local.

Let $\mu : \tilde{X} \to X$ be the covering with the covering transformation group . Then $\mu^*(\alpha) \in H^*(\tilde{X}, \partial_+ C)$ vanishes since $\tilde{X}$ is homotopy equivalent to C rel $\partial_+ C$ and the latter is a smooth manifold. The transfer $t : h^*(\tilde{X}, \partial_+ C) \to h^*(X, \partial_+ C/G)$ is defined and $t\mu^*$ is multiplication by q. The above lemma implies that μ^* is a monomorphism and $\alpha = 0$ as a consequence. Hence we can choose a lift of λ to B0 which is compatible with the given life for $\lambda | \partial_+ C/G$ so that the resulting normal map $f_1 : (V_1, \partial V_1) \to (X, Y)$ is rel $\partial_+ C/G$ as desired.

Let $K = \mathrm{Ker}(Wh_1(\pi \times G) \xrightarrow{\text{Transfer}} Wh_1(\pi))$. Although the Poincaré pair (X,Y) is not necessarily a simple Poincaré pair, the G-covering $(\tilde{X}, \tilde{Y})$ is a simple Poincaré pair since it is π-simple homotopy equivalent to the manifold pair $(C, \partial C)$. Consequently the Whitehead torsion of the duality isomorphism lies in K. Denote by L_*^K the surgery obstruction groups of Wall where the homotopy equivalences are required to have Whitehead torsion belonging to K. Then the π-π theorem of Wall can be modified slightly to show that $L_*^K(\pi', \pi') = 0$ for a finitely presented group π'. Since the hypotheses imply that $\pi_1(X) \cong \pi_1(Y - \partial_+ C/G)$, we may assume that f_1 is normally cobordant to a homotopy equivalence $f : (V, \partial V) \to (X, Y)$ rel $\partial_+ C/G$ with torsion in K.

Next, we can choose $(V, \partial V)$ such that the covering space $(V, \partial V)$ with group G is diffeomorphic to $(C, \partial C)$ rel $\partial_+ C$. We have the commutative diagram of surgery exact sequences of Sullivan-Wall corresponding to (X,Y) rel $\partial_+ C/G$ and $(\tilde{X}, \tilde{Y})$ rel $\partial_+ C$ and the maps induced by the covering projection:

$$\begin{array}{ccccccc}
L_{n+1}^S(\pi, \pi) & \longrightarrow & \mathcal{S}^S(\tilde{X}, \tilde{Y}) & \xrightarrow{\cong} & \mathcal{N}(\tilde{X}, \tilde{Y}) & \longrightarrow & L_n^S(\pi, \pi) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
L_{n+1}^K(\pi \times G, \pi \times G) & \to & \mathcal{S}^K(X, Y) & \xrightarrow{\cong} & \mathcal{N}(X, Y) & \longrightarrow & L_n^K(\pi \times G, \pi \times G)
\end{array}$$

The horizontal isomorphisms are due to Wall's (π-π) theorem. Now $\mathcal{N}(X, Y) \cong h^0(X, \partial_+ C/G)$ and $\mathcal{N}(\tilde{X}, \tilde{Y}) \cong h^0(\tilde{X}, \partial_+ C)$ where h^* is the cohomology theory associated with the $G/0$ spectrum.

Consider the Carton-Leray-Serre spectral sequence for the G-covering $\mu : \tilde{X} \to X$ in which the E_2-term is $H^*(BG; h^*(X, \partial_+C))$ and it converges to $h^*(X, \partial_+C/G) \cong h^*(X, \partial_+C)^G$. The only nonvanishing terms are $H^0(BG; h^*(\tilde{X}, \partial_+C) \cong h^*(\tilde{X}, \partial_+C)^G$. Hence the spectral sequence collapses and $h^*(X, \partial_+C/G) \cong h^*(\tilde{X}, \partial_+C)^G$. One can show that G acts trivially on $h^*(\tilde{X}, \partial_+C)$. Hence $h^*(\tilde{X}, \partial_+C) \cong h^*(X, \partial_+C/G)$ and μ^* induces the isomorphism. Thus we may choose the normal invariant in $N(X,Y)$ so that the corresponding homotopy smoothing $(V, \partial V)$ has the G-covering $(\tilde{V}, \partial\tilde{V})$ diffeomorphic to $(C, \partial C)$ rel ∂_+C.

G acts freely on $(\tilde{V}, \partial\tilde{V})$ as the group of covering transformations and this action extends the induced action on ∂_+C.

The idea of the proof of IV.1 is to construct a diagram

$$\begin{array}{ccc} \partial_0 C/G & \longrightarrow & \partial_+ C/G \\ \downarrow & & \downarrow \\ \partial_- X & \longrightarrow & X \end{array}$$

(In which ∂_-X, X and the dotted arrows are to be determined) with the property that the diagram $\tilde{\Delta}$ consisting of various G-covering spaces is (up to homotopy) the diagram D below:

$$\begin{array}{ccc} \partial_0 C & \longrightarrow & \partial_+ C \\ \downarrow & & \downarrow \\ \partial_- C & \longrightarrow & C \end{array}$$

In this vein, one constructs the diagrams Δ_q and $\Delta(\frac{1}{q})$ (of "localizations") such that there is a map $\Delta(\frac{1}{q}) \to \Delta_q(\frac{1}{q})$ which lifts to the appropriate maps of the G-coverings. The existence of such localized diagrams uses condition (1) and obstruction theory. A modified version of Lemma III.1 for diagrams yields the diagram Δ.

