In 1991 I distributed a preliminary version of some notes on Witten's recently discovered 3-manifold invariants. For various reasons the paper was never completed and published. Nevertheless, many people have told me that they still find the 1991 notes to be useful. For this reason, I have prepared this version of the notes which is distributable in electronic form.

I have not attempted to correct, complete or improve the 1991 version. In fact I have taken pains to make sure that all the page breaks occur in the same places — this version is essentially identical to the original one. The numerous hand-drawn figures were scanned in. Unfortunately, I could not find the original figures, so I had to scan from photocopies.

The postal and email addresses on page 1 are no longer current. I can be reached at `kevin@canyon23.net`.

Kevin Walker
March 29, 2001 (small changes August 3, 2003)

On Witten's 3-manifold Invariants

Kevin Walker *†

February 28, 1991

[PRELIMINARY VERSION #2]

[*Preintroduction.* Some sections are missing, and others are in rough draft form. The definition of a TQFT given in this version differs slightly from the one given in the previous version. (There are no more gluing coefficients; see (2.3), (2.9) and (2.13).)]

[*Introduction for experts.* The interesting (I hope) parts of this paper are: the definition of "extended" 2- and 3-manifolds, which allows one to resolve the projective ambiguity; gluing 3-manifolds "with corners"; a precise and rigorous version of Moore and Seibergs polynomial equations result; a proof that Witten-Jones TQFTs exist and satisfy all of the axioms which they ought to; a clarification of the relationship between Witten-Jones TQFTs and the invariants of Reshetikhin-Turaev; a proof that the Turaev-Viro invariant equal to the square of the norm of a W-J TQFT, for closed 3-manifolds.]

Contents

*Supported in part by an NSF fellowship.

†Current address: P.O.Box 945 / Moab, UT 84532; email: kwalker@math.ucsd.edu. (This email address might not be reliable.)

0 Introduction

In [Wi1] Witten introduced new 3-manifold invariants which "explain" the Jones polynomial and its generalizations. The main goal of this paper is to present a mathematically rigorous approach to these invariants which is palatable to low-dimensional topologists. (Consequently, very little emphasis is given to their relationship to physics.) Secondary goals are to show the relationships of other approaches to this problem ([RT2], [KM], [L], [MSt], [TV], ...) to the one given here, and to assemble various well-known results in one place. The long range goal is to lay the groundwork for the application of these invariants to problems in 3-manifold topology.

At first glance, this paper may seem unpleasantly long. At second glance, one notices that many of the sections are independent of one another (see Figure 1), so things aren't quite as bad as they seem. Readers may also find that there are unpleasantly many details at places, and that there is too much formalism. I hope that this is excused by the fact that there is no shortage of papers in circulation with the opposite faults.

Before explaining the contents of this paper in more detail, it will be necessary to give a very brief and selective history of the subject.

Given a compact Lie group G and an integer k Witten constructs, for each compact, oriented 3-manifold M, an invariant $Z(M)$ (the "partition function") which lies in a finite dimensional vector space $V(\partial M)$ functorially associated to the boundary of M. Z and V together comprise a "topological quantum field theory"* (TQFT). (If $G = SU(2)$ and M is the complement of a link in S^3, one can recover the Jones polynomial of the link from $Z(M)$ for various k. Similar things are true for generalizations of the Jones polynomial.) Witten's construction involves the use of the chimerical Feynmann path integral, and so cannot be made rigorous using current mathematical technology. Witten argues that Z and V have certain nice properties, the most important of which concern their behavior when manifolds are glued together. Most of the things which Witten proves about his invariants follow from these properties via clever but elementary arguments. Thus it seems natural to regard them as axioms, and to try to find alternative methods for proving the existence of Z's and V's which satisfy the axioms. (The axiomatic point of view has its origins in Segal's paper [S1], and was elaborated on by Atiyah [A2].)

