Princeton, 15th December, 1981

Dear Bruce,

Here are my ideas on the interlocking of the L^s-L^h and L^h-L^p exact sequences. Unfortunately, I am insufficiently familiar with the unpublished work of Vance and Giffen on $\mathbb{Z}_2$-equivariant algebraic K-theory, which is probably the best setting for all this. If you have a copy of Vance's thesis could you perhaps make a copy and send it to me?

Given a $\mathbb{Z}_2$-space X define abelian group morphisms

$$\Delta : \hat{H}^i(\mathbb{Z}_2;\pi_n(X)) \longrightarrow \hat{H}^i(\mathbb{Z}_2;\pi_{n+1}(X))$$

by sending $g:S^n \longrightarrow X$ to $h \cup (-)^i Th : S^{n+1} = D^{n+1}_+ \cup_{S^n} D^{n+1}_- \longrightarrow X$, for any null-homotopy $h : D^{n+1} \longrightarrow X$ of $g + (-)^{i+1}Tg : S^n \longrightarrow X$. The map Δ is universal for double connecting maps in Tate $\mathbb{Z}_2$-cohomology, in the following sense:

Given a $\mathbb{Z}_2$-map of $\mathbb{Z}_2$-spaces $f:X \longrightarrow Y$ there is defined a long exact sequence of $\mathbb{Z}_2$-modules

$$\dots \longrightarrow \pi_n(X) \xrightarrow{f} \pi_n(Y) \longrightarrow \pi_n(f) \longrightarrow \pi_{n-1}(X) \longrightarrow \dots .$$

Define

$$I_n = \ker(f:\pi_n(X) \longrightarrow \pi_n(Y)) \ , \ J_n = \operatorname{im}(\pi_n(Y) \longrightarrow \pi_n(f)) \ ,$$

so that there is defined a medium exact sequence

$$0 \longrightarrow \pi_n(X)/I_n \longrightarrow \pi_n(Y) \longrightarrow \pi_n(f) \longrightarrow I_{n-1} \longrightarrow 0$$

which breaks up into two short exact sequences

$$0 \longrightarrow \pi_n(X)/I_n \longrightarrow \pi_n(Y) \longrightarrow J_n \longrightarrow 0 \ ,$$

$$0 \longrightarrow J_n \longrightarrow \pi_n(f) \longrightarrow I_{n-1} \longrightarrow 0 \ .$$

Then the double connecting map

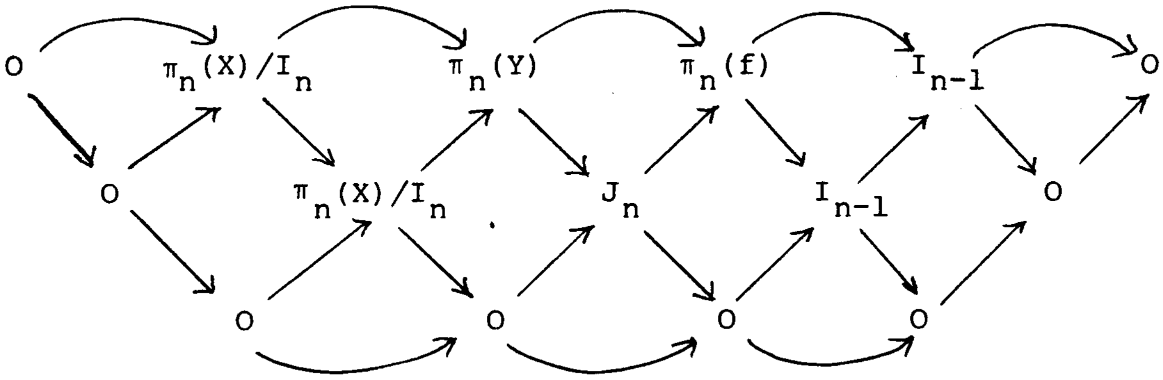
$$\delta^2 : \hat{H}^i(\mathbb{Z}_2;I_{n-1}) \xrightarrow{\delta} \hat{H}^{i+1}(\mathbb{Z}_2;J_n) \xrightarrow{\delta} \hat{H}^i(\mathbb{Z}_2;\pi_n(X)/I_n)$$