In Theorem IV.2, consider the diagram

$$\begin{array}{ccc} \partial_+ C & \longrightarrow & C \\ \downarrow & & \downarrow \\ \partial_+ C/G & \longrightarrow & X \end{array}$$

where X has been determined up to homotopy by Theorem IV.1. This diagram satisfies the hypothesis of Theorem 1.4, hence X is homotopy equivalent to a finite complex with $\tilde{X}$ being π-simple homotopy equivalent to C if and only if $\gamma\tau(C,\partial_+ C) \in Wh_1^T(\pi \subset \pi \times G)$ vanishes. In particular if (X,Y) is π-simple homotopy equivalent to $(C,\partial C)$ rel $\partial_+ C$ then $\gamma\tau(C,\partial_+ C) = 0$. (Observe that both X and $\partial_- X$ in the above diagram are finitely dominated.)

To prove the converse, suppose that $\gamma\tau(C,\partial_+ C) = 0$. Then by Theorem I.4, we can replace X in the above diagram by a finite complex whose covering with fundamental group π is π-simple homotopy equivalent to C. For simplicity of notation, assume that X is such a finite complex, so that $\tilde{X}$ is also finite. Consider the diagram

$$\begin{array}{ccc} \partial_- \tilde{X} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ \partial_- X & \xrightarrow{\eta} & X \end{array}$$

where $\partial_- X$ and η are determined by the diagram Δ above. By Poincaré duality, $C^*(X,\partial_+ C/G;\mathbb{Z}[\pi \times G])$ is chain homotopy equivalent to $C_*(X,\partial_- X;\mathbb{Z}[\pi \times G]) \equiv C_*(\eta;\mathbb{Z}[\pi \times G])$. Thus, there exists a $\pi \times G$-chain homotopy equivalence $f : C_*(X,\partial_- X;\mathbb{Z}[\pi \times G]) \to D_*$, where D_* is a finite $\pi \times G$-complex. Since $\pi_1(\partial_- X) \cong \pi_1(X) = \pi \times G$, by the additivity property of finiteness obstructions, it follows that the finiteness obstruction of $C_*(\partial_- X;\mathbb{Z}[\pi \times G])$ vanishes as well. Hence we may assume that $\partial_- X$ and its covering $\partial_- \tilde{X}$ are finite complexes. At this point, the argument of Theorem I.4 goes through to show that we can choose $\partial_- X$ to be a finite complex such that $\partial_- \tilde{X}$ is π-simple homotopy equivalent to $\partial_- C$. The additivity of Whitehead torsions shows that (X,Y) is a finite Poincaré pair such that $(\tilde{X},\tilde{Y})$ is π-simple homotopy equivalent to $(C,\partial C)$ rel $\partial_+ C$.

SECTION V. To formalize some of the results of Section IV, in this section we prove some general results for constructing quasi-simple free actions on a given homotopy type (see below). The question of choosing a particular simple-type can be treated using the algebraic theory developed in Section I. The analogs of theorem I.4 which describes the obstructions for the choice of a simple type (lying in $Wh_1^T(\pi \subset \pi \times G)$) are valid and may be formulated in the context of Theorems V.1 and V.2 below.

Definition. A free action $G \times X \to X$ is called quasi-simple if $\pi_1(X/G) = \pi_1(X) \times G$ and G acts trivially on $H_*(X;\mathbb{Z}[\frac{1}{q}])$, $q = |G|$.

V.1 Theorem. (Pushing forward actions.) Suppose $\phi : G \times A \to A$ is a free quasi-simple action, and $f : A \longrightarrow X$ induces an isomorphism $f_* : H_*(A;f^*\mathbb{Z}_q[\pi] \longrightarrow H_*(X;\mathbb{Z}_q[\pi])$, where $\pi = \pi_1(X)$. Then there exists a free quasi-simple G-action on a space X', an equivariant map $f : A \xrightarrow{\sim} X'$, and a homotopy equivalence $h : X \longrightarrow X'$ such that $h \circ f \longrightarrow f'$.

