Combinatorial Topology (Zeeman 1966)

Def: E^m = euclidean m-space

B^m = m-ball $= \{ (x_1, -, x_m) ; \; x_1^2 + \dots + x_m^2 \leq 1 \}$

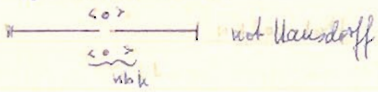
S^{m-1} = (m-1) sphere $= \{ (x_1, -, x_m) ; \; x_1^2 + \dots + x_m^2 = 1 \}$

Def: An m-manifold is a topological space that is

1) locally euclidean (each pt has a nbh homeom to $\overset{E^m}{\underset{or}{}} B^m$)

2) hausdorff

Later restricted by a 3. Axiom



Remark: i) We mostly assume connected mf. M

ii) The interior of $M = \mathring{M} = \{ x \in M ; x$ has a nbh homeom to $E^m \}$

iii) Boundary of $M = \dot{M} = \partial M = M - \mathring{M}$

Exerc: ∂M is a (m-1)-mf without boundary

If M is connected, ∂M is not in general

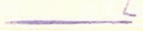
Remark: We call M closed if it is compact without boundary ($\partial M = \emptyset$)

" " open " " non-compact " "

Classification of 1-mf:

			compact	∂M
1)	S^1	closed	✓	$\emptyset$
2)	R	open	×	$\emptyset$
3)	I		✓	S^0
4)	$[0,1]$		×	1pt

5) Let Ω = {set of all ordinals}. Let $\Omega \times [0,1)$ order it by $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ or $x_1 = x_2$ & $y_1 < y_2$. Give it order topology that is all between $a < b$. This is the long line

5. Half a long line		non-compact	$\partial M = 1pt$
6. " open		non-compact	$\partial M = \emptyset$
7. Long line		non-compact	$\partial M = \emptyset$

To be able to work with mf we want a further axiom:

3) M^n has a countable base of open sets

Equivalent to 3)

i) Seperable (countable dense subset) metric

ii) Paracompact (every open cover has a locally finite refinement) ?

iii) Each open cover has a partition of unity

Classification of closed connected 2-manifolds

Def: If A, B are two 2-mf define $A \# B$, the connected sum as follows: Choose a 2-disk in each & ~~[illegible]~~ remove the interiors of the disks. Choose a homeom between the two circles & glue together #

Remark: i) $A \# B$ is unique up to homeom (only for 2-dim). (Hard to prove)

ii) # is associative and commutative, i.e.
$(A \# B) \# C \cong A \# (B \# C)$ $A \# B \cong B \# A$

iii) S^2 acts as a zero, i.e $A \# S^2 \cong A \cong S^2 \# A$

iv) Cancellation fails, i.e $A \# B \cong A \# C \not\Rightarrow B \cong C$

Def: M is prime if i) $M = A \# B \Rightarrow A$ or B is S^2
ii) $M \neq S^2$ prime

Theorem: $\exists''$ primes, the torus T and the real projective plane P. Any closed connected 2-mf is homeom to nT $(n \geq 0)$, orientable or kP $(k \geq 1)$, non orientable.

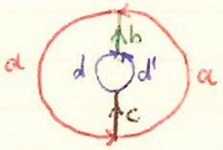
By the way $nT \# kP = (2n+k)P = 2nP \# kP$

This is a counterexpl ~~of~~ to remark (iv). Easiest expl $P \# T \cong P \#$ Kleinbottle

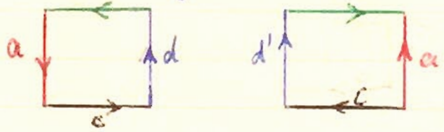
Proof of Theorem: i) Every 2-mf is homeom to one of these (geometry)
ii) These are all different (algebra)

Expl:

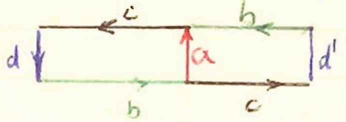
1) The real projective plane P - interior of a disk = Möbius strip
P = S^2 identifying diapodes ⇒ P  identifying diagonalpoints of equator ⇒ P = B^2_2 of Katuta.
So we have a disk, take out an interior of a disk ⇒

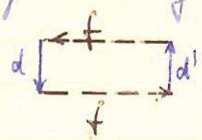


cut along c, d, b ⇒



Since a are to be identified, glue it together along a ⇒ Let $f = \overrightarrow{bc}$ ⇒





which is a Möbius strip

② Kleinbottle = P # P = M ∪ M

③ $P \# T = P$ with a handle sewn on

④ $P \# K = P$ with a handle sewn on, one end reversed

Homology groups

$H_0(nT) = Z$

$H_1(nT) =$ free abelian of rk $2n$

$H_2(nT) = Z$

$H_0(kP) = Z$

$H_1(kP) = \underbrace{Z + \cdots + Z}_{k-1} + Z_2$

$H_2(kP = 0$

Better distinction by Euler characteristic $\chi = 2 - 2n$ for nT

$\chi = 2 - k$ for kP

Unsolved Problem:

Classify closed connected 3-mf impossible for M^n, $n \geq 4$

S^3 acts as a zero, but one does not know whether it is the only one

Poincaré conjecture: If M^3 is a closed connected 3-mf $\ni \pi_1(M^3) = 0 \Rightarrow M^3 = S^3$

Poincaré thought: $H_*(M^3) = H_*(S^3) \Rightarrow M^3 = S^3$ wrong! 1898

therefore he invented the fundamental group 1899

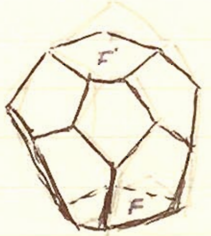
Expl 

$S^3 = K \cup_1 C \quad 1 : T \to T \quad T^2 = \partial K = \partial C$

$M^3 = K \cup_h C \quad 1 \neq h : T \to T$

Expl 2: The only known Poincaré sphere (M^3 closed connected 3-mf, $H_*(M^3) = H_*(S^3)$ with finite π_1 $\pi_1(M^3) \neq 0$).
It is called the dodecahedral space

Identify F with F' by $\frac{1}{10}$ twist

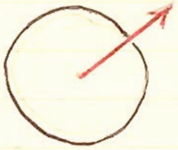


Other description: Consider $SO(3)$ = special orth gp
A_5 = alternating gp of degree 5
= symmetry gp of dodecahedron

$A_5 < SO(3)$ but not normal
Quotient space $SO(3)/A_5$ is the required 3-mf
(but not a gp)

$\pi_1(SO(3))$ order 2 $\quad$ $\pi_1(A_5)$ order 60 $\Rightarrow$ $\pi_1(M^3)$ order 120

$SO(3) \cong P^3$
rotations of S^2 = characterised by an axis and an angle $0 \leq \theta \leq \pi$. centre = identity
$\Rightarrow$ 3-ball and boundary identified antipodally



$$\Rightarrow$$
$$S^3 \longrightarrow P^3 = SO(3) \longrightarrow M^3$$
double covering $\quad$ 60-covering $\Rightarrow$ 120 covering

Liter. Seifert - Threlfall

Assume the following lemmas. Throughout the following M is a closed connected 2-mf

Lemma 1: Given $S^1 \subset M$, we can triangulate $M \ni S^1$ is a subcomplex (analysis)

Lemma 2: (geometry) $M = S^2 \Leftrightarrow S^1 \subset M$ seperates M (expl torus need not be seperated)

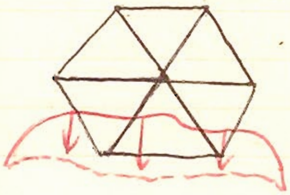
Lemma 3: (algebra) Let K triangulate M. Then χ = vertices - edges + faces $= \beta_0 - \beta_1 + \beta_2$
where β_i = Bettinumber of $H_i(M)$ = topol. invariant

Proof of the Classification theorems
Given $M \neq S^2$. Then by lemma 2 $\exists$ S^1 not seperating M. Cut it and fill in with 2 or one disk to get M_1

expl: 2 disks torus:  DI

1 disk Möbius 

These are the only possibilities, more than two is impossible because glueing two together closes it again, where to put a third circle. 0 is impossible, too. One immediately sees if using Lemma 1, that $\chi_1 = \chi + 1$ or $\chi + 2$ because

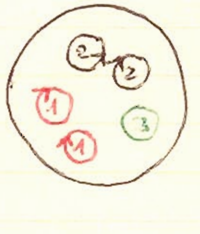


If $M_1 \neq S^2$ repeat & get M_2. If the circle hits the first disks push it out $\Rightarrow$ the disks are disjoint.

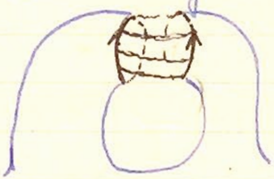
So we get a sequence of mfs $\{M_2\}$ and $\chi_{i+1} > \chi_i$

Now $\chi_i = \beta_0^i - \beta_1^i + \beta_2^i \le 2$ because $\beta_0^i = 1$ and $\beta_2^i = \begin{cases} 1 \\ 0 \end{cases}$ if orientable or not so after a finite number of steps we must stop. But then $M_n = S^2$

Now take the disks out and glue together again



Kleinbottle since twisted



Torus

Proj. plane

So $M = S^2 \# tT \# kK \# pP = \begin{cases} tT & \text{if } k = p = 0 \\ (2t+2k+p)P & \text{if } k \text{ or } p \neq 0 \end{cases}$ □

Lemma 2 will be proved later, Lemma 1 & 3 will not be proved.

Higher dimensions

$$\begin{tikzcd}
 & \text{top mf} \arrow[dl, "\text{triangulation}"'] \arrow[dr, "\text{smoothing}"] & \\
\text{combin mf} \arrow[dr] & & \text{diff mf} \arrow[dl] \\
 & \text{classification} & \\
 & \text{homology, homotopy} \arrow[u] &
\end{tikzcd}$$

dim	2	3	4	5
analysis	✓	Moise ✓ 1952	triangulation problem unsolved	
geometry	✓	X Poincaré unsolved	X	Poincaré conj solved Smale 1961
algebra	✓	nobody knows two mf which cannot proved homeo or not X	know 2 mf which we don't know whether they are homeom or not X	X
#	✓	1) wildness 2) orientation matters	1) & 2) 3) lack of triangul. 4) annulus conj.	

7 **Expl: Lens spaces $L(p,q)$:** Closed connected 3-mf. Take B^3 = 3-ball & identify ∂B^3 together by glueing the Northern hemisphere to the southern hemisphere by q/pth of a twist

To be presise: let λ be the longitude and φ be the magnitude, identify

$$(\lambda, \varphi) \sim (-\lambda, \varphi + \frac{2\pi q}{p})$$

Alternative definition:

$$S^3 = \{ (z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1 \} \quad \text{in } z \in \mathbb{C}$$

Let $h: S^3 \to S^3$ be the homeom given by $(z_1, z_2) \to (\omega z_1, \omega^q z_2)$ where $\omega = e^{2\pi i/p}$

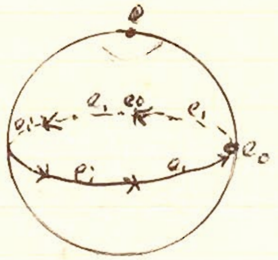
$h^p = 1 \Rightarrow h$ generates $\mathbb{Z}_p$ acting on S^3.

$$L(p,q) = S^3 / \mathbb{Z}_p \qquad p \geq 2 \quad 1 \leq q < p$$

Expl: Unsolved whether $L(7,1) \times S^1 \cong L(7,2) \times S^1$

Homology:

Expl: $L(5,1)$



Northern hemisphere identified with the southern hemisphere by twist is e_2
e_3 is the interior

cells $\quad e_3 \quad e_2 \quad e_1 \quad e_0$

chain gp $\quad \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{5} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$ $\quad$ one generator

$\partial e_1 = e_0 - e_0 = 0$

$\partial e_2 = 5 e_1$

$\partial e_3 = e_2 - e_2$ (southern + northern hemisphere, different orientation)

Homology: $\mathbb{Z} \quad 0 \quad \mathbb{Z}_5 \quad \mathbb{Z}$ $\quad$ by the chain gp

Cohomology: $\mathbb{Z} \quad \mathbb{Z}_5 \quad 0 \quad \mathbb{Z}$ $\quad$ reverse the arrows in the chain gp

To find the fundamental gp take a generator of 1-skeleton and attach 2-cell which gives the relation $\Rightarrow$

$$\pi_1(L(5,1)) = \mathbb{Z}_5$$

$$\left.\begin{array}{ll} \pi_1(L(5,1)) & = Z_5 \\ \pi_2 & = 0 \\ \pi_3 & = Z \\ \pi_4 & = Z_2 \\ \pi_5 & = \end{array}\right\}$$ because $\pi_i(L(p,q)) \cong \pi_i(S^3)$ $i \geq 2$ $\cong \pi_i(S^2)$ $i \geq 3$ since S^3 is the universal covering space of $L(p,q)$

$$H^p \otimes H^q \longrightarrow H^{p+q}$$

Multiplication in the cohomologyring trivial $\Rightarrow$ cohomologyring trivial

The lensspaces $L(5,1)$ and $L(5,2)$ are not homeom
but have the same homology, cohomology and homotopy gps

$L(5,3) \cong L(5,2)$ twist $3/5$ to the right $\equiv$ $2/5$ to the left
$L(5,4) \cong L(5,1)$ by the same reason

To distinguish them take homology over Z_5. We get $H_i(L, Z_5) = Z_5$ $\forall i = 0,-3$
compute it by the original homologygps
ringstructure again is trivial.
Let the generators be $1, x, y, z$ in each group.
In H^i, $i<3$ the choose of a generator is arbitrary. The choose of a generator in H^3 gives an orientation of the space. $\Rightarrow$ $z, 4z$ give orientation, $2z, 3z$ are different
In the cohomologyring: $x^2 = 0$, $y^2 = 0$, $xy = z$ by correct choose of y (theorem). This determines the ringstructure.
By ~~rings~~ the same argument we get the same ringstructure with $L(5,2)$

Passing from coeff Z to coeff Z_5 doesnot give the ringstructure in the new Cohom ring immediately

Bockstein operator: $\beta: H^1 \to H^2$
take exact sequence of coeff $0 \to Z_5 \to Z_{25} \to Z_5 \to 0$
β is the boundary in the corresponding exact sequence

Choose x and $x \cdot \beta x = \lambda z$ $(\lambda = \lambda(x))$, some λ after chosen z $\lambda = 1,4$
$2x$ $2x \cdot \beta(2x) = 4\lambda z$ " 4λ $\lambda = 2,3$

Choose $3x$	9λ	same 4λ
$4x$	16λ	λ

Blue for $L(5,1)$ red for $L(5,2)$. So we distinguish by λ
If we choose $-z$ instead of z we get $\lambda = 1,4$ for $L(5,1)$ but $\lambda = 2,3$ for $L(5,2)$

$L(7,1)$, $L(7,2)$

$L(7,1)$ and $L(7,2)$ have the same homotopy type
Therefore any algebraic invariants that are functors on the category of based spaces and homotopy classes of maps cannot distinguish between them

Two proofs for $L(7,1) \not\cong L(7,2)$

① Triangulate
Reidemeister or Whiteheadtorsions which are combin. invariants show different
Moise: any two triangulations were PL equiv.

② Brody (Annals 63): Choose embedding $S^1 \subset L$ unknotted in the sense that ~~the~~ $\pi_1(L - S^1) = Z$ and Alexander-polynomial $= 0$
Let λ be generator of Z. 4 possibilities $\lambda \to H(L, Z_7)$ gives two classes one for $(7,1)$ and one for $(7,2)$

Theorem: $L(p,q) \cong L(p',q') \iff \begin{cases} p = p' \\ qq' = \pm 1 \pmod p \end{cases}$ or $q = \pm q'$

$(p,q) = 1$
because $(\lambda p, \lambda q) \cong (p,q$

The proof $\Leftarrow$ is easy, we have done $\Rightarrow$
Expl: $L(7,2) \cong L(7,4)$
cone over p polygon in both directions, identify edge ~~a~~ with edge $n+q$ and identify lower cone with upper one.

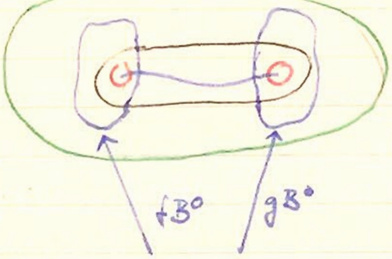
Unsolved problem: Theorem: $L_1 \times R^3 \cong L_2 \times R^3$ $(7,1), (7,2)$
Problem. Is 3 the least number $\ni L(7,1) \times R^n \cong L(7,2) \times R^n$
Equiv to question: $L(7,1) \times S^1 \cong L(7,2) \times S^1$?
$L(7,1) \times S^3 \cong L(7,2) \times S^3$?

$M_1^n \# M_2^n$ choose embeddings $B^n \overset{f_1}{\nearrow} M_1^n$, $B^n \overset{f_2}{\searrow} M_2^n$. Construct $(M_1 - f_1\mathring{B}) \cup (M_2 - f_2\mathring{B}) / \{f_1x = f_2x, x \in \partial B\}$ $\quad \#$

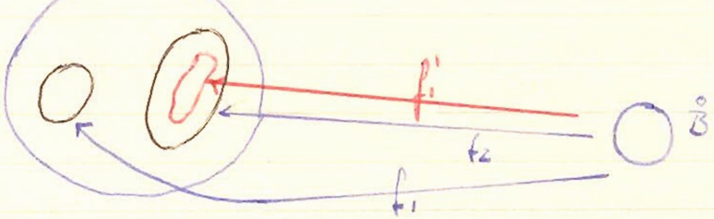
Quest: Given $f, g : B \to M$. Is $M - f\mathring{B} \cong M - g\mathring{B}$ (uniqueness of #)

Special: Assume further $gB \subset f\mathring{B}$

general case $\Longleftrightarrow$ special case



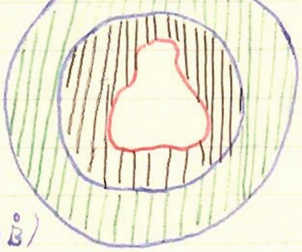
So if we discuss this question we assume that $gB \subset f\mathring{B}$



The special case would work if

① area between $fB - g\mathring{B} \cong S^{n-1} \times I$ and

② the larger sphere had a collar $S^{n-1} \times I$

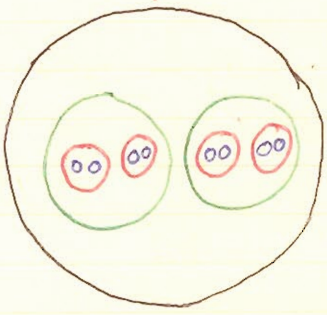


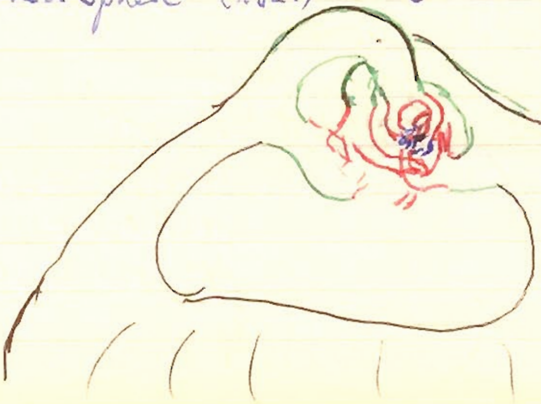
Then $\exists$ homeom h: green collar $\to$ green collar $\cup (fB - g\mathring{B})$

Extend to $\bar{h} : M - f\mathring{B} \to (M - f\mathring{B}) \cup (fB - g\mathring{B}) = M - g\mathring{B}$

① is in general false for $n \geq 3$

Counterexample: Alexander horned sphere (1924) $\quad S^2 \subset E^3$





" Take an embedding of S^2, take two disks and build them to horns, take to disks on the horn and make them to hornes and ~~[illegible]~~ link them.
This is of course an embedding and the fundamental gp of the outside is non trivial

Let A the outside of sphere then $\bar{A} = A \cup S^2$, $\pi_1(A) \neq 0$, $\pi_1(\bar{A}) = 0$, $\pi_1(B) = 0$ $B \cong E^3$ $\bar{B} = B \cup S^2 \cong B^3$

This stopped research till 1959 (arrogant Mazur and later Brown):
Given embedding $f: S^n \subset E^{n+1}$, supposed it is collared, that $\exists$ embedding $g: S^n \times I \to E^{n+1}$ $\ni f(x) = g(x, \frac{1}{2})$, then ex homeom $E^{n+1} \to E^{n+1}$ throughing $f S^n$ onto the standard sphere

Unsolved: Annulus Conjecture: Given collared $S^{n-1} \subset \mathring{B}^n$, we know closure of inside is ball
We do not know if closure of outside is an annulus, $n \geq 4$

Remark: Problem avoided by smooth or PL structure

Next problem: Suppose we glue the raw edges together with a different homeom $h_1: S^{n-1} \to S^{n-1}$ $\neq 1: S^{n-1} \to S^{n-1}$. Do we get the same result?

Remark: 1) Orientation problem $n \geq 3$
2) Unsolved in top case
3) False in smooth case
4) True in PL case

1) If we take an orientation reversing homeom, we might get another result.

Counterexample: Let $M^4 = CP^2 =$ complex projective plane
$H^0 = Z$, $H^1 = 0$, $H^2 = Z$, $H^3 = 0$, $H^4 = Z$
connected generator
compl proj line x
orientable mf y

In a ringstructure we have $x^2 = y$ (choosing y suitable)

y is preferred generator because $-y$ can not be a squared $\Rightarrow \nexists$ homeom of M onto itself reversing orientation. This cannot occur in a twodim. mf. $\Rightarrow$
$M \# M \ncong M \# (-M)$

2) given a homeom $h: S^{n-1} \to S^{n-1}$ orientation preserving. Is h then ~~homo~~isotopic to 1?
i.e. Is there a homeom $S^{n-1} \times I \xrightarrow{H} S^{n-1} \times I$ with $H(x,0) = (x,0)$, $H(x,1) = (hx,1)$

This is false in the smooth case: Counterexample S^6 Take a bad glue.