It turns out that the functor V is an example of a "modular functor" (a concept due to Segal and closely related to a "rational conformal field theory"), an object of independent interest. (Witten considered this coincidence (?) to be the most important observation of [Wi1], since, as modular functors were already reasonably well understood, it made concrete calculations possible.) By virtue of the axioms which they must satisfy, modular functors are determined by their values on certain simple surfaces. Moore and Seiberg [MS] worked out what relations this basic data must satisfy in order to consistently determine a modular functor. This result can be viewed as both a tool for classifying modular functors and a method for establishing their existence. In other words, if one can postulate basic data and show that it satisfies the relations, then one has proved the existence of a modular functor.

*More precisely, a "$2+1$-dimensional modular topological quantum field theory", but this seems too cumbersome.

A main idea of this paper is to adapt Moore and Seiberg's approach to the 3-dimensional invariant Z. This necessitates a couple of technical innovations. The first concerns the phase of the partition function Z. If one interprets Witten's definition of $Z(M)$ in a straight forward manner, it turns out to be well-defined only up to a unit complex number. Similarly, according to the usual definitions modular functors are only *projective* functors. To correct this defect Witten proposes (see [A1]) that instead of the category (loosely defined) of ordinary 3- and 2-manifolds, one should consider the extended category of 3-manifolds equiped with (appropriately defined equivalence classes of) framings of their tangent bundles and 2-manifolds equiped with framings of their stabilized tangent bundles. (This move should remind one of the fact that a projective representation of a group corresponds to an honest representation of a central extension of that group.) Unfortunately, framings are difficult to work with when making concrete calculations (for me, at least). So here framed 3- and 2-manifolds will be replaced by 3- and 2-manifolds equiped with (an abstract version of) a bordism class of null-bordisms. This leads to an extended category which is easier to work with. (The idea that a (closed) 3-manifold with a null-bordism can serve the same function as a framed 3-manifold seems to have occured to many people. I first heard of it from Andrew Casson.) The second innovation is to expand the axioms of the partition function to allow for gluing 3-manifolds "with corners" (e.g. attaching a handle). This added flexibility is crucial for applying Moore and Seiberg's approach to the partition function.

This paper is organized as follows. (More detailed introductory remarks can be found at the beginnings of the sections. (Some of them, anyway.)) Section 1 defines extended 2- and 3-manifolds. Section 2 defines modular functors and partition functions. Section 3 introduces basic data. In Section 4 we present some elementary consequences of the axioms of a TQFT. In Section 5 we show how to reconstruct a modular functor from its basic data. In Section 6 we derive a set of relations for basic data which guarantee that the reconstruction procedure gives consistent answers. (This is a version of Moore and Seiberg's result. The basic idea of the proof given here is due to them.) In Section 7 we show that a modular functor admits a compatible partition function if and only if it satisfies a certain relation. This reduces the problem of proving the existence of a partition function to proving the existence of a modular functor and verifying the relation.

The goal of Sections 8 through 11 is to clarify the relationship between the invariants of Reshetikhin and Turaev [RT2] and the invariants discussed here. In Section 8 we present a modified version of the some of the results of [RT2]. (Some of the new results in this section are joint work with Turaev.) In Section 9, we use these results to show that a modular Hopf algebra (as defined in [RT2]) leads to basic data which satisfy the relations of Sections 6 and 7, and thus to a TQFT. This raises the question of to what extent this process can be reversed. Section 10 contains the definition of a "modular reduced tangle functor", which, roughly, is the part of the tangle functors of Section 8 which can be recovered† from the corresponding TQFT. Section 11 shows how to effect this recovery.

In Section 12 we define the "Verlinde algebra" associated to a TQFT and derive some of its properties. Section 13 specifies the basic data of $\mathbf{sl}_2$-theories. In Section 14 we show how other surgery-based approaches ([KM], [L], [MSt]) fit into the framework developed here.

In Section 15 we derive, using the axioms of a TQFT, a state model for $|Z(M)|^2$, where Z is unitary. This state model is based on a triangulation of M. In the case where Z is one of the partition functions corresponding to $SU(2)$ and M is closed, this state model is the same as the one of Turaev and Viro [TV].