Outline of Proof: We need to construct a space Y and a map $A/G \xrightarrow{g} Y$ such that the G-covering $\tilde{Y}$ and the induced G-maps $\tilde{g} : A \to \tilde{Y}$ satisfy the property required for X' and f'. Let $g : (A/G)_q \longrightarrow Y$ be constructed as follows. Since $\pi_1(A) \longrightarrow \pi_1(X)$ is surjective with a q-perfect kernel, we can add free G-cells equivariantly to A to obtain $\tilde{Y}_q$ such that $\pi_1(\tilde{Y}_q) \cong \pi_1(X)$ and the inclusion $A \longrightarrow \tilde{Y}$ induces a $\mathbb{Z}_q[\pi]$-isomorphism (equivariant plus construction). Then define $Y_q \equiv \tilde{Y}_q/G$. Next, obstruction theory shows that $(A/G)(\frac{1}{q}) \simeq (A \times BG)(\frac{1}{q})$ since the action is quasisimple. Let $Y(\frac{1}{q}) = (X \times BG)(\frac{1}{q})$ and let $g(\frac{1}{q})$ be the composition $(A/G)(\frac{1}{q}) \xrightarrow{\sim} (A \times BG)(\frac{1}{q}) \longrightarrow (X \times BG)(\frac{1}{q})$. Then we have a map $Y(\frac{1}{q}) \xrightarrow{\alpha} Y_q(\frac{1}{q})$ by obstruction theory such that the pull-back diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y(\frac{1}{q}) \\ \downarrow & & \downarrow \\ Y_q & \longrightarrow & Y_q(\frac{1}{q}) \end{array}$$

has the G-covering $\tilde{Y}$ homotopy equivalent to X via $h : X \longrightarrow \tilde{Y}$. Let $X' = \tilde{Y}$. The G-action on X' is quasi-simple by construction. The maps g and $g(\frac{1}{q})$ pull back to give the map $g : A/G \longrightarrow Y$ and we let the lift $\tilde{g} : A \longrightarrow \tilde{Y}$ be $f : A \longrightarrow X'$. One verifies that f' and X' satisfy the required properties.

<u>V.2. Theorem (Pulling back actions)</u>. Let A be a free quasi-simple G-space with $\pi_1(A) = \pi$. Let $f : X \to A$ be such that $f_* : H_*(X; f^*\mathbb{Z}[\pi]) \to H_*(A; \mathbb{Z}[\pi])$ is an isomorphism. Then there exists a free quasi-simple G-space X' an equivariant map $f' : X' \to A'$ and a homotopy equivalence $h : X' \to X$ such that $f \circ h \simeq f'$.

<u>Outline of Proof:</u> As before, we need to construct the orbit space Y and $g : Y \to A/G$ satisfying the stated properties on the level of G-coverings. Let $Y_q = (A/G)_q$ with $g_q = \mathrm{id}$ and $Y(\frac{1}{q}) = X(\frac{1}{q}) \times BG$ with $g(\frac{1}{q}) = f(\frac{1}{q}) \times \mathrm{id}$. There exists a map $Y(\frac{1}{q}) \to Y_q(\frac{1}{q})$ which is up to homotopy the composition $X(\frac{1}{q}) \times BG \to X_q(\frac{1}{q}) \times BG \to A_q(\frac{1}{q}) \times BG \to (A/G)_q(\frac{1}{q})$. Let Y be pull-back of the diagram:

$$\begin{array}{ccc} Y & \longrightarrow & (A/G)_q \\ \downarrow & & \downarrow \\ X(\frac{1}{q}) \times BG & \longrightarrow & (A/G)_q(\frac{1}{q}) \end{array}$$

The Seifert-van Kampen theorem shows that $\pi_1(Y) \cong \pi_1(X) \times G$. Furthermore, functoriality of pull-backs and the Mayer-Vietoris theorem show that the G-covering Y is homotopy-equivalent to X. The maps g and $g(\frac{1}{q})$ yield $g : Y \to A/G$ (via the above pull-back) and we may define $X \to Y$ and $f' \equiv g : \tilde{Y} \to \tilde{A}$ the induced map on the covering spaces. One readily verifies that X' and f' satisfy the desired properties.

<u>V.3. Theorem</u>. (<u>The relative version</u>). Under the hypotheses of V.2 suppose that a subspace $X_0 \subset X$ is equipped with a quasi-simple free G-action such that $f|X_0$ is equivariant. Then it is possible to arrange for X_0 to be a G-invariant supspace of X', $f|X_0 = f|X_0$, and for h to be a homotopy equivalence rel X_0.

<u>Proof</u>: This is the quasi-simple analog of [A3] Proposotion 2.III with a similar obstruction theory argument.

Let A_1 and A_2 be the CW complexes. Call A_1 and A_2 "weakly $\mathbb{Z}_q$-

equivalent," if there exists a CW complex C and maps $f_1 : A_1 \to C$ such that $f_{i*} : (H_i(A_i; f_i^* \mathbb{Z}_q[\pi_1(C)]) \to H_i(C; \mathbb{Z}_q[\pi_1(C)])$ are isomorphisms. The equivalence relation generated by weak $\mathbb{Z}_q$-equivalence is simply called "$\mathbb{Z}_q$-equivalence".

V.4 Proposition. Suppose A_1 and A_2 are $\mathbb{Z}_q$-equivalent complexes. Then A_1 admits a free quasi-simple G-action if and only if A_2 does.

Proof: This follows from V.1, V.2, and the defintion of $\mathbb{Z}_q$-equivalence.

V.5 Remarks: The above results are valid for diagrams of spaces as it was needed in Section III.

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