HIGHER INVERSE LIMITS AND HOMOLOGY THEORIES

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Abstract

Part I is purely algebraic. We obtain the definition of the first derived functor $\varprojlim'$ of inverse limit and show that i) $\varprojlim'$ vanishes on all inverse systems (of abelian groups) which are "star-epimorphic." ii) $\varprojlim'$ is "right exact" on the category of inverse systems whose underlying index set is countable and "directed." iii) $\varprojlim'$ is preserved by cofinal inverse systems.

The definition of $\varprojlim'$ is obtained as follows: First we consider a trivial generalization of inverse limit, the relative inverse limit so-called. Then based on the relative inverse limit we define $\Lambda(B,V,A)$, a three-variable function of inverse systems with the condition $B \supset V \supset A$. From Λ we acquire a two-variable function by setting $\lambda(B,A) = \Lambda(B,A,A)$. Finally we ascertain that λ depends only on A as long as B is a star-epimorphic covering of A. This dependence of λ on A is the first derived functor of inverse limit. The laborious definition of $\varprojlim'$ has i) above as a by-product. This definition is also shown to be equivalent to one implied by Cartan-Eilenberg in <u>Homological Algebra</u>.

In order to show that $\varprojlim'$ is right exact on certain categories of inverse systems, we need the notion of

"derived inverse systems" as well as the following simple proposition: $\varprojlim(A + B) = \varprojlim A + \varprojlim B$ if $A, B \subset C$ and $A \cap B$ is star-epimorphic. The examples of star-epimorphic inverse systems are provided in Part I by some derived inverse systems and in Part II by certain inverse systems of chain groups.

Part II consists of applications of Part I to various homology and cohomology theories. First we consider infinite complexes as described by Lefschetz in Algebraic Topology without, however, the assumption that the complexes be star-finite or closure-finite. Certain infinite chains are selected as ∂-permissible or δ-permissible and the homology or cohomology based on such chains are considered. Then we establish a short exact sequence which relates the homology or cohomology of the infinite complex based on the group of permissible chains to the inverse system of the homology or cohomology groups of subcomplexes of the given infinite complex. This short exact sequence, named after Professor John Milnor, involves both $\varprojlim$ and $\varprojlim'$. By specializing the infinite complex to the simplicial, the singular, the Steenrod-Milnor, and the Eilenberg-MacLane complexes we obtain the corresponding Milnor short exact sequences. Of special interest is that for Steenrod-Milnor homology, which gives rise to a short exact sequence relating this homology to that of Čech.

In the appendix it is shown that $\varprojlim'$ is right exact on the category of inverse systems whose constituent groups are finitely generated free abelian groups. A number of theorems in Part II have a greater generality because of this result.

INTRODUCTION

Inverse limit as a functor on the category of inverse systems of abelian groups is not "right exact." Hence, according to Cartan-Eilenberg [1] it is possible to define the higher derived functors of inverse limit once the appropriate injective objects are found (cf. § 10, Part I). In Part I we study the first derived functor with the view of making applications in various homology and cohomology theories in Part II. For the study of the first derived functor a certain type of inverse system turned out to be useful in making the definition as well as acquiring an important property of the first derived functor. These inverse systems are called "star-epimorphic." It is shown that the first derived functor vanishes on all star-epimorphic inverse systems, and in Part II it is indicated that certain inverse systems of chain groups are star-epimorphic.

But for an effective application of the first derived functor to homology and cohomology theories, it is necessary to determine whether the first derived functor is right exact. This seems a difficult problem. In Part I only a restricted result is obtained: it is shown that the first derived functor is right exact, if the index set underlying the inverse systems is countable and directed, in particular if we are dealing with inverse sequences. A more universal result was obtained by Professor John Milnor. In the Appendix he ascertains the right exactness for all inverse systems whose constituent groups are finitely generated free abelian groups. The importance of this result becomes apparent in Part II (cf. Theorems 1, 2, 3, 4).

The purpose of Part II is to establish for various homology and cohomology theories certain short exact sequences. These sequences were first conjectured by Professor Milnor and were in fact the reasons for the study of the first derived functor of inverse limit in Part I. As we know, homology theory was first developed for finite complexes; and to extend it to infinite complexes, certain restrictions were imposed on the types of infinite complexes, such as "star-finiteness," "closure-finiteness," etc. However, such restrictions are not always necessary. This becomes clear as the result of studying Professor Milnor's modernized and generalized version of Steenrod's homology, for here one realizes that by restricting the type of infinite chains one may study the homology of an infinite complex which is not star-finite.

The relations between homology (cohomology) groups of infinite complexes based on suitable types of infinite chains and those of their (finite) subcomplexes are expressed by Milnor short exact sequences. In particular Steenrod-Milnor homology satisfies such a short exact sequence, which leads to another short exact sequence relating this homology to Čech homology. Other Milnor short exact sequences are obtained by specializing the infinite complex and its associated infinite chains.

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PART I

1. Basic Notions

Throughout all sections by group we shall always mean abelian group. And by inverse system always inverse system of abelian groups.

Let I be a partially ordered set ($\alpha < \alpha$ admitted). An inverse system (A;I) consists of a collection of groups $\{ A_\alpha \mid \alpha \in I \}$ and a collection of projection homomorphisms $\{ p_\alpha^\beta \mid \alpha < \beta \ , \ \alpha, \beta \in I \}$ where p_α^β is a homomorphism of A_β into A_α such that if $\alpha < \beta < \gamma$ then p_β^γ followed by p_α^β is identical with p_α^γ i.e. $p_\alpha^\beta \ p_\beta^\gamma = p_\alpha^\gamma$. The inverse limit denoted $\varprojlim(A;I)$ is the group each of whose element is a simultaneous selection of element a_α from A_α for all α such that $p_\alpha^\beta \ a_\beta = a_\alpha$ for $\alpha < \beta$.

A homomorphism h of (A;I) into (B;I) is a collection of homomorphisms $\{h_\alpha \mid \alpha \in I\}$ such that the diagram

$$\begin{tikzcd} A_\beta \arrow[r, "h_\beta"] \arrow[d, "p_\alpha^\beta"'] & B_\beta \arrow[d, "q_\alpha^\beta"] \\ A_\alpha \arrow[r, "h_\alpha"] & B_\alpha \end{tikzcd}$$

is commutative, i.e. $h_\alpha \ p_\alpha^\beta = q_\alpha^\beta \ h_\beta$.

A sequence of inverse systems

$$(A_1;I) \longrightarrow (A_2;I) \longrightarrow \cdot \ \cdot \ \cdot \longrightarrow (A_n;I)$$

where each arrow represents a homomorphism is said to be exact if for each pair of consecutive homomorphisms the image of the first is the kernel of the second. Now if $(A;I) \subset (B;I)$ i.e. A_α is a subgroup of B_α and p_α^β is the restriction of q_α^β for all $\alpha < \beta \in I$, and if $(C;I)$ denotes the usual quotient object arising from the inclusion $(A;I) \subset (B;I)$, then the following exact sequence holds

$$0 \longrightarrow (A;I) \xrightarrow{i} (B;I) \xrightarrow{j} (C;I) \longrightarrow 0$$

where i is the inclusion homomorphism and j is the cannonical homomorphism. Conversely, if such "short" exact sequence is given we may regard $(A;I)$ as a sub inverse system of $(B;I)$ and $(C;I)$ the quotient inverse system of $(B;I)$ by $(A;I)$.

For each fixed partially ordered set I, we may consider the category of inverse systems $\{(A;I)\}$, i.e. the totality of all inverse systems over the same index set I and all possible homomorphisms among these inverse systems. As long as we are considering only the inverse systems belonging to a fixed category, we may omit the letter "I" from the notations of inverse systems.

Inverse limit is a functor on the category of inverse systems. This means that for any given homomorphism $f:A \longrightarrow B$ there corresponds a natural induced homomorphism $f_*: \varprojlim A \longrightarrow \varprojlim B$ satisfying the following conditions.

1). If $f:A \longrightarrow A$ is an identity homomorphism then $f_*: \varprojlim A \longrightarrow \varprojlim A$ is also an identity homomorphism.

2). If $f:A \longrightarrow B$, $g:B \longrightarrow C$, then $(gf)_* = g_*f_*$.

Now, inverse limit is left exact as a functor. This means given a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of inverse systems, the functorial "action" of inverse limit gives rise to the following half "closed" exact sequence.

$$0 \longrightarrow \varprojlim A \xrightarrow{i_*} \varprojlim B \xrightarrow{j_*} \varprojlim C$$

By an example (§9) we shall show that j_* is in general not onto.

If we let k denote the quotient of $\varprojlim C$ by $j_*(\varprojlim B)$ then we may write

$$0 \longrightarrow \varprojlim A \longrightarrow \varprojlim B \longrightarrow \varprojlim C \longrightarrow k \longrightarrow 0$$

We will determine the nature of k's dependence on A and B in §8.

2. Star-epimorphic Inverse Systems

Let I' be a subset of I. Let

$$\mathfrak{D}(I') = \{ \alpha \mid \alpha \in I, \quad \alpha < \beta \quad \text{for some} \quad \beta \in I' \}.$$

We say that I' is a full subset of I if $\mathfrak{D}(I') = I'$. Unless otherwise stated we shall consider only subsets of I that are full.

Now if $I' \subset I$, we may consider the inverse system (A;I') obtained by restricting (A;I) to I' in an obvious way. Given $I' \subset I'' \subset I$ we consider the following restriction homomorphism,

$$P^{I''}_{I'}: \ \varprojlim(A;I'') \longrightarrow \varprojlim(A;I')$$

where $(P^{I''}_{I'}\xi)_\alpha = \xi_\alpha$ for $\xi \in \varprojlim(A;I'')$ and $\alpha \in I'$.

<u>Definition 1</u>. An inverse system $(A;I)$ is said to be star-epimorphic if $P^{I''}_{I'}$ is an epimorphism (i.e. onto-homomorphism) for every pair $I' \subset I''$.

We make the following remarks:

1). If A is star-epimorphic, then $\varprojlim A \neq 0$ unless $A = 0$.

2). An inverse system $(A;I)$ is star-epimorphic if and only if for any given $I' \subset I$ and $\xi \in \varprojlim(A;I')$ and for any given $\beta \in I - I'$

$$\bigcap_\alpha {p^\beta_\alpha}^{-1}(\xi_\alpha) \neq \phi$$

where α ranges in $I' \cap \mathcal{D}(\beta)$. ξ_α is an abbreviation for $\xi(\alpha) \in A_\alpha$.

3). The requirement that an inverse system be star-epimorphic is in general much stronger than merely requiring that every projection homomorphism p^β_α in the inverse system be epimorphic. If $I = Z^+$ (the set of positive integers) then the two requirements are equivalent. That is, "star-epimorphic inverse sequence" and "epimorphic inverse sequence" mean the same thing.

4). Any inverse system can be imbedded in a star-epimorphic inverse system. (See §5. Derived inverse systems.)

3. Relative Inverse Limits and Related Functors

Relative inverse limit is a generalization of inverse limit. Let ξ be a simultaneous choice of element from the system of groups, then the relative inverse limit is defined for each pair of inverse systems (B,A) with $A \subset B$ as follows:

Definition 2.

$$\varprojlim(B,A) = \{ \xi \mid p^{\beta}_{\alpha} \xi_{\beta} - \xi_{\alpha} \in A_{\alpha}, \alpha < \beta \}$$

Since $\varprojlim(B,A)$ will occur repeatedly, we will abbreviate it by [B,A]. Note $[B,0] = \varprojlim B$, $[B,B] = \prod_{\alpha} B_{\alpha}$, and that [B,A] is strictly monotone increasing on both variables, i.e. if $B' \supset B \supset A' \supset A$, then

$$\begin{array}{ccccc} & & [B',A] & & \\ & \rotatebox{45}{$\supset$} & & \rotatebox{-45}{$\supset$} & \\ [B',A'] & & & & [B,A] \\ & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\supset$} & \\ & & [B,A'] & & \end{array}$$

Lemma 1. Given a short exact sequence of inverse systems $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$, if $0 \longrightarrow \varprojlim A \xrightarrow{i_*} \varprojlim B \xrightarrow{j_*} \varprojlim C \longrightarrow k \longrightarrow 0$, then

$$k \approx [B,A] \,/\, \langle [A,A], [B,0] \rangle$$

where $\langle \cdots \rangle$ denote the group generated by $\cdots$.

Proof: i) $\varprojlim C \approx [B,A]/[A,A]$, for let $\pi : [B,A] \longrightarrow \varprojlim C$, where $\pi(\xi) = \{ j_{\alpha}(\xi_{\alpha}) \} \in \varprojlim C$, $\xi \in [B,A]$, then π is clearly onto and kernel of $\pi = [A,A]$.

ii) $j_*(\varprojlim B) \approx \varprojlim B / \varprojlim A \approx [B,0]/[A,0]$ by the exact sequence $0 \longrightarrow \varprojlim A \longrightarrow \varprojlim B \longrightarrow \varprojlim C$. Hence from i) and ii) follows

$$\varprojlim C/j_*(\varprojlim B) \approx ([B,A]/[A,A])/([B,0]/[A,0])$$
$$\approx [B,A]/\langle [A,A],[B,0] \rangle$$

We shall abbreviate this last expression by $\lambda(B,A)$. In general $\lambda(B,A)$ does not vanish. The only situation where we are sure of its vanishing at this moment is

<u>Lemma</u> <u>2</u>. $\lambda(B,A) = 0$ if $I = \mathcal{D}(\alpha)$ for some $\alpha \in I$.

<u>Proof</u>: Given $\xi \in [B,A]$, let $\zeta \in [B,0]$ be such that for $\beta < \alpha$ $\zeta_\beta = p^\alpha_\beta \xi_\alpha$, then $\xi_\beta - \zeta_\beta = \delta_\beta \in A_\beta$. Clearly $\delta = \{\delta_\beta\} \in [A,A]$ and $\xi = \zeta + \delta$.
(This lemma will be needed in proving some formulas of §5.)

$\lambda(B,A)$ can be generalized to the following useful form

$$\Lambda(B,V,A) = [B,A]/\langle [V,A],[B,0] \rangle \qquad A \subset V \subset B$$

As V varies from A to B, $\Lambda(B,V,A)$ "shrinks" from $\lambda(B,A)$ to zero. A theorem of Λ proved in §8 will trivially imply that $\lambda(B,A) = 0$ if A is a star-epimorphic inverse system. (See Corollary 2.)

4. Derived Inverse Systems

Given $(A;I)$ and $I' \subset I'' \subset I$ we have previously defined by restriction the homomorphism

$$P_{I'}^{I''}: \varprojlim(A;I'') \longrightarrow \varprojlim(A;I')$$

Now for given $(B;I) \supset (A;I)$ and $I' \subset I'' \subset I$ we can similarly define

$$P^{I''}_{I'}: \quad [B,A;I''] \longrightarrow [B,A;I']$$

by setting $(P^{I''}_{I'}\xi)_\alpha = \xi_\alpha$ for $\xi \in [B,A;I'']$ and $\alpha \in I'$.

<u>Definition 3</u>. Given an inverse system B, for each sub inverse system $A \subset B$ the derived inverse system of B with respect to A, which we denote B^A, is defined as follows:

$$B^A_\alpha = [B,A;\mathfrak{D}(\alpha)]$$
$$p^\beta_\alpha = P^{\mathfrak{D}(\beta)}_{\mathfrak{D}(\alpha)}: \quad [B,A;\mathfrak{D}(\beta)] \longrightarrow [B,A;\mathfrak{D}(\alpha)] \qquad \alpha < \beta$$

Clearly if $0 \subset A_1 \subset A_2 \subset \ldots \subset B$ then $B^0 \subset B^{A_1} \subset B^{A_2} \subset \ldots \subset B^B$. Also if $A \subset B$ then $A^A \subset B^B$.