The definition of a modular functor given in Section 2 differs from Segal's definition in that holomorphic surfaces are replaced by (extended) piecewise-linear surfaces. In Section 16 we show that any modular functor, as defined by Segal, gives rise to a modular functor as defined here, and vice-versa. Section 17 contains remarks on the 2-cocycles and central extensions arising from extended surfaces.

Section 18 contains background material on the nonadditivity of signature for 4-manifolds. Section 19 proves two topological results needed for Sections 6 and 7.

The interdependence of the sections is indicated in Figure 1.

It should be emphasized that most of the important ideas contained herein are not new. As explained above, they are due to Witten (mostly), Moore and Seiberg, Segal, Atiyah, and Reshetikhin and Turaev.

There have been many approaches to putting Witten's invariants on a rigorous footing. In addition to the ones mentioned above, I am aware of [Ko], [C], [CLM], [Koh] and [We]. (Any omissions from this list are, of course, unintentional.) Kontsevich's paper, in particular, has some overlap with Sections 1 and 7. (I was unaware of [Ko] until recently.)

Acknowledgements: I would like to thank Andrew Casson, Dan Freed and Rob Kirby for helpful conversations.

Conventions: All homology groups will have coefficients in $\mathbf{R}$ unless stated otherwise. Closely related maps which have distinct domains will often share the same notation. Maps will sometimes (though not usually) be confused with their isotopy classes.

†Perhaps more can be recovered, but this is all that can be recovered easily.

