THE POINCARÉ DUALITY THEOREM AND ITS CONVERSE I.

Andrew Ranicki (Edinburgh)

http://www.maths.ed.ac.uk/~aar

FLOER
CENTER OF
GEOMETRY

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Local to global and, if possible, global to local

- There are many theorems in TOPOLOGY of the type

  $$\text{local input} \Longrightarrow \text{global output}$$

- Theorems of the type

  $$\text{global input} \Longrightarrow \text{local output}$$

  are even more interesting, and correspondingly harder to prove! This frequently requires ALGEBRA.
- *Algebra is a pact one makes with the devil!*
  (Sir Michael Atiyah)
- *I rather think that algebra is the song that the angels sing!*
  (Barry Mazur)
- *One thing I've learned about algebra ... don't take it too seriously* (Peanuts cartoon)

Poincaré duality and its converse

- The Poincaré duality of an n-dimensional topological manifold M
$$H^*(M) \cong H_{n-*}(M)$$
is a local $\Longrightarrow$ global theorem.
- **Theorem** Let $n \geqslant 5$. A space X with n-dimensional Poincaré duality $H^*(X) \cong H_{n-*}(X)$ is homotopy equivalent to an n-dimensional topological manifold if and only if X has sufficient local Poincaré duality.
- Modern take on central result of the Browder-Novikov-Sullivan-Wall high-dimensional surgery theory for differentiable and PL manifolds, and its Kirby-Siebenmann extension to topological manifolds (1962-1970)
- Will explain "sufficient" over the course of the lectures!

The Seifert-van Kampen Theorem and its converse

- Local $\Longrightarrow$ global. The fundamental group of a union

$$X = X_1 \cup_Y X_2 \ , \ Y = X_1 \cap X_2$$

is an amalgamated free product

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \ .$$

- Global $\Longrightarrow$ local. Let $n \geqslant 6$. If X is an n-dimensional manifold such that $\pi_1(X) = G_1 *_H G_2$ then $X = X_1 \cup_Y X_2$ for codimension 0 submanifolds $X_1, X_2 \subset X$ with

$$\partial X_1 = \partial X_2 = Y = (n-1)\text{-dimensional manifold} \ ,$$
$$\pi_1(X_1) = G_1 \ , \ \pi_1(X_2) = G_2 \ , \ \pi_1(Y) = H \ .$$

The Vietoris Theorem and its converses

- **Theorem** If $f : X \to Y$ is a surjection of compact metric spaces such that for each $y \in Y$ the restriction
$$f| \ : \ f^{-1}(y) \to \{y\}$$
induces an isomorphisms in homology
$$H_*(f^{-1}(y)) \ \cong \ H_*(\{y\})$$
then f induces isomorphisms in homology
$$f_* \ : \ H_*(X) \ \cong \ H_*(Y) \ .$$
- Local input: each $f^{-1}(y)$ $(y \in Y)$ is acyclic
$$\widetilde{H}_*(f^{-1}(y)) \ = \ 0 \ .$$
- Global output: f_* is an isomorphism.
- Would like to have converses of the Vietoris theorem! For example, under what conditions is a homotopy equivalence homotopic to a homeomorphism?

Manifolds and homology manifolds

- An n-**dimensional topological manifold** is a topological space M such that each $x \in M$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$.
- An n-**dimensional homology manifold** is a topological space M such that the local homology groups of M at each $x \in M$ are isomorphic to the local homology groups of $\mathbb{R}^n$ at 0

$$H_*(M, M \backslash \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{if } * \neq n \end{cases}$$

- A topological manifold is a homology manifold.
- A homology manifold need not be a topological manifold.
- Will only consider compact M which can be realized as a subspace $M \subset \mathbb{R}^{n+k}$ for some large $k \geqslant 0$, i.e. a compact ENR. This is automatically the case for topological manifolds.

The triangulation of manifolds

- A **triangulation** of a space X is a simplicial complex K together with a homeomorphism

$$X \cong |K|$$

  with $|K|$ the polyhedron of K.
- X is compact if and only if K is finite.
- Triangulation of n-dimensional topological manifolds:
  - Exists and is unique for $n \leqslant 3$
  - Known: may not exist for $n = 4$
  - Unknown: if exists for $n \geqslant 5$
  - Differentiable and PL manifolds are triangulated for all $n \geqslant 0$
- Triangulation of n-dimensional homology manifolds:
  - Exists and is unique for $n \leqslant 3$
  - Known: may not exist for $n \geqslant 4$.