<u>Lemma 3</u>. $B \approx B^0$ for given inverse system B.

<u>Proof</u>: Let $\xi \in B$, define $\phi: B \longrightarrow B^0$ by

$$(\phi\xi)_\alpha = P^I_{\mathfrak{D}(\alpha)}\xi \in [B,0;\mathfrak{D}(\alpha)]$$

Then one can easily check the following commutative diagram

$$\begin{tikzcd}
\xi_\beta \arrow[r, "\phi_\beta"] \arrow[d, "p^\beta_\alpha"'] & (\phi\xi)_\beta \arrow[d, "P^{\mathfrak{D}(\beta)}_{\mathfrak{D}(\alpha)}"] \\
\xi_\alpha \arrow[r, "\phi_\alpha"] & (\phi\xi)_\alpha
\end{tikzcd}$$

<u>Lemma 4</u>. B^B is star-epimorphic for any given inverse system B.

<u>Proof</u>: Given $I' \subset I'' \subset I$ we must show

$$P_{I'}^{I''}: \varprojlim (B^B;I'') \longrightarrow \varprojlim (B^B;I') \quad \text{is onto.}$$

First we consider the following natural isomorphism

$$\psi_{I'}: \varprojlim (B^B;I') \longrightarrow \prod_{\gamma\in I'} B_\gamma$$

then we check the following commutative diagram

$$\begin{tikzcd} \varprojlim(B^B;I'') \arrow[r, "\approx"] \arrow[d, "P_{I'}^{I''}"] & \prod_{\gamma\in I''} B_\gamma \arrow[d, "\pi"] \\ \varprojlim(B^B;I') \arrow[r, "\approx"] & \prod_{\gamma\in I'} B_\gamma \end{tikzcd}$$

since π is onto, so is $P_{I'}^{I''}$.

<u>Theorem 1</u>. An inverse system can be imbedded in a star-epimorphic inverse system.

<u>Proof</u>: By Lemma 3 ϕ: $B \xrightarrow{\approx} B^0$. Let $i : B^0 \longrightarrow B^B$ be the inclusion, then $i\phi$ is the imbedding of B into B^B, where B^B is star-epimorphic by the previous lemma.

The following simple formulas of derived inverse systems will be needed later.

<u>Lemma 5</u>. Given $A, A' \subset B$, we have

i) $B^A = B^0 + A^A$

ii) $(A + A')^{(A+A')} = A^A + A'^{A'}$

iii) $B^{A+A'} = B^A + A'^{A'} = B^{A'} + A^A$

iv) $B^{A+A'} = B^A + B^{A'}$

<u>Proof</u>: i) It suffices to show $B_\alpha^A = B_\alpha^0 + A_\alpha^A$, but $B_\alpha^A = [B,A; \mathfrak{B}(\alpha)]$, $B_\alpha^0 = [B,0; \mathfrak{B}(\alpha)]$, $A_\alpha^A = [A,A; \mathfrak{B}(\alpha)]$,

and by Lemma 2 $[B,A] = [B,0] + [A,A]$ if the index set is of the form $\mathfrak{B}(\alpha)$. The appropriate restrictions of the projection homomorphism of B^A gives those of B^0 and A^A.

ii) To show $(A + A')_\alpha^{(A+A')} = A_\alpha^A + A'^{A'}_\alpha$ we note $(A + A')_\alpha^{(A+A')} = \prod_{\gamma<\alpha}(A_\gamma + A'_\gamma)$, $A_\alpha^A = \prod_{\gamma<\alpha} A_\gamma$, $A'^{A'}_\alpha = \prod_{\gamma<\alpha} A'_\gamma$, but $\prod_{\gamma<\alpha}(A_\gamma + A'_\gamma) = \prod_{\gamma<\alpha} A_\gamma + \prod_{\gamma<\alpha} A'_\gamma$.

iii) $B^{A+A'} = B^0 + (A + A')^{(A+A')}$ by i) and $(A + A')^{(A+A')} = A^A + A'^{A'}$ by ii), hence $B^{A+A'} = (B^0 + A^A) + A'^{A'} = B^A + A'^{A'}$. Similarly $B^{A+A'} = B^{A'} + A^A$.

iv) $B^{A+A'} = B^A + A'^{A'}$ by iii) but $B^{A+A'} \supset B^A + B^{A'} \supset B^A + A'^{A'}$, hence $B^{A+A'} = B^A + B^{A'}$.

5. Distributivity of Inverse Limit.

Lemma 6. The derived inverse system and the relative inverse limit are related by

$$\varprojlim B^A \approx [B, A]$$

Proof: By the definition of B^A we have a whole collection of commutative diagrams

$$\begin{tikzcd} B^A_\beta \arrow[r, "\approx"] \arrow[d, "p^\beta_\alpha"'] & {[B, A; \mathfrak{D}(\beta)]} \arrow[d, "P^{\mathfrak{D}(\beta)}_{\mathfrak{D}(\alpha)}"] \\ B^A_\alpha \arrow[r, "\approx"] & {[B, A; \mathfrak{D}(\alpha)]} \end{tikzcd}$$

ranging all pairs $\alpha < \beta \in I$. Through these diagrams it is evident that each element $\xi \in \varprojlim B^A$ determines a unique element $\phi(\xi) \in [B,A;I]$ and each element $\xi \in [B,A;I]$ determines a unique element $\psi(\xi) \in \varprojlim B^A$, and that ϕ and ψ are inverse of each other.

We make the following observations: Since we know from Lemma 5 $B^A = B^0 + A^A$ and now $\varprojlim B^A = [B,A]$, $\varprojlim B^0 = [B,0]$, $\varprojlim A^A = [A;A]$ and as remarked in §4 $\lambda(B,A) \neq 0$ in general, i.e.

$$[B,A] \supsetneqq [B,0] + [A,A]$$

hence we have the following negative statement.

Lemma 7. Given $A, A' \subset B$, $\varprojlim (A + A') \not\subset \varprojlim A + \varprojlim A'$ in general.

However we have the following positive case:

Theorem 2. If $A \cap A'$ is star-epimorphic, then

$$\varprojlim (A + A') = \varprojlim A + \varprojlim A'$$

Proof; It suffices to show $\varprojlim (A + A') \subset \varprojlim A + \varprojlim A'$. So given $\eta \in \varprojlim (A + A')$, let, say

$$\eta_\alpha = a_\alpha + a'_\alpha \ , \quad a_\alpha \in A_\alpha \ , \ a'_\alpha \in A'_\alpha \ .$$

$$\eta_\beta = a_\beta + a'_\beta \ , \quad a_\beta \in A_\beta \ , \ a'_\beta \in A'_\beta \ .$$

and suppose $\alpha < \beta$, then

$$p^\beta_\alpha (a_\beta + a'_\beta) = a_\alpha + a'_\alpha$$

Thus $\quad p^\beta_\alpha a_\beta - a_\alpha = -(p^\beta_\alpha a'_\beta - a'_\alpha)$

but $\quad p^\beta_\alpha a_\beta - a_\alpha \in A_\alpha$ and $p^\beta_\alpha a'_\beta - a'_\alpha \in A'_\alpha$

hence $\quad p^\beta_\alpha a_\beta - a_\alpha \, , \ p^\beta_\alpha a'_\beta - a'_\alpha \in A_\alpha \cap A'_\alpha$

Now let $A_\alpha \cap A'_\alpha$ be denoted by $\mathring{A}$, and suppose $\{a_\alpha\} = \xi$ and $\{a'_\alpha\} = \xi'$.

Then $\xi \in [A, \dot{A}]$ and $\xi' \in [A', \dot{A}]$

But since $\dot{A}$ is assumed star-epimorphic

$$\lambda(A, \dot{A}) = [A, \dot{A}] / \langle [\dot{A}, \dot{A}], [A, 0] \rangle = 0$$

i.e. $[A, \dot{A}] \doteq [\dot{A}, \dot{A}] + [A, 0]$.

Therefore there exist $\delta \in [\dot{A}, \dot{A}]$ and $\zeta \in [A, 0]$ such that

$$\xi = \delta + \zeta$$

Similarly there exist $\varepsilon \in [\dot{A}, \dot{A}]$ and $\zeta' \in [A', 0]$ such that

$$\xi' = \varepsilon + \zeta'$$

Thus $\eta = \xi + \xi' = \delta + \varepsilon + \zeta + \zeta'$,

but $\delta + \varepsilon \in [\dot{A}, \dot{A}]$ and also $\delta + \varepsilon \doteq \eta - (\zeta + \zeta') \in \underleftarrow{\lim}(A + A')$

hence $\delta + \varepsilon \in \underleftarrow{\lim}\, \dot{A}$, and hence $\delta + \varepsilon + \zeta \in \underleftarrow{\lim}\, A$.

Therefore $\eta = (\delta + \varepsilon + \zeta) + \zeta' \in \underleftarrow{\lim}\, A + \underleftarrow{\lim}\, A'$.

6. Lemmas from Group-theory

Many propositions in this paper will be proved by calculations involving the relative inverse limits. The calculations will be based on a few lemmas from the group-theory. These lemmas are stated here without proofs and will be applied repeatedly in the sections that follow. Other basic lemmas from the group-theory needed here and there are also gathered in this section.

Lemma 8. Suppose f is a homomorphism of a group-pair (G,G_0) into another (H,H_0), i.e. $f: G \longrightarrow H$ such that $f(G_0) \subset H_0$, then the induced homomorphism $f_*: G/G_0 \longrightarrow H/H_0$

can be completed into the following exact sequence

$$0 \longrightarrow f^{-1}(H_o)/G_o \longrightarrow G/G_o \longrightarrow H/H_o \longrightarrow H/\langle f(G),H_o\rangle \longrightarrow 0$$

$f^{-1}(H^o)/G$ will be called the kernel of f_* and $H/\langle f(G),H_o\rangle$ the cokernel of f_*.

Corollary 1. (Fundamental lemma of homomorphism) If $f\colon G \longrightarrow H$ is onto mod H_o, i.e. given $h \in H$ there exists $g \in G$ such that $f(g) - h \in H_o$ then $G/f^{-1}(H_o) \approx H/H_o$.

Lemma 9. Suppose $f\colon (G,G_o) \longrightarrow (H,H_o)$ so that $f_*\colon G/G_o \longrightarrow H/H_o$ then

$$\text{kernel } f_* = f^{-1}(H_o)/G_o$$

$$\text{image } f_* = \langle f(G),H_o\rangle /H_o$$

Lemma 10. Given $f\colon G \longrightarrow H$, and suppose $H_o,H_1 \subset H$. Then

$$f^{-1}(\langle H_o,H_1\rangle) = \langle f^{-1}(H_o),f^{-1}(H_1)\rangle$$

if $f(G) \supset H_o$ or $f(G) \supset H_1$, in particular if f is onto.

Lemma 11. Given groups and inclusion homomorphisms

$$\begin{tikzcd} & G_1 \arrow[dr] & \\ G_o \arrow[ur] \arrow[dr] & & G \\ & G_2 \arrow[ur] & \end{tikzcd}$$

where $G_o = G_1 \cap G_2$ we have the following isomorphism

$$(G/G_1)/(G_2/G_o) \approx G/\langle G_1,G_2\rangle$$

Lemma 12. Let $A_o \subset A$ and B be subgroups of G, then

$$\langle A,B\rangle / \langle A_o,B\rangle \approx A/\langle A \cap B,A_o\rangle$$

Lemma 13. Given the diagram of groups and inclusion homomorphisms

$$\begin{tikzcd} & D \\ A \arrow[ur, "i"] \arrow[dr, "i'"'] & \\ & D' \end{tikzcd}$$

There exists E so that the following diagram is commutative

$$\begin{tikzcd} & D \arrow[dr, "j"] & \\ A \arrow[ur, "i"] \arrow[dr, "i'"'] & & E \\ & D' \arrow[ur, "j'"'] & \end{tikzcd}$$

where j and j' are inclusion homomorphisms. The smallest such E exists and will be denoted D ⊕ D' rel A.

<u>Lemma 14</u>. Every group can be imbedded in an injective group. Q is called injective if given

$$\begin{tikzcd} B & \\ A \arrow[u] \arrow[r, "f"'] & Q \\ 0 \arrow[u] & \end{tikzcd}$$

There exists $\bar{f}$: B ⟶ Q such that

$$\begin{tikzcd} B \arrow[dr, "\bar{f}"] & \\ A \arrow[u] \arrow[r, "f"'] & Q \\ 0 \arrow[u] & \end{tikzcd}$$

is a commutative diagram.

7. Study of $\lambda(B,A)$ and $\Lambda(B,V,A)$

A homomorphism $f\colon (B,A) \longrightarrow (B',A')$ induces $f_*\colon \lambda(B,A) \longrightarrow \lambda(B',A')$. We compute the kernel and cokernel of f_*:

$$\begin{cases} 0 \longrightarrow \text{kernel } f_* \longrightarrow \lambda(B,A) \longrightarrow \lambda(B',A') \longrightarrow \text{cokernel } f_* \longrightarrow 0 \\ \text{kernel } f_* = \langle [f^{-1}A' \cap B, A], [B, f^{-1}0 \cap B] \rangle / \langle [A,A],[B,0] \rangle \\ \text{cokernel } f_* = [B',A'] / \langle [fB,fA],[A',A'],[B',0] \rangle \end{cases}$$

Lemma 15. Kernel $f_* = 0$ if f is a monomorphism of (B,A) into (B',A') which takes A onto A'.

Proof: $[f^{-1}A' \cap B, A]$ reduces to $[A,A]$, and $[B, f^{-1}0 \cap A]$ reduces to $[B,0]$.

Lemma 16. Cokernel $f_* = 0$ if B is a star-epimorphic inverse system.

Proof: A substitution of $fA = A'$ reduces the expression for cokernel to

$$[B',A']/\langle [fB,A'],[B',0]\rangle = \Lambda(B',fB,A')$$

where $fB \approx B$ is a star-epimorphic inverse system. We complete the proof by stating the following essential theorem of Λ :

Theorem 3. $\Lambda(B,V,A) = 0$ if V is a star-epimorphic inverse system.

Proof: Recalling $\Lambda(B,V,A) = [B,A]/\langle [V,A],[B,0]\rangle$ we will show by transfinite induction that given $\xi \in [B,A;I]$ we can find $\delta \in [V,A;I]$ and $\zeta \in [B,0;I]$ such that $\xi = \delta + \zeta$.

Suppose I is well-ordered, say $I = \{\alpha_1, \alpha_2, \ldots\}$. Let $I_1 = \mathfrak{D}(\alpha_1)$, $I_2 = \mathfrak{D}(\alpha_1, \alpha_2)$, ..., then $I_1, I_2, \ldots$ is an increasing sequence of subsets of I such that $\bigcup_i I_i = I$.

In general for $I' \subset I$ let $(\xi; I')$ denote $P^I_{I'}\xi$.

i) Since $I_1 = \mathfrak{D}(\alpha_1)$, by Lemma 2 there exist $(\delta; I_1) \in [A,A;I_1]$ and $(\zeta; I_1) \in [B,0;I_1]$ such that $(\delta; I_1) + (\zeta; I_1) = (\xi; I_1)$.

ii) Suppose we have shown for $j < n$ that there exist $(\delta;I_j) \in [A,A;I_j]$ and $(\zeta;I_j) \in [B,0;I_j]$ such that $(\delta;I_j) + (\zeta;I_j) = (\xi;I_j)$ and that $P^{I_j}_{I_i}(\delta;I_j) = (\delta;I_i)$; $P^{I_j}_{I_i}(\zeta;I_j) = (\zeta;I_i)$. Now this implies that for $I^{\#} = \bigcup_{j<n} I_j$ there exist $(\delta;I^{\#}) \in [A,A;I^{\#}]$ and $(\zeta;I^{\#}) \in [B,0;I^{\#}]$ such that $(\delta;I^{\#}) + (\zeta;I^{\#}) = (\xi;I^{\#})$.