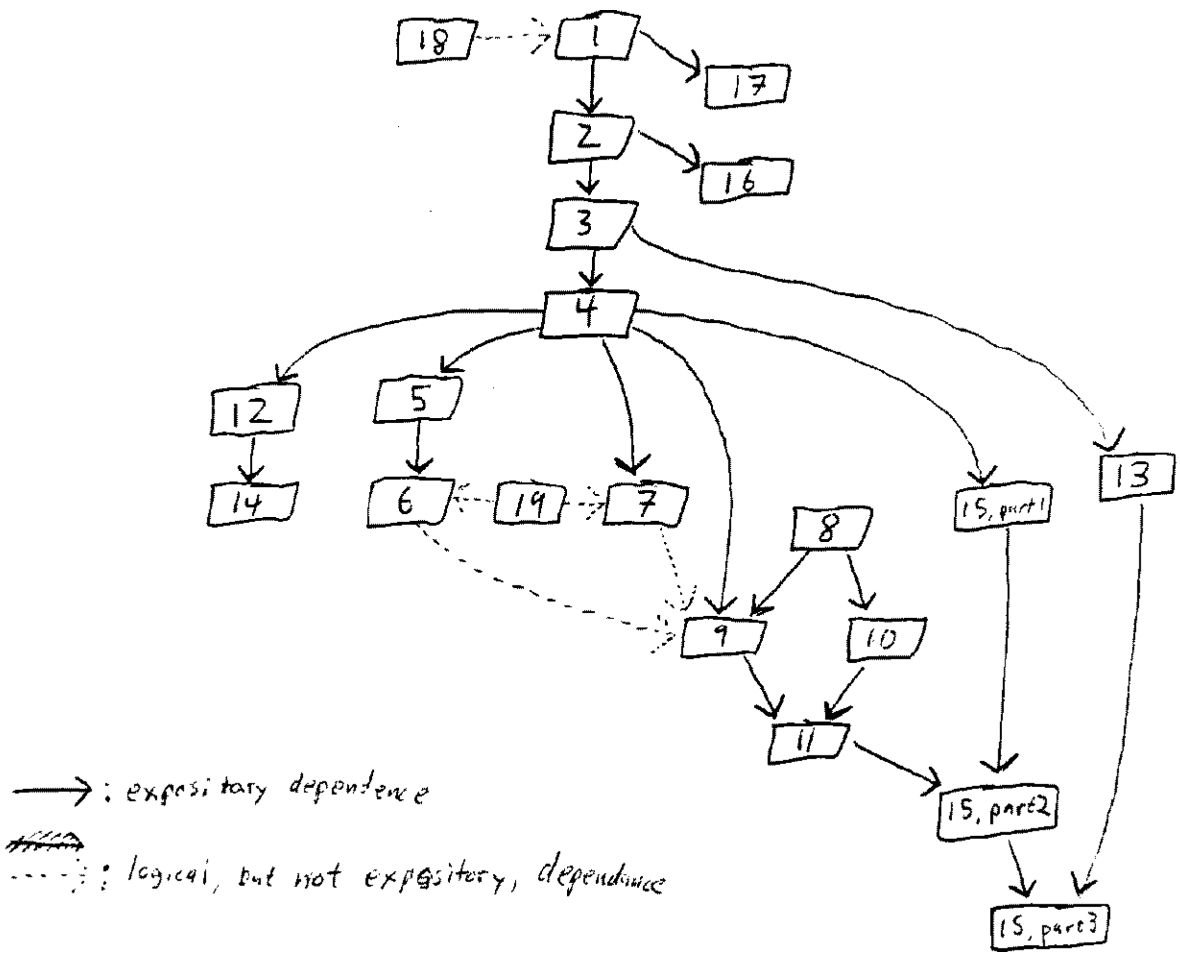

Figure 1: Boxes, numbers, arrows.

1 Extended 2- and 3-manifolds

This section contains the definitions of extended 2- and 3-manifolds, their morphisms, and their gluing operations. We begin with some lengthy motivational remarks, saving the precise definitions for the end.

In order to resolve phase ambiguities in the partition function, Witten considers framed 3-manifolds rather than garden variety 3-manifolds (see [Wi1, A1]). In other words, the partition function in an invariant of a 3-manifold M together with a section of its frame bundle $F(TM)$. Two framings are considered equivalent if they are isotopic (rel boundary, if $\partial M \neq \emptyset$) after being included diagonally into $F(TM \oplus TM)$. This equivalence relation sees $\pi_3(SO(3))$ but not $\pi_1(SO(3))$, since the diagonal inclusion $SO(3) \hookrightarrow SO(3) \oplus SO(3) \subset SO(6)$ is zero on π_1.

Another way of accomplishing the same thing is to redefine a framing of M to be a real singular cycle in $C_3(F(TM), \partial F(TM); \mathbf{R})$ which projects to a representative of the fundamental class in $H_3(M, \partial M; \mathbf{R})$. (A section of $F(TM)$ clearly gives rise to such a cycle.) Framings a and b are considered equivalent if $a - b$ is a null-homologous cycle in $C_3(F(TM); \mathbf{R})$. Note that in order for $a - b$ to be a cycle we must have $\partial a = \partial b$. It is not hard to see that the set of all equivalence classes of framings with a fixed boundary is affinely equivalent to $H_3(SO(3); \mathbf{R}) \cong \mathbf{R}$. Define a framed 3-manifold to be a smooth, compact, oriented 3-manifold equiped with an equivalence class of framings.

A framed 2-manifold is defined to be the sort of thing which one gets if one takes the boundary of a framed 3-manifold. In other words, a framed 2-manifold is a smooth, closed, oriented 2-manifold Y together with a cycle in $C_2(F(TY \oplus \epsilon); \mathbf{R})$ which projects to a representative of the fundamental class in $H_2(Y; \mathbf{R})$. (Here ϵ denotes the trivial $\mathbf{R}$ bundle over Y, which can be identified with the normal bundle of Y in the tangent bundle of a 3-manifold which Y bounds.) A morphism of framed 2-manifolds is defined to be the sort of thing which can be used to glue framed 3-manifolds together along their boundaries. More precisely, a morphism from (Y_1, a_1) to (Y_2, a_2) is a pair (f, b), where $f : Y_1 \to Y_2$ is a diffeomorphism and b is a chain in $C_3(F(TY_2 \oplus \epsilon); \mathbf{R})$ such that $\partial b = a_2 - f_*(a_1)$. (f_0, b_0) is considered equivalent to (f_1, b_1) if there is an isotopy $\{f_t\}$ connecting f_0 to f_1 and if the cycle $b_1 - b_0 + \{(f_t)_*(a_1)\}$ is null-homologous. The group of (equivalence classes of) morphisms of a framed surface is a central extension by $\mathbf{R}$ of the mapping class group of the underlying unframed surface.