Chapter 1: Combinatorial Category

In E^p let $a_0, a_1, \dots, a_n$ be n+1 linearly independent points.
The simplex A spanning $a_0, -, a_n$ is defined to be the smallest convex set containing them. simplex
$A = \{x; ~~x\in~~ x = \sum \lambda_i a_i \ , \sum \lambda_i = 1 \ , \lambda_i \geq 0\}$
Interior $\mathring{A} = \{x \in A ; \lambda_i > 0\}$ Interior
Boundary $\partial A = A - \mathring{A}$ Boundary
Barycentre $\hat{A} = \frac{1}{n+1} \sum a_i$ barycentre
A face is any simplex B spanning a subset of a_i. Write $A > B$. B is a proper face if $A \neq B$ face
$\emptyset$ is (-1)-simplex is face of A $\emptyset$

Def: A finite simplicial complex, or more briefly a complex, is a finite collection of simplexes in E^p ∋ complex
i) $A \in K \Rightarrow$ all faces of A are in K
ii) $A, B \in K \Rightarrow A \cap B$ = common face (possibly empty)

dim K = max(dim A), $A \in K$ dim

Subcomplex L is a subset of K satisfying (i) subcomplex

Euclidean polyhedron |K| underlying K is $\bigcup_{A \in K} K$ eucl. polyh. |K|

We shall frequently abuse notation by using K for both K and |K|

A subdivision K' of K is a complex ∋ |K'| = |K| and every simplex of K' is subdivision contained in some simplex of K.

map

simplicial map

Maps: Abusing notation: $K \subset E^p$, $L \subset E^q$. A map $f: K \to L$ is a continuous map $f: |K| \to |L|$
Call f simplicial if it maps vertices to vertices and simplexes linearly to simplexes
i.e. $f(\Sigma \lambda_i a_i) = \Sigma \lambda_i f(a_i)$

isomorphism

graph $\Gamma(f)$

piecewise lin

Call f an isomorphism if it is a simplicial isom
The graph $\Gamma(f)$ is defined by $\Gamma(f) = \{(x, f(x));\ x \in |K|\} \subset |K| \times |L| \subset E^{p+q}$
Call f piecewise linear if $\Gamma(f)$ is a euclidean polyhedron in E^{p+q}
Call f piecewise linear embedding if $f(x) = f(y) \Rightarrow x = y$

Remark: Usually 1-1 does not imply f is an embedding in the topology category. If the sourcespace is compact, 1-1 is sufficient for embedding (i.e. homeom onto a certain subspace). Complexes are compact

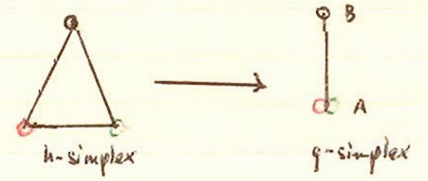
Disadvantage of simplicial maps: One can not slide around. Embedding $S^1 \to S^1 \times S^1$ keeps fixed since vertices to vertices.

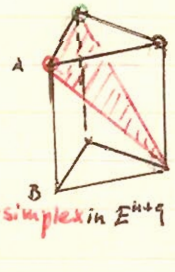
Def: The combinatorial category has $\begin{cases} \text{objects: finite simplicial complexes} \\ \text{morphisms: PL maps} \end{cases}$

We have to prove 1_K is PL. (trivial)
The composition of two pl maps is piecewise lin.

Lemma 1: A simplicial map is PL

Proof: The graph of a linear map from an n-simplex to a q-simplex is an n-simplex.



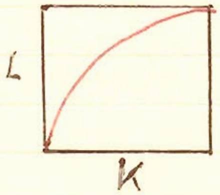


Given $f: K \to L$ simplicial
$\Gamma f = \bigcup_{A \in K} \Gamma(f|A)$ = union of simplexes
= euclidean polyhedron

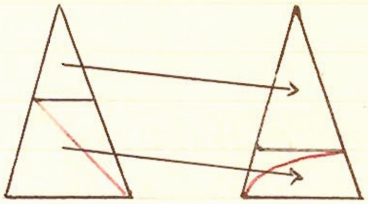
Standard Mistake:

The radial map $f: K \to L$ is not PL

$K \| L$



Since it is curved it can't underly a 1-dim euclidean polyhedron



top part pL, bottom part not

Attention with Guggenheim

Joins: $x, y \in E^p$, xy interval
Given $X, Y \subset E^p$ we say X, Y joinable if all intervals xy, $x \in X, y \in Y$ have disjoint interiors
If joinable, define join $XY = \bigcup_{x \in X, y \in Y} xy$. If X is a point call this a cone

joins

XY, cone

Expl: 

not joinable in E^3

If A^p, B^p are joinable simplexes then AB is a $p+q+1$ simplex
If K, L are joinable, the join complex $KL = \{AB;\ A \in K, B \in L\}$. Then $|KL| = |K||L|$

join complex

Def: A ~~complex~~ convex linear cell, or a cell, $A \subset E^p$ is a compact subset given by

$$\begin{cases} \text{linear equations } f_1 = 0, f_2 = 0, \dots, f_r = 0 & + \\ \text{linear inequalities } g_1 \geq 0, g_2 \geq 0, \dots, g_s \geq 0 \end{cases}$$

convex lin cell

Notice A is convex because it is the intersection of convex sets.

Def: $\dim A = n$, A cell, if the maximal number of independent points in A is $n+1$.
A face $B < A$ is a cell obtained by replacing some of the $g_i \geq 0$ by $g_i = 0$
∂A = union of all proper faces, interior $\mathring{A} = A - \partial A$

EXC: ① A is a cell, $x \in A$. Let B = union of all faces not containing x $\Rightarrow A = xB$, i.e a linear cell is a cone

② A = convex hull of its vertices Do vertices exist?

Solution: cell = compact subset of E^p def by $\begin{cases} f_1 = 0, \dots, f_r = 0 \\ g_1 \geq 0, \dots, g_s \geq 0 \end{cases}$

n-cell Face $B < A$. A is an n-cell if $\exists$ $n+1$ lin indep. pts in A and no more

0-cell = pt

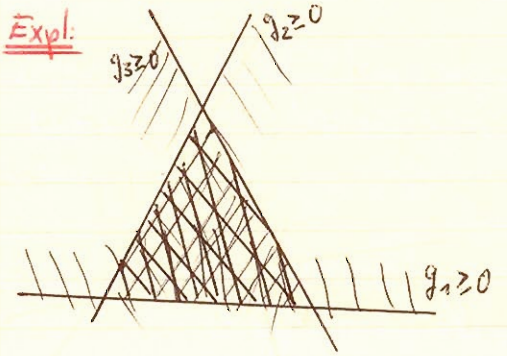
1-cell = 1-simplex

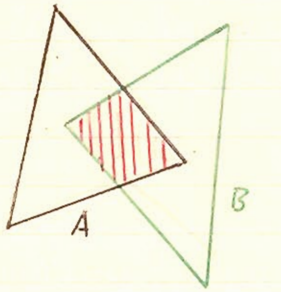
① $A \cap C$ = another cell, because $A \cap B$ is given by the union of the defining (in-)equalities

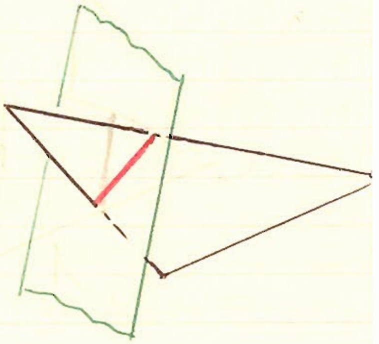
② $A \cap$ linear subspace = another cell

③ $A \subset E^p$, $B \subset E^q$ then $A \times B$ is a cell in $E^p \times E^q$. The defining equations give $(A \times E^q) \cap (E^p \times B)$

Expl:



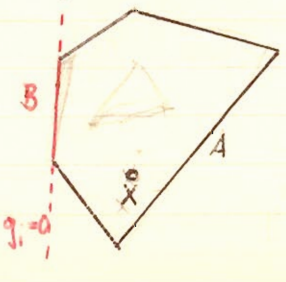




Lemma 2: $B < A$, $B \neq A \Rightarrow \dim B < \dim A$

not nec { **Proof** Let $a = \dim A$, $b = \dim B$. In A select $a+1$ indep points. Let Δ span these. Let E^a be lin space spanning them. So ~~$\Delta^a \subset E^a$~~ $\Delta^a \subset A \subset E^a$ $\therefore$ top dim $A = a$

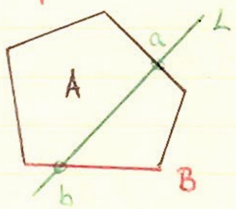
Since $B \neq A$ choose $x \in A - B$. Since B is given by replacing $g_i \geq 0$ by $g_i = 0$ choose $x \ni$ $g_i(x) \neq 0$



Choose $b+1$ indep pts in $B \subset [g_i = 0]$. Adding x gives one more indep point since $g_i(x) \neq 0$

Lemma 3: Let $A \cap$ line $L = ab \Rightarrow \exists\ B < A,\ B \neq A \ni B \cap L = b$

Proof



Let $g_i \geq 0$ be the inequalities in the def of A. Let $L_i = L \cap [g_i \geq 0]$ $= L$ or half line, each inequality exists, otherwise $a = b$ or $A = L$
it can't be empty since $g_i \geq 0$ contains A
Since $A \cap L_i = ab$ we have $\partial L_i = b$ for some i. Let B = face got by putting $g_i = 0$, $B \ni b$ and $L \cap B = b$ since $\partial L_i = \partial(L \cap g_i \geq 0) = b$ >>

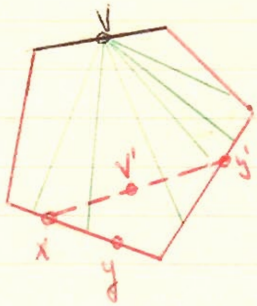
Lemma 4: Cell A = convex hull of its vertices = smallest convex set containing the vertices.
compact set

Cor: A^n must have at least n+1 vertices

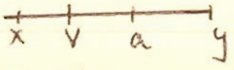
Proof: Let HA denote the convex hull of its vertices. But $HA \subset A$ because A is convex
We prove the other way by induction on n
True for $n = 0$. Assume $< n$.
We want to show $A \subset HA$. Choose $x \in A$ and a line through x meeting A in bc (interval bc), $b \in$ face $B = HB$ by lemma 2, 3 & induction, $HB = B \subset HA$. similarly $c \subset HA$
$x \in bc \subset HA$ since convex >>

Lemma 5: V point in A, X = union of faces not containing V then A = cone VX

Proof:



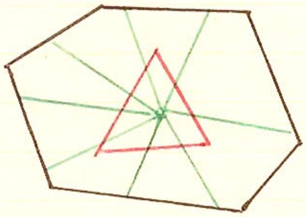
First show V joinable to X.
Suppose not, i.e. Vxy is a line L with $x \neq y$ and $x, y \in X$. Now $x \ni$ face $B \not\ni V$ by assumption. $A \cap L > B \cap L$, face and $A \cap L \neq B \cap L$ since $y \notin B \cap L$ but in $A \cap L$.* The only intersections with a line can be an intervall or a point, since dim $A \cap L >$ dim $B \cap L$, $B \cap L$ vertex and $A \cap L$ interval, (or empty) But $V, x, y \in B \cap L \Rightarrow$ contradiction
$\Rightarrow$ VX exists, $VX \subset A$ because A convex
Conversely we want to show $A \subset VX$. Given $a \in A$, $a \neq V$, let L be line joining Va
$L \cap A$ = 1-simplex because it contains two points
= xy say (possibly $x = V$, $y = a$)
choose $y \ni y \neq V$



* if $y \in B$ then $xy \in B$ since B convex, if then $V \notin B$, x, y, V lin indep ↯

By lemma 3 $\exists$ face $B \ni B \cap L = y$ $\therefore B =$ face of $A \not\ni V$ $\therefore y \in B \subset X$ $\therefore a \in Vy \subset VX$ >>

Cor: A is a top. n-ball, $(A, \partial A) \cong$ n-ball, (n-1) sphere (since it can be joined to a point)



Radial maps from an interior point of a simplex in the interior of A Possible since cone

{: Not PL map

cell complex

Cell complexes: A cell complex K in E^p is a finite collection of cells $\ni$

① If $A \in K$, all its faces are

② $A, B \in K$ then $A \cap B =$ common face

$|K|$

vertices

subdivision

$|K| = \bigcup_{A \in K} A$, vertices of K = 0-cells

K' subdivision of K if $|K'| = |K|$ + every cell of K' $\subset$ some cell of K

Lemma 6: Any cell complex can be subdivided into a simplicial complex without introducing any more vertices

Cor: (i) K cell complex $\Rightarrow$ $|K|$ euclidean polyhedron

(ii) The intersection of two euclidean polyhedra is another

Proof: Cor ii follows from Cor(i), $|K| = X$, $|L| = Y$, $K \cap L =$ cell complex$\{A \cap B;\ A \in K, B \in L\}$.

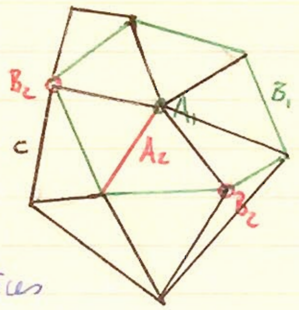
Cor iii: The product of two polyhedra is a polyhedron ($X \subset E^p$, $Y \subset E^q$, $X \times Y \subset E^{p+q}$)

Proof of Lemma 6: Order the vertices. By lemma 5 write each cell A = cone VB where V is the first cell in the ordering and B union of faces $\not\ni$ V. Subdivide cells inductively ↑ in dim, beginning trivially with vertices.
Given A then B = union of cells of lower dim (lemma 2), + so by induction B' is a simplicial complex + we can define $A' = VB'$ (this is a simplicial complex). This def is compatible with any face C of A, because either $C \subset B$ or C has V as his first vertex + $\therefore$ is already a cone with vertex V.

Theorem I: The composition of two PL maps is PL.

Proof: Given $K \xrightarrow{f} L \xrightarrow{g} M$, $|K| \subset E^p$, $|L| \subset E^q$, $|M| \subset E^r$. $\Gamma f \subset |K| \times |L| \subset E^{p+q}$ is an eucl. polyhedron, recall $\Gamma f = \{(x, f(x)); x \in |K|\}$
similarly Γg
we want to show that $\Gamma(gf)$ is an eucl polyhedron in $E^p \times E^r$.
Let $\Gamma = \{(x, fx, gfx); x \in |K|\} \subset E^p \times E^q \times E^r$
$= (\Gamma f \times |M|) \cap (|K| \times \Gamma g)$
$\subset$ trivial. Suppe $x, y, z \in$ RHS $\Rightarrow y = fx, z = gfx \Rightarrow \supset$ $z = gy = gfx$
RHS euclidean polyhedron by ~~collo~~ corollaries above $\therefore$
$\exists$ complex $J \ni |J| = \Gamma$
Let $\pi: E^p \times E^q \times E^r \to E^p \times E^r$ be the projection. In particular π maps Γ homeom onto $\Gamma(gf)$, because there is a 1-1 correspondence between (x, fx, gfx) and (x, gfx) parametrisised by x.
$\therefore \pi$ maps J homeom onto a complex $\pi J \subset E^p \times E^r$ and $|\pi J| = \Gamma(gf)$ $\therefore$ eucl polyhedron $\gg$
Work was done in Γ polyhedron + $\exists$ of J

$\Rightarrow$ We have the PL category



Def: Given a complex K and a simplex $A \in K$ the link of A in K is defined by
$lk(A,K) = \{B;\ AB \in K\}$
$A \subset |K|$ but not in lk

link

Remark: One cannot join a simplex to one of its vertices

Exc: $B = Av \Rightarrow lk(B,K) = lk(v, lk(A,K))$

Def: The closed star $\overline{st}(A,K) = A \cdot lk(A,K)$ a complex
the open star $st(A,K) = \bigcup_{A<B} \mathring{B}$ (not a complex)

star

Remark: $|\overline{st}(A,K)| = \bigcup_{A<B} B$. We often use $\overline{st}$ for the underlying polyhedron.
$\overline{st}(A,K) - lk(A,K) = st(A,K)$

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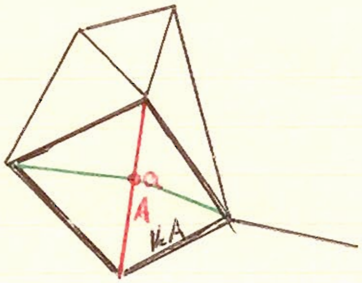
Expls of subdivisions:

stellar subdiv. (elementary)

① An elementary stellar subdivision
Given $A \in K$ simplex, $a \in \mathring{A}$, let $L = lk(A, K)$

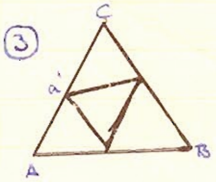
Define $K' = (K - AL) \cup a(\partial A)L$



stellar subdiv.

② A stellar subdivision of K, written σK, is the result of a finite no of elementary ones.
If $L \subset K$, then a stellar subdivision σK determines a unique stellar subdivision σL

Not stellar

③ 

Not stellar since stellar you must start with a vertex that gives you a line a'B which you cannot get rid of.

derivatives stellar

④ A first derived $K^{(1)}$ of K is a stellar subdivision obtained by stellar subdividing all simplexes of K in some decreasing order
Alternatively define $K^{(1)}$ upwards on the simplexes by inductive rule $A^{(1)} = a(\partial A)^{(1)}$

An r^{th} derived $K^{(r)}$ is $(K^{(r-1)})^{(1)}$

barycentric

The barycentric first derived is got by using barycentres (any two r^{th} derived barycentrics are isomorphic)

Lemma 7: $K \supset L \Rightarrow$ (i) any subdivision K' of K induces a unique subdivision L' of L
(ii) any subdivision L' of L can be extended (not uniquely) to a subdivision K' of K

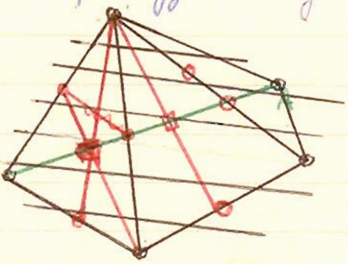
Proof: (i) trivial. (ii) Subdivide simplexes of $K - L$ inductively ↑ (upwards in dim) by the rule $A' = a(\partial A)'$

Cor Given $f: K \to L$ simplicial embedding and given subdivision K' of $K \Rightarrow \exists$ subdivision L' of $L \ni f: K' \to L'$ simplicial

Lemma 8: $|K| \supset |L| \Rightarrow \exists$ r^{th} derived $K^{(r)}$ of K and a subdivision L' of $L \ni L'$ a subcomplex of $K^{(r)}$

Proof: Induction on the no of simplexes in L. Begins trivially with $L = \emptyset$
Choose A = principal simplex in L i.e. a simplex which is not a face of a bigger one. By induction choose $K^{(r-1)} \ni$ subdivision $L - A$. $A \cap B$ = cell $\forall$ $B \in K^{(r-1)}$. So form $K^{(r)}$ by staring each simplex B in $K^{(r-1)}$ at a point b in $\mathring{A} \cap \mathring{B}$ if $\mathring{A}$ meets $\mathring{B}$ and at any point otherwise

principal simplex.



In particular the cell complex $A \cap K^{(r-1)}$ has been subdivided into a simplicial complex $A' \subset K^{(r)}$ $\therefore$ $L' \subset K^{(r)}$
If A had been a face it is subdivided by the induced subdivision of the bigger cell

Cor 1: $|K| = |L| \Rightarrow$ they have a common subdivision $K^{(r)} = L'$

Unsolved problem: $|K| = |L| \Rightarrow \exists$ stellar subdivision $\ni \sigma K = \sigma L$

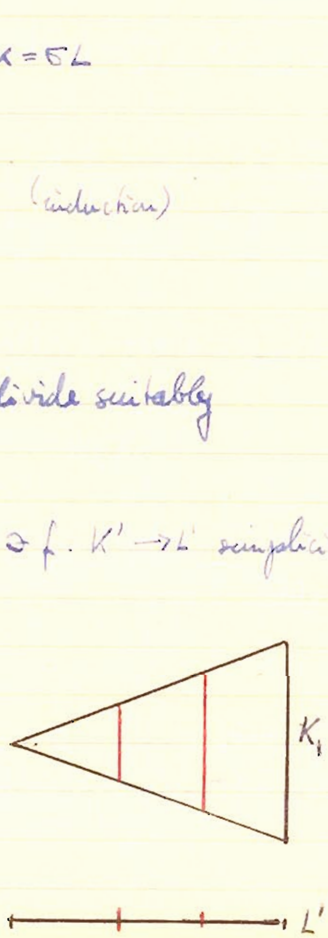
Cor 2: $|K| \supset |L_i|$ $i = 1, 2, \dots r \Rightarrow \exists K^{(r)} \supset L_i$ $\forall i$ (induction)

Cor 3: The union of two polyhedra is another (incl.)

Regard both polyhedra in a large simplex. Then subdivide suitably

Lemma 9: $f: K \to L$ simplicial, subdivision $L' \Rightarrow \exists K' \ni f. K' \to L'$ simplicial

Proof: Let $K_1 = f^{-1} L'$, a cellcomplex subdividing K
By Lemma 6 subdivide K_1 into a simplicial complex K' introducing no new vertices.
This gives a simplicial map $K' \to L'$
(since for each two points the intervall between them is mapped linearly

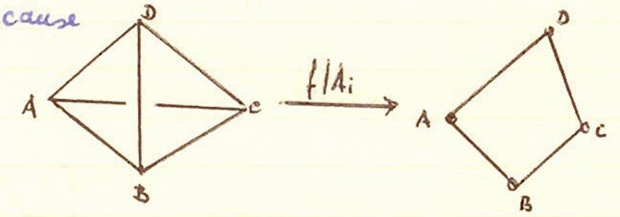


Remark: The dual is not true, i.e. a subdivision K' cannot be pressed onto L

Def: A map $f: K \to E^q$ is linear if each simplex is mapped linearly (not nec embedded)

Lemma 10: Given a map $f: K \to L \ni$ the composition $K \longrightarrow L \subset E^q$ linear $\Rightarrow \exists K', L' \ni$
$f: K' \longrightarrow L'$ simplicial

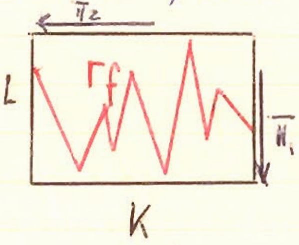
Proof: For each simplex $A_i \in K$ let $B_i = f A_i$ which is a cell in E^q (need not be a simplex because



But it is a cell because it is the convex hull of the vertices since linear $\therefore |B_i| \subset |L|$
by lemma 8
Choose $L' \ni$ simplicial complex B_i' subdiv. B_i, each i. Then $f^{-1}B_i'$ is a cell complex subdiv A_i (prisms intersecting with A_i). Better $(f|A_i)^{-1}B_i'$
The union $f^{-1}L'$ is a cell complex subdiv. K. By lemma 6 choose a simplicial subdivision K' of $f^{-1}L'$ with the same vertices $\Rightarrow f: K' \to L'$ simplicial

Theorem 2: $f: K \longrightarrow L$ is piecewise linear $\Leftrightarrow \exists K', L' \ni f: K' \longrightarrow L'$ simplicial.