<u>Notations</u>: let $(\xi;I^{\#}) = \{\xi_\alpha \mid \alpha \in I^{\#}\}$, $(\delta;I^{\#}) = \{\delta_\alpha \mid \alpha \in I^{\#}\}$, $(\zeta;I^{\#}) = \{\zeta_\alpha \mid \alpha \in I^{\#}\}$, then $\xi_\alpha = \delta_\alpha + \zeta_\alpha$ for $\alpha \in I^{\#}$.

To show there exist $(\delta;I_n) \in [A,A;I_n]$ and $(\zeta;I_n) \in [B,0;I_n]$ such that

(1) $\quad (\delta;I_n) + (\zeta;I_n) = (\xi;I_n)$

(2) $\quad P^{I_n}_{I^{\#}}(\zeta;I_n) = (\zeta;I^{\#})$; $\quad P^{I_n}_{I^{\#}}(\delta;I_n) = (\delta;I^{\#})$

first note $I_n = I^{\#} \cup \mathfrak{D}(\alpha_n)$, and let $I^* = I^{\#} \cap \mathfrak{D}(\alpha_n)$.
As usual, decompose in the manner of Lemma 2.

$$(\xi;\mathfrak{D}(\alpha_n)) = (\delta';\mathfrak{D}(\alpha_n)) + (\zeta';\mathfrak{D}(\alpha_n)) \qquad [1]$$

Let $\quad P^{I^{\#}}_{I^*}(\zeta;I^{\#}) - P^{\mathfrak{D}(\alpha_n)}_{I^*}(\zeta';\mathfrak{D}(\alpha_n)) = (\varepsilon;I^*) \in [B,0;I^*] \qquad [2]$

But since for $\alpha \in I^*$

$$\begin{aligned}(\varepsilon;I^*)_\alpha &= (P^{I^{\#}}_{I^*}(\zeta;I^{\#}))_\alpha - (P^{\mathfrak{D}(\alpha_n)}_{I^*}(\zeta';\mathfrak{D}(\alpha_n)))_\alpha \\ &= (\xi_\alpha - \delta_\alpha) - p^{\alpha_n}_\alpha\, \xi_{\alpha_n} \\ &= (\xi_\alpha - p^{\alpha_n}_\alpha\, \xi_{\alpha_n}) - \delta_\alpha \in A_\alpha\end{aligned}$$

hence $\quad (\varepsilon;I^*) \in [A,0;I^*] \subset [V,0;I^*]$

Now since V is star-epimorphic, there exists $(\delta'';\mathfrak{D}(\alpha_n)) \in [V,0;\mathfrak{D}(\alpha_n)]$ such that

$$P_{I^*}^{\mathfrak{D}(\alpha_n)}(\delta'';\mathfrak{D}(\alpha_n)) = (\varepsilon;I^*). \qquad [3]$$

Define $(\zeta;\mathfrak{D}(\alpha_n)) = (\zeta';\mathfrak{D}(\alpha_n)) + (\delta'';\mathfrak{D}(\alpha_n))$ [4]

and $(\delta;\mathfrak{D}(\alpha_n)) = (\delta';\mathfrak{D}(\alpha_n)) - (\delta'';\mathfrak{D}(\alpha_n))$ [5]

Then clearly $(\delta;\mathfrak{D}(\alpha_n)) + (\zeta;\mathfrak{D}(\alpha_n)) = (\xi;\mathfrak{D}(\alpha_n))$

It remains to check that $(\zeta;\mathfrak{D}(\alpha_n))$ and $(\delta;\mathfrak{D}(\alpha_n))$ are consistent with $(\zeta;I^{\#})$ and $(\delta;I^{\#})$ respectively:

(a) $P_{I^*}^{\mathfrak{D}(\alpha_n)}(\zeta;\mathfrak{D}(\alpha_n)) = P_{I^*}^{I^{\#}}(\zeta;I^{\#})$, but

$P_{I^*}^{\mathfrak{D}(\alpha_n)}(\zeta;\mathfrak{D}(\alpha_n)) = P_{I^*}^{\mathfrak{D}(\alpha_n)}((\zeta';\mathfrak{D}(\alpha_n)) + (\delta'';\mathfrak{D}(\alpha_n)))$ by [4]

$= [P_{I^*}^{I^{\#}}(\zeta;I^{\#}) - (\varepsilon;I^*)] - (\varepsilon;I^*)$ by [2],[3]

$= P_{I^*}^{I^{\#}}(\zeta;I^{\#})$.

Thus $(\zeta;\mathfrak{D}(\alpha_n))$ and $(\zeta;I^{\#})$ agree on I^*, hence together make up $(\zeta;I^n)$ which we want in (1), (2).

(b) $P_{I^*}^{\mathfrak{D}(\alpha_n)}(\delta;\mathfrak{D}(\alpha_n)) = P_{I^*}^{I^{\#}}(\delta;I^{\#})$, but

$P_{I^*}^{\mathfrak{D}(\alpha_n)}(\delta;\mathfrak{D}(\alpha_n)) = P_{I^*}^{\mathfrak{D}(\alpha_n)}(\delta';\mathfrak{D}(\alpha_n)) - P_{I^*}^{\mathfrak{D}(\alpha_n)}(\delta'';\mathfrak{D}(\alpha_n))$ by [5]

$= [P_{I^*}^{\mathfrak{D}(\alpha_n)}(\xi;\mathfrak{D}(\alpha_n)) - P_{I^*}^{\mathfrak{D}(\alpha_n)}(\zeta';\mathfrak{D}(\alpha_n))] - (\varepsilon;I^*)$ by [1],[3]

$= [P_{I^*}^{I^{\#}}(\zeta;I^{\#}) + P_{I^*}^{I^{\#}}(\delta;I^{\#})] - P_{I^*}^{\mathfrak{D}(\alpha_n)}(\zeta';\mathfrak{D}(\alpha_n)) - (\varepsilon;I^*)$

by the induction hypothesis

$= [P_{I^*}^{I^{\#}}(\zeta;I^{\#}) - P_{I^*}^{\mathfrak{D}(\alpha_n)}(\zeta';\mathfrak{D}(\alpha_n))] + P_{I^*}^{I^{\#}}(\delta;I^{\#}) - (\varepsilon;I^*)$

$= (\varepsilon;I^*) + P_{I^*}^{I^{\#}}(\delta;I^{\#}) - (\varepsilon;I^*)$ by [2]

$= P_{I^*}^{I^{\#}}(\delta;I^{\#})$

Thus $(\delta;\mathfrak{D}(\alpha_n))$ and $(\delta;I^{\#})$ agree on I^*, hence together make up $(\delta;I^n)$ which we want in (1), (2). Q.E.D.

Corollary 2. $\lambda(B,A) = 0$ if A is star-epimorphic.

Proof: $\lambda(B,A) = \Lambda(B,A,A) = 0$ since A is star-epimorphic.

Corollary 3. $\Lambda(B,W,A) = 0$ if there exists a star-epimorphic V such that $W \supset V \supset A$.

Proof: $[B,A] = \langle [V,A],[B,0] \rangle \subset \langle [W,A],[B,0] \rangle$, hence $\Lambda(V,W,A) = 0$.

Theorem 4. Given $A \subset B$, $A \subset B'$, if there exists a star-epimorphic V such that $A \subset V \subset B \cap B'$, then

$$\lambda(B,A) \approx \lambda(B',A).$$

Proof: Consider the inclusion homomorphisms induced by $(V,A) \subset (B,A)$ and $(V,A) \subset (B',A)$

$$i_*: \lambda(V,A) \longrightarrow \lambda(B,A)$$

$$i_*': \lambda(V,A) \longrightarrow \lambda(B',A).$$

Then by Lemma 15 and Lemma 16,

$$\text{kernel } i_* = 0; \text{ cokernel } i_* = 0$$

$$\text{kernel } i_*' = 0; \text{ cokernel } i_*' = 0$$

Hence $\lambda(B,A) \approx \lambda(V,A) \approx \lambda(B',A)$. In particular we have

Corollary 4. If $A \subset V \subset B \subset B'$ with V star-epimorphic, then $\lambda(B,A) \approx \lambda(B',A)$.

8. The First Derived Functor of Inverse Limit

Definition 4. A star-epimorphic covering of a given inverse system A is an inverse system which contains a star-epimorphic inverse system which contains A.

Let B and B' be star-epimorphic coverings of A, then the diagram

$$\begin{cases} 0 \longrightarrow A \longrightarrow B \\ 0 \longrightarrow A \longrightarrow B' \end{cases}$$

can be imbedded in a commutative diagram

$$\begin{tikzcd} & & B \arrow[dr] & \\ 0 \arrow[r] & A \arrow[ur] \arrow[dr] & & B \oplus B' \text{ rel } A \\ & & B' \arrow[ur] & \end{tikzcd}$$

where B ⊕ B' rel A is the (smallest) inverse system containing both B and B' preserving their common inclusion of A. (Cf. Lemma 13.)

Theorem 5. If B and B' are star-epimorphic coverings of A then

$$\lambda(B,A) \approx \lambda(B',A).$$

Proof: By Corollary 4,

$$\lambda(B,A) \approx \lambda(B \oplus B' \text{ rel } A, A)$$

$$\lambda(B',A) \approx \lambda(B \oplus B' \text{ rel } A, A)$$

Hence $\lambda(B,A) \approx \lambda(B',A)$. Thus we see $\lambda(B,A)$ has a certain degree of invariance with respect to B, and this enables us to obtain a functor of A from λ.

<u>Definition 5</u>. The first derived functor of inverse limit, which we denote by $\varprojlim'$, is defined by

$$\varprojlim' A = \lambda(B,A)$$

where B is a star-epimorphic covering of A.

From this definition, since $\lambda(A,A) = 0$, we have the following

<u>Theorem 6</u>. $\varprojlim' A = 0$ if A is star-epimorphic.

In §5 "Derived inverse systems" we noted that for any inverse system A, we have the inclusion relation

$$A \approx A^{0} \subset A^{A}$$

where A^A is a star-epimorphic inverse system, hence A^A is a star-epimorphic covering of A. Thus we may formulate the preceding definition as follows:

<u>Definition 6</u>. $\varprojlim' A = [A^A, A^0]/\langle [A^0, A^0], [A^A, 0] \rangle$.

From this definition it is seen that $\varprojlim'$ is a functor on the category of inverse systems.

<u>Example</u>. Let $(A;Z^+)$ be such that A_i is the additive group of integers for each $i \in Z^+$, and p_i^j be a multiplication by 2^{j-i}. It can be seen that $\varprojlim A = 0$, but we will show that $\varprojlim' A \neq 0$.

First we note that $(A;Z^+)$ is isomorphic to $(A^*;Z^+)$ where A_i^* is the additive group of all integers divisible by 2^i, and p_i^j is the inclusion of A_j in A_i. Now we let

$(B;Z^+)$ be such that B_i is the group of integers and p_i^j is the identity, then B is an epimorphic inverse sequence containing A. Let $C = B/A^*$. By Lemma 19 we have

$$0 \longrightarrow \varprojlim A^* \longrightarrow \varprojlim B \xrightarrow{j_*} \varprojlim C \longrightarrow \varprojlim{}' A^* \longrightarrow \varprojlim{}' B \longrightarrow \varprojlim{}' C \longrightarrow 0$$

but $\varprojlim A^* = \varprojlim A = 0$, $\varprojlim B = Z$, $\varprojlim{}' B = 0$, and it is not hard to see that $\varprojlim C$ has as many elements as a cantor set.

Hence $\varprojlim{}' A = \varprojlim{}' A^* = \varprojlim C/j_*(Z) \neq 0$, a huge group. (In fact it can be seen that $\varprojlim{}' A$ is a divisible group and therefore is a direct sum of groups each isomorphic to the additive group of rationals or to the group of p-adic rationals mod the additive group of integers for various primes p. Cf. Kaplansky, Infinite Abelian Groups, Ann Arbor, 1954, Theorem 4.)

<u>Theorem 7</u>. $\varprojlim{}'$ is a half-exact functor, i.e. given

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have an exact sequence

$$\varprojlim{}' A \longrightarrow \varprojlim{}' B \longrightarrow \varprojlim{}' C$$

<u>Proof</u>: Consider the following commutative diagram

$$\begin{tikzcd}
\lambda(A^A, A^o) \arrow[r, "i_*"] \arrow[d, "\cong"] & \lambda(B^B, B^o) \arrow[r, "i_*"] & \lambda(C^C, C^o) \\
\lambda(B^B, A^o) \arrow[ur, "i_*"] & &
\end{tikzcd}$$

where $\lambda(A^A, A^o) \approx \lambda(B^B, A^o)$ by Corollary 4. We will show by straight forward computation that

$$\text{image } i_* = \text{kernel } j_*$$

image $i_*^! = \langle [B^B,A^o],[B^o,B^o]\rangle / \langle [B^o,B^o],[B^B,0]\rangle$

kernel $j_* = j^{-1}\langle [C^o,C^o],[C^C,0]\rangle \cap [B^B,B^o]/\langle [B^o,B^o],[B^B,0]\rangle$

but $\quad j^{-1}\langle [C^o,C^o],[C^C,0]\rangle \cap [B^B,B^o]$

$$= \langle j^{-1}[C^o,C^o] \cap [B^B,B^o],\quad j^{-1}[C^C,0] \cap [B^B,B^o]\rangle$$

$$= \langle [j^{-1}C^o \cap B^B,\ j^{-1}C^o \cap B^o],[j^{-1}C^C \cap B^B,\ j^{-1}0 \cap B^o]\rangle$$

$$= \langle [B^A,B^o],[B^B,A^o]\rangle$$

but $\quad [B^A,B^o] = [B^o + A^A,B^o] = \langle [B^o,B^o],[A^A,A^o]\rangle$. (In general it is not difficult to see $[B + B',B] = [B,B] + [B',B' \cap B]$.)

Hence kernel $j_* = \langle [B^B,A^o],[B^o,B^o]\rangle / \langle [B^o,B^o],[B^B,0]\rangle$

$=$ image $i_*^! =$ image i_*. Q.E.D.

Now we recall Lemma 1.

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim A \arrow[r] \arrow[d, "\cong"] & \varprojlim B \arrow[r] \arrow[d, "\cong"] & \varprojlim C \arrow[r] \arrow[d, "\cong"] & \lambda(B,A) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & {[A^o,0]} \arrow[r] & {[B^o,0]} \arrow[r] & {[B^o,A^o]/[A^o,A^o]} \arrow[r, "h"] & \lambda(B^o,A^o) \arrow[r] & 0
\end{tikzcd}$$

But by Theorem 4, $0 \longrightarrow \lambda(B^o,A^o) \xrightarrow{i} \lambda(B^B,A^o) = \varprojlim{}' A$

so the composition of h and i gives

$$\delta_*: \varprojlim C \longrightarrow \varprojlim{}' A$$

thus δ_* connects the two exact sequences

$$0 \longrightarrow \varprojlim A \xrightarrow{i_*} \varprojlim B \xrightarrow{j_*} \varprojlim C$$

$$\varprojlim{}'' A \xrightarrow{i_*^!} \varprojlim{}' B \xrightarrow{j_*^!} \varprojlim{}' C$$

and by straightforward calculation as in the previous theorem we can show

$$\text{image } j_* = \text{kernel } \delta_*$$

$$\text{image } \delta_* = \text{kernel } i_*^! .$$

Hence we have

<u>Theorem</u> <u>8</u>.

$$0 \longrightarrow \varprojlim A \xrightarrow{i_*} \varprojlim B \xrightarrow{j_*} \varprojlim C \xrightarrow{\delta_*} \varprojlim{}' A \xrightarrow{i_*^!} \varprojlim{}' B \xrightarrow{j_*^!} \varprojlim{}' C$$

is an exact sequence.