This is all well and good, but not quite well enough or good enough. Less obscurely, it would be nice (for purposes of computation) to replace framed 2- and 3-manifolds with simpler objects. Furthermore, it would nice to be able to glue framed 3-manifolds together "with corners", that is, glue them together along codimension zero submanifolds of their boundaries which themselves have boundary. (An example of this operation would be a framed version of attaching a 1- or 2-handle to a 3-manifold.) But with the above definitions such gluings are awkward, since the tangent bundle is badly behaved at the corners.

Our goal, then, is to transmogrify framed 2- and 3-manifolds into something more to our liking. This will be done in two stages, the intermediate stage motivating the final one.

Consider first the case of a closed, oriented 3-manifold M. Let W be an oriented 4-manifold with $\partial W = M$. Given a framing a of M (as defined above), one can compute the relative first Pontryagin class* $p_1(W, a) \in \mathbf{R}$. This is an affine function of a (with slope ??? (see [A1])), so there is a unique framing a_W of M such that $p_1(W, a_W) = 0$. Thus the pair (M, W) determines the framed 3-manifold (M, a_W), and the intermediate transmogrification of a framed 3-manifold is such a pair.

If $\partial W' = M$ also, then W and W' determine the same framing of M if and only if $\sigma(W) = \sigma(W')$, where $\sigma(W)$ denotes the signature of the cup product on $H^2(W)$. (This follows from the fact that $3\sigma(X) = p_1(X)$ if X is a closed 4-manifold and the additivity properties of σ and p_1.) So the only essential information W contributes is its signature. Note also that any value for the signature is possible. Hence we define a closed extended 3-manifold to be a pair (M, n), where M is a closed, oriented 3-manifold and n is an integer which can be thought of as the signature of a 4-manifold bounded by M.

What sorts of objects should extended 3-manifolds with boundary be? They certainly ought to have the property that they can be glued together (via "extended diffeomorphisms" of their "extended boundaries") to get a closed extended 3-manifold.

Let M_1 and M_2 be compact, oriented 3-manifolds. Let $f : \partial M_1 \to -\partial M_2$ be a diffeomorphism of their boundaries. ("$-$" indicates a reversal of orientation.) Let M be the closed 3-manifold $M_1 \cup_f M_2$. Since M_i ($i = 1, 2$) is not closed, it cannot bound a 4-manifold. This can remedied by capping off ∂M_i with another 3-manifold $M_i^\dagger$. In other words, $\partial M_i^\dagger = -\partial M_i$ and $M_i \cup M_i^\dagger$ is a closed 3-manifold. Let W_i be a 4-manifold bounded by $M_i \cup M_i^\dagger$. We wish to combine the triples $(M_1, M_1^\dagger, W_1)$ and $(M_2, M_2^\dagger, W_2)$ and get a 4-manifold bounded by M. Let I_f be the mapping cylinder of f (i.e. $I_f = \partial M_1 \times I$ with $\partial M_1 \times \{1\}$ identified with ∂M_2 via f and $\partial M_1 \times \{0\}$ identified with ∂M_1 via id). Let W be a 4-manifold with $\partial W = (-\partial M_1^\dagger) \cup I_f \cup (-\partial M_2^\dagger)$. Then $W_1 \cup W \cup W_2$ is a 4-manifold with

$$\partial(W_1 \cup W \cup W_2) = M_1 \cup I_f \cup M_2 \cong M_1 \cup M_2 = M$$

*$p_1(W, a)$ is defined to be the integral over W of the first Pontryagin form of a connection whose restriction to a collar of ∂W is a product connection such that the cohomology class of the Chern-Simons form of the ∂W factor of the product is Poincaré dual to a.