Proof: $\Leftarrow$ lemma 1. Proof $\Rightarrow$: Given $f: K \to L$ piecewise linear i.e. $\Gamma f \subset |K| \times |L|$ is a eucl. polyhedron in E^{p+q}, i.e. $\exists$ complex $M \ni |M| = \Gamma f$. Projection π_1 onto the first factor $\pi_1: E^{p+q} \to E^p$



is a linear map being a projection and maps Γf homeom onto $|K|$. i.e. $\pi_1: M \to E^p$ is linear. By lemma 10 $\exists$ subdivision M', K' $\ni \pi_1: M' \to K'$ is a simplicial isomorphism
Similarly $\exists$ subdivisions M'', L'' of $M, L \ni \pi_2: M'' \to L''$ is simplicial map. Let K'' be the isomorphic subdiv of K'. The result is a simplicial isom $K'' \longrightarrow M''$ and a simplicial map $M'' \longrightarrow L''$. The composition is f made to a simplicial map.

Remark: If we had used Theorem 2 as definition of PL maps we would have had trouble with theorem 1

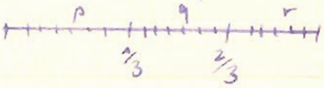
Counter example: Given a homeom $h: M \to M$ then in general $\nexists$ subdivision $\ni h: M' \to M'$ is simplicial.

Exc: If h periodic, i.e $h^p = 1$ for $p \in \mathbb{N}$, then $\exists$ subdivision as above (right)

Counterexample:

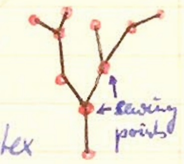
$1/3 \to 2/3$, $[0, \frac{1}{3}]$ linearly to $[0, \frac{2}{3}]$ and $[\frac{1}{3}, 1]$ to $[\frac{2}{3}, 1]$

$\nexists$ subdivision: suppose $\exists M'$



between $0, \frac{2}{3}$ $p+q+1$ points

Def: A tree is a 1-complex defined inductively:
a tree with 0-edges = point
a tree with n-edges = tree with $(n-1)$-edges together 1 edge sewn on by 1 vertex



tree

Lemma 11: Following 4 conditions are equivalent for T

① T is a tree
② T is a contractible 1-complex
③ T is a connected 1-complex containing no loops
④ T is a connected 1-complex ~~contain~~ and $\chi(T) = 1$

Proof: ① ⇒ ② induction on n, contract along the 1-edges
② ⇒ ③ since contractible ⇒ connected ⇒ $\pi_1 = 0$ ⇒ no loops
③ ⇒ ① induction: true $n = 0$. $\exists$ at least one free vertex at the end of only 1 edge, otherwise $\exists$ loop. Remove this vertex and edge, by induction what is left is a tree ∴ whole thing is tree
① ⇒ ④ χ = vertices − edges $= v - e = 1$ (for point $\chi = 1$, by induction we add a vertex with each edge ⇒ stays 1)
④ ⇒ ① $\exists$ one free vertex for if not then $e \geq \frac{2v}{2}$ ⇒ $\chi \leq 0$ rest by induction

oriented tree

one-way tree

Def: An oriented tree is got by sticking arrows on. A one-way tree is an oriented tree in which each vertex tails at most one edge

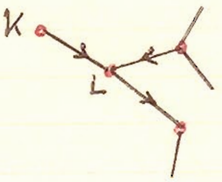
Lemma 12: In a one-way tree ∃ exactly one vertex that tails no edge + all other vertices tail one edge
If tree ≠ point ∃ at least one vertex that tails one edge + heads no edge

Proof: 

map each edge to its tail ⇒ monom. Now $v \to v+1$. Start anywhere and go backwards >>

Consider the categ $\mathcal{C}$ of complexes and PL maps
Given a finite set T in $\mathcal{C}$, the diagram of T in $\mathcal{C}$ is got by replacing complex – point and PL map – oriented edge
We call T simplicial if the maps are simplicial
We call T a (one-way) tree if the diagram is a (one-way) tree
" " T' a subdivision of T if have the same diagram, same maps + complexes of T' are subdivisions of complexes of T

Theorem 3: If T is a one-way tree in $\mathcal{C}$ then ∃ simplicial subdivision T'
If all the maps are embeddings we can drop the "one-way" hypothesis

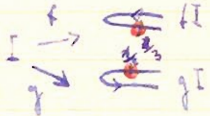


Proof: By induction. Trivially true for $n=0$
Assume for $n-1$. Select $f: K \to L$ + K not involved in any other maps (∃ by lemma 12).
Let T_* be the subtree obtained by forgetting K, f.
By theorem 2 choose subdivisions K', L' ∋ $f: K' \to L'$ simplicial. Let T_*' be the sudivision of T_* got by replacing L by L'. Let T_*'' be a simplicial subdivision of T_*' by induction
Let L'' be the corresponding subdivision of L'. Lemma 9 is carefully tailored for this purpose
Choose K'' ∋ $f: K'' \to L''$ simplicial. Then $T'' = T_*'' \cup (K'', f)$ simplicial >>

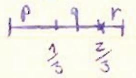
Part 2) the same proof except of f going the other way. Instead of lemma 9 use Cor to lemma 7 because f is an embedding >>

Counter example:

Given two maps $$\begin{tikzcd} & I \\ I \arrow[ur] \arrow[r] & I \end{tikzcd}$$ one cannot subdivide in general ∋ both are simplicial

fold it 

subdivide again $p = q + 1 + r$ above, $\cancel{p}\ r = p + 1 + q$ below $\Big\} \Rightarrow q = -1$

$I =$ 

$f: 0 \to 0 \quad \frac{1}{3} \to 1 \quad [\frac{1}{3}, 1]$ back again
$g: 0 \to 0 \quad \frac{2}{3} \to 1 \quad [\frac{2}{3}, 1]$ " "

Chapter II Manifolds

Def: A PL manifold is (for this course) a compact mf with an atlas $f_i : \Delta^n \to M^n$ (Δ simplex) ∋ every point has some $f_i\Delta$ as a nbh and the overlaps are PL, i.e. the f_i are embeddings and $P_{ij} = f_i^{-1}(f_i\Delta \cap f_j\Delta)$ is a polyhedron in Δ and $f_j^{-1} f_i : P_{ij} \to P_{ji}$ is PL — PL mf

X top space. A triangulation of X is a homeom $t: K \to X$ from a euclidean polyhedron $|K|$ onto X. If $\exists t$, call X triangulable ($\Rightarrow$ X compact Hausdorff) — triangulation, triangulable

Two triangulations t_1, t_2 are compatible if $t_2^{-1} t_1$ is PL — compatible

An equivalence class of triangulations is called a PL structure on X — PL structure

A polyhedron (not nec eucl), or PL-space, is a top space X together with a PL-structure — PL-space, polyhedron

Hauptvermutung: Given two structures on X, equivalently two non-compatible triangulations $\begin{smallmatrix} K_1 \searrow \\ \quad X \\ K_2 \nearrow \end{smallmatrix}$. Then does there exist some PL homeom $h: K_1 \to K_2$?

False : Milnor 1961: Counterexample $L(7,1) \times D^3 \cup$ cone on boundary $L(7,2) \times D^3 \cup$ cone on the boundary. They are top homeom. These are mf except of the conepoint. The proof was done with bad point

Unsolved: Hauptvermutung for mf: Let M^3 = closed 3-mf, homology S^3, $\pi_1(M^3) \neq 0$
Join $S^1 \cdot M^3 = S^5$ topologically.

25 Shift from the Hauptvermutung to other questions, Čech-homology 1930, singular Homology 1942

Čech: Free gp of corresponding generators for each circle
Singular: Wild gp

PL-invariant **Def:** A property of X is a PL-invariant if it depends only on the structure & not on the particular triangulation

PL map A PL-map $f: X \to Y$ is a map ∃ for some chosen triangulation $t_2^{-1} f t_1^{+1}$ is PL

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ K_1 \arrow[u, "t_1"] & K_2 \arrow[u, "t_2"'] \end{tikzcd}$$

Lemma 13: The definition of PL-maps is invariant

Proof: Given other triangulations (K_1', t_1') and (K_2', t_2') then $t_2'^{-1} f t_1' = (t_2'^{-1} t_2)(t_2^{-1} f t_1)(t_1^{-1} t_1')$ PL by theorem I since all three maps are PL »

triangulate f Given PL-map $f: X \to Y$. When we say 'triangulate f' this means choose triangulations $K_1 \xrightarrow{t_1} X \xrightarrow{f} Y \xleftarrow{t_2} K_2$ ∋ $t_2^{-1} f t_1$ is simplicial

PL-category **Def:** The category of PL-spaces (polyhedra) and PL-maps is called PL-category

In the PL-category we can define joins and products

Expl: top spaces X, Y which are triangulable
Take the top join $X * Y$ and the simplicial join $K_1 \cdot K_2$
The join in $K_1 \cdot K_2$ gives three coord.
$x \in K_1$, $y \in K_2$, $t \in xy$ ⇒ homeo to $X * Y$

$$\begin{tikzcd} X & & Y \\ K_1 \arrow[u] & & K_2 \arrow[u] \\ E^p \arrow[u, hook] & & E^q \arrow[u] \\ & E^{p+q+1} \arrow[ul, hook] \arrow[ur] & \end{tikzcd}$$

K_i ~~top~~ eucl. polyhedra

Join

Product: $K \to |K_1| \times |K_2| \to X \times Y$
cell compl in E^{p+q} top prod

If $X = |K|$ a euclidean polyhedron. Then the natural structure of X is given by identity triangulation $K \xrightarrow{1} X$ 26

Remark: The mapping cylinder has no natural structure

Def: A PL n-ball$^{B^n}$ is a PL-space ~~which is a~~ whose structure contains a triangulation by an n-simplex PL n-ball B^n

A PL n-sphere S^n " " " by the boundary of an (n+1)-simplex PL-n sphere S^n

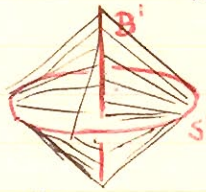
Lemma 14: $B^p \cdot B^q = B^{p+q+1}$, $B^p \cdot S^q = B^{p+q+1}$, $S^p \cdot S^q = S^{p+q+1}$

Proof: 1) $\Delta^p \cdot \Delta^q = \Delta^{p+q+1}$ see expl of join

2) We have to show $\Delta^p \cdot \partial \Delta^{q+1} \cong \Delta^{p+q+1}$. ~~By induction up~~

$\Delta^p \cdot \partial \Delta^{q+1} = \Delta^{p-1} \cdot v \cdot \partial \Delta^{q+1} \cong \Delta^{p-1} \Delta^{q+1} = \Delta^{p+q+1}$

PL but not iso since  different triangulations

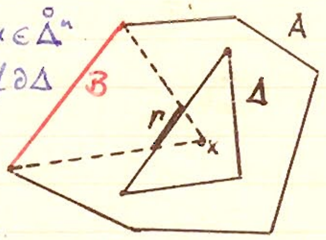
Expl for 2) 

$\Rightarrow$ 3 ball other configurations not joinable 

3) $\partial \Delta^p \cdot \partial \Delta^{q+1} \overset{2}{\cong} \partial(\Delta^p(\partial \Delta^{q+1})) \cong \partial \Delta^{p+q+1}$

Lemma 15: A convex linear n-cell, with its natural structure, is an n-ball

Proof: Choose an n-simplex in the interior of the n-cell A, choose $x \in \mathring{\Delta}^n$.



Take radial map ($\nexists$ not PL) let $(\partial\Delta)'$ be the cell subdivision of $\partial\Delta$ consisting of all cells of the form $\Gamma \cap xB$, $\Gamma < \Delta$, $B < A$ faces.

Let $(\partial A')$ be the isomorphic subdivision of ∂A, cells $x\Gamma \cap B$

Let $(\partial\Delta)''$ be the simplicial subdivision using same vertices.

Let $(\partial A)''$ be the isomorphic subdivision. Then let $f : (\partial\Delta)'' \to (\partial A)''$ be the simplicial isom. "pseudo radial projection". Extend $x(\partial\Delta)'' \to x(\partial A)''$. This is a PL pseudo radial homeom $\Delta \to A$

Lemma: The join XY of two polyhedra is invariant, i.e. it does not depend on the special triangulation of X and Y

Proof: $KL = \{AB;\ A \in K, B \in L\}$ ~~then~~. K, L joinable in E^p if the intervals xy, $x \in |K|$, $y \in |L|$ has disjoint interiors

Deduce $|KL| = \{ty + (1-t)x;\ x \in |K|, y \in |L|, t \in I\}$

Def of the top join (for emphasis we write $*$), $X * Y = X \cup X \times I \times Y \cup Y / \varrho$

where ϱ is the equivalence $x = (x, 0, y)$, $y = (x, 1, y)$

For $|KL|$ we used the ~~top~~ vectorspace structure of E^p, not so for $X * Y$

Deduce

$$|K| * |L| \longrightarrow KL$$

by

$$(x, t, y) \longmapsto (1-t)x + ty$$

is a homeom.

given $f: X \to X_1$ $g: Y \to Y_1$. Define $f * g$: $X * Y \to X_1 * Y_1$

$$(x, t, y) \mapsto (fx, t, gy)$$

Deduce if $\left.\begin{matrix} f: K \longrightarrow K_1 \\ g: L \longrightarrow L_1 \end{matrix}\right\}$ are PL then $f * g$ is also

because $\Gamma(f*g) = \{((1-t)x + ty, (1-t)fx + tgy);\ x \in |K|, y \in |L|, t \in I\} < E^{p+q}\}$

$= \{(1-t)(x, fx) + t(y, gy)$; " " $\}$ vectorspace

$= (\Gamma f)(\Gamma g)$ euclidean join & so $(\Gamma f)(\Gamma g)$ eucl polyhedron

Finally give polyhedra X, Y. Choose triangulations $t: K \to X$, $u: L \to Y$ in the structures. Then $t * u: KL \to X * Y$ is a triangulation of the top. join and hence determines a unique structure. This is invariant because if t_1, u_1 are other choices then

$$(t_1 * u_1)^{-1}(t * u): KL \to X * Y \longrightarrow K_1 L_1$$

$$\shortparallel$$

$$(t_1^{-1} * u_1^{-1})(t * u) = (t_1^{-1} t * u_1^{-1} u) \text{ which is PL}$$

>>

Notation: A simplicial isom between complexes will be $K \equiv L$

A PL homeom between polyhedra $X \cong Y$

Remark: $K \equiv L \overset{\Longrightarrow}{\not\Longleftarrow} |K| \cong |L| \Rightarrow K \cong L$

comb mf **Def:** A combinatorial manifold K of dim n is a complex in which the link of every vertex is an $(n-1)$-ball or sphere.

Expl: Δ and $\partial\Delta$

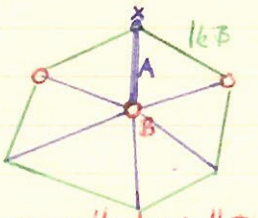
Lemma 16: Any triangulation of an n-ball or n-sphere is a combinatorial n-manifold

You take a PL-ball, i.e. $\exists$ one triangulation with just Δ. Take any other triangulation then it is a n-comb mf

Lemma 17: In a comb n-mf, the link of any p-simplex is an $(n-p-1)$-ball or sphere

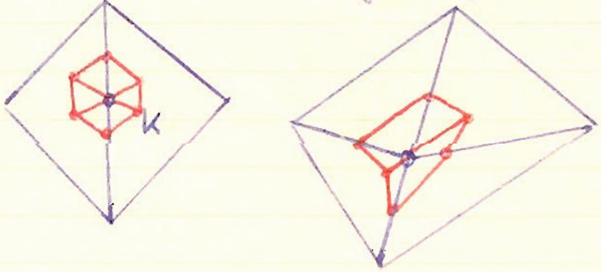
Lemma 18: If K is a combinatorial n-mf + $f: K \cong L$, then L is also

Proof: $(16(n-1)) \Rightarrow 17n \Rightarrow 18n \Rightarrow 16n$. Proof by induction, starting trivially with $n=0$ and $17n$



16(n-1) ⇒ 17n : By induction on p. True for $p=0$ by def^n of $\emptyset$
Assume for dimensions less than p true:
Given A^p, $p>0$, write $A = xB$. Then $lk(A,K) = lk(x, lk(B,K))$
RHS. $lk(B,K)$ is a $(n-p)$-ball or sphere by induction on p
and so an $(n-p)$-~~ball~~ mf by our main ind^n $16(n-p)$

17n ⇒ 18n: Given vertex $y \in L$, we have to prove $lk(y,L) = (n-1)$ ball or sphere.
By theorem 2 choose subdivisions $\ni K' \equiv L'$
Now y is a vertex of L', $f^{-1}y = x$, say, is a vertex of K'. But x may not be a vertex of K.



Suppose $x \in \mathring{A}$, $A \in K$.

$$\partial A \cdot lk(A,K) \underset{\uparrow \text{pseudo radial proj.}}{\cong} lk(x,K') \equiv lk(y,L') \cong lk(y,L)$$

Now $lk(A,K)$ is a ball or sphere by $17n$
∂A is a sphere. So $\partial A \cdot lk(A,K)$ is a ball or sphere by lemma 14

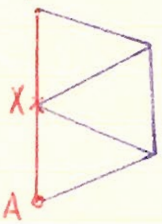
combin. m-mf ∂M **Def:** If M is a comb m-mf, define $\partial M = \{A \in M;\ lk(A,M) = \text{ball}\}$, $\mathring{M} = M - \partial M$

Lemma 18: $f: M \cong Q \Rightarrow f(\partial M) = \partial Q$. Hence the boundary is invariant

Proof: Given $x \in |M|$, then $x \in |\partial M|$ iff $lk(x, M') =$ ball in any subdivision in which x is a vertex.
~~Choose M'~~ iff $lk(fx, Q') =$ " " " " fx "
iff $fx \in |\partial Q|$ $\gg$

Lemma 20: If M is a comb m-mf, then ∂M is a closed comb $(m-1)$-mf (closed = without boundary)

Proof: To show $x \in \partial M \Rightarrow lk(x, \partial M) = (m-2)$-sphere. This would prove the whole lemma.



Now $lk(x, \partial M) = \{A \in lk(x,M);\ xA \in \partial M\}$
$= \{A \in lk(x,M);\ lk(xA, M) = \text{ball}\}$
But $lk(xA, M) = lk(A, lk(x, M))$
so $lk(x, \partial M) = \{A \in lk(x,M);\ A \in \partial(lk(x,M))\}$
$= \partial(lk(x,M))$
But because $x \in \partial M$, $lk(x,M) = (m-1)$ball and so $\partial(lk(x,M)) = S^{m-2}$ by lemma 19.

PL-mf **Def:** A PL-mf = PL-space, whose structure contains one (and $\therefore$ all by the lemma) comb. mf (all since $K \cong L$ implies ~~$K' \cong L'$ by~~ L' comb too by lemma 18)

compatible Let X be a Hausdorff space. Let $f: K \to X$, $g: L \to X$ be embeddings of Euclidean polyhedra. Say f, g compatible if $g^{-1}fK$, $f^{-1}gL$ are subpolyhedra of L, K and $g^{-1}f: f^{-1}gL \to g^{-1}fK$ is PL

n-atlas An n-atlas on M is a compatible family (i.e. pairwise compatible maps) $f_i: \Delta^n \to M$ that covers in the following sense:
$\forall x \in M\ \exists i \ni f_i\Delta$ is a closed nbh of x in M.

Theorem: ① A PL-mf has a finite atlas compatible with the structure (this atlas is not unique)

② If the topol. space M has a finite atlas then this atlas determines a unique PL structure on M, and M is a PL-mf

Remark: Extends to non cpt case with countable triangulation and contable atlas

Proof of theorem ① Given PL-mf, choose triangulation vertex stars give atlas

② Given atlas; we shall construct inductively an embedding $g_i : K_i \to M$ compatible with the given atlas $\mathfrak{A}$, $\ni g_i K_i = \bigcup_{1 \leq j \leq i} f_j \Delta_j$

Start with $g_1 = f_1 : \Delta \to M$, $K_1 = \Delta$. We want to finish with a triangulation

Lemma: Given $f : K \to M$ ~~and $g : L \to X$ which are compa~~, f embedding compatible with the atlas $\mathfrak{A}$, then $\exists g : L \to M$ compatible with $\mathfrak{A}$ $\ni gL = fK \cup f_i\Delta$
(i.e. g helps by means of a new complex to glue i^{th} balls onto $(i-1)$-balls)

Proof: Let $K^* = f^{-1} f_i \Delta$, $\Delta_* = f_i^{-1} f K$. Let $h = f_i^{-1} f_1 : K_* \to \Delta_*$. By compatability the diagram of embeddings

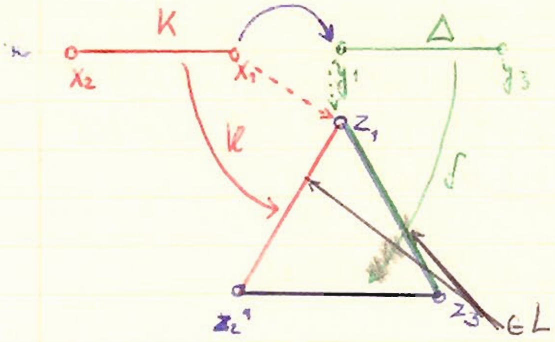
$$\begin{tikzcd} K & K_* \arrow[l, "\supset"'] \arrow[r, "h"] & \Delta_* \arrow[r, "\subset"] & \Delta \end{tikzcd}$$

(no one way tree)

is PL. By Theorem 3 choose subdiv K', Δ' with subcomplexes $K_*, \Delta'_* \ni h : K'_* \cong \Delta'_*$
(What we are doing is mainly based on the fact that the union of two PL subspace is again a PL subspace)

Let K'_* have vertices $x_1, \dots, x_\lambda$
K' $\quad x_1, \dots, x_\lambda, \dots, x_\mu$
Δ'_* $\quad y_1, \dots y_\lambda \qquad y_i = hx_i$
Δ' $\quad y_1, \dots y_\lambda \qquad y_{\mu+1}, \dots y_\theta$

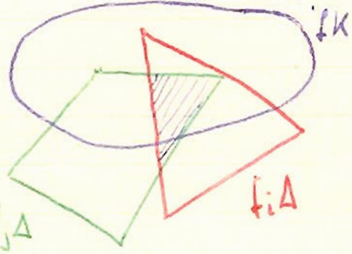
Choose new simplex A with vertices $z_1, \dots, z_\theta$
Let $\kappa : K \to A$ be the simplicial embedding given by $x_i \mapsto z_i$.
Let $\delta : \Delta \to A$ " " " $y_i \mapsto z_i$ Hence:
Let $L = \kappa K \cup \delta \Delta$, (We glue together in following way)



Define $g: L \to M \ni$ following diagram commutes

$$\begin{tikzcd} K \arrow[d, "\kappa"'] \arrow[rd] & \\ L \arrow[r] & M \\ \Delta \arrow[u, "\delta"] \arrow[ru] & \end{tikzcd}$$

Remains to verify that g is compatible with $\mathcal{A}$.