is given by the composite

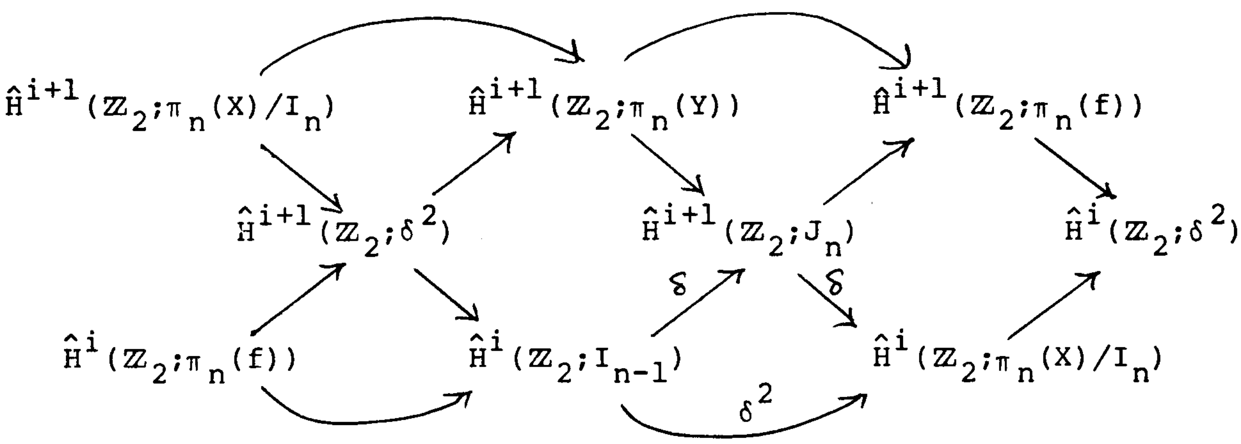
$$\delta^2 : \hat{H}^i(\mathbb{Z}_2;I_{n-1}) \xrightarrow{\text{inclusion}_*} \hat{H}^i(\mathbb{Z}_2;\pi_{n-1}(X)) \xrightarrow{\Delta} \hat{H}^i(\mathbb{Z}_2;\pi_n(X))$$

$$\xrightarrow{\text{projection}_*} \hat{H}^i(\mathbb{Z}_2;\pi_n(X)/I_n) \ .$$

Furthermore, the commutative braid of exact sequences of $\mathbb{Z}_2$-modules



gives rise to commutative braids of Tate $\mathbb{Z}_2$-cohomology groups, one for each n



Given a ring A with antistructure (β,u) let $X = \tilde{K}(A,\beta,u)$ be the $\mathbb{Z}_2$-space with homotopy groups $\pi_n(X) = \tilde{K}_n(A)$ $(n \geqslant 0)$ such that

$$T = T_{\beta,u} : \tilde{K}_1(A) \longrightarrow \tilde{K}_1(A) \ ; \ \tau(\alpha_{ij}) \longmapsto \tau(\beta(a_{ji})u)$$

is the usual (β,u)-duality involution, in which case

$$T = T_{\beta,u} : \tilde{K}_0(A) \longrightarrow \tilde{K}_0(A) \ ; \ [P] \longmapsto -[P^*] \ , \ P^* = \mathrm{Hom}_A(P,A)$$

is the opposite of the usual duality involution. Let $\tilde{K}_i(A,\beta,u)$ $(i = 0,1)$ denote $\tilde{K}_i(A)$ with this $\mathbb{Z}_2$-action, so that the Tate cohomology groups fit into the exact sequences