The naked homeomorphism

- *Poincaré, for one, was emphatic about the importance of the naked homeomorphism - when writing philosophically - yet his memoirs treat DIFF or PL manifolds only.* in L. Siebenmann's 1970 ICM lecture on topological manifolds.
- ... *topological manifolds bear the simplest possible relation to their underlying homotopy types. This is a broad statement worth testing.* (ibid.)
- Will describe how surgery theory manufactures the homotopy theory of topological manifolds of dimension > 4 from Poincaré duality spaces and chain complexes.
- Poincaré duality is the most important property of the algebraic topology of manifolds.

The original statement of Poincaré duality

- Analysis Situs and its Five Supplements (1892–1904)
- 228 ANALYSIS SITUS.

  Donc

  $$P_p = P_{h-p}.$$

  Par conséquent, pour une variété fermée, les nombres de Betti également distants des extrêmes sont égaux.

  Ce théorème n'a, je crois, jamais été énoncé; il était cependant connu de plusieurs personnes qui en ont même fait des applications.
- Originally proved for a differentiable manifold M, but long since established for topological and homology manifolds.
- $h = n$, the dimension of M.
- $P_p = \dim_{\mathbb{Z}} H_p(M)$, the pth Betti number of M.
- Happy birthday! 2011 is the 100th anniversary of Brouwer's proof that homeomorphic manifolds have the same dimension. Also true for homology manifolds.

Orientation

- A **local fundamental class** of an n-dimensional homology manifold M at $x \in M$ is a choice of generator

$$[M]_x \in \{1, -1\} \subset H_n(M, M\backslash\{x\}) \; = \; \mathbb{Z} \; .$$

- The local Poincaré duality isomorphisms are defined by

$$[M]_x \cap - \; : \; H^*(\{x\}) \; \cong \; H_{n-*}(M, M\backslash\{x\}) \; .$$

- An n-dimensional homology manifold M is **orientable** if there exists a fundamental homology class $[M] \in H_n(M)$ such that for each $x \in M$ the image

$$[M]_x \in H_n(M, M\backslash\{x\}) \; = \; \mathbb{Z}$$

  is a local fundamental class.
- We shall only consider manifolds which are orientable, together with a choice of fundamental class $[M] \in H_n(M)$.

Poincaré duality in modern terminology

- **Theorem** For an n-dimensional manifold M the cap products with the orientation $[M] \in H_n(M)$ are Poincaré duality isomorphisms

$$[M] \cap - \ : \ H^*(M) \ \cong \ H_{n-*}(M) \ .$$

- **Idea of proof** Glue together the local Poincaré duality isomorphisms

$$[M]_x \cap - \ : \ H^*(\{x\}) \ \cong \ H_{n-*}(M, M\backslash\{x\}) \ (x \in M)$$

to obtain the global Poincaré duality isomorphisms

$$[M] \cap - \ = \ \varprojlim_{x \in M} [M]_x \cap - \ :$$

$$H^*(M) \ = \ \varprojlim_{x \in M} H^*(\{x\}) \ \cong \ H_{n-*}(M) \ = \ \varprojlim_{x \in M} H_{n-*}(M, M\backslash\{x\})$$

- Need to work on the chain level, rather than directly with homology.

Poincaré duality spaces

- **Definition** An n-**dimensional Poincaré duality space** X is a finite CW complex X with a homology class $[X] \in H_n(X)$ such that cap product with $[X]$ defines Poincaré duality isomorphism
$$[X] \cap - \;:\; H^*(X;\mathbb{Z}[\pi_1(X)]) \cong H_{n-*}(X;\mathbb{Z}[\pi_1(X)]) \;.$$
- In the simply-connected case $\pi_1(X) = \{1\}$ just
$$[X] \cap - \;:\; H^*(X) \cong H_{n-*}(X) \;.$$

- Homotopy invariant: any finite CW complex homotopy equivalent to an n-dimensional Poincaré duality space is an n-dimensional Poincaré duality space.
- A triangulable n-dimensional homology manifold is an n-dimensional Poincaré duality space.
- A nontriangulable n-dimensional homology manifold is homotopy equivalent to an n-dimensional Poincaré duality

Floer's Diplom thesis

- Floer's 1982 Bochum Diplom thesis (under the supervision of Ralph Stöcker) was on the homotopy-theoretic classification of $(n-1)$-connected $(2n+1)$-dimensional Poincaré duality spaces for $n > 1$.
- http://www.maths.ed.ac.uk/˜aar/papers/floer.pdf

Klassifikation hochzusammenhängender Poincaré-Räume

Andreas Floer

Diplomarbeit

Ruhr-Universität Bochum

Abteilung für Mathematik

1982

Manifold structures in the homotopy type of a Poincaré duality space

- (Existence) When is an n-dimensional Poincaré duality space homotopy equivalent to an n-dimensional topological manifold?
- (Uniqueness) When is a homotopy equivalence of n-dimensional topological manifolds homotopic to a homeomorphism?
- There are also versions of these questions for differentiable and PL manifolds, and also for homology manifolds.
- But it is the topological manifold version which is the most interesting! Both intrinsically, and because most susceptible to algebra, at least for $n > 4$.