We know f, f_j compatible, f_i, f_j compatible. Are f_j and f compatible?

given f_j, the graph $\Gamma(f_j^{-1}g)$ of the overlap map in $L \times \Delta$ is the union of two subgraphs in $\kappa K \times \Delta$ and $\delta\Delta \times \Delta$. In each graph of the ~~subbets~~ we have graphs of compat of f, f_i + f_i, f_j

$\kappa K \times \Delta$ and $\delta\Delta \times \Delta$ are already PL, but the union of two polyhedra is another.

Verify the uniqueness of the structure i.e we veryfy that if two triangulations $f: K \to M$, $g: K \to M$ are both compatible with $\mathcal{A}$, then they are ~~[illegible]~~ compatible ∴ determine the same structure.

Proof: Let $K_i = f^{-1}f_i\Delta$ (which is a ball in K). Let $h = g^{-1}f: K \to L$. Then $h|K_i = g^{-1}f_if_i^{-1}f|K_i$ = product of PL map by the compatability of f and g with the atlas ∴ $h|K_i$ is PL. Then $\Gamma(h) = \bigcup_i$ ~~PL~~ $\Gamma h_i|K_i$ = union of polyhedra = polyhedron ∴ h PL >

Last thing to check is that M is a PL-mf, i.e. the structure is a PL-mf structure, i.e that some triangulation and hence all is a comb. mf.

$$\begin{tikzcd} \Delta \arrow[rd] & \\ \Delta \arrow[r] & K \\ \Delta \arrow[ru] & \end{tikzcd}$$

Choose triangulation $f: K \to M$. since this is a one-way tree subdivide $K' \ni$ each ball K_i is a subcomplex. Each vertex $x \in K'$ has some K_i' as a nbh. ∴ $lk(x, K') \cong lk(x, K_i')$ since K_i' nbh of x. But this is a $(m-1)$ sphere or ball because K_i is a m-ball >>

ambient isotopy **Def:** An ambient isotopy of M is a PL-homeom $h: M \times I \to M \times I$ that is levelpreserving, i.e $h(x,t) = (h_tx, t)$ + $h_0 = 1_M$

Other possibility

$$\begin{tikzcd} M \times I \arrow[rr, "h"] \arrow[rd, "proj"'] & \text{©} & M \times I \arrow[ld, "proj"] \\ & I & \end{tikzcd}$$

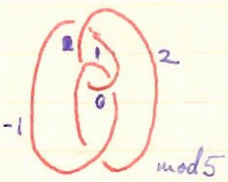
and $h_0 = 1_M$

PL is selfunderstood in the following.
We need levelpreserving to get a PL homeom in the x and t coord., otherwise it need not be PL in the t-coord. If we want ~~only~~ PL hom only on the top and the bottom, this is called a cob~~or~~dism. In codim≥3 isotopy = cobordism (see Hudson seminar)

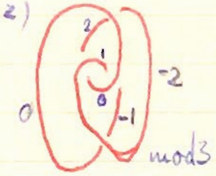
ambient isotopy

Def: We say the isotopy h keeps X fixed, if $h_t | X = 1$, $X \subset M$
Two embeddings are are ambient isotopy if ex an ambient isotopy combining them $(h_1 f = g)$
Two subspaces $X, Y \subset M$ are called ambient isotopy if $\exists h \ni h_1 X = Y$

1)



2)



These knots are not ambient isotopic subspaces
Not ambient isotopic embeddings are homotopic, shrink to a point ~~and open~~ both of them.
Distinguish these knots by the fundamental grp of the complementary space. 1) π_1(complement) = {a, b; } 2) π_1(complement) =
Quicker way, start somewhere label with 0 and 1 the next crossing label from that on so that the avarage of following arcs is the label of the ~~one~~ crossing. You have to work mod something to finish up in a good way; if different modolo then different knots

Knot K, regular presentation P (no triplepoints)

n-colourable

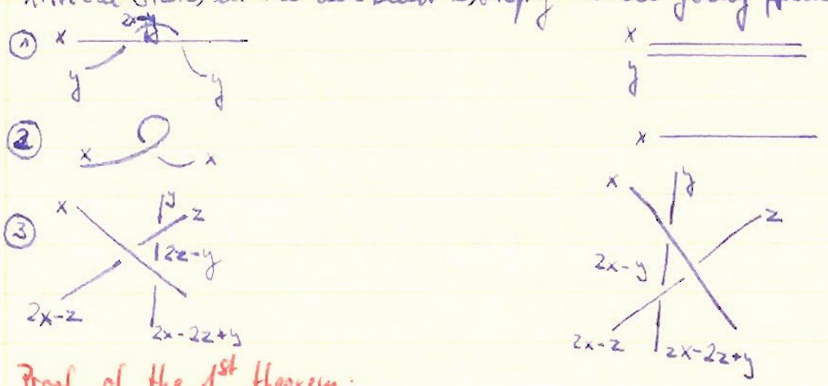
Def: P is n-colourable if ∃ map $\varphi: P \to Z_n \ni$
(i) im φ contains at least two numbers
(ii) each overpass is mapped by φ to the avarage of the two adjacent underpasses.

Theorem: n-colourability is an invariant of K (indep of P)

Theorem: K n-colourable ⇔ n divides $\Delta(-1)$, when $\Delta(t)$ Alexanderpolinomial.
⇔ ∃ homom $\pi_1(E^3 - K) \to S_n$ with non-abelian image
⇒ π_1 not abelian ⇒ $\pi_1 \neq Z$ ⇒ knotted

Alexanderpol: 1) $1 - 3t + t^2$ 2) $1 - t + t^2$

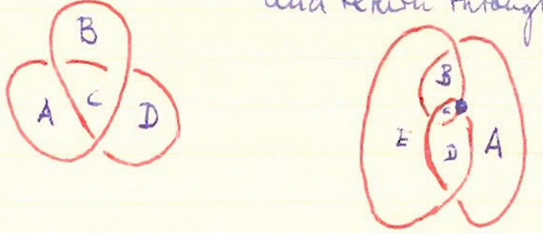
Kritical states in the ambient isotopy while going from one presentation P to another P'



Proof of the 1st theorem:

We start labeling as wanted in description above. So we get the same colouring after one critical step. So if we go to another representation we get the same colouring since these 3 critical cases are the only ones.

To get generators and relations, number the regions of the knot. Pass through one region and return through another. If you can slip, then you get a relation.



$ab^{-1} = dc^{-1}$ because you can slip over the circled crossing

Theorem 5: (Alexander) Any homeom of a ball onto itself keeping the boundary fixed is ambient isotopic to 1 keeping the boundary fixed.

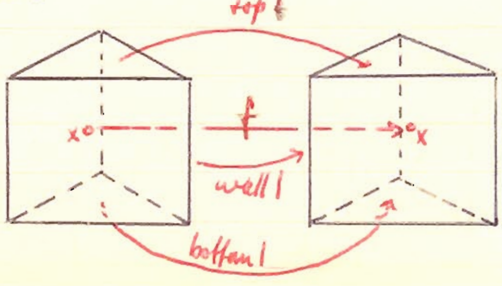
Proof: Suffices to prove for a simplex, because given $B \xrightarrow{h} B$ Choose $g: B \to \Delta$, let $f = g h g^{-1}$

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "h"] & \Delta \arrow[d, "f"] \\ B \arrow[r, "g"] & \Delta \end{tikzcd}$$

Suppose we find ambient isotopy $F: \Delta \times I \to \Delta \times I$, $F_1 = f$ then define H by

$$\begin{tikzcd} B \times I \arrow[r, "g \times 1"] \arrow[d, "H"'] & \Delta \times I \arrow[d, "F"] \\ B \times I \arrow[r, "g \times 1"] & \Delta \times I \end{tikzcd}$$

Given $f: \Delta \to \Delta$. We have defined $\partial(\Delta \times I) \to \partial(\Delta \times I)$ level preserving
map $x \to x$ & join the rest. This gives F
of course PL-homeom



False: in differentiable theory.

Proof: Suppose true for $\mathbb{B}_6$. Then any homeom $S^6 \to S^6$ of degree $+1$ is diffeomorphic to 1. Take two seven balls and glue them together along their boundary, this can be queer with the diff structure. But the theorem this queer 7 sphere can be made to a good 7-sphere in contrary to former examples. You get usually one bad point.

Exc: Try an analogy for S^4 for maps of degree 1, and for $\mathbb{P}^3-\mathring{D}^3$, $P^4-\mathring{D}^n$

Lemma 23: Let $f: K \to K$ be a homeom mapping each simplex into itself and keeping a subcomplex L fixed then f isotopic to 1, the isotopy keeping each simplex in itself + keeping L fixed (false in the smooth cat, right in the top cat)

Proof: Define $F: K \times I \to K \times I$, $F_0 = 1$, $F_1 = f$. Build F inductively $\uparrow$dim on the prisms $A \times I \to A \times I$, $A \in K$. By mapping centre to centre, join to the boundary where the map is defined by induction. Proof the same as th. 5

Lemma 24: Any homeomorphism between the boundaries of two balls can be extended to the interiors (true in top cat, false in smooth)

Proof:

$$\begin{tikzcd}
\partial B_1 \arrow[r, "h"] & \partial B_2 \\
\partial\Delta \arrow[u, "t_1/\partial\Delta"] \arrow[r] \arrow[ur, phantom, "©"] & \partial\Delta \arrow[u, swap, "t_2/\partial\Delta"]
\end{tikzcd}$$

Extend
conewise
∃ comm

$$\begin{tikzcd}
B_1 \arrow[r, "\bar{h}"] & B_2 \\
\Delta \arrow[u, "t_1"] \arrow[r, "\text{extension}"] \arrow[ur, phantom, "©"] & \Delta \arrow[u, swap, "t_2"]
\end{tikzcd}$$

Chapter III. Regular Neighbourhoods

face of M^m

Def: If $B^{m-1} \subset \partial M^m$ then we call B^{m-1} a face of M

Theorem 6: (A) $B^n \subset S^n \rightarrow \overline{S^n - B^n}$ is a ball

(B) B^{n-1} face of B^n, Δ^{n-1} face of Δ^n $\Rightarrow$ any homeom $B^{n-1} \rightarrow \Delta^{n-1}$ can be extended to a homeom $B^n \rightarrow \Delta^n$

(C) If two balls meet in a common face then the union is a ball (face exactly one dimension lower than the ball)

(D) If $B_1^n \subset B_2^n$ + $\partial B_1^n \cap \partial B_2^n$ = common face then $\overline{B_2^n - B_1^n}$ is a ball

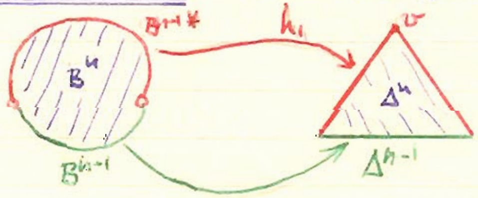
(These things are false in the top + smooth cat). Top cat Alexander horned sphere for all 4 cases. In the smooth cat trouble with the edges. If theorems work in the top cat they usually work in the PL cat. We have an ordered finite structure here, hence we ~~can~~ have advantage to the top cat.

Unsolved problem: (PL-Schönflies) Given $S^{n-1} \subset S^n$ then are the closures of each of the components balls?
(True for $n=1,2,3$, the proof ~~for~~ $n=4$ implies the proof for $n \geq 4$, by handlebody theory). We know that the closures are top balls but we do not know whether they are PL-balls (Proved by using Brown's result)

Proof: $A(n-1) \Leftrightarrow B(n) \Rightarrow C(n)$ $A(n) + C(n) \Rightarrow D(n)$ $B(\leq n) + C(\leq n) \Rightarrow A(n)$

(A), –, (D) are trivially true for $n=0$

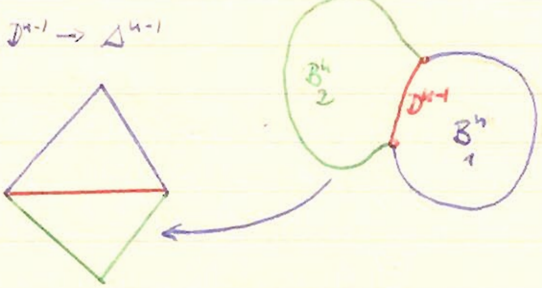
$\underline{A(n-1) \Rightarrow B(n)}$: $B^{n-1 *}$ is a ball by $A(n-1)$. Given $h: B^{n-1} \rightarrow \Delta^{n-1}$. Extend $h | \partial B^{n-1}$ to $h_1: B_*^{n-1} \rightarrow v\, \partial \Delta^{n-1}$ by lemma 24. ∴ we have $h \cup h_1: \partial B^n \rightarrow \partial \Delta^n$. Extend this by lemma 24 to $h_2: B^n \rightarrow \Delta^n$



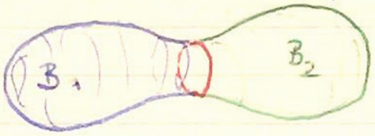
$\underline{B(n) \Rightarrow A(n-1)}$ [not necessary]: Given $B^{n-1} \subset S^{n-1}$. Let B^n = cone on S^{n-1}

Choose any homeom $B^{n-1} \to \Delta^{n-1}$. Extend to $B^n \to \Delta^n$ by B(n). Then
$\overline{S^{n-1} - B^{n-1}} \cong v(\partial\Delta^{n-1}) = \text{ball}$

B(n) ⇒ C(n): Given B^n_1, B^n_2 like picture. Choose $h: D^{n-1} \to \Delta^{n-1}$
Extend to $h_1: B_1^n \longrightarrow \Delta_1^n$
$h_2: B_2^n \longrightarrow \Delta_2^n$
$\therefore\ h_1 \cup h_2 : B_1^n \cup B_2^n \longrightarrow \text{ball}$



Corollary to 24: The union of two balls sewn along their boundaries is a sphere.



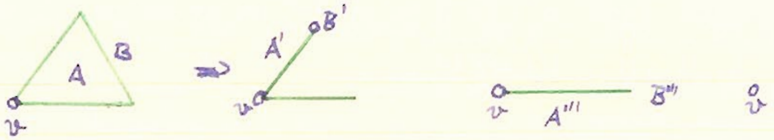
Proof: Given $\partial B_1 = \partial B_2$. Choose
$h: \partial B_1 \to \partial\Delta$, Extend to $h_1: B_1 \to v_1 \partial\Delta$
$h_2: B_2 \to v_2 \partial\Delta$
$\therefore$
$h_1 \cup h_2 : B_1 \cup B_2 \longrightarrow$

A(n) + B(n) ⇒ D(n): Given $B_1 \subset B_2$. The trick is to glue on B_3 by $\partial B_3 = \partial B_2$.
$\overline{B_2 - B_1} = \overline{(B_2 \cup B_3) - (B_1 \cup B_3)} = \overline{S^n - B^n}$ by cor since $\overline{B_2 \cup B_3} = S^n$ and
$B_1 \cup B_3$ ball by C(n) since glued on a common face, hence $\overline{S^n - B^n}$ by A(n)

Throughout the rest of this chapter we assume B + C. Our aim is to prove the last inductive steps.

Def of Collapsing: Suppose $K = L \cup A$, L subcomplex, A simplex $= aB$, $a(\partial B) = L \cap A$
In other words A is principal in K (not the face of any other simplex). B is a free face of A (not the face of any other simplex than A)
We say there is an elementary simplicial collapse from K to L
We say collapse A from B or collapse A onto B
We say there is a simplicial collapse from K to L, written $K \overset{s}{\searrow} L$ if ∃ a sequence of elementary simplicial collapses — simplicial collapse
If L = pt we write $K \overset{s}{\searrow} 0$ + call K collapsible (simplicial collapsible)

Expl: ①

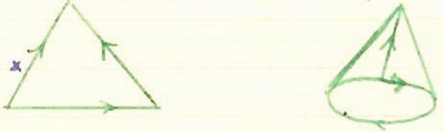


$\Delta \overset{s}{\downarrow} O$

② A cone collapses onto any subcone

③ 1dim complex collapsible $\Leftrightarrow$ tree $\overset{\text{lemma 11}}{\Leftrightarrow}$ contractible

④ The Dunce Hat is not collapsible (but contractible)



Not collapsible since ∄ free face

$\pi_1(X)$ given by: 1 generator x and the relation $x^2x^{-1}=1$ ∴ $x=1$ ∴ $\pi_1(X)=1$
$H_1=0$, $H_2=0$ ∴ $\pi_2(X)=0$ ∴ By Whitehead's theorem for CW-complexes this is the homotopy-type of a point ∴ contractible

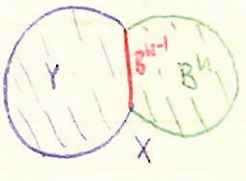
Exc: Duncehat × I $\downarrow$ 0 with suitable triangulation (Zeeman Topol 1963)

Remark: X collapsible $\Rightarrow$ X contractible
$\not\Leftarrow$ Counterexample Duncehat

Collapsing in the PL-category:

X, Y polyhedra, ∃ $X = Y \cup B^n$ and face $B^{n-1} = Y \cap B^n$ B^n - n-ball
Then we say there is an elementary collapse from X to Y across B^n onto B^{n-1}
across B^n from $\overline{\partial B^n - B^{n-1}}$, ball

collapse



We say X collapses to Y, written $X \downarrow Y$, if ∃ a sequence of elementary collapses $X = X_0 \downarrow X_1 \downarrow X_2 \downarrow \cdots X_r = Y$
A particular case is when X, Y and all X_i are n-mf + all the collapses are n-dimensional. In this case we say X shells to Y.

shelling

If $K \supset L$ are complexes we write $K \downarrow L$ if $|K| \downarrow |L|$
If Y is a point we call X collapsible

Remark: The balls can cover different simplices, they can go through them, they do not have anything in common with simplexes

Theorem 7: $K \searrow L \Rightarrow \exists$ subdivision $K' \overset{s}{\searrow} L'$

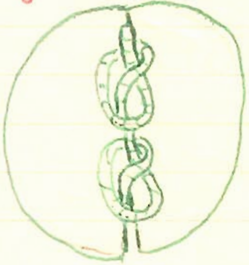
Cor 1: $X \searrow Y \Rightarrow \exists$ triangulation $\ni K \overset{s}{\searrow} L$

Cor 2: $X \searrow 0 \Rightarrow \exists$ triangulation $\ni K \overset{s}{\searrow} 0$

Remark: $K \searrow 0 \not\Rightarrow K \overset{s}{\searrow} 0$
counterexample by Bing: $\exists$ a comb 3-~~of~~ ball that is not simpl collapsible
(counterexample to Chris's conjecture in his seminarnotes)
Hence $\searrow$ invariant but $\overset{s}{\searrow}$ not invariant.

Bings example:

3 ball. Push hole into it along the knot and stop after the 2nd knot Now you have a triangulable 3 ball where the rest-piece of the knot is a nice simplex.



Remark: $X \searrow Y$ ordered relation. Make it to an unordered relation by $X \sim Y$ if X collapses into Y ~~or~~ Y collapses into X or X and Y are connected by $\searrow \nearrow$ up + down orderings
This is called simple homotopy type

simple hpy type

$L(7,1) \simeq L(7,2)$ but $L(7,1) \not\sim L(7,2)$

Remark: If $\pi_1 = 0, Z, Z_2, Z_3, Z_5$ then hpy type $\Leftrightarrow$ simple hpy type

Remark: Duncehat $D^2 \not\searrow 0$ but $D^2 \nearrow B^3 \searrow 0 \Rightarrow D^2 \sim 0$

Conjecture: K is contractible 2-complex $\Rightarrow K \times I \searrow 0$

∴ This conjecture implies the Poincaré-conjecture (M^3 closed $M^3 \simeq$ hpy $S^3 \Rightarrow M^3 = S^3$)

Proof: Remove a little open 3-ball, get left V^3 contractible. Now $V^3 \downarrow K^2$ hence by our conjecture $V^3 \times I \downarrow K^2 \times I \downarrow 0$. By theorem 10 (proved in future) $M^4 \downarrow 0 \Leftrightarrow M^4 = B^4$
$\therefore V^3 \times I = B^4$
$V^3 < E^3$, $\partial V^3 = S^2$ ∴ by the Schoenflies theorem in PL (which is true in this dimension proved by Alexander ~1920). $V^3 = B^3$. Glue back the ball, then $M^3 = S^3$

Lemma 25: If $K \overset{s}{\downarrow} L$, we can order the collapses in order of ↓ dimensions

Proof Suppose we have elementary simpl collapses $K_1 \overset{s}{\downarrow} K_2$ across A^p from B^{p-1} and $K_2 \overset{s}{\downarrow} K_3$ across C^p from D^{q-1}. Proof by induction
If $p<q$ we can interchange the order as follows: (We cannot interchange in general if $p \geq q$), eg if is a simplex, we cannot collapse a nonprinciple ~~side~~ face.
First $C^q \neq A$ or B and $C \nless A, B$ because of dimension. ∴ C is principal in K_1
Next $D^{q-1} \nless A$ or B. Worst that ~~q~~ can happen $p = q-1$, if $D^{q-1} < A^p$ then $D^{q-1} = A^p$ but A^p has been removed and of course $D^{q-1} \nless B^{p-1}$ ∴ D^{q-1} is a free face of C in K_1. (by induction)
∴ collapse $K \downarrow K_2^*$ across C^q, from D^{q-1}. A remains principal in K_2^*, B remains free face in K_2^* ∴ ∃ collapse. By a finite number of interchanges lemma follows

Remark: We cannot order arbitrarily but there exists an ordering.