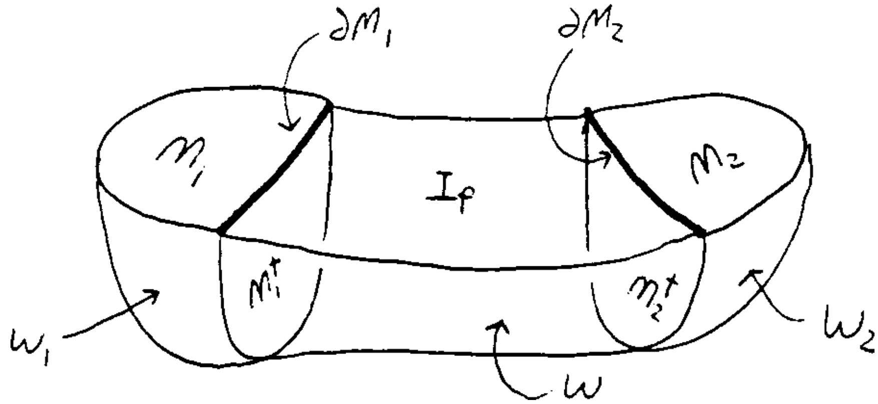

Figure 2: Some manifolds glued together.

(see Figure 2).

The above discussion suggests the following provisional definitions: that an extended 3-manifold should be a triple $(M, M^\dagger, W)$, where M is a compact 3-manifold, $\partial M^\dagger = -\partial M$ and $\partial W = M \cup M^\dagger$; that an extended surface should be a pair $(Y, M^\dagger)$, where $\partial M^\dagger = -Y$; that $\partial(M, M^\dagger, W) = (\partial M, M^\dagger)$; and that an extended diffeomorphism from $(Y_1, M_1^\dagger)$ to $(Y_2, M_2^\dagger)$ is a pair (f, W), where $f : Y_1 \to Y_2$ is a diffeomorphism and $\partial W = (-M_1^\dagger) \cup I_f \cup (-M_2^\dagger)$.

The above definitions can clearly be pared down. All one ultimately cares about is the signature of $W_1 \cup W \cup W_2$. To compute this one only needs to know $\sigma(W_1)$, $\sigma(W)$, $\sigma(W_2)$, $K_i \stackrel{\text{def}}{=} \ker(H_1(\partial M_i) \to H_1(M_i^\dagger))$ and $L_i \stackrel{\text{def}}{=} \ker(H_1(\partial M_i) \to H_1(M_i^\dagger))$ (see Section 18 or [Wa]). This suggests that the final definitions of extended objects should be as follows. An extended 3-manifold is a triple (M, L, n), where M is a compact 3-manifold, L is a lagrangian subspace of $H_1(\partial M)$ (with respect to the intersection pairing on $H_1(\partial M)$), and n is an integer. (Needless to say, one should think of L as being $\ker(H_1(\partial M) \to H_1(M^\dagger))$ and n as being $\sigma(W)$, where $\partial W = M \cup M^\dagger$.) An extended surface is a pair (Y, L), where Y is a surface and L is a lagrangian subspace of $H_1(Y)$. $\partial(M, L, n)$ is defined to be $(\partial M, L)$. A morphism from (Y_1, L_1) to (Y_2, L_2) is a pair (f, n), where $f : Y_1 \to Y_2$ is a diffeomorphism and n is an integer (thought of as being the signature of a 4-manifold with appropriate boundary).

Enough motivation. Now for the official definitions (the first few of which will simply be restatements of the above).

(1.1) *Definition.* An *extended 3-manifold* is a triple (M, L, n), where M is a piecewise-linear, compact, oriented 3-manifold, L is a maximal isotropic (i.e. lagrangian) subspace of $H_1(\partial M)$, and n is an integer.