9. Cofinal Inverse Systems

A partially ordered set I is said to be directed if for any given pair $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha < \gamma$, $\beta < \gamma$. Let $I^{\#} \subset I$, $I^{\#}$ is said to be cofinal in I if $\mathfrak{D}(I^{\#}) = I$. Given (A;I) and $I^{\#}$ cofinal in I, we call $(A;I^{\#})$ a cofinal inverse subsystem of (A;I).. As before we may consider the homomorphism by restriction:

$$P^{I}_{I^{\#}}: [B,A;I] \longrightarrow [B,A;I^{\#}]$$

Lemma 15.

$$\text{kernel } P^{I}_{I^{\#}} \subset [A,A;I]$$

$$\text{cokernel } P^{I}_{I^{\#}} = 0 \quad \text{if I is directed}$$

1. Suppose $\xi \in [B,A;I]$ such that $P^{I}_{I^{\#}}\xi \in [0,0;I^{\#}]$ then

$$\text{if } \alpha \in I, \quad \xi_\alpha = 0 \in A_\alpha$$

$$\text{if } \alpha \in I - I^{\#}, \text{ there exists } \beta \in I^{\#},$$

so that $p_\alpha^\beta \xi_\beta - \xi_\alpha \in A_\alpha$,

but $\xi_\beta = 0 \quad \therefore \quad \xi_\alpha \in A_\alpha$.

2. To show cokernel $P^I_{I^\#} = 0$, i.e. $P^I_{I^\#}$ is onto, given $\xi^\# \in [B,A;I^\#]$ choose $\xi \in [B,A;I]$ as follows:

if $\alpha \in I^\#$ let $\xi_\alpha = \xi^\#_\alpha$

if $\alpha \in I - I^\#$ let $\xi_\alpha = p_\alpha^\gamma \xi_\gamma$ for some arbitrary $\gamma \in I^\#$.

To see $\{\xi_\alpha\}$ indeed determines an element of $[B,A;I]$ we must check

$$p_\alpha^\beta \xi_\beta - \xi_\alpha \in A_\alpha \quad \text{for any } \alpha, \beta \in I.$$

Case I. $\alpha, \beta \in I^\#$ $\quad p_\alpha^\beta \xi_\beta - \xi_\alpha = p_\alpha^\beta \xi_\beta^\# - \xi_\alpha^\# \in A_\alpha$

Case II. $\alpha \in I^\#, \beta \in I - I^\#$, $\quad p_\alpha^\beta \xi_\beta - \xi_\alpha = p_\alpha^\beta (p_\beta^\gamma \xi_\gamma) - \xi_\alpha$ for some $\gamma \in I^\#$

$= p_\alpha^\gamma \xi_\gamma - \xi_\alpha$ reducing to Case I.

Case III. $\alpha \in I - I^\#$, $\beta \in I^\#$

$$p_\alpha^\beta \xi_\beta - \xi_\alpha = p_\alpha^\beta \xi_\beta - p_\alpha^\gamma \xi_\gamma \quad \text{for some } \gamma \in I^\#$$

Now since I is directed and $I^\#$ is cofinal in I there exists $\delta \in I^\#$ such that $\beta < \delta$, $\gamma < \delta$

so that $p_\alpha^\beta \xi_\beta - p_\alpha^\gamma \xi_\gamma = p_\alpha^\beta (p_\beta^\delta \xi_\delta + a_\beta) - p_\alpha^\gamma (p_\gamma^\delta \xi_\delta + a_\gamma)$

$$= p_\alpha^\delta \xi_\delta + p_\alpha^\beta a_\beta - p_\gamma^\delta \xi_\delta - p_\alpha^\gamma a_\gamma \in A_\alpha$$

Case IV. $\alpha, \beta \in I - I^\#$

$$p_\alpha^\beta \xi_\beta - \xi_\alpha = p_\alpha^\beta (p_\beta^\gamma \xi_\gamma) - p_\alpha^\delta \xi_\delta \quad \text{for some } \gamma, \delta \in I^\#$$

$$= p_\alpha^\gamma \xi_\gamma - p_\alpha^\delta \xi_\delta$$

Now since I is directed and $I^\#$ cofinal in I, etc., like Case III.

The lemma implies in particular that $[B,0;I] \approx [B,0;I^{\#}]$ i.e.

Corollary 5. The inverse limit is preserved by cofinal inverse subsystems if I is directed.

Throughout the rest of this section we will assume that I is directed.

Lemma 16. Let $\overline{P}^{I}_{I^{\#}}(\ldots.)$ denote the inverse image of .... under $P^{I}_{I^{\#}}$,

then $\qquad \overline{P}^{I}_{I^{\#}}([A,A;I^{\#}]) = [A,A;I]$

$\qquad \overline{P}^{I}_{I^{\#}}([B,0;I^{\#}]) \subset \langle [A,A;I],[B,0;I] \rangle$

1. Let $\xi \in [B,A;I]$ be such that $P^{I}_{I^{\#}}\xi \in [A,A;I^{\#}]$

then if $\alpha \in I^{\#}$ $\qquad \xi_\alpha \in A_\alpha$

and if $\alpha \in I - I^{\#}$ there exists $\beta \in I^{\#}$ such that

$$p^{\beta}_{\alpha}\,\xi_\beta - \xi_\alpha \in A_\alpha$$

but $\xi_\beta \in A_\beta$, hence $p^{\beta}_{\alpha}\xi_\beta - \xi_\alpha \in A_\alpha$ and $\xi_\alpha \in A_\alpha$

2. Let $\xi \in [B,A;I]$ be such that $P^{I}_{I^{\#}}\xi = \zeta \in [B,0;I^{\#}]$;

then since $[B,0;I] \approx [B,0;I^{\#}]$ by previous lemma, ζ can be extended in a unique way to $\bar{\zeta} \in [B,0;I]$. Now

if $\alpha \in I^{\#}$ $\qquad \xi_\alpha - \bar{\zeta}_\alpha = \xi_\alpha - \zeta_\alpha = 0 \in A_\alpha$

and if $\alpha \in I - I^{\#}$ there exists $\beta \in I^{\#}$ such that

$$\xi_\alpha - \bar{\zeta}_\alpha = (p^{\beta}_{\alpha}\,\xi_\beta + a_\alpha) - p^{\beta}_{\alpha}\,\zeta_\beta \quad \text{for some } a_\alpha \in A_\alpha$$

but $\zeta_\beta = \xi_\beta$ hence $\xi_\alpha - \bar{\zeta}_\alpha \in A_\alpha$

therefore $\xi - \bar{\zeta} \in [A,A;I]$

$P^{I}_{I^\#}$ now induces the following homomorphism P_*

$P_*: [A^A, A^o; I]/\langle [A^o, A^o; I], [A^A, 0; I]\rangle$

$\longrightarrow [A^A, A^o; I^\#]/\langle [A^o, A^o; I^\#], [A^A, 0; I^\#]\rangle$

Theorem 9. Let $(A; I^\#)$ be a cofinal inverse subsystem of $(A; I)$. Then

$$\varprojlim{}'(A;I) \approx \varprojlim{}'(A;I^\#)$$

Proof: It suffices to check the vanishing of kernel and cokernel of P_*.

1. kernel $P_* = \overline{P}^{I}_{I^\#}(\langle [A^o, A^o; I^\#], [A^A, 0; I^\#]\rangle)/\langle [A^o, A^o; I], [A^A, 0; I]\rangle$

$= \langle \overline{P}^{I}_{I^\#}([A^o, A^o; I^\#]), \overline{P}^{I}_{I^\#}([A^A, 0; I^\#])\rangle / \langle [A^o, A^o; I], [A^A, 0; I]\rangle$

$\subset \langle [A^o, A^o; I], [A^A, 0; I]\rangle / \langle [A^o, A^o; I], [A^A, 0; I]\rangle = 0$ by lemma 16.

2. cokernel $P_* = 0$ since $P^{I}_{I^\#}([A^A, A^o; I]) = [A^A, A^o; I^\#]$ by lemma 15.

Lemma 17. If I is countable and directed, then there exists $I^\# \subset I$ with $\mathfrak{D}(I^\#) = I$ and $I^\# = Z^+$, the set of positive integers.

Proof: Since I is countable let $I = \{\alpha_1, \alpha_2, \alpha_3, \dots\}$. Since I is directed, given $\alpha, \beta \in I$, let $\langle \alpha, \beta\rangle \in I$ such that $\alpha < \langle \alpha, \beta\rangle$ and $\beta < \langle \alpha, \beta\rangle$. Let $\gamma_1 = \alpha_1$, $\gamma_2 = \langle \gamma_1, \alpha_2\rangle$, $\gamma_3 = \langle \gamma_2, \alpha_3\rangle$, $\gamma_4 = \langle \gamma_3, \alpha_4\rangle$, ..., then let $I^\# = \{\gamma_1, \gamma_2, \gamma_3, \dots\}$. Clearly $\mathfrak{D}(I^\#) = I$ and $I^\# = Z^+$ because $\gamma_1 < \gamma_2 < \gamma_3 < \dots$.

10. Injective Inverse Systems

The objective of this section is to relate our definition of $\varprojlim{}'$ to that implied in Homological Algebra of Eilenberg-Cartan. The results of this section are primarily due to Professor J. Milnor.

Definition 7. An inverse system Q is said to be injective if for any given pair of inverse systems $A \subset B$ the diagram

$$\begin{tikzcd} B & \\ A \arrow[u, "i"] \arrow[r, "f"'] & Q \\ 0 \arrow[u] & \end{tikzcd}$$

can be imbedded in a commutative diagram

$$\begin{tikzcd} B \arrow[dr, "\bar{f}"] & \\ A \arrow[u, "i"] \arrow[r, "f"'] & Q \\ 0 \arrow[u] & \end{tikzcd}$$

Theorem 10. There exists an injective inverse system. For each $\alpha \in I$, let $\dot{Q}_\alpha$ be an arbitrary injective group (Cf. Lemmas from group theory) and for each pair $\alpha < \beta \in I$ let $p_\alpha^\beta = 0$. Let $(\dot{Q};I)$ denote the resulting inverse system. Let $Q = \dot{Q}^{\dot{Q}}$. To show Q is an injective inverse system, let

$$f_\beta : A_\beta \longrightarrow \prod_{\gamma<\beta} \dot{Q}_\gamma = \dot{Q}$$

and suppose $\pi_\gamma: \prod_{\gamma<\beta} \dot{Q}_\gamma \longrightarrow \dot{Q}_\gamma$ is the projection homomorphism.

Let $\qquad (f_\beta)_\gamma = \pi_\gamma f_\beta : A_\beta \longrightarrow \dot{Q}_\gamma$

In particular consider for each $\beta \in I$

$$(f_\beta)_\beta : A_\beta \longrightarrow \dot{Q}_\beta$$

Then since $\dot{Q}_\beta$ is injective $(f_\beta)_\beta$ can be extended to F_β so that the following diagram is commutative

$$\begin{tikzcd} B_\beta \arrow[dr, "F_\beta"] & \\ A_\beta \arrow[u] \arrow[r, "(f_\beta)_\beta"'] & \dot{Q}_\beta \\ 0 \arrow[u] & \end{tikzcd}$$

Now we are ready to define

$$\overline{f}_\alpha : B_\alpha \longrightarrow Q_\alpha = \prod_{\beta<\alpha} \dot{Q}_\beta$$

by $\qquad (\overline{f}_\alpha)_\beta = F_\beta \; \overline{p}^\alpha_\beta$

We must check the following

1) $\{ \overline{f}_\alpha \mid \alpha \in I \}$ give rise to $\overline{f}$, a homomorphism of $B \longrightarrow Q$. It suffices to show the following diagram is commutative

$$\begin{tikzcd} B_{\alpha'} \arrow[r] \arrow[d, "\overline{p}^{\alpha'}_\alpha"'] & Q_{\alpha'} \arrow[d, "p^{\mathfrak{D}(\alpha')}_{\mathfrak{D}(\alpha)}"] & = \prod_{\beta<\alpha'} \dot{Q}_\beta \\ B_\alpha \arrow[r] & Q_\alpha & = \prod_{\beta<\alpha} \dot{Q}_\beta \end{tikzcd}$$

i.e. for each $\beta < \alpha$

$$\begin{tikzcd} B_{\alpha'} \arrow[dr, "(\overline{f}_{\alpha'})_\beta"] \arrow[dd, "\overline{p}^{\alpha'}_\alpha"'] & \\ & \dot{Q}_\beta \\ B_\alpha \arrow[ur, "(\overline{f}_\alpha)_\beta"'] & \end{tikzcd}$$

is commutative, but $(\overline{f}_\alpha)_\beta \; \overline{p}^{\alpha'}_\alpha = F_\beta \; \overline{p}^\alpha_\beta \; \overline{p}^{\alpha'}_\alpha = F_\beta \; p^{\alpha'}_\beta = (\overline{f}_{\alpha'})_\beta$

2) $\bar{f}\underline{i} = f$ i.e. the following diagram is commutative

$$\begin{tikzcd} B_\alpha \arrow[drr, "\bar{f}_\alpha"] & & \\ A_\alpha \arrow[u, "i_\alpha"] \arrow[rr, "f_\alpha"] & & Q_\alpha = \prod_{\beta<\alpha} \dot{Q}_\beta \end{tikzcd}$$

for this it suffices to know

$$\begin{tikzcd} B_\alpha \arrow[drr, "(\bar{f}_\alpha)_\beta"] & & \\ A_\alpha \arrow[u, "i_\alpha"] \arrow[rr, "(f_\alpha)_\beta"'] & & \dot{Q}_\beta \end{tikzcd}$$

is commutative, but $(f_\alpha)_\beta = (f_\beta)_\beta \; p^\alpha_\beta$ because of the commutative diagram

$$\begin{tikzcd} A_\alpha \arrow[d, "p^\alpha_\beta"'] \arrow[drr, "(f_\alpha)_\beta"] & & & & A_\alpha \arrow[d, "p^\alpha_\beta"'] \arrow[r, "f_\alpha"] & Q_\alpha = \prod_{\gamma<\alpha} \dot{Q}_\gamma \arrow[d] \\ A_\beta \arrow[rr, "(f_\beta)_\beta"'] & & \dot{Q}_\beta & \Leftarrow & A_\beta \arrow[r, "f_\beta"'] & Q_\beta = \prod_{\gamma<\beta} \dot{Q}_\gamma \end{tikzcd}$$

and $(f_\beta)_\beta \cdot p^\alpha_\beta = F_\beta \cdot p^\alpha_\beta$.

On the other hand

$$(\bar{f}_\alpha)_\beta \cdot i_\alpha = F_\beta \cdot \bar{p}^\alpha_\beta \cdot i_\alpha = F_\beta \cdot p^\alpha_\beta$$

Remark. In forming $\dot{Q}^{\dot{Q}}$ from $\dot{Q}$ the nature of $\{p\}$ of $(\dot{Q};I)$ is entirely irrelavant. In fact one can form $\dot{Q}^{\dot{Q}}$ without talking about $\{p\}$ at all.

<u>Theorem</u> <u>11</u>. Every inverse system can be imbedded in an injective inverse system.

Given $(A;I)$ let A_α be imbedded in some injective $\dot{Q}_\alpha$, and let $Q = \dot{Q}^{\dot{Q}}$ then $A \cong A^o \subset Q$ where Q is injective by the proof of the previous theorem. Incidentally we remark that Q is also a star-epimorphic inverse system.

<u>Theorem</u> <u>12</u>. If Q is an injective inverse system then $\varprojlim' Q = 0$.