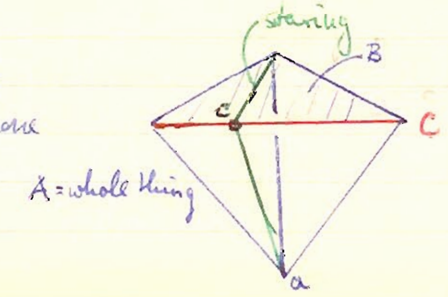
Lemma 26: $K \overset{s}{\downarrow} L \Rightarrow$ for any stellar subdivision $\sigma K \overset{s}{\downarrow} \sigma L$

Proof By induction on the # of elementary simpl. collapses + stellar subdivisions. Assume $K \overset{s}{\downarrow} L$ elem. across A from B. Assume that σK is obtained by staring C at c, assume that $A = aB$.

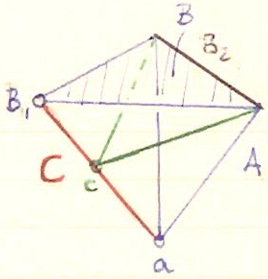
1. $C \nless A$
2. $C < B$
3. $C \nless B$ but $C < A$

} 3 possibilities

1. trivially true because you do not interfere with A
2. $\sigma K \overset{s}{\searrow} \sigma L$ by collapsing cone $a(\sigma B)$ onto subcone $a\partial(\sigma B)$



3.



$A = aB$, $C = aB_1$, $B = B_1B_2$

$\sigma A \downarrow$? across cB from B (B not touched yet since $c \notin B$) to get the upper walls in, we get

$\sigma K = \sigma L \cup \sigma A \downarrow^{s} \sigma L \cup acB_2$ (across cB from B) $\downarrow^{s} \sigma L$

by collapsing cone $a(cB_2) \downarrow$ subcone $a\partial(cB_2)$ ≫

Unsolved problem: $K \downarrow^{s} L \Rightarrow$ any subdivision $K' \downarrow^{s} L'$

Lemma 27: If $K \downarrow L$ is an elementary collapse (not in general simplicial), then $\exists$ subdivision K' & stellar subdivision $\sigma L \ni K' \downarrow^{s} \sigma L$

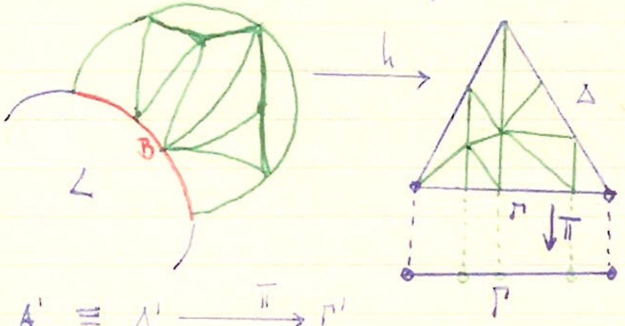
Proof: Let $A = \overline{K-L}$, $B = A \cap L$. By our mamoth induction (theorem 6) we can choose a homeom.

$h.\ A, B \cong \Delta, \Gamma$

Let $\pi: \Delta \to \Gamma$ be the orthog projection

Choose subdivision $\ni A' \xrightarrow{h} \Delta' \xrightarrow{\Delta} \Gamma'$ simplicial.

So we get a cylindrical subdivision of Δ to Δ' ∴



$$\begin{tikzcd}
A' \arrow[r, phantom, "\cong"] & \Delta' \arrow[r, "\pi"] & \Gamma' \\
& \cup & \\
& \Gamma' \arrow[uur, "1"'] &
\end{tikzcd}$$

Let B' be the subdivision of B given by $h^{-1}\Gamma'$. By lemma 9 we ~~de~~ can choose a stellar subdivision σB of B. that is at the same time a subdivision of B'. Let Γ'' be the $h(\sigma B)$, the corresponding subdivision of Γ'. By lemma 8 let Δ'' be a subdivision of $\Delta' \ni \pi: \Delta'' \to \Gamma''$ remains simplicial

Let $A'' = h^{-1}\Delta''$. Hence we a diagram of simplicial maps

$$\begin{tikzcd}
A'' \arrow[r, phantom, "\overset{h}{\cong}"] & \Delta'' \arrow[r, "\pi"] & \Gamma'' \\
\cup & \cup & \\
\sigma B \arrow[r, phantom, "\overset{h}{\cong}"] & \Gamma'' \arrow[uur, "1"'] &
\end{tikzcd}$$

σB extends to σL. Let $K'' = A'' \cup \sigma L$ (agree on overlap $\Delta'' \cap \sigma L = \sigma B$). Then $K' \downarrow^{s} \sigma L$ because

$A'' \downarrow^{s} \sigma B$ because $\Delta'' \downarrow^{s} \Gamma''$ cylinderwise (i.e we collapse each cylinder alone) ≫

Proof of theorem 7: Hypothesis means $\exists$ sequence of elementary collapses $|K| = X_r \downarrow X_{r-1} \downarrow \cdots X_0 = |L|$ where X_i are subpolyhedra maybe nothing to do with simplicial complex K. (they may criss-cross the triangulation). In fell swoop triangulate the lot $\ni K_r$ subdivision of K

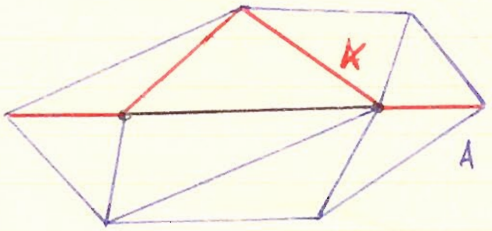
$K_r \downarrow \ldots \downarrow K_0$ (not simplicial collapses), elem collapses

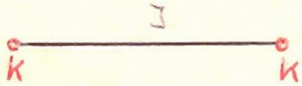
By induction on r. If $r=1$ result is true by lemma ~~27~~ 27.
Assume the theorem for $r-1$, i.e $\exists$ a subdivision K'_{r-1} of K_{r-1} $\ni$ $K'_{r-1} \downarrow^{s} K'_0$. By lemma 8 extend K'_{r-1} to a subdivision K'_r of K_r. Apply lemma 27 to the elementary collapses $K'_r \downarrow K'_{r-1}$ and get $K''_r \downarrow^{s} \sigma K'_{r-1} \downarrow^{S} \sigma K'_0$ by lemma 26. Now the composition of simplicial ~~complexes~~ collapses is simplicial $\therefore$ $K''_r \downarrow^{s} \sigma K'_0$ >>

J complex $\supset$ K subcomplex

full subcompl. **Def:** We say K is full in J if $A \in J$, all vertices of $A \in K \Rightarrow A \in K$
(i.e. fullness: The vertices of K in J span K and each simplex spanned by them in J is in K)

Expls: These are expls of K not full, the black lines are the bad lines





Elementary properties:

① $K < J$, J' is first derived $\Rightarrow$ K' is full in J' (Do it yourself)

② K full in J, J_* subdivision of J $\Rightarrow$ K_* full in J_*

③ K full in J, $A \in$ ~~K~~ $J \Rightarrow A \cap K$ is a face (Could be two points or edges otherwise)

④ K full in J $\Rightarrow$ $\exists$ unique simplicial map $f: J \longrightarrow I \ni f^{-1}0 = K$ I = unit interval
(map vertices of K to 0, rest to 1)

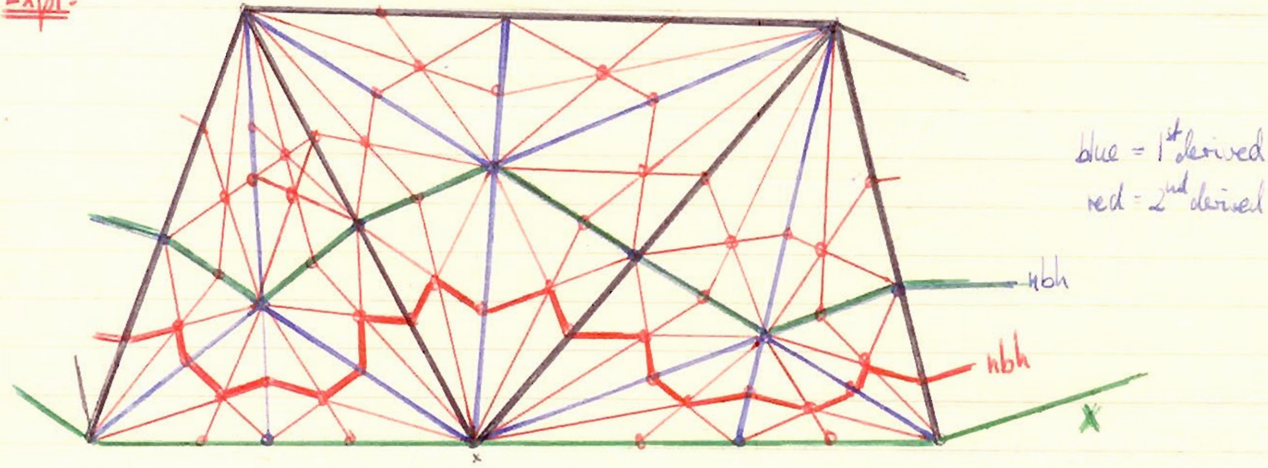
neighbourhoods **Neighbourhoods:** J complex and $X \subset |J|$ ($\therefore$ not nec complex).

simpl nbh **Def:** The simplicial nbh $N(X, J)$ is the smallest subcomplex of J containing a topol nbh is the union of all closed simplexes meeting X derived nbh

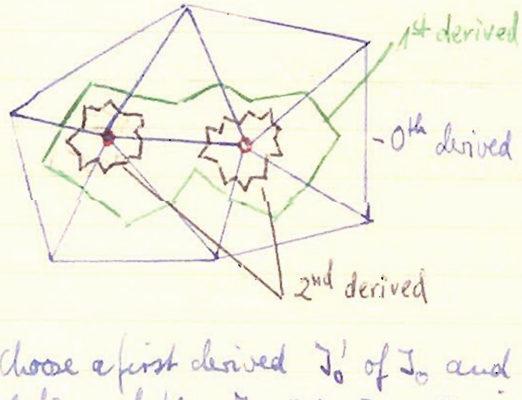
Def: M PL-mf, X a subpolyhedron. As derived nbh of X in M is obtained by choosing a triangulation J, K of M, X ∋ K is full in J and then choosing a first derived J' of J, and defining $N = N(X, J')$
An r^{th} derived nbh N is obtained by choosing a triangulation J, K of M, X choosing an r^{th} derived $J^{(r)}$ and defining $N = N(X, J^{(r)})$)

Remark: 1) An r^{th} derived nbh is a derived nbh for $r \geq 2$ but not for $r = 1$ in general.
2) Let $J' = 1^{st}$ derived, $J'' = 2^{nd}$ derived. $N(X, J') = \bigcup_{x \in K} st(x, J')$ and
$N(X, J'') = \bigcup_{\text{simpl} A \in K} st(\hat{A}, J'')$ $\hat{A}$ = barycentre (Do it yourself)

Expl:



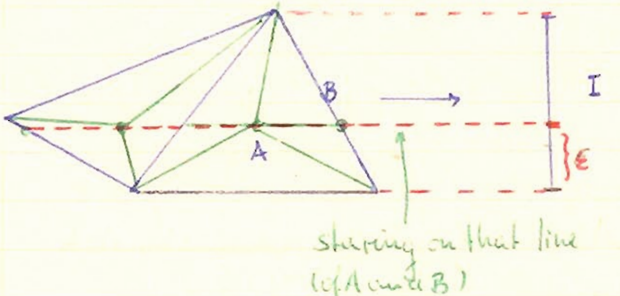
Lemma 28: Any two derived nbhs of X in M are ambient isotopic keeping X fixed. If, further, $X \subset \mathring{M}$, the isotopy can be chosen ∋ ∂M is kept fixed



The sketch gives the reason why we have to go to the 2^{nd} derived or choose fullness (the two vertices glued together) and use 1^{st} derived.

Proof: Let $N_1 = N(X, J_1')$, $N_2 = N(X, J_2')$ be the two given derived nbhs
Let J_0 be a common subdivision of J_1, J_2 choose a first derived J_0' of J_0 and let $N_0 = N(X, J_0')$. By fullness of the triangulation of $K_1 \subset J_1$ $\exists f: J_1 \to I$ simplicial ∋ $f^{-1} 0 = X$

Choose $\varepsilon > 0 \ni \varepsilon < fx$ ∀vertices $x \in J_0$, $x \notin X$. Let J_1^ε denote a 1st derived of J_1 obtained by staring A in J_1 on $f^{-1}\varepsilon$ if $fA = I$ + arbitrarily otherwise



Let J_0^ε denote a first derived of J_0 obtained by staring $A \in J_0$ on $f^{-1}\varepsilon$ if $fA \ni \varepsilon$ + arbitrarily otherwise

$N(X, J_1^\varepsilon) = f^{-1}[0,\varepsilon] - N(X, J_0^\varepsilon)$

Use lemma 23. Any two 1st derived J_1', J_1^ε of J_1 are isomorphic and the isom is ambient isotopic to the identity.

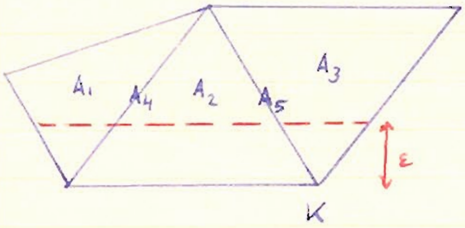
This isom is a homeom $J_1 \to J_1$ mapping each simplex to itself)

To make sure that X is kept fixed we restrict the arbitrary staring of simplices in J_1, J_0: We star ∋ they agree with J' on X (and ∂M if $X \subset \mathring{M}$)

$N_1 = N(X, J_1') \cong N(X, J_1^\varepsilon)$ (amb isot) $- N(X, J_0^\varepsilon) \cong N_0$ (ambistop by lemma 23) $\cong N_2$ (ambisot) by symmetry (analogously to N_1)

Lemma 29: Any derived nbh of X in M collapses to X.

Proof Suffices to prove for one nbh by lemma 28. Choose triangulation J, K of M, X with K full in J. Let $N = N(K, J^\varepsilon)$ where J^ε defined as before (by map to the unit interval). Order the simplexes of $J - K$ that meet K in ↓dim. Then $A_i \cap N$ is a cell and $A_i \cap f^{-1}\varepsilon$ is a face. Collapse across $A_i \cap N$ from $A_i \cap f^{-1}\varepsilon$, $i = 1, \dots, r$ where A_i is one of the ordered simplices. Start collapsing with A_1



Theorem 8: A derived nbh of a collapsible polyhedron in an n-mf is an n-ball

Corollary: A mf is collapsible ⟺ its a ball

Proof: ball collapsible since simplex collapsible (since simplex $\searrow^s 0$, any PL to this is $\searrow 0$) $\Rightarrow M^n$ is a derived nbh of M^n in M^n, since its collapsible its a ball by the theorem

Remark: Collapsible characterized the ball, contractible not (∃ counterexpls)

Proof of the theorem: By ind on n up to the Mamoth-induction $\therefore$ suffices to prove for one particular derived nbh by lemma 28.

Let J, K triangulate M, X, let J'' be the (barycentric) second derived. By theorem 7 we can choose $K \ni K \overset{s}{\searrow} 0$ by theorem 7. Let $N = N(X, J'')$

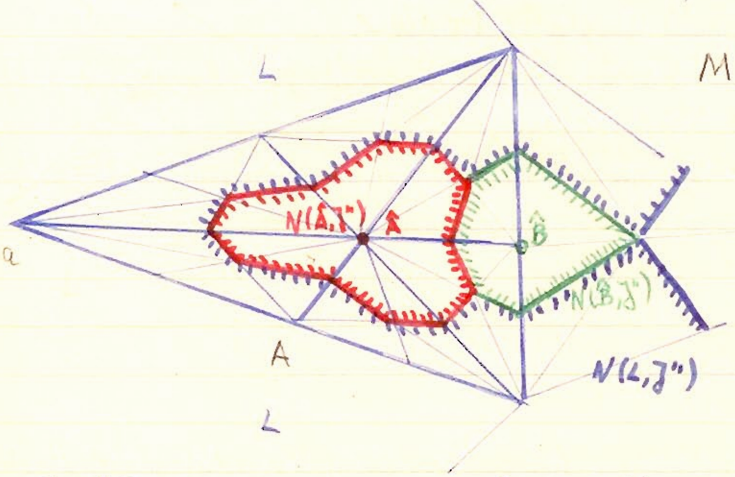
Let $r =$ # of elementary simplicial collapses. We show that N is a ball by induction on r.

If $r=0$, $K=pt$, $N =$ its closed star – ball since we are in a mf ✓

Inductive step. Let $K \overset{s}{\searrow} L$ be the first elementary simpl collapse across $A = aB$ from B

Let $\hat{A}, \hat{B}$ be the barycentres

Claim $N = N(K, J'') = P \cup Q \cup R$ where $P = N(L, J'')$, $Q = N(\hat{A}, J'')$, $R = N(\hat{B}, J'')$



P ball by induction, Q, R balls being closed stars (ind on r)

By Mamoth induction Thm 6(c)ii it suffices to show that $P \cap Q$ is an $(n-1)$ ball, whence $P \cup Q$ is an n-ball, and $(P \cup Q) \cap R$ is an $(n-1)$ ball, whence $(P \cup Q) \cup R$ is an n-ball by the Mamoth induction

Now $P \cap Q \subset \partial Q$. Let $J_* = lk(\hat{A}, J') = (\partial A)' I^*$, where I^* is PL isom to $(lk(A,J))'$, $I^* \equiv (lk(A,J))'$ even simpl. isom (look at the star of A) This simpl isom is given by $\widehat{AC} \to \hat{C}$ (natural map by dropping A)

$\therefore J_*$ ball or sphere of dim $(n-1)$

Pseudoradial projection gives a simplicial isom $\partial Q \xrightarrow{\equiv} J_*'$ ($\partial Q = lk(\hat{A}, J'')$, blow it up to $lk(\hat{A}, J')$, typical vertex $\in \partial Q$ is barycenter of 2nd derived. Hence this map is given by $\widehat{\hat{A}D} \to \hat{D}$ for $D \in \partial A$ simplex)

Under this isom $P \cap Q \xrightarrow{\equiv} N(a(\partial B), J_*')$

Now $a(\partial B)$ is collapsible, being a cone, and $(a(\partial B))'$ is full in J_*

$a(\partial B)$ not full in ∂A but $(a(\partial B))'$ is full in $(\partial A)'$ $\therefore$ joining up remains full since no more vertices get involved

Hence $N(a(\partial B), J_*')$ is a derived nbh. By induction on the theorem this is a ball since J_*' is $(n-1)$ ball or sphere (ind on n)

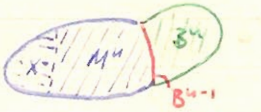
Blow ∂R up in P to the first derived $\therefore (P \cup Q) \cap R \subset \partial R$ similarly. Choose $J_* = lk(\hat{B}, J')$

$\exists$ isom $\partial R \xrightarrow{\equiv} J_*'$ throwing $(P \cup Q) \cap R \xrightarrow{\equiv} N(\hat{A}(\partial B), J_*')$ which is an $(n-1)$-ball by induction on n

≫

Cont. of previous page

Lemma 30: Suppose $M^n \cap B^n$ = common face B^{n-1}, $X \subset M^n$ and X does not meet B^{n-1} (M^n mf, B^n ball) $\Rightarrow \exists$ homeom $M^n \to M^n \cup B^n$, keeping X fixed (everything PL)



Proof: Triangulate everything and call it by the same names. Let A^n be a 2nd derived nbh of B^{n-1} in M^n. A^n is a ball by theorem 8. $\mathring{A}^n \subset \mathring{M}$ ∴ B^{n-1} is a face ∴ $A^n \cup B^n$ = ball (mamoth induction Thm 6 (C) n). Hence we localized the problem to $A^n \cup B^n$. Let $h = 1$ on $\partial A^n - \mathring{B}^{n-1}$. By lemma 24 extend h to homeom $h: B^{n-1} \to \partial B^n - \mathring{B}^{n-1}$ i.e. $h: \partial A \cong \partial(A \cup B)$; again extend h to $h: A \cong A \cup B$

$\parallel$ ball by mamoth ind Thm 6 A(n-1) (allowed to use)

Then extend h to the rest of M by the identity $\gg$

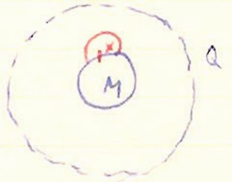
Lemma 31: $M^n \subset \mathring{Q}^n \Rightarrow \overline{Q-M}$ is an n-mf (False in top cat (Alexander horned sphere))

Proof: Let $M_1 = \overline{Q-M}$. $lk(x, M_1) = \overline{lk(x,Q) - lk(x,M)}$

$= \overline{S^{n-1} - B^{n-1}}$

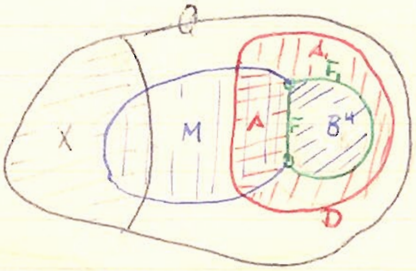
$=$ ball by Mamoth induction

(Th 6 A(n-1))



Lemma 32: $M^n \cup B^n \subset Q^n$, $M^n \cap B^n$ = common face, $X \subset Q^n$, $X \cap B^n = \emptyset \Rightarrow \exists$ ambient isotopy of Q, keeping $X \cup \partial Q$ fixed + moving M onto $M \cup B$.