(1.2) *Definition.* An *extended surface* is a pair (Y, L), where Y is a piecewise-linear, compact, oriented surface with parameterized boundary, and L is a maximal isotropic subspace of $H_1(Y)$.

To say that Y has parameterized boundary means that Y is equiped with orientation preserving piecewise-linear homeomorphisms from the standard S^1 to each boundary component of Y. An isotopy of a surface with parameterized boundary is, by definition, fixed on the boundary. (One could alternatively replace the parameterizations with base points.)

Note that if $L \subset H_1(Y)$ is maximal isotropic, then $\operatorname{im}(H_1(\partial Y) \to H_1(Y)) \subset L$. Note also that L maps to a lagrangian subspace of $H_1(\hat{Y})$, where $\hat{Y}$ denotes Y with its boundary capped of by disks.

(1.3) *Definition.* $\partial(M, L, n) \stackrel{\text{def}}{=} (\partial M, L)$.

(1.4) *Definition.* An *extended morphism* from an extended surface (Y_1, L_1) to an extended surface (Y_2, L_2) is a pair (f, n), where $f : Y_1 \to Y_2$ is an isotopy class of orientation preserving piecewise-linear homeomorphisms which also preserve boundary parameterizations, and n is an integer.

Let $\hat{Y}_i$ denote Y_i with its boundary capped off by disks, and let $\hat{f} : \hat{Y}_1 \to \hat{Y}_2$ be the induced map. Let $M_i^\dagger$ be a 3-manifold such that $\partial M_i^\dagger = -\hat{Y}_i$ and $L_i = \ker(H_1(Y_i) \to H_1(M_i^\dagger))$. Then (f, n) should be thought of as corresponding to a 4-manifold W such that $\partial W = (-M_1^\dagger) \cup I_{\hat{f}} \cup (M_2^\dagger)$ and $\sigma(W) = n$. In fact, we make the following definition.

(1.5) *Definition.* The *mapping cylinder* of an extended morphism $(f, n) : (Y_1, L_1) \to (Y_2, L_2)$ is the extended 3-manifold

$$I_{(f,n)} \stackrel{\text{def}}{=} (I_f, L, n),$$

where L is induced from the inclusions of L_1 and L_2 into $H_1(\partial I_f)$.

Note that in the above definition the boundary of Y_i was not capped off.

(1.6) *Definition. Composition of extended morphisms.* Let $(f_1, n_1) : (Y_1, L_1) \to (Y_2, L_2)$

and $(f_2, n_2) : (Y_2, L_2) \to (Y_3, L_3)$ be extended morphisms. The composition of (f_2, n_2) and (f_1, n_1) is defined to be

$$(f_2, n_2)(f_1, n_1) \stackrel{\text{def}}{=} (f_2 f_1,\ n_2 + n_1 + \sigma((f_2 f_1)_* L_1, (f_2)_* L_2, L_3)),$$

where $\sigma(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is Wall's non-additivity function (see [Wa] or Section 18).

Let W_i be a 4-manifold corresponding to (f_i, n_i), as described above. Then $W_2 \cup W_1$ is a 4-manifold corresponding to $(f_2, n_2)(f_1, n_1)$. (It clearly has the right boundary, and its signature is easily (in view of [Wa] or ???) seen to be $n_2+n_1+\sigma((f_2 f_1)_* L_1, (f_2)_* L_2, L_3)$.) This viewpoint makes it clear that composition of extended morphisms is associative. One can easily verify that morphisms of the form $(\mathrm{id}, 0)$ are left- and right-sided identities, and that $(f, n)^{-1} = (f^{-1}, -n)$. (Hint: use the fact that $\sigma(A, A, B) = \sigma(A, B, B) = \sigma(A, B, A) = 0$ for all A, B.) Thus extended morphisms form a groupoid.