Since $Q^0 \subset Q^Q$ and Q^0 is injective

$$Q^Q = Q^0 \oplus X.$$

Since $\varprojlim'$ is a functor

$$\varprojlim' Q^Q = \varprojlim' Q^0 \oplus \varprojlim' X$$

But $\varprojlim' Q^Q = 0$ because Q^Q is star-epimorphic. We conclude

$$\varprojlim' Q^0 = 0$$

<u>Theorem</u> <u>13</u>. Given $0 \longrightarrow A \longrightarrow Q \longrightarrow C \longrightarrow 0$ where Q is injective, consider $0 \longrightarrow \varprojlim A \xrightarrow{i_*} \varprojlim Q \xrightarrow{j_*} \varprojlim C$, if we let

$$F = \varprojlim C / j_*(\varprojlim Q), \text{ i.e.}$$

$$0 \longrightarrow \varprojlim A \longrightarrow \varprojlim Q \longrightarrow \varprojlim C \longrightarrow F \longrightarrow 0$$

Then F depends only on A as long as Q is injective.

<u>Proof</u>: By the preceding Theorem and Theorem 8,

$$\begin{cases} 0 \longrightarrow \varprojlim A \longrightarrow \varprojlim Q \longrightarrow \varprojlim C \longrightarrow \varprojlim' A \longrightarrow \varprojlim' Q \longrightarrow \varprojlim' C \\ \varprojlim' Q = 0 \end{cases}$$

Hence $F = \varprojlim' A$.

This Theorem may be regarded as an alternative definition of the first derived functor of inverse limit.

11. Right Exactness of $\varprojlim'$

Lemma 18. Given as usual $0 \to A \to B \to C \to 0$, if k' is such that $\varprojlim' A \to \varprojlim' B \xrightarrow{j_*} \varprojlim' C \to k' \to 0$ then

$$k' \approx [B^B, B^A]/\langle [B^B, B^o], [B^B, A^A]\rangle$$

Proof: i) $\varprojlim' C = [C^C, C^o]/\langle [C^o, C^o], [C^C, 0]\rangle$, but it is not difficult to see the natural homomorphism $f\colon [B^B, B^A] \to [C^C, C^o]$ is onto, hence by the Fundamental Lemma of Homomorphism (Cf. §6, Corollary 1)

$$\varprojlim' C \approx [B^B, B^A]/f^{-1}\langle [C^o, C^o], [C^C, 0]\rangle$$

Since $f^{-1}\langle [C^o, C^o], [C^C, 0]\rangle = \langle f^{-1}[C^o, C^o], f^{-1}[C^C, 0]\rangle$

$$= \langle [f^{-1}C^o, f^{-1}C^o], [f^{-1}C^C, f^{-1}0]\rangle = \langle [B^A, B^A], [B^B, A^A]\rangle$$

we have $\varprojlim' C \approx [B^B, B^A]/\langle [B^A, B^A], [B^B, A^A]\rangle$

ii) $j_*\colon [B^B, B^o]/\langle [B^o, B^o], [B^B, 0]\rangle \longrightarrow$

$[B^B, B^A]/\langle [B^A, B^A], [B^B, A^A]\rangle$ hence

$$k' \approx \text{cokernel } j_* = [B^B, B^A]/\langle [B^B, B^o], [B^A, B^A], [B^B, A^A]\rangle$$

iii) Note however $[B^A, B^A] = [B^o + A^A, B^o + A^A]$

$$= \prod_\alpha (B^o_\alpha + A^A_\alpha) = \prod_\alpha B^o_\alpha + \prod_\alpha A^A_\alpha$$

Therefore $[B^A, B^A] = \langle [B^o, B^o], [A^A, A^A]\rangle \subset \langle [B^B, B^o], [B^B, A^A]\rangle$

Thus $\quad k' \approx [B^B, B^A]/\langle [B^B, B^o], [B^B, A^A]\rangle$.

Now suppose we let

$$\mathfrak{b} = B^B$$

$$\mathfrak{U} = A^A$$

$$\mathfrak{U}' = B^0$$

Then $$\mathfrak{U} + \mathfrak{U}' = B^A \quad \text{(Cf. Formula 1), Lemma 5)}$$

so that $$k' = [\mathfrak{b}, \mathfrak{U} + \mathfrak{U}']/\langle [\mathfrak{b}, \mathfrak{U}], [\mathfrak{b}, \mathfrak{U}'] \rangle$$

Hence the vanishing of k' depends on whether the following equality holds:

$$[\mathfrak{b}, \mathfrak{U} + \mathfrak{U}'] = [\mathfrak{b}, \mathfrak{U}] + [\mathfrak{b}, \mathfrak{U}']$$

But by Lemma 6 $$[\mathfrak{b}, \mathfrak{U} + \mathfrak{U}'] = \varprojlim \mathfrak{b}^{\mathfrak{U}+\mathfrak{U}'}$$

$$[\mathfrak{b}, \mathfrak{U}] = \varprojlim \mathfrak{b}^{\mathfrak{U}}$$

$$[\mathfrak{b}, \mathfrak{U}'] = \varprojlim \mathfrak{b}^{\mathfrak{U}'}$$

and by Lemma 5 iv) $$\mathfrak{b}^{\mathfrak{U}+\mathfrak{U}'} = \mathfrak{b}^{\mathfrak{U}} + \mathfrak{b}^{\mathfrak{U}'}$$

Hence the equality above is equivalent to the following distributivity of $\varprojlim$:

$$\varprojlim(\mathfrak{b}^{\mathfrak{U}} + \mathfrak{b}^{\mathfrak{U}'}) = \varprojlim \mathfrak{b}^{\mathfrak{U}} + \varprojlim \mathfrak{b}^{\mathfrak{U}'}$$

Now by Theorem 2 the distributivity holds if $\mathfrak{b}^{\mathfrak{U}} \cap \mathfrak{b}^{\mathfrak{U}'}$ is star-epimorphic, i.e. if $(B^B)^{A^0}$ is star-epimorphic. But this is true if I is the set of positive integers, for in general it is not hard to see that C^A is star-epimorphic if C is and if $I = Z^+$. Thus we have the following

<u>Lemma</u> <u>19</u>. $\varprojlim'$ is right exact if $I = Z^+$, i.e. given $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $\varprojlim' A \longrightarrow \varprojlim' B \longrightarrow \varprojlim' C \longrightarrow 0$ holds.

Hence for $I = Z^+$, $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ implies

$$0 \longrightarrow \varprojlim A \longrightarrow \varprojlim B \longrightarrow \varprojlim C \longrightarrow \varprojlim{}' A \longrightarrow \varprojlim{}' B \longrightarrow \varprojlim{}' C \longrightarrow 0$$

<u>Theorem 14</u>. If I is countable and directed then we have

$$0 \longrightarrow \varprojlim A \longrightarrow \varprojlim B \longrightarrow \varprojlim C \longrightarrow \varprojlim{}' A \longrightarrow \varprojlim{}' B \longrightarrow \varprojlim{}' C \longrightarrow 0$$

<u>Proof</u>: It suffices to show that $\varprojlim{}'$ is right exact for I countable and directed. But this is clear in view of Lemmas 17, 19 and Theorem 9.

PART II

1. Milnor Short Exact Sequences

In this section we state in general terms two dual theorems, which will be specialized in the subsequent section into special theorems corresponding to various homology and cohomology theories.

By complex we mean the type described by Lefschetz (Cf. Algebraic Topology, 1942). Let K be an infinite complex, i.e. a complex with an infinite number of cells. Let ∂ be the cell boundary operator of K, which assigns to each q-cell of K a linear combination of (q-1)-cells of K according to the incidence numbers. An infinite q-chain c of K is said to be ∂-permissible if ∂ can be linearly extended on c to give a well-defined infinite linear combination of (q-1) cells, the boundary of c denoted ∂c. For example, if K is a star-finite complex then every infinite chain of K is ∂-permissible. But if K is not star-finite it may still be possible to have a group of infinite chains which are ∂-permissible. This group will lie between the group of finite chains and the group of all infinite chains. (e.g. cf. §3, §5)

Given an infinite complex K, let C denote a group of

∂-permissible infinite chains, and let Z, B, and H respectively denote the corresponding group of cycles, group of boundaries, and homology group. Let K_α and K_β be open subcomplexes with $K_\alpha \subset K_\beta$, then there is a chain homomorphism $p_\alpha^\beta : C(K_\beta) \to C(K_\alpha)$ sending $Z(K_\beta)$ into $Z(K_\alpha)$, $B(K_\beta)$ into $B(K_\alpha)$ and consequently we have

$$p_\alpha^\beta : H(K_\beta) \to H(K_\alpha)$$

If $I = \{K_\alpha\}$ is a family of subcomplexes of K, then we have an inverse system

$$(H;I) = \{ H(K_\alpha), p_\alpha^\beta \}$$

In general it does not hold that $\varprojlim (H;I) = H(K)$, but when the family of subcomplexes satisfies certain conditions, there is a relation between H(K) and (H;I). This relation is expressed by Milnor short exact sequence involving $\varprojlim$ and $\varprojlim'$.

<u>Theorem 1</u>. Given an infinite complex K, let $I = \{K_\alpha\}$ be a family of subcomplexes with the following properties:

(1) (C;I) is a star-epimorphic inverse system.

(2) $\varprojlim (C;I) \approx C(K)$

(3) I has a cofinal subsequence $I^{\#}$ or

(3)' $C(K_\alpha)$ is finitely generated for every $K_\alpha \in I$

then

$$0 \to \varprojlim{}'(H_{q+1};I) \to H_q(K) \to \varprojlim(H_q;I) \to 0 \qquad 0 \leqq i.$$

Remark I. Condition (1) holds whenever I has the "finite intersection" property, i.e. if K_α, $K_{\alpha'} \in I$ then $K_\alpha \cap K_{\alpha'} \in I$. To see this let $I' \subset I$ and $\beta \in I - I'$ be given, let $I^* = I' \cap \mathfrak{D}(\beta)$, and suppose $\zeta \in \varprojlim(C(K_\alpha);I^*)$, then ζ determines uniquely an element $X_\zeta \in C(\overline{K})$, $\overline{K} = \bigcup_{\alpha \in I^*} K_\alpha$, as follows: let $\sigma \in K_\alpha$, $\alpha \in I^*$, then $X_\zeta(\sigma) = \zeta_\alpha(\sigma)$. X_ζ is well-defined, for if $\sigma \in K_{\alpha'}$, $\alpha' \in I^*$, then $\zeta_{\alpha'}(\sigma) = \zeta_\alpha(\sigma) = \zeta_\gamma(\sigma)$ where $K_\alpha \cap K_{\alpha'} = K_\gamma$, $\gamma \in I^*$ since I^* is full. Now since $\overline{K}$ is a subcomplex of K_β, X_ζ can be extended to an element y of $C(K_\beta)$. Clearly then $p_\alpha^\beta\, y = \zeta_\alpha$ for any $\alpha \in I^*$. Hence $\{C(K_\alpha) \mid \alpha \in I\}$ is a star-epimorphic inverse system (Cf. Def. 1, Remark 2 of Part I). Condition (1) holds also whenever I has the following property: each $\sigma \in K$ is contained in some smallest subcomplex $K_\mu \in I$, i.e. $\sigma \in K_\alpha \iff \mu < \alpha$. This statement is needed for Lemma 1 of the Appendix.

Remark II. It is not hard to check that (2) implies

(2)' $\quad \varprojlim(Z;I) \approx Z(K)$

We point out that in contrast to (2) and (2)' a similar statement can not be made for B and that herein lies the origin of Milnor short exact sequence as we shall see in the proof of the theorem.

<u>Proof</u>: For each $K_\alpha \in I^\#$ consider the following diagram

$$\begin{tikzcd}
 & & 0 & & \\
 & & B'(K_\alpha) \arrow[u] \arrow[dl, "d"'] & & \\
 & & C(K_\alpha) \arrow[u] & & \\
0 \arrow[r] & B(K_\alpha) \arrow[r] & Z(K_\alpha) \arrow[u] \arrow[r] & H(K_\alpha) \arrow[r] & 0 \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

where $B'(K_\alpha)$ is by definition $C(K_\alpha)\,/\,Z(K_\alpha)$. Each arrow except d represents a homomorphism preserving degree, i.e. $Z_q(K_\alpha)$ goes into $C_q(K_\alpha)$ for each q, etc. d is an isomorphism with degree -1, i.e. $B'_{q+1}(K_\alpha)$ goes into $B_q(K_\alpha)$ for each q.

By Theorem 14, Part I, and Appendix we have in view of the condition (3)

$$0 \longrightarrow \varprojlim(Z;I^{\#}) \longrightarrow \varprojlim(C;I^{\#}) \longrightarrow \varprojlim(B';I^{\#}) \longrightarrow \varprojlim{}'(Z;I^{\#}) \longrightarrow \varprojlim{}'(C;I^{\#}) \longrightarrow \varprojlim{}'(B';I^{\#}) \longrightarrow 0$$

and $$0 \longrightarrow \varprojlim(B;I^{\#}) \longrightarrow \varprojlim(Z;I^{\#}) \longrightarrow \varprojlim(H;I^{\#}) \longrightarrow \varprojlim{}'(B;I^{\#}) \longrightarrow \varprojlim{}'(Z;I^{\#}) \longrightarrow \varprojlim{}'(H;I^{\#}) \longrightarrow 0$$

Hence substituting the conditions $(1)^{(2)}$ and $(2)'$ in the above sequences we have

$$0 \longrightarrow Z(K) \longrightarrow C(K) \longrightarrow \varprojlim(B';I^{\#}) \longrightarrow \varprojlim{}'(Z;I^{\#}) \longrightarrow 0 \longrightarrow \varprojlim{}'(B';I^{\#}) \longrightarrow 0$$

and $$0 \longrightarrow \varprojlim(B;I^{\#}) \longrightarrow Z(K) \longrightarrow \varprojlim(H;I^{\#}) \longrightarrow 0 \longrightarrow \varprojlim{}'(Z;I^{\#}) \longrightarrow \varprojlim(H;I^{\#}) \longrightarrow 0$$

or simply

$$0 \longrightarrow Z(K) \longrightarrow C(K) \longrightarrow \varprojlim(B';I^{\#}) \longrightarrow \varprojlim{}'(H;I^{\#}) \longrightarrow 0$$

and $$0 \longrightarrow \varprojlim(B;I^{\#}) \longrightarrow Z(K) \longrightarrow \varprojlim{}'(H;I^{\#}) \longrightarrow 0$$

Now it is possible to combine the two exact sequences into the following diagram

$$
\begin{tikzcd}
 & & 0 \\
 & & \varprojlim{}'(H;I^{\#}) \arrow[u] \\
 & & \varprojlim(B';I^{\#}) \arrow[u] \arrow[ddl, "d_*"', "\approx"] \\
 & & C(K) \arrow[u] \\
0 \arrow[r] & \varprojlim(B;I^{\#}) \arrow[r] & Z(K) \arrow[r] \arrow[u] & \varprojlim(H;I^{\#}) \arrow[r] & 0 \\
 & & 0 \arrow[u]
\end{tikzcd}
$$

We shall call $\varprojlim(B;I^{\#})$ the group of local boundaries of K (with respect to I) and denote it by $\tilde{B}(K)$.

Next we expand the above diagram as follows:

$$
\begin{tikzcd}
 & & 0 & & & \\
0 \arrow[dr] & 0 \arrow[r] \arrow[d] & X' \arrow[r] \arrow[u] \arrow[dl, "d_*"', "\approx"] & \varprojlim(B';I^{\#}) \arrow[r] & \varprojlim{}'(H;I^{\#}) \arrow[r] & 0 \\
 & X \arrow[d] \arrow[dr] & C(K) \arrow[u] & 0 & & \\
0 \arrow[r] & \tilde{B}(K) \arrow[r] \arrow[d] & Z(K) \arrow[r] \arrow[u] \arrow[dr] & \varprojlim(H;I^{\#}) \arrow[r] \arrow[u] & 0 & \\
 & Y \arrow[d] & 0 \arrow[u] & W \arrow[u] \arrow[dr] & & \\
 & 0 & & Y \arrow[u] & 0 & \\
 & & & 0 \arrow[u] & &
\end{tikzcd}
$$

d_* is an isomorphism with degree -1.