Surfaces

- Surface = 2-dimensional topological manifold.
- Every orientable surface is homeomorphic to the standard surface Σ_g of genus $g \geqslant 0$.
- Every 2-dimensional Poincaré duality space is homotopy equivalent to a surface.
- A homotopy equivalence of surfaces is homotopic to a homeomorphism.
- In general, the analogous statements for false for n-dimensional manifolds with $n > 2$.

Bundle theories

	spaces	bundles	classifying spaces
differentiable	manifolds	vector bundles	BO $\pi_*(BO)$ infinite
topological	manifolds	topological bundles	$BTOP$ $\pi_*(BTOP)$ infinite
homotopy theory	Poincaré duality spaces	spherical fibrations	BG $\pi_*(BG) = \pi^S_{*-1}$ finite
- Forgetful maps downwards. Difference between the first two rows = finite (but non-zero) = exotic spheres (Milnor).
- An n-dimensional differentiable manifold M has a tangent bundle $\tau_M : M \to BO(n)$ and a stable normal bundle $\nu_M : M \to BO$.
- Similarly for a topological manifold M, with $BTOP(n)$.
- An n-dimensional Poincaré duality space X has a Spivak normal fibration $\nu_X : X \to BG$.

The Hirzebruch signature theorem

- The **signature** of a $4k$-dimensional Poincaré duality space X is

$$\sigma(X) \;=\; \text{signature}(H^{2k}(X), \text{intersection form}) \in \mathbb{Z}$$

- The **Hirzebruch $\mathcal{L}$-genus** of a vector bundle η over a space X is a certain polynomial $\mathcal{L}(\eta) \in H^{4*}(X;\mathbb{Q})$ in the Pontrjagin classes $p_*(\eta) \in H^{4*}(M)$.
- **Signature Theorem** (1953) If M is a $4k$-dimensional differentiable manifold then

$$\sigma(M) \;=\; \langle \mathcal{L}(\tau_M), [M] \rangle \in \mathbb{Z}$$

- There have been many extensions of the theorem since 1953!

The Browder converse of the Hirzebruch signature theorem

- **Theorem** (Browder, 1962) For $k > 1$ a simply-connected $4k$-dimensional Poincaré duality space X is homotopy equivalent to a $4k$-dimensional differentiable manifold M if and only if $\nu_X : X \to BG$ lifts to a vector bundle $\eta : X \to BO$ such that
$$\sigma(X) \;=\; \langle \mathcal{L}(-\eta), [X] \rangle \in \mathbb{Z} \;.$$
- Novikov (1962) initiated the complementary theory of necessary and sufficient conditions for a homotopy equivalence of simply-connected differentiable manifolds to be homotopic to a diffeomorphism.
- Many developments in the last 50 years, including versions for topological manifolds and homeomorphisms.

The Browder-Novikov-Sullivan-Wall surgery theory I.

- Is an n-dimensional Poincaré duality space X homotopy equivalent to an n-dimensional topological manifold?
- The surgery theory provides a 2-stage obstruction for $n > 4$, working outside of X, involving normal maps $(f, b) : M \to X$ from manifolds M, with b a bundle map.
- Primary obstruction in the topological K-theory of vector bundles to the existence of a normal map $(f, b) : M \to X$.
- Secondary obstruction $\sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ in the Wall surgery obstruction group, depending on the choice of (f, b) in resolving the primary obstruction. The algebraic L-groups defined algebraically using quadratic forms over $\mathbb{Z}[\pi_1(X)]$.
- The mixture of topological K-theory and algebraic L-theory not very satisfactory!

The Browder-Novikov-Sullivan-Wall surgery theory II.

- Is a homotopy equivalence $f : M \to N$ of n-dimensional topological manifolds homotopic to a homeomorphism?
- For $n > 4$ similar 2-stage obstruction theory for deciding if f is homotopic to a homeomorphism.
- The mapping cylinder of f

  $$L = M \times [0,1] \cup_{(x,1) \sim f(x)} N$$

  defines an $(n+1)$-dimensional Poincaré pair $(L, M \sqcup N)$ with manifold boundary. The 2-stage obstruction for uniqueness is the 2-stage obstruction for relative existence.
- Again, the mixture of topological K-theory and algebraic L-theory not very satisfactory!