Proof: Let $F = M \cap B$, $F_1 = \overline{\partial B - F}$ = (n-1) ball by M. Ind A(n-1)
Let D = derived nbh of B in Q = n-ball. $A = D \cap M$ = derived nbh of F in M = n ball. $A_1 = D \cap M_1$, where $M_1 = \overline{Q - (M \cup B)}$ = mf by lemma 31 $\Rightarrow$
A_1 = derived nbh of F_1 in M_1 ∴ A_1 = n ball



Define homeom $h: Q \to Q$ by $h = 1$ outside D. $h = 1$ on $\partial D \cup (\partial A - \mathring{F})$
$h: \partial F \xrightarrow[\cong]{1} \partial F_1$, extend $h: F \xrightarrow{} F_1$. $A \cup B$ = n-ball because $A \cap B = F$, common face, by Mam Ind^n C(n). Analogously $B \cup A_1$ = n-ball because $B \cap A_1 = F_1$, common face
$h: \partial A \to \partial(A \cup B)$ + so extend to interiors
$h: \partial(B \cup A_1) \to \partial A_1$ + extend to interiors. This completes def of h.
By Alexander's theorem 5, any homeom $h: D \to D$ keeping ∂D fixed is ambient isotopic to 1 keeping ∂D fixed. So keep h fixed outside D + finished $\gg$
(Pushing up by polyhedra-balls)

Def of regular nbh: ($\hat{=}$ tubular nbhs in diff cat, and normal bundle) Trouble here $\nexists$ natural fibering. In PL one does not want to have a structure. It's an unsolved problem whether you can put a structure in it.

Let X be a polyhedron in M^n. A <u>regular nbh</u> N of X in M is a polyhedron that is — regular nbh

(1) a topological nbh of X in M — n mf

(2) an n-mf

(3) $N \searrow X$

Theorem 9: 1) Any derived nbh is regular (=> existence)

2) Any two $\overset{\text{regul}}{}$ nbhs are homeomorphic, keeping X fixed

3) If $X \subset \mathring{M}$, then any two regular nbhs in the interior. $\mathring{M}$ are ambient isotopic keeping $X \cup \partial M$ fixed

To 3: We must have in $\mathring{M}$ otherwise take $\overset{\text{one of}}{\text{nbhs}}$ which meets the boundary. Since amb. isotopy throws boundary to boundary ~~this~~ we cannot get the required result.

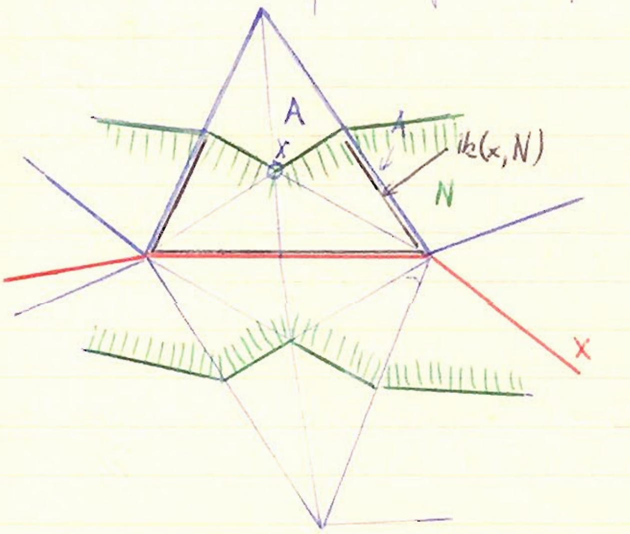
Proof of 9.

(1). Choose triangulations J, K of M, X with K full in J. Choose a first derived J' define $N = N(X, J') = N(K', J')$

We've got to ~~very~~ verify 3 conditions for regularity:

(1) easy (3) right by Lemma 29. Remains to prove (2), i.e. the lk of any vertex^x in N is an $(n-1)$-ball or sphere. If ~~W~~ $x \in X$, then $lk(x, N) = lk(x, J')$ ~~since~~ $=$ ball or sphere

because $x \in \partial M$ or $\mathring{M}$



If $x \notin X \Rightarrow$ then $x \in \mathring{A}$ where A is a unique simplex of $J - X$

By fullness $A \cap K =$ face B

~~Let $L = lk(x, J)$~~ Let $L = lk(x, J') = (\partial A)' S$

where $S \equiv (lk(A, J))'$ (Go again in 3 dimensions)

$S \subset st(A, K)$, which does not meet X, since it is the open star. $\therefore L \cap X = B'$

$\therefore lk(x, N) = N(B', L)$ (N nbh)

$= N(B', (\partial A)' S) = N(B', \partial A') \cdot S$

Any simplex of $(\partial B)'$ meeting B' joined to S is included in $N(B', (\partial A)'S)$. The other way analogously
Now by theorem 8 $N(B', (\partial A)')$ is a ball since B is full in ∂A. S is a sphere or ball since it is inside the int M and $S \equiv (lk(A,J))'$ $\therefore$ $lk(x, N)$ is a ball. >

Part 2: It suffices to prove: Any reg nbh is homeom to some derived nbh, because by lemma 28 all derived nbhs are homeom (keeping X fixed)
Given N. Triangulate N, X by $J, K \ni J \overset{s}{\searrow} K$ by thm 7, we may do this by (3) $\therefore$ $J = K_r \searrow K_{r-1} \searrow \cdots \searrow K_0 = K$
Let J'' be the (barycentric) 2nd derived. Let $N_i = N(K_i, J'')$ simplicial nbh. Get a shelling
$J \searrow N_r \searrow N_{r-1} \searrow \cdots \searrow N(K, J'')$ = derived nbh (shelling = little balls which might collapse)

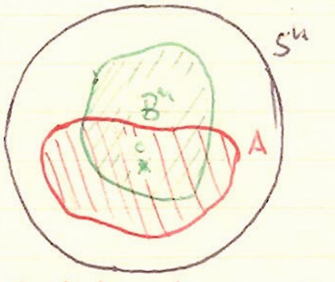

= shelling

By lemma 30 we have homeoms $J = N_r \cong N_{r-1} \cong \cdots \cong N_0$
Hence N_0 derived nbh >

Part 3: When everything is in the interior, use lemma 32 instead of lemma 30. So we get N ambient isotopic to a derived nbh. But any 2 derived nbhs are ambient isotopic $\therefore$ any 2 regular nbhs in $\mathring{M}$ are ambient isotopic >>

Exc: M^2 spherelike if every closed curve separates. Show M^2 = sphere using 2nd derived.

Theorem 6 A: $\overline{S^n - B^n} = B^n$.
By induction we can assume B, C, D in dimension n.
Given S^n and $B^n \subset S^n$. Let $\Delta = (n+1)$-simplex $= v \cdot \Gamma$, Γ face. Choose homeom $h: \partial\Delta, v \to S^n, x$ where $x \in S^n$ pt, $x \in \mathring{B}^n$. Let $A = h[v(\partial\Gamma)]$, then A, B are both regular nbh of x in S^n,

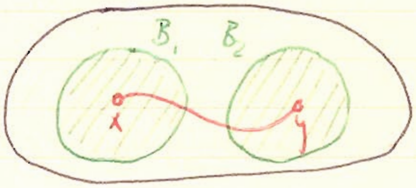


because any ball is collapsible to any of its points. $\therefore \exists$ ambient isotopy A onto B, by theorem 9.
$\therefore$ $\overline{S^n - A} \cong \overline{S^n - B^n}$. But $\overline{S^n - A} \cong \Gamma$ under h and Γ is a ball. >>

Cor 1: (Point homogeneity theorem) Assume M is connected then any two points in $\mathring{M}$ are ambient isotopic

Connect them by an arc, remove crossings, choose regular nbh

Cor 2: (Ball homogeneity theorem): Any two balls in $\mathring{M}^n$; M^n connected, are ambient isot.



Move x to y ∴ two reg nbh at y, move one to the other by ambient isot.

Cor 3: (Regular neighbourhoods annulus theorem): Given $X \subset \mathring{M}$, N, N_1 reg nbhs of X, $N_1 \subset \mathring{N}$ $\Longrightarrow N - \mathring{N}_1 \cong \partial N \times I$

Proof: Recall the proof of lemma 28. We constructed ε-nbhs. Choose a triangulation X full subcomplex of M. ∃ simplicial maps $f: M \to I$ ∋ $f^{-1}(0) = X$. Choose $\varepsilon_1 < \varepsilon$ · intervals $[0, \varepsilon] \subset I$. Let $Q = f^{-1}[0, \varepsilon]$, $Q_1 = f^{-1}[0, \varepsilon_1]$. Since these are particularly nice, $Q - \mathring{Q}_1 \cong \partial Q \times I$ (using the cellcomplexstructure of $Q - \mathring{Q}_1$). By theorem 9(2) choose homeom $h: N \to Q$ keeping X fixed. hN_1 is a regular nbh of X in $\mathring{Q}$, but $Q_1 \subset \mathring{Q}$ too and Q_1 regular ∴ by thm 9(3) ∃ amb isot of Q throwing hN_1 onto Q_1 keeping $X \cup \partial Q$ fixed. ∴ by composition of homeom + amb. isotopy $N - \mathring{N}_1 \cong Q - \mathring{Q}_1 \cong \partial Q \times I \cong \partial N \times I$ >> ⇒

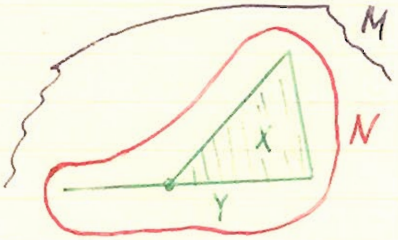
Cor 4 (Annulus theorem): Ball $B_1^n \subset \mathring{B}^n \Rightarrow B^n - \mathring{B}_1^n \cong S^{n-1} \times I$

Remark: Unsolved problem in the top. cat.

Def: If M is a bounded mf and $M \searrow X$ we call X a *spine* of M spine

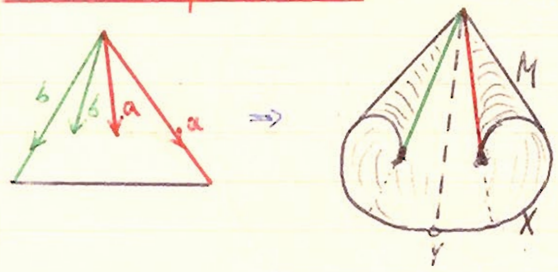
Expls: ① A pt is a spine of a ball
② The duncehat is a spine of B^3
③ S^p is the spine of the solid torus $S^p \times D^q$
④ ∃ 4mf $M^4 \neq B^4$ having the duncehat as spine but not having a pt as spine (proved later)

Cor 5: $X, Y \subset \mathring{M}$, $X \downarrow Y \Rightarrow$ { X is a spine iff Y is a spine } (M must be mf otherwise false



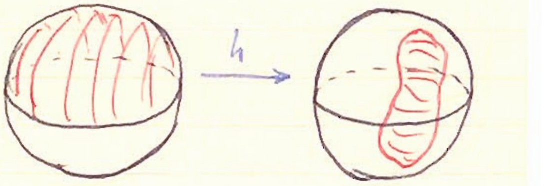
Proof: suppose X a spine ∴ $M \downarrow X \downarrow Y$ ∴ Y is a spine
Y spine. Let N be a reg nbh of X, $N \subset \mathring{M}$. Now $N \downarrow X \downarrow Y$
∴ N reg nbh of Y but M also reg nbh of Y since $M \downarrow Y$, Y being a spine ∴ $M - \mathring{N}$ annulus ∴ $M \downarrow N$ cylinderwise (triang M, and N, push in the ~~triangl~~ simplaces of uppo dim) ∴ $M \downarrow X$ ∴ X spine »

Counterexample to Cor 5: Let M be no mf. Take triangle and identify like picture.



$X \downarrow Y$. We can collapse "conewise" first uppwards ~~and~~ to subcones and then down the middle line to Y. But $M \not\downarrow X$

Theorem 10: Any homeom of S^n onto itself of degree 1 is ambient isotopic to the identity i.e. the isotopyclasses are Z_2. ∃ natural homeom onto homotopyclasses and from them to $Out(H_n(S^n)) = Z_2$ which is onto ∴
Isotopyclasse ≅ Homotopyclasses



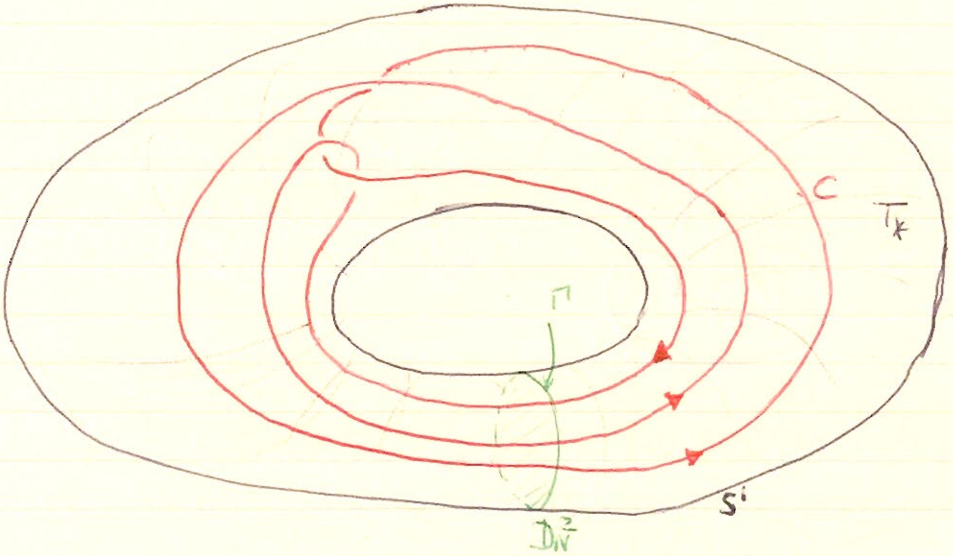
Proof: By induct^n on n. True for n = 0.
Assume for n-1. Given $h: S^n \to S^n$ of degree 1
By cor 2 ~~is~~ ∃ amb isot. of image of northern hemisphere onto the northern hemisphere ∴ h isot to h_1 : N-hemisphere $\circlearrowleft$. Since h of degree 1
h|equator of degree 1 (proof follows immediately from homology exact sequence)
By induction this is isot to the identity. Suspend the isotopy ∴ isotop h_1 to h_2 where h_2|equat. has degree 1. Use theorem 5.

Expl (Mazur Ann 1961) ∃ mf M^4 ∋ (1) bounded, cpct, PL, contractible
(2) $\pi_1(\partial M^4) \neq 0$ (hence $M^4 \ncong B^4$)
(3) $M^4 \times I = B^5$ (the double of M^4 = glued along boundary

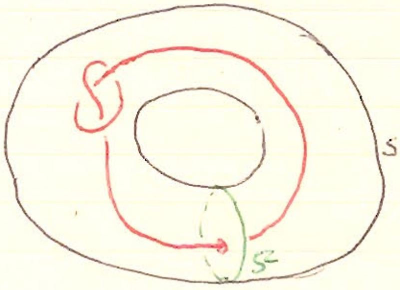
is S^4 (follows from 3) and ∃ involution S^4 with fixedpoint set $=\partial M^4$ which is not simply connected.

Smith: Any periodic homeom of S^4 has fixedpoint set a homology sphere (≙ ∄ excision for hpy)

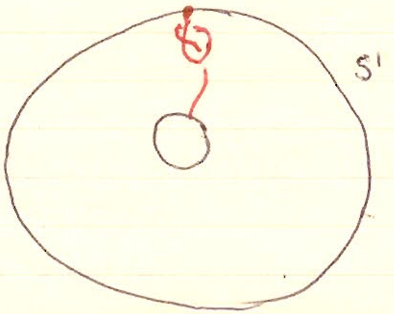
Construction: Take $S^1 \times B^3$, take $\partial(S^1\times B^3) = S^1 \times S^2$. choose curve C in boundary as given



Note: $C \simeq S'\times pt$ (homotopic) one can prove it using generators (see arrows)
$xxx^{-1} = x$ hence what's claimed
+ (first unknot by hpy which is allowed, you may cross, then you get a circle)
Hence $C \sim S'\times pt$ homologous
But $C \not\approx S'\times pt$ not isotopic This is to prove because



can be unknotted: Cut along $S^2 \Rightarrow$

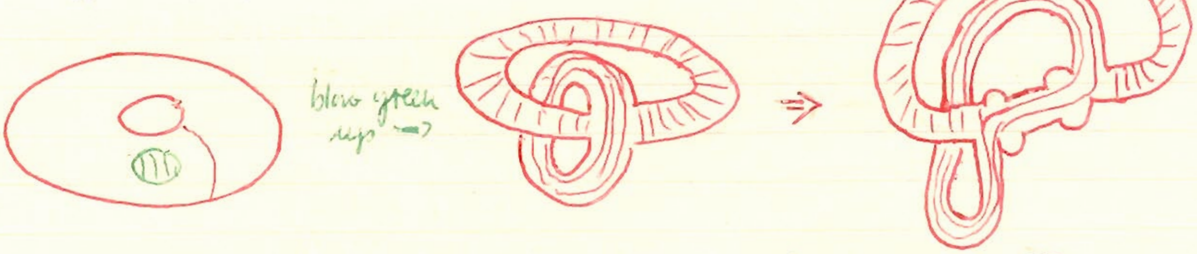


This is a lamp hanging on a knotted string in a room. Just unknot it.

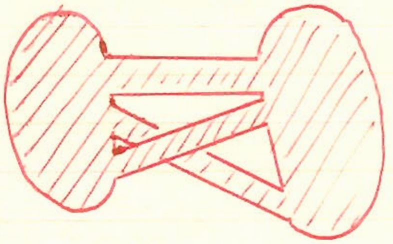
Now take $B^4 = D^2 \times D^2$. Boundary $\partial B^4 = D^2 \times \partial D^2 \cup \partial D^2 \times D^2$
Let T^2 = ~~tak~~ torus $\partial D^2 \times D^2$ (solid torus). Choose homeom $h: N \xrightarrow{\cong} \partial D^2 \times D^2$
where N is a chosen regular nbh of C
Now glue along h: $M^4 = S^1 \times B^3 \cup_h B^4$
We fix h by translation of C onto the boundary of N, that prevents h from twisting

Remark: This M^4 is the lowest dimensional mf $\ni M^4 \times I = B^5$ and $M^4 \not\cong B^4$

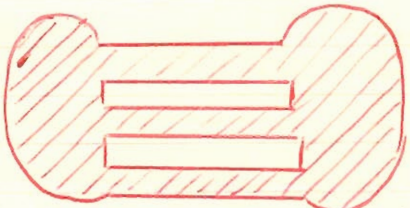
Expl: Let M^2 = torus - little hole, Q^2 = disk - 2 little hole. Then $M^2 \times I \cong Q^2 \times I$
∂M^2 = 1 circle, ∂Q^2 = 3 circles $\therefore Q^2 \cong M^2$



Hence the punctured torus is ambient isotopic to



And Q^2

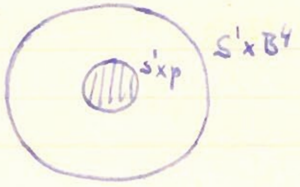


Now cross it with I and big rubber then unwist $M^2 \times I$; this gives the homeom.

Back to our example: $M^4 \times I = \underset{S^1 \times B^4}{\underset{\shortparallel}{S^1 \times B^3 \times I}} \cup_{h \times 1} \underset{D^2 \times D^3}{\underset{\shortparallel}{B^4 \times I}}$

Choose curve $C \times \frac{1}{2}$, choose nbh $N \times I$. Choose homeom $h \times 1: N \times 1 \to \partial D^2 \times D^3$ (First multiplying by I and then glueing = glueing the multiplying !
Now $C \times \frac{1}{2} \subset \partial(S^1 \times B^4) = S^1 \times S^3$
Suppose we choose C_*, N_* amb isotop to first choice, i.e. $\exists$ ~~amb~~ homeom $f: S^1 \times B^3 \supset$ throwing N_*, C_* onto N, C. $M_*^4 = S^1 \times B^3 \cup_{hf} B^4$
claim $M_* \cong M$: by taking $f: S^1 \times B^3$ and $1: B^4 \to B^4$ glue together you get

$$\begin{tikzcd}
M_4^* \arrow[d, hook] & = S^1\times B^3 \arrow[d, "f"] & \cup_{h\circ f} & B^4 \\
M_4 & = S^1\times B^3 & \cup_h \arrow[u] & B^4
\end{tikzcd}$$



So if you isotop the curve and its "h" you get the same M_4

$\pi_1(M^4) \xrightarrow{\text{homom}} S_7$. $\partial M^4 = (S^1\times S^2 - \mathring{N}) \cup_{h|\partial N} D^2\times\partial D^2$ $\qquad N \overset{h}{\cong} \partial D^2\times D^2$

but ∂N = torus, hence glueing along torus

$\partial H^4 = (S^1\times D_S^2) \cup_{T_*} (S^1\times D_N^2 - \mathring{N}) \cup_T D^2\times\partial D^2$ $\qquad D_N$ = northern hemisphere, D_N southern

T_* equatortorus

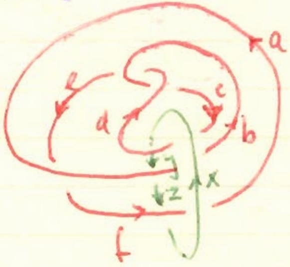
Map torus into solid torus: kills one direct summand. By van Campen then

$$\begin{tikzcd}
\pi_1(S^1\times D_S^2) & & \pi_1(S^1\times D_N^2-\mathring{N}) & & \pi_1(D^2\times\partial D^2) \\
Z & \pi_1(T_*) \arrow[lu] & & \pi_1(T) & Z \\
 & Z\times Z & & Z\times Z &
\end{tikzcd}$$

Hence gp with amalgamations: $\pi_1(\partial M^4) = \pi_1(S^1\times D_N^2-\mathring{N})$ / relations $C=1, \Gamma=1$

$= \pi_1(S3 - U_\varepsilon(C,\Gamma))$ / rel^s $C=1=\Gamma$

S^3 = ~~two~~ solid torus + S^1



Relations of this knot in the usual way.