X', X, Y, W emerge in that order during the expansion. And we see that

(1) Since $X' \approx B'(K)$, $d_*(X') = X$ must be $B(K)$, the group of boundaries of K.

(2) From $0 \to X \to Z(K) \to W \to 0$, $W = H(K)$.

(3) $Y \approx \varprojlim{}'(H;I^{\#})$ with shift of degree, i.e. $Y_i \approx \varprojlim{}'(H_{i+1};I^{\#})$. Substitution of (2) and (3) in $0 \to Y \to W \to \varprojlim(H;I^{\#}) \to 0$ gives

$$0 \to \varprojlim{}'(H_{q+1};I^{\#}) \to H_q(K) \to \varprojlim(H_q;I^{\#}) \to 0$$

But since $\varprojlim{}'(H_{q+1};I^{\#}) = \varprojlim{}'(H_{q+1};I)$ and $\varprojlim(H_q;I^{\#}) = \varprojlim(H_q;I)$ the proof is complete.

The preceding has the following dual for cohomology: Let K be an infinite complex. Let δ be the cell coboundary operator, which assigns to each q-cell of K a linear combination of (q+1)-cells of K according to the incidence numbers. An infinite q-chain of K is called δ-permissible if δ can be linearly extended on it to give a well-defined (infinite) linear combination of (q+1)-cells. Given an infinite complex K, let C denote a group of δ-permissible (co-) chains, and let Z, B, H respectively denote the corresponding group of cocycles, group of coboundaries and cohomology group. Let K_α and K_β be closed subcomplexes with $K_\alpha \subset K_\beta$, then there is a chain homomorphism $p_\alpha^\beta : C(K_\beta) \to C(K_\alpha)$ sending $Z(K_\beta)$ into $Z(K_\alpha)$, $B(K_\beta)$ into $B(K_\alpha)$ and consequently we have $p_\alpha^\beta : H(K_\beta) \to H(K_\alpha)$. If $I = \{K_\alpha\}$ is a family of subcomplexes of K, then we have an inverse system $(H;I) = \{ H(K_\alpha);p_\alpha^\beta \}$.

<u>Theorem 2</u>. Given an infinite complex K, let $I = \{K_\alpha\}$ be a family of closed subcomplexes with the following conditions:

(1) $(C;I)$ is a star-epimorphic inverse system.

(2) $$\varprojlim(C;I) \approx C(K)$$

(3) I has a cofinal subsequence $I^{\#}$ or

(3)' $C(K_\alpha)$ is finitely generated for $K_\alpha \in I$.

Then

$$\begin{cases} 0 \longrightarrow \varprojlim{}'(H^{q-1};I) \longrightarrow H^q(K) \longrightarrow \varprojlim(H^q;I) \longrightarrow 0 & 1 \leq q \\ H^0(K) \approx \varprojlim(H^0;I) & \end{cases}$$

For $q = 0$ $H^0(K) = Z^0(K) = \varprojlim(Z^0;I) = \varprojlim(H^0;I)$.

2. Star-finite Simplicial Complex and Closure-finite Cellular Complex

If a simplicial complex K is star-finite[1] (or equivalently if each vertex of K belongs to at most a finite number of simplices) then every infinite chain of K is ∂-permissible. Hence we may consider the group of all infinite

[1] A complex is locally finite if it is both star-finite and closure finite. Since a simplicial complex is always closure-finite, a star-finite simplicial complex is also referred to as a locally finite simplicial complex.

chains of K and study its homology. Theorem 1 specializes into the following

<u>Theorem 3</u>. Let K be a star-finite simplicial complex. Let $I = \{K_\alpha\}$ be the family of all the finite open subcomplexes of K, then

$$0 \to \varprojlim{}'(H_{q+1};I) \to H_q(K) \to \varprojlim(H_q;I) \to 0 \qquad 0 \leqq q$$

<u>Proof</u>: Conditions (1), (2), (3)' of Theorem 1 are easily verified.

If a cellular complex K is closure-finite (or equivalently if each cell is incident to at most a finite number of vertices), in particular if K is simplicial, then every infinite chain is δ-permissible, and the group of all infinite chains gives rise to the corresponding cohomology group H(K). Theorem 2 specializes into the following

<u>Theorem 4</u>. Let K be a closure-finite cellular complex. Let $I = \{K_\alpha\}$ be the family of all finite closed subcomplexes of K, then

$$0 \to \varprojlim{}'(H^{q-1};I) \to H^q(K) \to \varprojlim(H^q;I) \to 0 \qquad 0 \leqq q$$

where we set formally $H^{-1}(K_\alpha) = 0 \qquad \alpha \epsilon I$.

<u>Proof</u>: Conditions (1), (2), (3)' of Theorem 2 are easily verified.

Remark I. In both Theorem 3 and Theorem 4 the condition that $\{K_\alpha\}$ be the family of all finite (open or closed, respectively) subcompleses of K may be weakened to the condition $\bigcup K_\alpha = K$.

Remark II. In Theorems 3 and 4 subcomplexes K_α are finite so that their chain groups are finitely generated. The theorems will still hold even if K_α are not finite as long as their chain groups are finitely generated.

<u>Example</u>. Let S denote circle (1-dimensional sphere) in general. Let X be the space built inductively by

$$X = X_1 \cup X_2 \cup X_3 \cup \ldots, \quad X_1 \subset X_2 \subset X_3 \subset \ldots$$

where $X_1 = S_1$ (circle) and X_2 = the mapping cylinder of

$$f_{1,2}: S_1 \longrightarrow S_2 \quad \text{with the degree of } f_{1,2} = 2$$

and in general let X_{i+1} be obtained from X_i by adjoining a mapping cylinder as follows: let S_i be the last circle in X_i, consider

$$f_{i,i+1}: S_i \longrightarrow S_{i+1} \quad \text{with } \deg(f_{i,i+1}) = 2$$

and adjoin the mapping cylinder of $f_{i,i+1}$ to X_i by identification along S_i.

It is not hard to see that there is a triangulation K of X such that its restriction to X_i gives a triangulation K_i of X_i. The cochain complexes generated by K_i for all i form an inverse system of cochain complexes which is cofinal in the inverse system of all finite cochain subcomplexes of K. Hence by Theorem 15, we obtain in particular:

$$0 \to \varprojlim{}' \{ H^1(X_i, G)\} \longrightarrow H^2(X, G) \longrightarrow \varprojlim \{ H^2(X_i, G)\} \longrightarrow 0$$

where G is some chosen coefficient group.

To go further we examine the inverse sequences $\{H^1(X_i,G)\}$ and $\{H^2(X_i,G)\}$. First the homologies of X_i are

$$H_0(X_i) = Z \qquad \text{for all } i$$
$$H_1(X_i) = Z \qquad \text{for all } i$$
$$H_2(X_i) = 0 \qquad \text{for all } i$$

and the inclusion $i: K_i \longrightarrow K_{i+1}$ induces

$$i_*: H_0(X_i) \longrightarrow H_0(X_{i+1}) \qquad \text{identity}$$
$$i_*: H_1(X_i) \longrightarrow H_1(X_{i+1}) \qquad \text{multiplicative by 2}$$
$$i_*: H_2(X_i) \longrightarrow H_2(X_{i+1}) \qquad \text{trivial}$$

By the Universal Coefficient Theorem (Cf. Eilenberg, "Group Extentions and Homology," Annals of Mathematics,, 1942):

$$H^1(X_i,G) = \mathrm{Hom}(H_1(X_i),G) + \mathrm{Ext}(H_0(X_i),G) = G + 0 = G$$

$$H^2(X_i,G) = \mathrm{Hom}(H_2(X_i),G) + \mathrm{Ext}(H_1(X_i),G) = 0 + 0 = 0$$

and the inclusion $K_i \subset K_{i+1}$ induces

$$i_*: H1(X_i,G) \longleftarrow H1(X_{i+1},G) \qquad \text{multiplication by 2}$$

$$i_*: H^2(X_i,G) \longleftarrow H^2(X_{i+1},G) \qquad \text{trivial}$$

Hence $\{H^2(X_i)\} = 0$ and $\{H^1(X_i)\}$ is an inverse sequence similar to (A,Z^+) in Example A. (Cf. p. 22.)

So if in general we let $G^{(n)} = \underleftarrow{\lim}' \{ G \xleftarrow{xn} G \xleftarrow{xn} G \xleftarrow{xn} \ldots \}$

then $\qquad H^2(X,G) \approx \underleftarrow{\lim} \{ H'(X_i,G)\} \approx G^{(2)}$

3. Singular Homology Based on Locally Finite Singular Chains

Let X be a topological space then a singular chain of X is said to be locally finite if each point $x \in X$ has a neighborhood U_x which intersects with at most finite number of singular simplexes with non-zero coefficient in the given singular chain. Unless a singular chain is locally finite it is impossible to define its boundary. A locally finite singular chain is ∂-permissible. If a space is compact its singular chain is locally finite if and only if it is finite.

An open subset of X is said to be bounded if its closure is compact in X. For each bounded open set U we may consider $C(X, X-U) = C(X)/C(X-U)$. The totality $\{C(X, X-U)\}$ of such singular chain complexes constitute an inverse system with obvious homomorphism $p_U^V\colon C(X,X-V) \to C(X,X-U)$. And letting $I = \{U\}$ be the family of all bounded open sets of X, we may denote by $(H;I)$ the resulting inverse system of singular homology groups.

__Lemma 1__. Let $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is a bounded open subset of X, then the set $I = \{U \mid U \text{ bounded in } X\}$, partially ordered by $U < V$ if $U \subset V$, has a cofinal subsequence $I^{\#}$.

__Proof__: Let U be an arbitrary bounded open set of X. Since $\overline{U}$ is compact and $\{X_i \mid i=1,2,\ldots\}$ covers $\overline{U}$, there is a finite subcovering, say by $X_{i_1}, X_{i_2}, \ldots X_{i_n}$, of U. So if we let $Y_1 = X_1$, $Y_2 = Y_1 \cup X_2$, and in general $Y_i = Y_{i-1} \cup X_i$, then $U \subset Y_{i_n}$. Therefore $I^{\#} = \{Y_1, Y_2, \ldots\}$ is a cofinal subsequence of I.

__Corollary 1__. Let X be locally compact, and suppose X has a countable basis $\mathfrak{X} = \{X_i \mid i=1,2,\ldots\}$. Then $I = \{U \mid U \text{ bounded in } X\}$ has a cofinal subsequence $I^{\#}$.

__Proof__: Let $\mathfrak{X}'$ consist of all open sets of $\mathfrak{X}$ which are bounded. Denote $\mathfrak{X}' = \{X_i' \mid i=1,2,\ldots\}$. In view of the previous lemma it suffices to show $X = \bigcup_{i=1}^{\infty} X_i'$. Let $x \in X$ be an arbitrary point, then since X is locally compact, there is an open set U_x containing x with $\overline{U}_x$ compact. Since $\mathfrak{X}$ is a basic of X, there is an X_i such that $x \in X_i \subset U_x$, so that $\overline{X}_i \subset \overline{U}_x$ is compact. Hence $X = \bigcup_{i=1}^{\infty} X_i'$.

__Theorem 5__. Let X be locally compact and suppose X is a union of countable number of bounded open subsets, then

$$0 \to \varprojlim{}'(H_{q+1};I) \to H_q(X) \to \varprojlim(H_q;I) \to 0$$

where $H_q(X)$ is the q^{th} singular homology group of X based on locally finite singular chains, and $I = \{U \mid U \text{ open and bounded in } X\}$

__Proof__: First we remark that $C(X, X-U)$ can be interpreted as the free abelian group generated by singular simplexes of X which intersect with U. Under this interpretation p_U^V is achieved by merely deleting singular simplexes which intersect V but which do not intersect U, $U \subset V$. Then it is not hard to see that $\{ C(X,X-U) \mid U \in I \}$ constitute a star-epimorphic inverse system. That $\varprojlim(C;I) \subset C(X)$ is clear, but for $\varprojlim(C;I) \supset C(X)$ we need the assumption that X is locally compact. Hence by Lemma 1 and Theorem 1, the theorem is proved.

<u>Corollary 2</u>. Let X be locally compact and separable (i.e. with a countable basis) then the Milnor short exact sequence of Theorem 5 holds.

<u>Proof</u>: Cf. <u>Corollary 1</u>.

<u>Corollary 3</u>. Let X be such that for any bounded open set $U \subset X$ there exists $U' \supset U$ such that

$$H_q(X, X-U') = \begin{cases} Z & \text{for } q = n \\ 0 & \text{for } q > n \end{cases}$$

Then
$$H_q(X) = \begin{cases} Z \text{ or } 0 & \text{for } q = n \\ 0 & \text{for } q > n \end{cases}$$

4. Singular Cohomology

Let X be a topological space. Since the singular complex of X is closure-finite, every singular chain of X is δ-permissible. The resulting cohomology is the singular cohomology of X, denoted here $H(X)$.

Lemma 2. Let X be a locally finite union of countable number of compact subsets, i.e. there exist compact subsets $X_1, X_2, \ldots$ such that $X = \bigcup_{i=1}^{\infty} X_i$ and for any $x \in X$ there exists a neighborhood U_x of x such that U_x intersects with at most finite number of X_i's, then $I = \{ F | F \text{ compact in } X \}$ has a cofinal subsequence $I^{\#}$.

<u>Proof</u>: Let F be an arbitrary compact subset of X, then for each $f \in F$ there is U_f intersecting with at most finite number of X_i's. $\{ U_f \mid f \in F \}$ is a covering of F, compact; hence there is a finite subcovering of F by say $U_{f_1}, U_{f_2}, \ldots, U_{f_n}$. $\bigcup_{i=1}^{n} U_{f_i}$ intersects with at most finite number of X_i's, hence the same can be said of $F \subset \bigcup_{i=1}^{n} U_{f_i}$, so that if we let $Y_1 = X_1$, $Y_2 = Y_1 \cup X_2, \ldots$ then $I^{\#} = \{ Y_1, Y_2, \ldots \}$ is cofinal in I.

<u>Corollary 3</u>. Let X be a locally compact, paracompact Lindelöf space, then $I = \{ F \mid F \text{ compact in } X \}$ has a cofinal subsequence.

<u>Proof</u>: Since X is locally compact, for each $x \in X$ there exists an open set U_x containing x such that $\overline{U}_x$ is compact in X. $\mathfrak{X} = \{ U_x \mid x \in X \}$ is a covering of X. But since X is paracompact $\mathfrak{X}$ has a locally finite refinement $\mathfrak{X}'$, which in turn has a countable subcovering $\mathfrak{X}'' = \{V_x\}$ since X is Lindelöf. It is easily seen that X is a locally finite union of the compact sets $\{ \overline{V}_x \}$. Hence by Lemma 2 the corollary follows.

Remark related to Lemma 2. Given X let $Y_1 \subset Y_2 \subset Y_3 \subset \ldots$

be compact subsets of X, then $X = \bigcup_{i=1}^{\infty} Y_i$ does not guarantee that $I^{\#} = \{Y_1, Y_2, \ldots\}$ constitutes a cofinal subsequence of $I = \{F \mid F \text{ compact}\}$. As an example let X be the subset of the Euclidean plane determined by $X = \{(x,\frac{1}{n}) \mid 0 \leq x \leq 1, n=1,2,3,\ldots\} \cup (0,0)$. Let $X_i = \{(x,\frac{1}{i}) \mid 0 \leq x \leq 1\}$ for $i=1,2,3, \ldots$.[1] Let $Y_1 = X_1$, $Y_2 = Y_1 \cup X_2$, ... then clearly $Y_1 \subset Y_2 \subset Y_3 \subset \ldots$ and $X = \bigcup_{i=1}^{\infty} Y_i$, but $F = (0,0) \cup \{(x,\frac{1}{n}) \mid 0 \leq x \leq \frac{1}{n}, n=1,2,\ldots\}$ is not contained in any Y_i, hence $I^{\#} = \{Y_1, Y_2, \ldots\}$ is not a cofinal subsequence of I. In fact we will show that I does not have any cofinal subsequence.