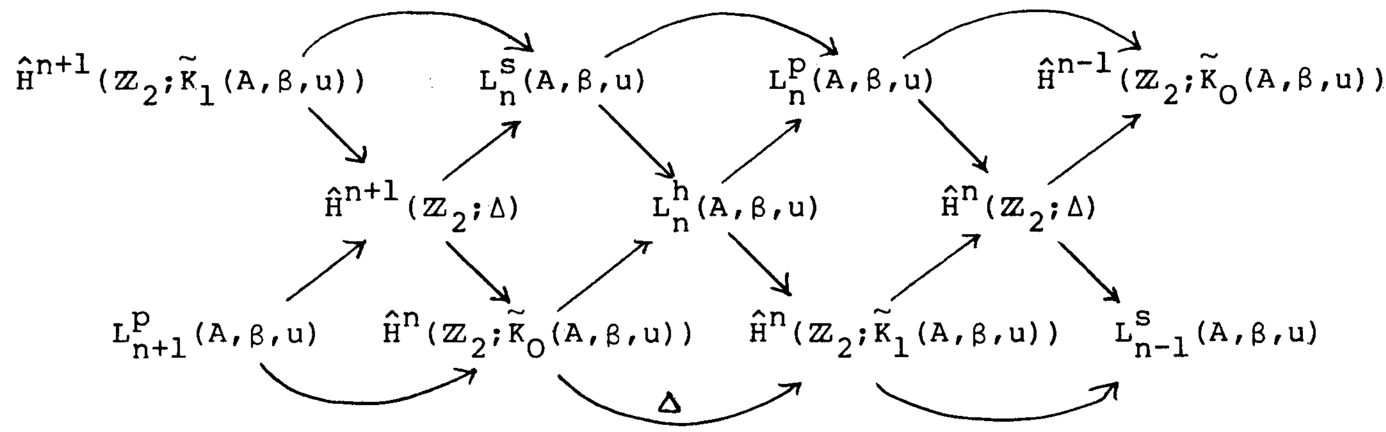
$$\ldots \longrightarrow L_n^s(A,\beta,u) \longrightarrow L_n^h(A,\beta,u) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\tilde{K}_1(A,\beta,u)) \longrightarrow L_{n-1}^s(A,\beta,u) \longrightarrow \ldots$$

$$\ldots \longrightarrow L_n^h(A,\beta,u) \longrightarrow L_n^p(A,\beta,u) \longrightarrow \hat{H}^{n-1}(\mathbb{Z}_2;\tilde{K}_0(A,\beta,u)) \longrightarrow L_{n-1}^h(A,\beta,u) \longrightarrow \ldots$$

such that the composite

$$\hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(A,\beta,u)) \longrightarrow L_n^h(A,\beta,u) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\tilde{K}_1(A,\beta,u))$$

is the map $\Delta:\hat{H}^n(\mathbb{Z}_2;\pi_0(X)) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\pi_1(X))$ for $X = \tilde{K}(A,\beta,u)$. (Warning: the map $\Delta:\hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(A,\beta,u)) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\tilde{K}_1(A,\beta,u))$ is not the same as $\Delta:\hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(A,\beta,-u)) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\tilde{K}_1(A,\beta,-u))$, although they do have the same domain and target. They differ by the map induced by $\tau(-1:\mathbb{Z} \longrightarrow \mathbb{Z})\otimes- : \tilde{K}_0(A) \longrightarrow \tilde{K}_1(A)$). The two exact sequences interlock in a commutative braid



It would be nice if the localization situation you were dealing with last summer were helped along by the above. Let me know if it does.

With best wishes

Andrew

P.S. Here is a related braid. Given a $\mathbb{Z}_2$-map $f: X = \hat{K}(A,\beta,u) \longrightarrow Y$ to any $\mathbb{Z}_2$-space Y define T-invariant subgroups $I_n = \ker(f_*: \hat{K}_n(A) \to \pi_n(Y)) \subseteq \hat{K}_n(A)$ $(n=0,1)$. The braid is