$x^{-1}ax = a^x = b$

$b^d = c \qquad c^{ax^{-1}} = d \qquad d^b = e \qquad e^a = f \qquad f^x = a$

$x^{d^{-1}} = y \qquad y^a = z \qquad z^f = x$

(relations given by crossings, generators by underlying crossings)

$\Gamma = d^{-1}af = 1 \qquad C = x^2dx^{-1}ba = 1$

In the symmetric gp we take the cycles $a^x = b = (265734)$

$b^d = c = (7524613)$

$d = (7413652)$

$e = (3124576)$

$f = 5637124$

$a = 1647235$

$y = 45176$

$z = 71674$

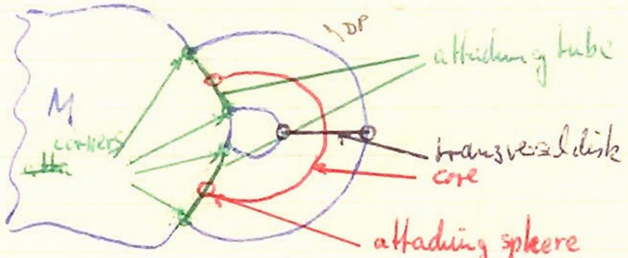
$x = 12345$

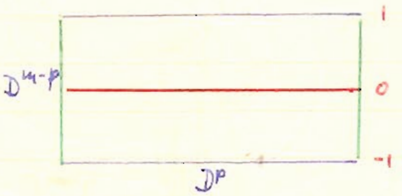
These satisfy the relations and hence the subgp of S_7 generated by them is the required gps.

Chapter 4: Handlebody theory

Everything in this chapter is PL.

h^p Let M = m-mf, bounded, let h^p be a m-ball.
Suppose $M \cap h^p = \partial M \cap \partial h^p$, suppose $\exists$ homeom $D^p \times D^{m-p}, \partial D^p \times D^{m-p} \longrightarrow h^p, M \cap h^p$. We then
p-handle say, $M \cup h^p$ is obtained from M by <u>attaching the p-handle</u> h^p



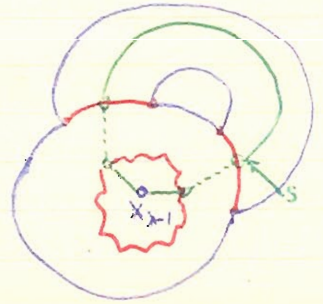


transverse disk
core $D^p \times 0$ is called the <u>core</u> of handle. $\partial D^p \times 0$ is called the <u>attaching sphere</u>, $\partial D^p \times D^{m-p}$ is called the
attaching sphere <u>attaching tube</u>, $0 \times D^{m-p}$ the <u>transverse disk</u> and $0 \times \partial D^{m-p} = S^{m-p-1}$ the <u>transverse sphere</u>

Def: A <u>handlebody structure</u> on M is a homeom $M \cong h_0 \cup h_1 \cup \cdots \cup h_\lambda$, $h_\mu = h_\mu^{n_\mu}$
h_μ handle (n_μ-handle)

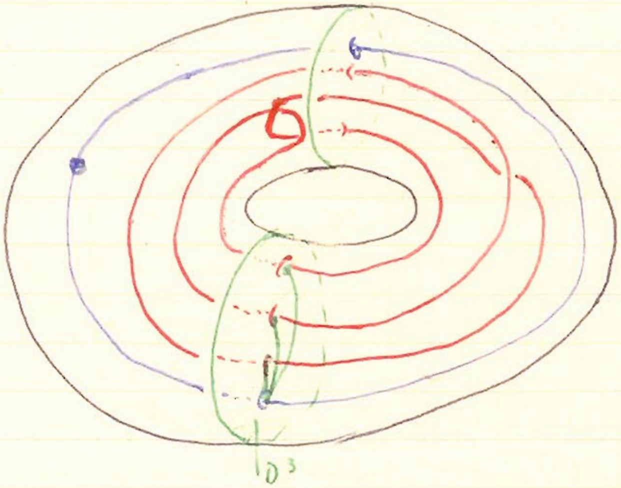
Lemma 33: Associated with each handle structure is a spine of M with cell structure $X = e_0 \cup e_1 \cdots \cup e_\lambda$
$\dim e_\mu = n_\mu$ (not unique, unique up to hpy)

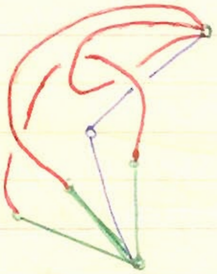
Proof: X spine: $X \subset \mathring{M}$, $M \searrow X$. Proof by induction
$\lambda^0 \Rightarrow X = pt$, $h^0 = m$-ball, $h^0 \searrow pt$
Assume we have $M_{\lambda-1} \searrow X_{\lambda-1}$ where $M_{\lambda-1} \cong h_0 \cup \cdots \cup h_{\lambda-1}$
since $M_{\lambda-1} \searrow X_{\lambda-1}$ $\exists$ derived nbh of $X_{\lambda-1} \ni M_{\lambda-1} \cong$ regular derived nbh, keeping $X_{\lambda-1}$ fixed.



Claim: $M_{\lambda-1} \searrow X_{\lambda-1} \cup_f S\times I$ where $f: S\times I \to X_{\lambda-1}$, S attaching sphere
Use ε-nbh and collapse from highest dimension, leaving S in the reg nbh and its simplices fixed. Then blow up by radial (or quasiradial maps) to the attaching sphere
$M = M_{\lambda-1} \cup h_\lambda \searrow M_{\lambda-1} \cup$ ~~core~~ core $h_\lambda \searrow X_{\lambda-1} \cup_f S\times I \cup \text{core} = X_{\lambda-1} \cup e_\lambda$ >>

Expl: $S^m = h^0 \cup h^m$
$P^m = h^0 \cup h^1 \cup h^2 \cup \dots \cup h^m$
Mazur mf $M^4 = h^0 \cup h^1 \cup h^2$ by taking different homeom to attach (different twists) so you get countably many expl's of Mazur mfs.





spine of Mazur-mf is the Dunce hat, since using generators you get the relation x^2x^{-1}

Conjectures: $M^3 \searrow K^2$ contractible $\Rightarrow M^3 = $ ball (true for $K^2 = $ duncehat)
$M^5 \searrow K^2$ contractible $\Rightarrow M^5 = $ ball

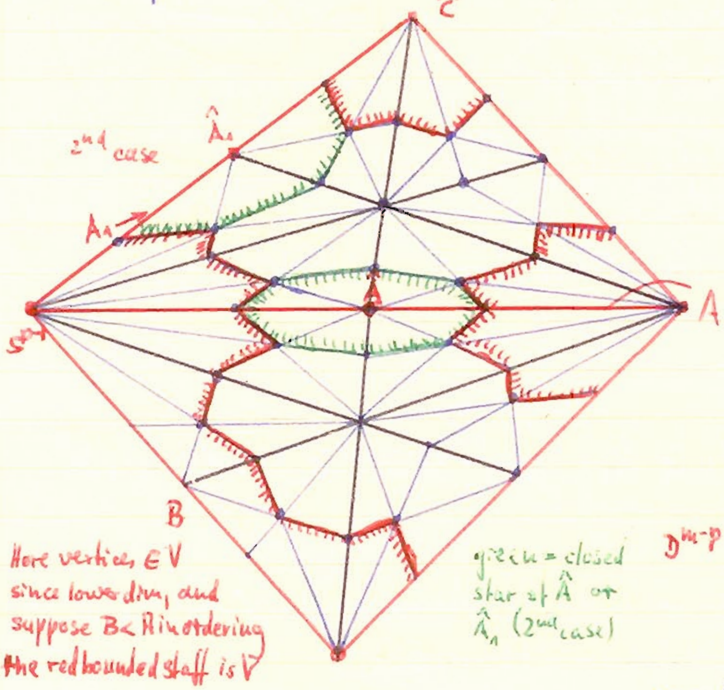
False: $M^4 \searrow K^2$ contractible $\Rightarrow M^4 = $ ball (counterexample Mazur mf)

True: $m \geq 6$, $M^m \searrow K^2$ contractible $\Rightarrow M^m = $ ball

Lemma 34: Every cpct PL-mf has a handle-structure

Proof: Choose a triangulation M, order the simplices A^p in order of ↑ dim. Let h^p_A be the p-handle corresponding to A^p. Let M'' = (barycentric) 2nd derived), let $h^p_A = \overline{st}(\hat{A}, M'')$ where $\hat{A}$ is the barycentre. We have to show that the glue is correct.
Let $V = \bigcup_{\substack{B<A \\ \uparrow\text{order}}} h_B$ $\forall B < A$ in this ordering (do not mix up with face). pt = 0-handle:

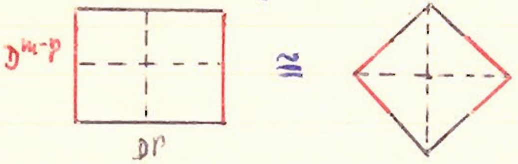
induction basis ex. We have got to produce a homeom $D^p \times D^{m-p}, \partial D^p \times D^{m-p} \longrightarrow h^p_A, h^p_A \cap V$
$V =$ nbh of $\cup B$ and since 2nd derived nbh, it is regular $\therefore$ V inf



Here vertices $\in V$ since lower dim, and suppose $B < A$ in ordering the red bounded staff is V

green = closed star of $\hat{A}$ or $\hat{A}_1$ (2nd case)

$lk(\hat{A}, M') = (\partial A') S^{m-p-1}$
where $S^{m-p-1} \equiv lk(A,M)'$
$\widehat{AC} \longmapsto \hat{C}$
$lk(\hat{A}, M') = S^{p-1} S^{m-p-1}$
$\partial h \cdot lk(\hat{A}, M'') \equiv (S^{p-1} S^{m-p-1})'$
$\widehat{AD} \longmapsto \hat{D}$
$V \cap h = \partial V \cap \partial h \equiv N(S^{p-1}, (S^{p-1} S^{m-p-1})')$
In our picture $S^{p-1} = (\partial A)'$
Now $N(S^{p-1}, (S^{p-1} \cdot S^{m-p-1})') \cong S^{p-1} \times D^{m-p-1}$
What we say is



see def of handle. In this case we used $A \subset \mathring{M}$

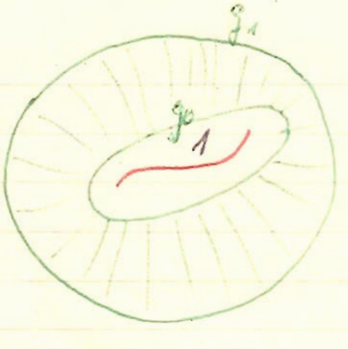
<u>Case 2:</u> $A \subset \partial M \therefore lk(A, M') =$ ball $\therefore h \equiv \hat{A}(S^{p-1} B^{m-p-1})'$, $V \cap h \equiv N(S^{p-1}, (S^{p-1} B^{m-p-1}))'$
$\not\equiv S^{p-1} \times D^{m-p-1}$ so the same formula is true
No!!
$\gg$

In differential theory one can do similar things using Morse-theory. ~~T~~ E.g. going along intersections on a torus each critical pt corresponds to sewing on a handle.

We had last time: $M \cup h^p$ is an abbreviation for $M \cup_f D^p \times D^{m-p}$, $f: \partial D^p \times D^{m-p} \subset \partial M$
$S^{p-1} = f(\partial D^p \times 0)$ attaching sphere of the handle h^p
$N = f(\partial D \times D^{m-p})$ attaching tube

<u>Lemma 35:</u> S_* is ambient isotopic to S in ∂M and N_* is a regular nbh of S_* in ∂M. Then $\exists$ handle h^p_* with attaching sphere S_* and attaching tube $N_* \ni M \cup h^p_* \cong M \cup h^p$

Proof: let $g_t: \partial M \to \partial M$ ambient isot S to S_* + then isot. the image of the attaching tube N of h^p onto N_* keeping S_* fixed. This is possible since N_* and im of N are regular nbhs of S_* and hence amb. isot. Now extend $g_1: \partial M \to \partial M$ to homeom $g: M \to M$



Take nf, put collar at outside which gives reg nbh of new boundary, take reg nbh of collar which is again reg nbh of boundary, ∃ amb isotopy which throws the big one to the small. This gives spine Now ambient isotop in the collar to get g.

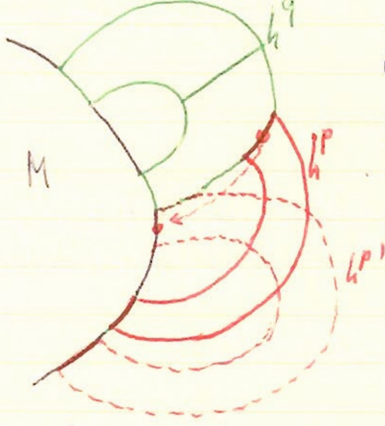
Let $f_* = g_1 f : \partial D^p \times D^{m-p} \longrightarrow \partial M$. This defines $M \cup h^p_*$

$$\begin{tikzcd}
\partial D^p \times D^{m-p} \arrow[r, "1"] \arrow[d, "f"] \arrow[dr, phantom, "\copyright"] & \partial D^p \times D^{m-p} \arrow[d, "f_*"] \\
\partial M \arrow[r, "g_1"] & \partial M
\end{tikzcd}$$

Hence $M \cup h^p \xrightarrow[g \cup 1]{\cong} M \cup h^p_*$

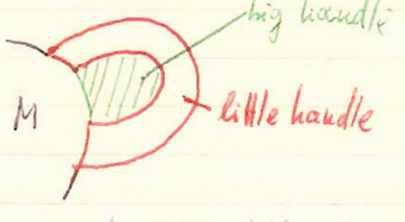
Lemma 36: Let $Q = M \cup h^q \cup h^p$, $p \leq q$. Then ∃ h^p_* disjoint from h^q ∋ $Q \cong M \cup h^p_* \cup h^q$

Cor: Given any handle structure we can rearrange the handles in order of ↑ dim + ∋ any two handles of the same dimension are disjoint.



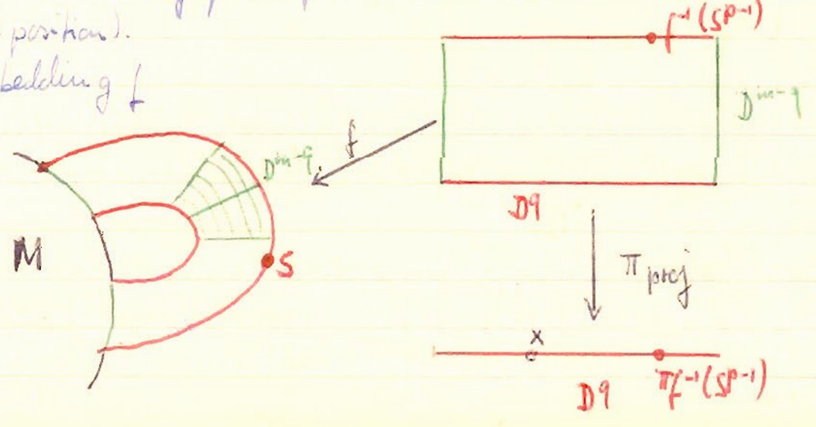
We first move the attaching sphere off and then the attaching tube.

The decomposition is i.g. impossible if the little handle comes first



Proof: Let S^{p-1} = attaching sphere of h^p, let choose transverse disk D^{m-q} of h^q not meeting S^{p-1} (general position).

There ∃ embedding f

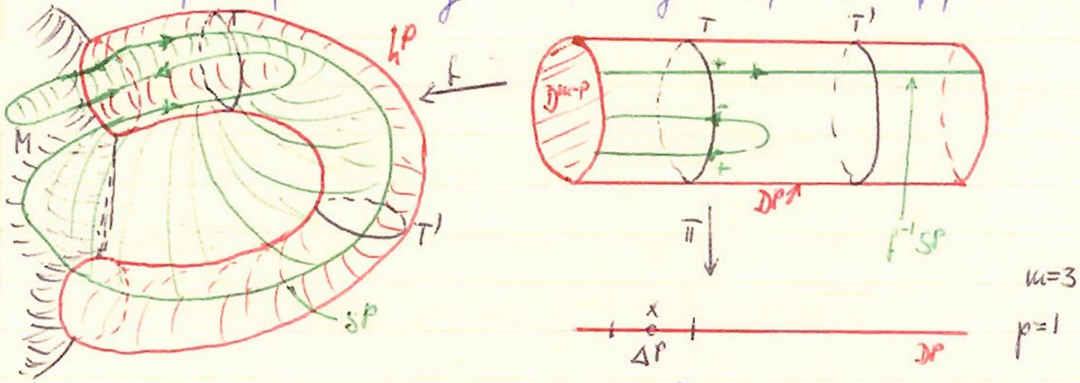


The proj cannot cover the whole thing so $p-1<q$

Choose a reg. nbh of D^{m-q} in h^q not meeting S^{p-1}. Both N, h^q are reg nbhs of D in $M \cup h^q$ ∴ ambient isot. S off h^q + isotopes h^p. Call the new position S_*.
Now choose reg nbh N_* of S_* not meeting h^q, possible since S_* does not meet h^q. Now isotop h^p further ∋ its attaching tube $= N_*$. Call this h^p_*

Handle cancellation

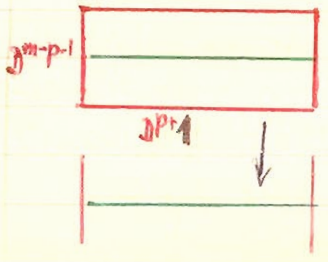
Let $Q = M \cup h^p \cup h^{p+1}$. Let S^p = attaching sphere of h^{p+1}. Choose T^{m-p-1} to be a transverse sphere of h^p cutting S transversely in a finite # of pts.

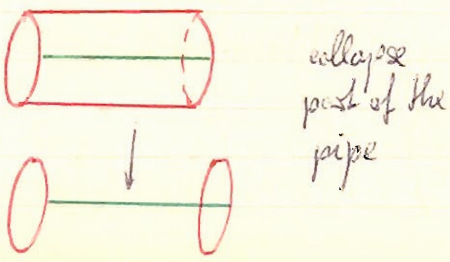


Choose triangulation of $f^{-1}S^p \subset D^p \times D^{m-p} \xrightarrow{\pi} D^p$. Choose $\Delta^p \in D^p$
One can choose triangulations ∋ maps simplicial then (equiv to Sard's theorem in diff top) take $x \in \mathring{\Delta}^p$
The geometric intersection $S \underset{G}{\cap} T$ = number of points
The algebraic intersection $S \underset{A}{\cap} T = \sum$ algebr. intersections.
To get the 2nd choose orientation of D^p and S^p + assign +1 or −1 at each point according whether the orientations agree or not. (unique up to sign in the sum)

Lemma 37: (1st cancellation lemma) If $S \underset{G}{\cap} T = 1 \Rightarrow M \cup h^p \cup h^{p+1} \cong M$

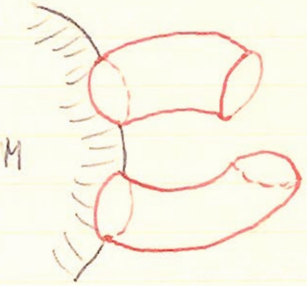
Proof: See picture above T'. Idea: $Q = M \cup h^p \cup h^{p+1} \searrow M \cup h^p \cup D^{p+1}$. Reducing the dimension the blue thing gives





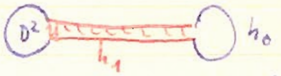
Hence $Q \downarrow M \cup [h^p - f(\mathring{\Delta}^p \times D^{m-p})] \cup D^{p+1} \downarrow M \cup f[I(D^p - \mathring{\Delta}^p) \times D^{m-p}]$. Hence we get

$\cong$ attaching tube × I



This collapses to M cylinderwise ∴ the result.
$Q^m \downarrow M^m$ ∴ both Q, M are reg nbhs in Q of a spine of M ∴ $Q \cong M$ (to be more accurate the spine should lie in $\mathring{M}$)
Difficulties if the attaching tube is twisted around the attaching sphere. E.g. $1 : (S^p \times D^{m-p}) \to (S^p \times D^{m-p})$ cannot be extended to $S^p \times D^{m-p} \cup h^{p+1} \to S^p \times D^{m-p} \cup h^p_*$

Expl



trivial glue ∴ trivial homeom

since twist ∄ trivial homeom

Theorem 11: If $f, g : S^p \to \mathring{M}^m$ are hpic embeddings, if $p \le m-3$ and M $(2p-m+2)$-connected, Then f and g are ambient isotopic

Without Proof   Db case in Ann Math 41 p 809   (PL Zeeman $\overline{\text{VIII}}$, Cor 3)

Cor 1: Let M^m be closed and p-connected $p \le m-3$. Then any two S^p's are ambient isotopic in M^m

Proof: $p + p - m + 2 \le p + (m-3) - m + 2 = p - 1$

Cor 2: If $p \le m-3$ then any S^p in S^m is unknotted (i.e. ambient isotopic to a standard $S^p \subset S^m$. The same is true for balls if $B^p \subset B^m$ properly embedded.