Example of a space whose partially ordered family of compact subsets does not have a cofinal subsequence: Let

$$X = \{(x,\tfrac{1}{n}) \mid 0 \leq x \leq 1, n=1,2,3,\ldots\} \cup (0,0)$$

let $\quad X_i = \{(x,\frac{1}{i}) \mid 0 \leq x \leq 1\} \quad i=1,2,\ldots$

Let $A_1 \subset A_2 \subset \ldots$ be an arbitrarily given sequence of compact subsets of X,[2] we will show that $I^{\#} = \{A_n\}$ is not cofinal in $I = \{F \mid F \text{ compact in } X\}$ by exhibiting a compact subset B of X which is not contained in any A_n. Consider $X_1 \cap A_1$. Since A_1 is compact, $X_1 \cap A_1$ must have the greatest point (i.e. the point with largest x-coordinate) say a_{11}.[3] Similarly $X_2 \cap A_1$ must have the greatest point

1) And include (0,0) in X_i.

2) And assume without the loss of generality that each A_i contains (0,0).

3) In case $X_1 \cap A_1 = \phi$, set $a_{11} = 0$.

a_{21}, etc. . . so that we can associate with A_1 a column of real numbers $(a_{11}, a_{21}, a_{31}, \ldots)$. Similarly we can associate with A_2 a column of real numbers $(a_{12}, a_{22}, a_{32}, \ldots)$ etc. We contend that the set of points $\{(1,a_{11}), (\frac{1}{2}, a_{21}), (\frac{1}{3}, a_{31}), \ldots\}$ has one and only one limit point which is the origin $(0,0)$, for if it should have another limit point say p it would have to be on the x-axis so that we could select a set of points of A_1 within a small neighborhood of p clustering at p, and A_1 would no longer be compact. Similar contentions can be made for $\{(1, a_{12}), (\frac{1}{2}, a_{22}), \ldots\}$ etc. Now define B as follows: Let $b_1 = 1$, $b_2 = 1$, ... until i_1 is such that $a_{i,1} < 1$ when we set $b_{i_1} = 1$ and continue with $b_{i_1+1} = 1$, $b_{i_1+2} = 1$, ... until $i_2 > i_1$ is such that $a_{i_2 2} < \frac{1}{2}$ when we set $b_{i_2} = \frac{1}{2}$ and continue with $b_{i_2+1} = \frac{1}{2}$, $b_{i_2+2} = \frac{1}{2}$, ... until $i_3 > i_2$ is such that $a_{i_3 3} < \frac{1}{3}$ when we set $b_{i_3} = \frac{1}{3}$ and continue with $b_{i_3} + 1 = \frac{1}{3}$, $b_{i_3} + 2 = \frac{1}{3}$, ... etc. If we let now $B = \{(x, \frac{1}{n}) \mid 0 \leqq x \leqq b_n\} \cup (0,0)$ then clearly B is compact and is not contained in any A_n.

<u>Theorem</u> <u>6</u>. Let X be such that $I = \{F \mid F \text{ compact in } X\}$ has a cofinal subsequence. Then the following Milnor short exact sequence holds

$$0 \longrightarrow \varprojlim{}'(H_{q-1};I) \longrightarrow H_q(X) \longrightarrow \varprojlim(H_q;I) \longrightarrow 0$$

where $H_q(X)$ is the q^{th} singular cohomology of X.

<u>Proof</u>: This is the dual of Theorem 5. Here we merely

point out that $P^{F_2}_{F_1}: C(F_2) \longrightarrow C(F_1)$ for $F_1 \subset F_2$ can be described as deleting all "co-simplexes" in F_2 which do not lie entirely in F_1.

5. Steenrod-Milnor Homology

Steenrod [7] introduced a new type of homology for compact metric spaces. The following presentation is based on Professor Milnor's generalization.

Let X be a topological space satisfying the T_1 separation axiom. Let K be the simplicial complex whose n-simplex is an (n+1)-tuple of points in X.[1] Given a covering α of X a simplex of K is said to be contained in α if there exists an open set belonging to α which contains the simplex. Two points p,q of X are said to be separated by the covering α if none of the open sets of α contains p,q simultaneously. We remark that for any given $p,q \in X$ there exists a covering which separates them provided that X satisfies the T_1 separation axiom, for we can modify any covering α by replacing all the open sets U_1, U_2, ... which contain both p and q by $U_1 - p$, $U_2 - p$, ... and then add to α $U_1 - q$, an open set because X is a T_1-space. We note that this new covering is a refinement of α.

1) If all n+1 points are identical, such n-simplex shall be treated as null simplex. All 0-simplexes shall be treated as null simplexes.

Now let $\hat{C}(X)$ denote the group of all infinite chains of K satisfying the following "regularity" condition.

<u>Definition 1</u>. An infinite chain c of K is regular if for any covering α of X at most a finite number of simplexes outside α have non-zero coefficients in c. While K is not star-finite the regular chains are ∂-permissible. To see this let c be any regular q-chain of K, $q \geq 2$. Then any (q-1)-simplex $A_0A_1\ldots A_{q-1} = \sigma$ of K is incident to at most a finite number of q-simplexes of K with non-zero coefficients in c. Reason: T_1 separation axiom insures a covering α which separates say A_0 and A_1, so that σ is outside α. Now all the n-simplexes to which σ is incident will also be outside α. The regularity of c then guarantees that at most a finite number of these n-simplexes have non-zero coefficients in c. If c is a regular 1-chain its boundary shall be zero, $\partial c = 0$ for $c \in \hat{C}_1(X)$.[1)]

It is not difficult to see that the boundary of a regular chain is again regular, and that $\partial\partial = 0$ as usual, so that $\hat{C}(X)$ is a chain complex, with $\hat{Z}(X)$, $\hat{B}(X)$ and $\hat{H}(X)$ defined in the usual way. $\hat{H}(X)$ is called the Steenrod-Milnor homology.

<u>Proposition</u>. Let $A \subset X$. Let $\hat{C}(X \bmod A) = \hat{C}(X)/\hat{C}(A)$. The boundary operator ∂ of $\hat{C}(X)$ together with its restriction to $\hat{C}(A)$ induces a boundary operator for $\hat{C}(X \bmod A)$,

1) We set formally $\hat{C}_0(X) = 0$.

which gives rise to $\hat{H}(X \bmod A)$, the relative Steenrod-Milnor homology. From the short exact sequence $0 \to \hat{C}(A) \xrightarrow{i} \hat{C}(X) \xrightarrow{j} \hat{C}(X \bmod A) \to 0$ we immediately obtain the exact sequence for Steenrod -Milnor homology

$$\cdots \xleftarrow{i_*} H_{q-1}(A) \xleftarrow{\partial} H_q(X \bmod A) \xleftarrow{j_*} H_q(X) \xleftarrow{i_*} H_q(A) \xleftarrow{\partial} \cdots$$

Let $C(K)$ be the finite-chain complex of K.[1] Given a covering α of X let K_α be those simplexes of K which are contained in α and let $C(K_\alpha)$ be the finite-chain complex of K_α. Let $C(K \bmod K_\alpha) = C(K)/C(K_\alpha)$. The boundary operator of $C(K)$ and $C(K_\alpha)$ induces the boundary operator of $C(K \bmod K_\alpha)$ and make $C(K \bmod K_\alpha)$ a chain complex with the usual $Z(K \bmod K_\alpha)$, $B(K \bmod K_\alpha)$, $H(K \bmod K_\alpha)$, etc.

Now suppose β is a refinement of α then a unique chain homomorphism $p^\beta_\alpha : C(K \bmod K_\beta) \to C(K \bmod K_\alpha)$ is defined by making the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(K_\beta) & \longrightarrow & C(K) & \longrightarrow & C(K \bmod K_\beta) & \longrightarrow & 0 \\
 & & \downarrow \cap & & \downarrow \mathrm{id} & & \downarrow p^\beta_\alpha & & \\
0 & \longrightarrow & C(K_\alpha) & \longrightarrow & C(K) & \longrightarrow & C(K \bmod K_\alpha) & \longrightarrow & 0
\end{array}$$

Let I be the partially ordered set of all covering of X with $\alpha < \beta$ if β is a refinement of α. With p^β_α defined for every pair $\alpha < \beta$ the family $\{C(K \bmod K_\alpha) \mid \alpha \in I\}$ becomes an inverse system which we will denote by $(C;I)$ in the sequel. Similarly the notation $(Z;I)$, $(B;I)$, $(H;I)$, etc., will be used.

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1) i.e. the free abelian group generated by simplexes of K.

Lemma 3. i) $(C;I)$ is star-epimorphic.

ii) $\varprojlim(C;I) = \hat{C}(X)$.

Proof: First $C(K \bmod K_\alpha)$ may be interpreted as the free abelian group generated by simplexes of K which lie outside of α, and $p^\beta_\alpha : C(K \bmod K_\beta) \longrightarrow C(K \bmod K_\alpha)$ may be interpreted as deletion of simplexes lying outside of β but contained in α. With this interpretation we easily check i), ii).

Lemma 4. I has a cofinal subsequence if X is a compact metric space.

Proof: For each integer n, cover each $p \in X$ by an open sphere of radius $\frac{1}{n}$ centered at p. Let the resulting covering be denoted by α_n. Let $I^\# = \{ \alpha_n \mid n=1,2,3,\ldots \}$. Now let any covering α of X be given. Since X is compact, α has a finite subcovering which determines a certain number ε known as Lebesgue number so that if n is such that $\frac{1}{n} < \varepsilon$ then α_n is a refinement of α. Hence $I^\#$ is cofinal in I.

With Lemma 3 and Lemma 4 Theorem 1 specializes into

Theorem 7. Steenrod-Milnor homology $\hat{H}(X)$ satisfies the following Milnor short exact sequence if X is compact metric

$$0 \longrightarrow \varprojlim{}'(H_{q+1};I) \longrightarrow \hat{H}_q(X) \longrightarrow \varprojlim(H_q;I) \longrightarrow 0 \qquad q \geqq 0$$

or $$0 \rightarrow \varprojlim{}' \{ H_{q+1}(K \bmod K_\alpha) \mid \alpha \in I \} \longrightarrow \hat{H}_q(X) \rightarrow \varprojlim \{ H_q(K \bmod K_\alpha) \mid \alpha \in I \} \rightarrow 0 \qquad q \geqq 0$$

Remark I. Homology functor applied on

$0 \longrightarrow C(K_\alpha) \longrightarrow C(K) \longrightarrow C(K \bmod K_\alpha) \longrightarrow 0$ gives

$0 \leftarrow H_0(K \bmod K_\alpha) \leftarrow H_0(K) \leftarrow H_0(K_\alpha) \leftarrow H_1(K \bmod K_\alpha) \leftarrow H_1(K) \leftarrow H_1(K_\alpha) \leftarrow \cdots$

But clearly $H_q(K) = 0$ for $q \geqq 1$ hence

$$H_q(K_\alpha) \approx H_{q+1}(K \bmod K_\alpha) \qquad q \geqq 1$$

and
$$\overline{H}_0(K_\alpha) \approx H_1(K \bmod K_\alpha)$$

where $\overline{H}_0(K_\alpha) \subset H_0(K_\alpha)$ is the group of homology classes of 0-cycles with Kronecker index equal to 0. So that we have the following alternative Milnor short exact sequence for Steenrod-Milnor homology

$$0 \to \varprojlim{}' \{H_q(K_\alpha) \mid \alpha \in I\} \to \widehat{H}_q(X) \to \varprojlim \{H_{q-1}(K_\alpha) \mid \alpha \in I\} \to 0 \qquad q \geqq 2$$

and $$0 \to \varprojlim{}' \{H_1(K_\alpha) \mid \alpha \in I\} \to \widehat{H}_1(X) \to \varprojlim \{\overline{H}_0(K_\alpha) \mid \alpha \in I\} \to 0$$

Remark II. As was pointed out in the proof of Theorem 1 we may define the group of local boundaries by setting

$$\widetilde{B}(X) = \varprojlim (B; I)$$

In Steenrod's paper quoted earlier the notion of the group of weak boundaries (weakly bounding cycles) was introduced. Denoting this group by $\dot{B}(X)$. We have the following diagram

$$\begin{tikzcd}
 & \widetilde{B}(X) \arrow[dr, phantom, "\subset" sloped] & \\
\widehat{B}(X) \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \widehat{Z}(X) \\
 & \dot{B}(X) \arrow[ur, phantom, "\subset" sloped] &
\end{tikzcd}$$

$\widetilde{B}(X)$ and $\dot{B}(X)$ are not apparently related. (But cf. §5a.) Steenrod defined the weak homology group by setting $\dot{H}(X) = \dot{B}(X)/\widehat{B}(X)$.

He also showed that $\hat{Z}_q(X) / \dot{B}_q(X) \approx V_{q-1}(X)$ where $V_{q-1}(X)$ is the $(q-1)^{th}$ Vietoris group, so that he had in effect the following short exact sequence

$$0 \longrightarrow \dot{H}_q(X) \longrightarrow \hat{H}_q(X) \longrightarrow V_{q-1}(X) \longrightarrow 0$$

5a. Relation Between Steenrod-Milnor Homology and Čech Homology

The definition of Čech homology is based on nerves of coverings and inverse limits. (Cf. Eilenberg-Steenrod, Foundations of Algebraic Topology.)

Let α be a covering of X, then the nerve of α, N_α, is a simplicial complex whose n-simplex is an (n+1)-tuple of open sets of α with non-vacuous intersection. If $\beta = \{V\}$ is a refinement of $\alpha = \{U\}$, then we may choose a chain mapping $p^\beta_\alpha : N_\beta \longrightarrow N_\alpha$ by assigning to each vertex V of N_β a vertex U of N_α where $V \subset U$. It is known that the induced homomorphism $p^\beta_\alpha : H(N_\beta) \longrightarrow H(N_\alpha)$ is independent of the choices of chain mappings described above; so that if we let I be the set of all coverings of X partially ordered by $\alpha < \beta \iff \beta$ is a refinement of α, then to every X is associated a unique inverse system $\{ H(N_\alpha) \mid \alpha \in I \}$. Čech homology of X is defined by $H(X) = \varprojlim \{ H(N_\alpha) \}$. If X is compact it suffices to consider only finite coverings of X, since the set of such coverings is known to be cofinal in I.

In order to establish the relation between Steenrod-Milnor and Čech homology we need the theory of carrier. We state below a few lemmas without proof. (Cf. [4] or [8].)

A carrier C from a complex K to a complex K' is a function which assigns to each cell σ of K a non-vacuous subcomplex $C(\sigma)$ of K' such that $\sigma < \tau$ implies $C(\sigma) \subset C(\tau)$. The carrier is acyclic if $C(\sigma)$ is acyclic for every σ, i.e. each q-cycle, $q > 0$, of $C(\sigma)$ is a boundary and each 0-cycle of index 0 is a boundary.

A chain transformation Φ of K into K' is a sequence of homomorphisms $\Phi\colon C_q(K) \longrightarrow C_q(K')$ (for all q) such that

$$\partial\Phi c = \Phi\partial c, \qquad c \in C_q(K)$$
$$\text{Index } (\Phi c) = \text{Index } (c), \qquad c \in C_0(K)$$

<u>Lemma A</u>. If C is an acyclic carrier $K \longrightarrow K'$, then C carries some chain transformation ϕ; and if ϕ, ψ are two chain transformations carried by C, then C carries a chain homotopy D of ϕ into ψ. Thus C gives rise to a unique homomorphism $C^*\colon H(K) \longrightarrow H(K')$.

<u>Lemma B</u>. Let C_1 be an acyclic carrier $K \longrightarrow K'$. Let C_2 be an acyclic carrier $K' \longrightarrow K''$. Let $C_2(C_1(\sigma))$, $\sigma \in K$, denote $\bigcup_{\tau \in C_1(\sigma)} C_2(\tau)$. If C is an acyclic carrier $K \longrightarrow K''$ such that $C_2(C_1(\sigma)) \subset C(\sigma)$ for every $\sigma \in K$, then the composition $C_2^* \cdot C_1^*: H(K) \longrightarrow H(K') \longrightarrow H(K'')$ is identical with $C^*: H(K) \longrightarrow H(K'')$.

A cell transformation $f: K \longrightarrow K'$ is a function which assigns to each cell $\sigma \in K$ a cell $f(\sigma) \in K'$ in such a way that if $\sigma, \tau \in K$ are incident, so are $f(\sigma), f(\tau) \in K'$. Thus f induces a homomorphism $f^*: H(K) \longrightarrow H(K')$. The cell transformation f is said to be carried by a carrier C if $f(\sigma) \in C(\sigma)$ for each $\sigma \in K$.

<u>Lemma C</u>. If a cell transformation $f: K \longrightarrow K'$ is carried by an acyclic carrier $C: K \longrightarrow K'$ then $f_*: H(K) \longrightarrow H(K')$ and $C^*: H(K) \longrightarrow H(K')$ are identical.

<u>Lemma D</u>. If f is a cell transformation $K \longrightarrow K'$ such that for each cell $\sigma \in K'$ $f^{-1}(\sigma)$ is non-vacuous and acyclic in K, then $f_*: H(K) \longrightarrow H(K')$ is an isomorphism.

Given a covering α of X consider the product complex $N_\alpha \times K_\alpha$. Let Γ_α be the set of pairs of the type $(U_0, U_1, \ldots U_m; x_0, x_1, \ldots x_n)$ with $x_0, x_1, \ldots, x_n \in U_0 \cap U_1 \cap \ldots \cap U_m$, then Γ_α is a subcomplex of $N_\alpha \times K_\alpha$. Consider the following cell-transformations f_1, f_2 by projection.

$$\begin{tikzcd} & \Gamma_\alpha \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ N_\alpha & & K_\alpha \end{tikzcd}$$

It is not difficult to check that both f_1 and f_2 satisfy the condition of Lemma D, hence $H(N_\alpha) \approx H(\Gamma_\alpha) \approx H(K_\alpha)$. To see $\{H(N_\alpha) \mid \alpha \in I\} \approx \{H(K_\alpha) \mid \alpha \in I\}$, we need only check the following commutative diagram.

$$\begin{array}{ccccc} H(N_\beta) & \xleftarrow{\approx} & H(\Gamma_\beta) & \xrightarrow{\approx} & H(K_\beta) \\ \downarrow & & \downarrow & & \downarrow \\ H(N_\alpha) & \xleftarrow{\approx} & H(\Gamma_\alpha) & \xrightarrow{\approx} & H(K_\alpha) \end{array}$$

Milnor short exact sequence for Steenrod-Milnor homology now assumes the following form. (Cf. Remark I of §5.)

$$0 \longrightarrow \underleftarrow{\lim}' \{H_q(N_\alpha)\} \longrightarrow \hat{H}_q(X) \longrightarrow \underleftarrow{\lim} \{H_{q-1}(N_\alpha)\} \longrightarrow 0 \qquad q \geqq 1$$

where $H_0(N_\alpha)$ is understood to be the group of homology classes of 0-cycles with index 0. Recalling the definition of Čech homology we immediately have the following theorem.

<u>Theorem 8</u>. Given X let $\hat{H}(X)$ be Steenrod-Milnor homology, $H(X)$ be Čech homology, and $H'(X) = \underleftarrow{\lim}' \{H(N_\alpha)\}$, then the following short exact sequence holds

$$0 \longrightarrow H'_q(X) \longrightarrow \hat{H}_q(X) \longrightarrow H_{q-1}(X) \longrightarrow 0$$

Remark I. Steenrod-Milnor homology satisfies Exactness Axiom [4] as indicated in §5 while Čech homology does not. Thus although Steenrod-Milnor homology is no new invariant in terms of Čech homology (Cf. [2]) it has this formal advantage.

Remark II. Eilenberg-MacLane [2] had implicitly the following exact sequence (page 825, §45)

$$0 \longrightarrow \dot{H}_q(X) \longrightarrow \hat{H}_q(X) \longrightarrow H_{q-1}(X) \longrightarrow 0$$

where $\dot{H}_q(X)$ is the q^{th} weak homology group of Steenrod [7]. Thus we see that $\underline{H}'(X)$ and $\dot{H}(X)$ are equivalent, and the evaluation of the weak homology group $\dot{H}(X)$ amounts to computing $\varprojlim'$ of some inverse system. Steenrod [7] evaluated $\dot{H}(X)$ for some particular cases, and Eilenberg-MacLane [2] equated $\dot{H}(X)$ with certain group of group extensions involving Čech cohomology group with integer coefficients.

6. Cohomology of Eilenberg-MacLane Complexes

Eilenberg-MacLane [3] defined in a purely algebraic fashion the complexes $K(\pi, n)$ for any abelian group π and any integer $n = 1, 2, \ldots$. The topological significance of these complexes rests on the fact that their homology (and cohomology) groups $H(\pi, n)$ are those of any arcwise connected space X with homotopy groups $\pi_n(X) \approx \pi$, $\pi_i(X) = 0$ for $i \neq n$.

A brief definition of $K(\pi, n)$ is as follows: Let Δ_q denote the standard q-simplex. Let $K_q(\pi, n) = Z^n(\Delta_q; \pi)$, where Δ_q as a complex admits all faces of Δ_q including degenerate ones. Let $f_i: \Delta_{q-1} \longrightarrow \Delta_q$ be defined by $(0, \ldots q-1) \rightarrow (0, \ldots i-1, i+1, \ldots q)$. f_i induces $f_i^*: Z^n(\Delta_q, \pi) \longrightarrow Z^n(\Delta_{q-1}, \pi)$ which gives rise to $\partial_i: K_q(\pi, n) \rightarrow K_{q-1}(\pi, n)$. $\partial : K_q(\pi, n) \rightarrow K_{q-1}(\pi, n)$ is then defined by $\partial = \sum_{i=0}^{q} (-1)^i \partial_i$.

Now let $I = \{\pi_\alpha\}$ be the family of all finitely generated subgroups of π, then $\{ Z^n(\Delta_q, \pi_\alpha) \mid \pi_\alpha \in I\}$ form a direct system of groups with $\alpha < \beta$ if $\pi_\alpha \subset \pi_\beta$, and it is not difficult to see $\varinjlim \{Z^n(\Delta_q, \pi)\} \approx Z^n(\Delta_q, \pi)$ since Δ_q is a finite complex and each π_α is finitely generated. In fact since for $\pi_\alpha \subset \pi_\beta$ $Z^n(\Delta_q \pi_\alpha) \subset Z^n(\Delta_q, \pi_\beta)$ we may write

$$\bigcup_{\pi_\alpha \in I} Z^n(\Delta_q, \pi_\alpha) = Z^n(\Delta_q, \pi)$$

i.e. we have $\bigcup_{\pi_\alpha \in I} K_q(\pi_\alpha, n) = K_q(\pi, n)$ or $\bigcup_{\pi_\alpha \in I} K(\pi_\alpha, n) = K(\pi, n)$

Furthermore we see $\partial(K_q(\pi_\alpha, n)) \subset K_{q-1}(\pi_\alpha, n)$ i.e. each $K(\pi_\alpha, n)$ is a closed subcomplex of $K(\pi, n)$.

<u>Theorem</u> 9. Given a countable abelian group π, let $I = \{\pi_\alpha\}$ be the family of all the finitely generated subgroups of π, then

$$0 \to \lim{}' \{ H^{q-1}(\pi_\alpha, n)\} \longrightarrow H^q(\pi, n) \longrightarrow \lim \{H^q(\pi_\alpha, n)\} \longrightarrow 0 \qquad 0 \leq q$$

<u>Proof</u>: We need only check condions (1), (2), (3) of Theorem 2.

Appendix

We shall consider closure finite complexes, with finite chains, infinite cochains and closed subcomplexes. Let a directed family of subcomplexes K_α with union K satisfy the following condition:

(*) Each cell σ is contained in some smallest subcomplex K_μ; i.e. $\sigma \in K_\alpha \Leftrightarrow \mu \le \alpha$

The higher derived functors of inverse limit shall be denoted by $\varprojlim' = \varprojlim^{(1)}, \varprojlim^{(2)}, \varprojlim^{(3)}, \ldots$.

Lemma 1. $\varprojlim^{(n)} C^i(K_\alpha ;G) = 0$ for $n > 0$.

Proof: Case i) $n=1$. The assertion follows since $\{C^i(K_\alpha ;G)\}$ is star-epimorphic. (Cf. Theorem 6, Part I.)

Case ii) The coefficient group G is injective. We will show in this case the inverse system $\{C^i(K_\alpha ;G)\}$ is injective. Let $C = \{C(K_\alpha ;G)\}$, and let inverse systems $A \subset B$ be given with

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[dl, dashed, "?"] \\ & C & \end{tikzcd}$$

then for each α we have

$$\begin{tikzcd} 0 \arrow[r] & A_\alpha \arrow[r] \arrow[d, "f_\alpha"] & B_\alpha \\ & C_\alpha & \end{tikzcd}$$

Since C_α is a direct product of copies of G, i.e. $C_\alpha = \prod_{\sigma \in K_\alpha} C(\sigma ;G)$ where $C(\sigma ;G) \approx G$, it suffices to consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & A_\alpha \arrow[r] \arrow[d, "f_{\alpha,\sigma}"'] & B_\alpha \\ & C(\sigma ;G) & \end{tikzcd}$$

Since G is injective we can find $\overline{f}_{\alpha,\sigma} : B_\alpha \longrightarrow C(\sigma ;G)$ such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & A_\alpha \arrow[r] \arrow[d, "f_{\alpha,\sigma}"'] & B_\alpha \arrow[dl, "\overline{f}_{\alpha,\sigma}"] \\ & C(\sigma ;G) & \end{tikzcd}$$

But we must choose $\overline{f}_{\alpha,\sigma}$ in such a way that for $\alpha < \beta$ $\overline{f}_{\alpha,\sigma}$ and $\overline{f}_{\beta,\sigma}$ commute with the projections p_α^β of B and C. In order to achieve this let K_μ be the minimum subcomplex containing σ, then we have the following commutative diagram by the preceding argument.

$$\begin{tikzcd} 0 \arrow[r] & A_\mu \arrow[r] \arrow[d, "f_\mu"'] & B_\mu \arrow[dl, "\overline{f}_\mu"] \\ & C_\mu & \end{tikzcd}$$

We combine the last two diagrams as follows

$$\begin{tikzcd} A_\alpha \arrow[rr] \arrow[d] \arrow[dd, bend left, "f_{\alpha,\sigma}"] & & B_\alpha \arrow[d, "p_\mu^\alpha"] \\ A_\mu \arrow[rr] \arrow[ddd, bend right, "f_\mu"'] & & B_\mu \arrow[ddll, bend left, "\overline{f}_\mu"] \\ C(\sigma, G) \arrow[d, "\pi"] & & \\ C_\mu & & \end{tikzcd}$$

where r is the obvious restriction homomorphism. We let

$$\bar{f}_{\alpha,\sigma} = r\bar{f}_{\mu}\, p^{\alpha}_{\mu}.$$

Thus C is injective, hence $\varprojlim^{(n)} C = 0$ for $n > 0$. The assertion that higher derived functors vanish on injective objects can be found in Chapter 3 of Cartan-Eilenberg.

Case iii) $n > 1$ and $G \subset Q$ injective, then Q/G is also injective. This is so because for abelian groups injectivity is equivalent to infinite divisibility and the latter is clearly preserved by quotient. The following sequence

$$\cdots \varprojlim^{(n-1)} \underset{\substack{\| \\ 0}}{C^i(K_\alpha;Q/G)} \longrightarrow \varprojlim^{(n)} C^i(K_\alpha;G) \longrightarrow \varprojlim^{(n)} \underset{\substack{\| \\ 0}}{C^i(K_\alpha;Q)} \longrightarrow \cdots$$

completes the proof.

Lemma 2. Let $\{K_\alpha\}$ satisfy (*). If $H^i(K_\alpha) = 0$ for $i \neq 0$ and $C^i(K) = 0$ for $i < 0$ then

$$H^n(K) = \varprojlim^{(n)} H^0$$

[any coefficient group].

Proof: From the sequence $0 \to Z^i \to C^i \to B^{i+1} \to 0$ it follows that

1) $\tilde{B}^{i+1}(K) / B^{i+1}(K) = \varprojlim{}' Z^i$ (Cf. Proof of Theorem 1, Part II.)

2) $\varprojlim^{(n)} B^{i+1} = \varprojlim^{(n+1)} Z^i$ for $n > 1$

From the sequence $0 \to B^i \to Z^i \to H^i \to 0$ it follows that $B^i = Z^i$ for $i \neq 0$ hence $\tilde{B}^i(K) = Z^i(K)$ and

3) $H^i(K) = \widehat{B}^i(K) / B^i(K)$ for $i \neq 0$

Furthermore

4) $H^o = Z^o$

Now $\varprojlim^{(n)} H^o = \varprojlim^{(n)} Z^o = \varprojlim^{(n-1)} B^1 = \varprojlim^{(n-1)} Z' = \ldots$

$= \varprojlim' Z^{n-1} = \widehat{B}^n(K) / B^n(K) = H^n(K)$ for $n > 0$.

For $n = 0$, clearly $\varprojlim H^o = \varprojlim Z^o = Z^o(K) = H^o(K)$. Q.E.D.

Corollary. If furthermore $H_i = 0$ for $i \neq 0$ then $\lim^{(n)} H^o = 0$ for $n > 1$.

Proof: Then $H_i(K) = \varinjlim H_i = 0$ for $i \neq 0$; hence by the universal coefficient theorem, $H^i(K) = 0$ for $i \neq 0, 1$.

Now let $\{A_\alpha\}$ be an inverse system of finitely generated free abelian groups.

Theorem. $\varprojlim^{(n)} A = 0$ for $n > 1$.

Proof: Construct a complex K as follows. Let $A^\alpha = \mathrm{Hom}(A_\alpha; Z)$ so that $\{A^\alpha\}$ is a direct system. Define $A^{\alpha_0 \cdots \alpha_n}$ as a copy of A^{α_0} and choose a fixed basis for each such. Let

$$\partial_i : A^{\alpha_0 \cdots \alpha_n} \longrightarrow A^{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_n}$$

be the identity for $i > 0$; and the appropriate homomorphism from the direct system for $i = 0$, $\alpha_0 < \alpha_1$. Define

$$C_n(K_\alpha) = \bigoplus_{\alpha_0 < \cdots < \alpha_n < \alpha} A^{\alpha_0 \cdots \alpha_n}$$

with the induced preferred basis, and define $\partial : C_n \longrightarrow C_{n-1}$ by $\partial = \partial_0 - \partial_1 + \ldots \pm \partial_n$. Then $\partial^2 = 0$ and $\{K_\alpha\}$ satisfies (*).

Assertion: $H_i(K_\alpha) = 0$ for $i \neq 0$ and $H_0(K_\alpha)$ is naturally isomorphic to A^α. This follows by considering K_α as a "mapping cylinder." Now $\{A_\alpha\}$ is isomorphic to $\{H^0(K_\alpha)\}$, and the corollary above implies that $\varprojlim^{(n)} A_\alpha = 0$ for $n > 1$.

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