Def: Given two disjoint spheres in a sphere. We say they are
① homologically unlinked if each is homologous to zero in the complement of the other
② homotopically unlinked if each is hpic to the constant in the complement of the other
③ geometrically unlinked if we can ambient isotope one into the Northern hemisphere + the other into the southern hemisphere

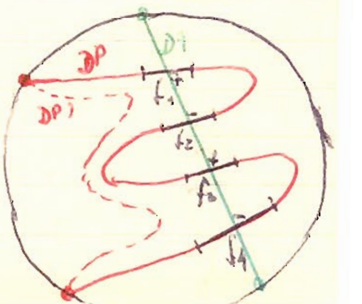
Expls: Cases				When can linking occur
1)	L	U	U	$p+q=m$
2)	L	L	L, U	$p+q \geq m$ and $p,q \leq m-2$
3)	L	L	L	

geometrically symmetric $\underset{\not\Leftarrow}{\Rightarrow}$ homotopically asymmetric $\underset{\not\Leftarrow}{\Rightarrow}$ homologically symmetric

Lemma 38: $p+q=m$, $p,q \geq 3$, $S^{p-1}, S^{q-1} \subset S^{m-1}$ disjoint and homologically unlinked $\Rightarrow$ geometrically unlinked $\therefore$ all 3) equivalent in this case

Proof: Let N = reg nbh of S^{q-1}, $V = S^{m-1} - N$. Then $V \cong S^{p-1} \times D^{m-p}$ because S^{q-1} unknotted, because $(m-1)-(q-1) = p \geq 3$ by one of the cor.
Now S^{p-1} determines an element $\xi \in \pi_{p-1}(V) \cong H_{p-1}(V) \cong \mathbb{Z}$. $\xi = 0$ by the hypothesis of homol unlinked $\therefore S^{p-1} \simeq 0$ in V. Now ambient isotop S^{p-1}, N in Northern hemisphere $\therefore V \supset$ Southern hemisphere. Choose $S^{p-1}_* \subset$ Southern hemisphere $\therefore S^{p-1}_* \simeq 0$ in V. $S^{p-1} \simeq S^{p-1}_*$ in V. Now apply theorem 11, codim $= (m-1)-(p-1) = q \geq 3$
$2(p-1)-(m-1)+2 = 2p-m+1 \leq p+(m-3)+m+1 \leq p-2$ since $q \geq 3$ and $p+q=m$
$\therefore$ V is $(p-2)$-connected $\therefore$ by theorem 11 S^{p-1} amb isot S^{p-1}_* in V, $p+q \leq m-3+\frac{1}{2}\min(p,q)$ $p,q \leq m-3$ keeping ∂V fixed (V $p-2$ connected since of the hpy type of S^{p-1} since $V \cong S^{p-1} \times D^{m-p}$), Now ambient isotop S^{p-1} to S^{p-1}_* in S^{m-1} keeping S^{q-1} fixed. >>

Lemma 39: $p+q=m$, $p,q \geq 3$ and $D^p, D^q \subset D^m$ proper embeddings (i.e. boundary to boundary, interior to interior), boundaries disjoint, interiors transversal and algebraic intersection 0. Then we can ambient isotop D^p off D^q keeping the boundaries fixed.



Proof: Since $q \leq m-3$ D^q is unknotted in D^m, i.e. $D^m \cong D^q * S^{p-1}$ $\partial D^m \cong \partial D^q * S^{p-1}$ $\therefore$ we have hpy equivalences $\partial D^m - \partial D^q \subsetneq D^m - D^q$ and $D^m - D^q \to S^{p-1}$ by retraction. $\partial D^p \subset \partial D^m - \partial D^q$ (no hpy eq^{ce}) ∂D^p determines $\xi \in H_{p-1}(\partial D^m - \partial D^q) \cong H_{p-1}(D^m - D^q) \cong H_{p-1}(S^{p-1}) \cong \mathbb{Z}$

$\cong$ because of the hpyeq^{ce}. Now $f \sim f_1 + f_2 + f_3 + \cdots$ ∴ $[f] = [f_1] + [f_2] + \cdots$
$= D^p \underset{A}{\cap} D^q = 0$ by hypthesis ($\underset{A}{\cap}$ algebr. intersection) ∴ $\xi = 0$ ∴ ∂D^p is homologous unlinked ~~by lemma~~ from ∂D^q in ∂D^m ∴ by lemma 38 geometrically unlinked.

1) Ambient isotop D^m moving $\begin{cases} \partial D^q \text{ into the Northern hemisphere} \\ \partial D^p \text{ into the southern hemisphere} \end{cases}$

possible since geom unlinked

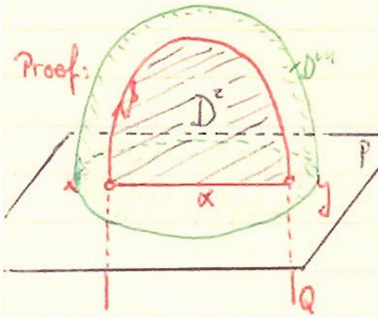
2) Ambient isotop D^m leaving ∂D^m fixed into the northern hemiball

3) Ambient isotop D^m keeping ∂D^m fixed to move D^p into the southern hemisphere (ignore D^q)

Now apply 1), 2), 2)$^{-1}$, 1)$^{-1}$ to D^q and leave
then 1), 2), 3), 2)$^{-1}$, 1)$^{-1}$ to D^p leave ∂D^p fixed.

Whitney lemma 40: $p+q=m$, $p,q \geq 3$, $P, Q \subset M$ closed mfs, P, Q connected, oriented, and transversal, M one connected. Then we can ambient isotop P until $P \underset{G}{\cap} Q = P \underset{A}{\cap} Q$

Proof:



Let x and y have opposite sign. Join with arcs $\alpha \in P$, $\beta \in Q$. The homeom $S^1 \to \alpha \cup \beta$ can be extended to an embedding $D^2 \subset M$ because M is 1-connected ($m \geq 6$, by moving of the map into general position (chapter $\underline{VI}$) we get the embedding) $\ni D \cap P = \alpha$, $D \cap Q = \beta$ By moving $\mathring{D}$ into general position ($2_p + p < p+q = m$) Triangulate everything inside + let $D^m = 2^{nd}$ derived nbh of D^2 in M = ball, $D^P = D^m \cap P$ = 2^{nd} derived nbh of α in P = ball
$D^q = D^m \cap Q$ similar. Use lemma 39

>>

Lemma 41: **(2nd cancellation lemma)** : Let S^p = attaching sphere of h^{p+1} and T^{m-p-1} be a transverse sphere of h^p and let $S \underset{A}{\cap} T = 1$. Then if ∂M^m is 1-connected and $3 \leq p \leq m-4$. Then $M \cup h^p \cup h^{p+1} \cong M$
it is enough that $\partial(M \cup h^p)$

Proof: Isotop S to $S_* \ni S_* \underset{G}{\cap} T = 1$. Isotop h^{p+1} to h^{p+1}_* with S_* = attaching sphere. Cancell by first cancellation lemma.

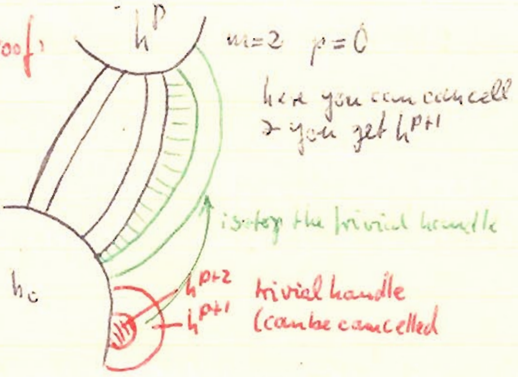
$\langle , \rangle$, $(\cap)$ Notation: Geometric intersection $S \cap T$, algebraic intersection $\langle S, T \rangle$

3rd cancellation lemma: Let $(h^p)_\lambda = h_1^p \cup h_2^p \cup \dots \cup h_\lambda^p$. Let $M = h^0 \cup (h^p)_\lambda \cup (h^{p+1})_\mu$, let M be p-connected, $p \geq 0$ and $2p+3 \leq m$. Then $M \cong h^0 \cup (h^{p+1})_\lambda \cup (h^{p+1})_\mu$

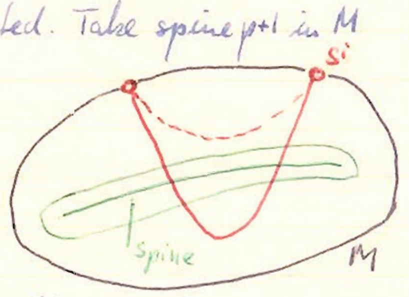
$M \cong h^0 \cup (h^{p+1})_\mu \cup (h^{p+2})_\lambda$

Remark: The Eulercharacteristic is not changed under this isotopy

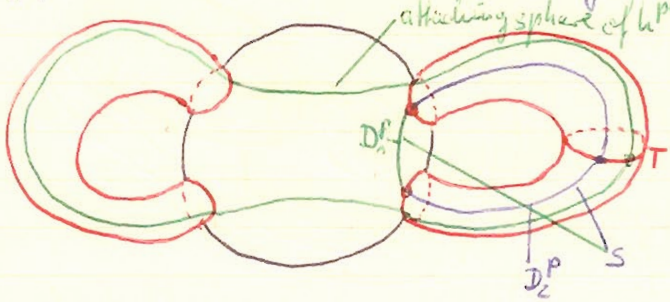
Proof:

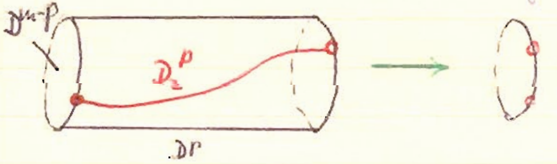


1) ∂M is also p-connected. Take spine $p+1$ in M. Given $S^i \subset \partial M$, $i \leq p$, then S^i is spanned by $D^{i+1} \subset M$. By general position homotop D^{i+1} off spine; because $(i+1)+(p+1) < m$, this is possible. Then homotop D^{i+1} collarwise into ∂M ∴ $\pi_i(\partial M) = 0$



2) Isotop all h^p's disjoint (lemma 36). Consider the last h^p. Choose a transverse sphere T of h^p. Choose $S \subset \partial M \ni S$ cuts T transversely exactly once. Proof of the $\exists$ of S: Notice that all the attaching spheres of the h^p's are disjoint and unlinked, because they are $p-1$ in ~~dim~~ M, and $2(p-1) + 1$ (+1 for linking) $< m-1$ by assumption ∴ they are unlinked. They are disjoint by lemma 36. Hence $\exists$ disk $D_1^p \subset \partial h^0 \ni \mathring{D}_1^p$ does not meet the h^p's. $\partial D_1^p \subset \partial$(attaching tube of h^p).

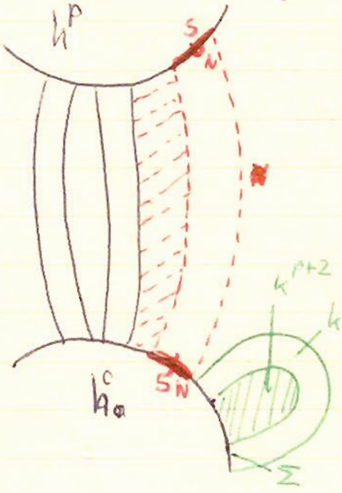




$$\begin{tikzcd}
\partial D_1^p \arrow[r, phantom, "\subset"] \arrow[rr, bend right, "f"'] & \partial D^p \times \partial D^{m-p} \arrow[r, "proj"] & \partial D^{m-p}
\end{tikzcd}$$

Extend f to $D^p \longrightarrow \partial D^{m-p}$, possible because $p-1 < m-p$. Then the graph $D^p \xrightarrow[\subset]{\Gamma f} D^p \times \partial D^{m-p}$

Let $D_2^p = \operatorname{im} \Gamma f$. D_2^p cuts T transversally. Now let $S^p = D_1^p \cup D_2^p$ ∴ S^p exists. We had to do this construction, because the attaching maps can be twisted around wildly. Hence we proved $\exists$ $S^p \subset \partial(h^0 \cup (h^p))$ because h^p is disjoint from all other handles. Now isotop the attaching spheres and tubes of h^{p+1} off S^p (start with attach spheres (Possible because $p+p<m-1$). Finally choose a reg. nbh N of S^p not meeting any h^{p+1}.

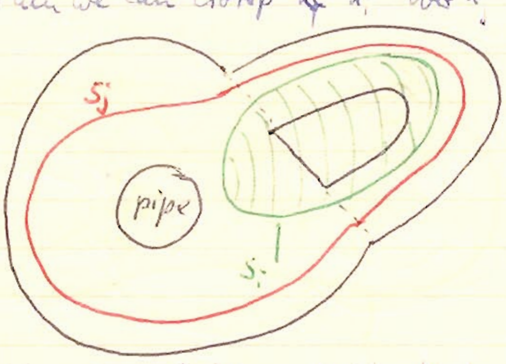


3) Add trivial $k^{p+1} \cup k^{p+2}_i$ on some free piece of ∂h^0. $M = h^0 \cup (h^p) \cup (h^{p+1}) \cong h^0 \cup (h^p) \cup (h^{p+1}) \cup k^{p+1} \cup k^{p+2}$. Let Σ = attaching sphere of k^{p+1}. S, Σ are p-spheres in the p-connected closed mf ∂M. By Cor 1 of Thm 11 these are ambient isotopic if $p \leq m-3$ ∴ isotop k^{p+1} ∋ its attaching sphere + tube are $S + N$, k^{p+2} gets carried along. k^{p+1} is disjoint from (h^{p+1}) by construction above. Hence we can attach k^{p+1} first, next to h^p ∴

$M \cong h^0 \cup (h^p) \cup (k^{p+1}) \cup (h^{p+1}) \cup k^{p+2}$ and cancell by the first cancellation lemma. Now show the lemma inductively.

Fourth cancellation lemma: Let $Q = M \cup h^p \cup (h^{p+1})_{\lambda}$, let the homology of h^p be killed, i.e. $H_p(M \cup h^p, M) \xrightarrow{i} H_p(Q, M)$ is zero, let ∂M be 1-connected and $3 \leq p \leq m-4$. Then $Q \cong M \cup (h^{p+1})_{\lambda-1}$

Proof: Let $T = T^{m-p-1}$ = transverse sphere of $(h^p) \subset \partial(M \cup h^p)$. Let $S_i = S_i^p$ = attach $(h_i^{p+1}) \subset \partial M \cup h^p$)
Let $b_i = \langle S_i, T \rangle$ Suppose $|b_i| > |b_j| > 0$ for some i, j. Then we can isotop ~~by~~ h_i^{p+1} over h_j^{p+1} to h_i^* say ∋ $|b_i^*| = |b_i| - |b_j|$, $< |b_i|$ as follows:
Choose a tiny $S^p \subset \partial h_j$ linking the transverse sphere.
~~Isotop~~ S off h_j onto $\Sigma \subset \partial(M \cup h^p)$, near S_j ∴
$\langle \Sigma, T \rangle = \langle S_j, T \rangle = b_j$. Let $S_i \# S$ (connected sum) be S_i joined to S by a little pipe. S_i isotopic to $S_i \# S$ isotopic to $S_i \# \Sigma = S_i^*$. Isotop h_i to h_i^*. $b_i^* = \langle S_i \# \Sigma, T \rangle = b_i \pm b_j$. Sign from orientation of the pipe, choose the sign by giving the pipe a twist ∋ $|b_i^*| < |b_i|$ possible since we are in codim 3, i.e. $|b_i^*| = |b_i| - |b_j| < |b_i|$



Remark: The 4th cancellation lemma can be improved to $2 \leq p \leq m-4$. A proof can be found in Smale's papers.
If ∂M has several components and each is 1-connected, we can work on each component separately.

Continuation of the proof: Consider the exact homologysequence of the triple $Q, M\cup h^p, M$

$$H_{p+1}(Q, M\cup h^p) \xrightarrow{\partial} H_p(M\cup h^p, M) \xrightarrow{j} H_p(Q,M)$$

By hypothesis $j=0$, and $H_p(M\cup h^p, M)=Z$ since just one handle added $\therefore$ ∂ epim.
$H_{p+1}(Q, M\cup h^p)$ is free abelian of rk λ with generators k_i, $1\le i\le\lambda$ corresponding to the handles h_i^{p+1}, and $\partial k_i = b_i$ by looking at the intersections with T. Hence the highest common factor of the b_i's is 1 since ∂ is epi.
By the soap bubble process of the first part of the proof we can reduce some b_j, b_1 say, to 1. $\therefore$ $\langle S_1, T\rangle = 1$ $\therefore$ By 2nd cancellation lemma we can cancell $h^p\cup h_1^{p+1}$!
We have to ~~verf~~ verify the various assumptions in the 2nd cancellation lemma:
$\partial(M\cup h^p)$ 1-connected: Because $\partial(M\cup h^p) = (\partial M - \text{attach tube } h^p)\cup(\text{boundary tube } h^p)$
$\simeq (\partial M - \text{attach sphere } h^p)\cup T$ $\Rightarrow$ van Campens theorem gives 1-connec=
1 conn because codim ≥ 4 — 1-conn dim $m-p-1$
tedness of $\partial(M\cup h^p)$

»

Remark: The 3rd cancellation lemma can be proved for $Q = M\cup(h^p)_\lambda\cup(h^{p+1})_\mu$ where M need not be a h^0

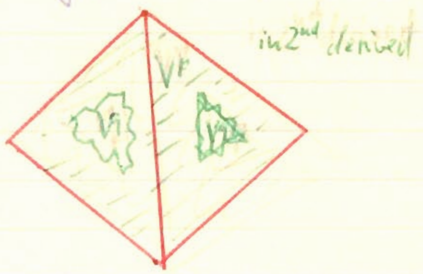
Poincaré conjecture: Smale had a gap in his proof. So proved by Stallings for dim M ≥ 7. Pushed down by Zeeman to dim M ≥ 5, using engulfings. Smale could fill the gap in his proof, to get a stronger result, a PL-sphere whilst Stallings + Zeeman got a top. sphere only.

Theorem 12: Let M^m be a PL-manifold which is a homotopy S^m-sphere (i.e. $H_i(S^m)=\delta_{im}Z$ for $i>0$, $\pi_1(S^m)=0$; 1-connected), $m\ge5$. Then $M^m \cong S^m$ where $\cong$ is a PL homeom.

Proof: $m\ge6$ Triangulate M, let M' be the 1st derived, M'' the 2nd derived. Let M^p be the p-skeleton and M_*^q the dual q-skeleton $=\{A\in M' ; A\cap M^{m-q-1}=\emptyset\}$. Let $V^p = N(M^p, M'')$, $V_*^q = N(M_*^q, M'')$
$V^1 = h^0\cup(h^0)\cup(h^1)$, 0-connected, $\cong h^0\cup(h^1)\cup(h^2)$ by (3) glue on 2-handles
$V^2 = h^0\cup(h^1)\cup(h^2)$, 1-connected $\cong h^0\cup(h^2)\cup(h^3)$ glue on 3-handles
$V^3 = h^0\cup(h^2)\cup(h^3)$, 2-connected $\therefore$ 2 dim hol. killed in handles. Use 4th lemma $\therefore$

$V^3 \cong h^0 \cup (h^3)$ (must be enough ~~p~~ 3-handles, otherwise hand $h^0 \cup (h^2)$ not killed)

Now by induction $V^p = h^0 \cup (h^p)$, $3 \leq p \leq m-3$. Similarly $V_*^q = h_*^0 \cup (h_*^q)$ $3 \leq q \leq m-3$



$M = V^2 \cup V_*^{m-3} \cong [h^0 \cup (h^2) \cup (h^3)] \cup [h_*^0 \cup (h_*^{m-3})]$

Unglue h_*^{m-3} in the 2nd factor and glue them onto the 1st factor with gluing disk D^3 insted of D^{m-3}. So we get some more 3 handles on the 1st factor:

$$M \cong \underbrace{[h^0 \cup (h^2) \cup (h^3)]}_{\text{~~hand~~ ~~hpy~~ ball}} \cup h_*^0$$

$\therefore$ 2nd hand killed, we must not run out to early of three handles, otherwise we cannot kill the hand. Since 2nd hand killed:

$$M \cong [h^0 \cancel{h_*^0} \cup (h^3)] \cup h_*^0$$

Now h^3 knock out because otherwise it would not be a hpy ball $\therefore M \cong h^0 \cup h_*^0 = S^m$ »

Def: W^{m+1} is an <u>h-cobordism</u> ~~of~~ between M_0^m and M_1^m, where M_1, M_0 are closed mfs, if $\partial W^{m+1} = M_0 \cup M_1$ and $M_i \subset W$ is a hpy eq^{ce}. h-cobordism

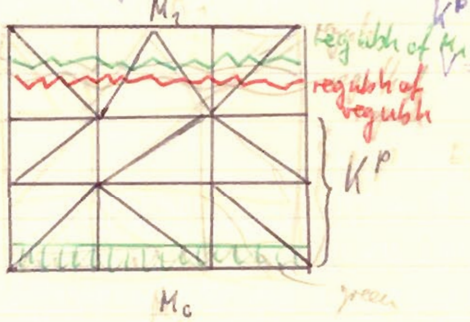
Expl. ~~Co~~Bordism which is no h-cobordism

Theorem: W^{m+1} is h-cobordism between M_0, M_1; M_i is closed and 1-connected, $m \geq 5$. Then $M_0 \cong M_1$ and $W \cong M_0 \times I \cong M_1 \times I$

Proof: Triangulate W^{m+1}, with M_i a full subcomplex. W^p = p-skeleton, W_*^q dual q-skeleton

$K^p = \{A \in W^p;\ A \cap M_1 = \emptyset\}$, $K_*^q = \{A \in W_*^q, A \cap M_0 = \emptyset\}$

$V = N(M_0, W'') =$ reg nbh of M_0 in W; $\therefore \cong M_0 \times I$ by collar thm



$V_* = N(N(M_1, W'), W'') \cong M_1 \times I.$

$V^p = V \cup N(K^p, W'')$ $\qquad V_*^q = V_* \cup N(K_*^q, W'')$

$V^2 \cong V \cup (h^2) \cup (h^3)$ by (3) + 1-connectedness

$V^3 \cong V \cup (h^2) \cup (h^3)$ + since $V \subset W$ hpy eq^{ce}, $V \subset V^3$ induces isom of hpy in dim ≤ 2

$\therefore$ 2-dim homology is killed. $\therefore$

$V^3 \cong V \cup (h^3)$ by 4 $\therefore$

$V^p \cong V \cup (h^p)$ $\qquad 3 \leq p \leq m-3$ $\therefore$

$W = V^2 \cup V_*^{m-2}$ $\qquad$ but dim $W = m+1$

$\cong [V \cup (h^2) \cup (h^3)] \cup [V_* \cup h_*^{m-2}]$ $\qquad$ transfer (h_*^{m-2}) to (h^3) as in the proof of the Poincaré conjecture

but $[V \cup (h^2) \cup (h^3)]$ hpy eq^{ce} to V $\therefore$ 2nd homol killed $\therefore$

$W \cong V \cup V_*$ $\qquad$ because if there were any handles left it would not ~~be~~ have 2nd homol killed $\therefore$

$W \cong V \cup V_* \cong (M_0 \times I) \cup (M_1 \times I)$ with $M_0 \times 1 \cong M_1 \times 1$