For (Y, L) an extended surface, define the mapping class group of (Y, L), $\mathcal{M}(Y, L)$, to be the group of extended automorphisms of (Y, L). If $\mathcal{Y}$ is a collection of extended surfaces, $\mathcal{M}(\mathcal{Y})$ will denote the corresponding mapping class groupoid. The set of extended morphisms from (Y_1, L_1) to (Y_2, L_2) will be denoted by $\mathcal{M}((Y_1, L_1), (Y_2, L_2))$. Note that $\mathcal{M}(\emptyset) \cong \mathbf{Z}$.

The mapping class group of an extended surface (Y, L) is a central extension by $\mathbf{Z}$ of the mapping class group of Y. The 2-cocycle which defines this extension is

$$c(f, g) \stackrel{\text{def}}{=} \sigma((fg)_* L, f_* L, L). \tag{1.7}$$

This is just the well-known Shale-Weil cocycle. In Section 16 we will show that this cocycle arises naturally from considering the action of the mapping class group of Y on its determinant line bundle. In Section 17 we discuss this cocycle and central extension further.

(1.8) *Definition. Gluing extended surfaces* . Let (Y, L) be an extended surface and let g be a fixed point free involution of a closed submanifold of ∂Y which fails to preserve parameterizations by the standard reflection of S^1. Define the gluing of (Y, L) by g to be

$$(Y, L)_g \stackrel{\text{def}}{=} (Y_g, L_g),$$

where Y_g is Y with part of its boundary identified by g, $L_g = q_*(L)$, and $q : Y \to Y_g$ is the quotient map. For (f, n) an automorphism of (Y, L) such that f commutes with g, define $(f, n)_g : (Y, L)_g \to (Y, L)_g$ by

$$(f, n)_g \stackrel{\text{def}}{=} (qfq^{-1}, n).$$

(1.9) *Definition.* Let (Y_1, L_1) and (Y_2, L_2) be extended surfaces. The *disjoint union* of (Y_1, L_1) and (Y_2, L_2) is

$$(Y_1, L_1)\coprod(Y_2, L_2) \stackrel{\text{def}}{=} (Y_1\coprod Y_2, L_1 \oplus L_2).$$

(Here we are identifying $H_1(Y_1\coprod Y_2)$ with $H_1(Y_1) \oplus H_1(Y_2)$.) If (f_i, n_i) $(i = 1, 2)$ are extended morphisms, then

$$(f_1, n_1)\coprod(f_2, n_2) \stackrel{\text{def}}{=} (f_1\coprod f_2, n_1 + n_2).$$

(1.10) *Definition.* (Y_1, L_1) *contains* (Y_2, L_2) (or $(Y_2, L_2) \subset (Y_1, L_1)$) if $Y_2 \subset Y_1$, $i_*(L_2) \subset L_1$ (where $i : Y_2 \hookrightarrow Y_1$), each component of $\partial Y_1 \cap \partial Y_2$ is a component of ∂Y_1 and ∂Y_2, and the parameterizations agree on each such component.

(1.11) *Definition.* Let (M_1, L_1, n_1) and (M_2, L_2, n_2) be extended 3-manifolds . The *disjoint union* of (M_1, L_1, n_1) and (M_2, L_2, n_2) is

$$(M_1, L_1, n_1)\coprod(M_2, L_2, n_2) \stackrel{\text{def}}{=} (M_1\coprod M_2, L_1 \oplus L_2, n_1 + n_2).$$

(1.12) *Definition. Gluing extended 3-manifolds.* Let (M, L, n) be an extended 3-manifold (possibly disconnected). Let $(Y_i, L_i) \subset \partial(M, L, n)$, $i = 1, 2$. Assume $Y_1 \cap Y_2 = \emptyset$. Let $(f, m) : (Y_1, L_1) \to (-Y_2, L_2)$ be an extended morphism. Let $(Z, J) = \partial(M, L, n) \setminus ((Y_1, L_1) \cup (Y_2, L_2))$. (In other words, $\partial(M, L, n) = (Z, J) \cup (Y_1, L_1) \cup (Y_2, L_2)$, where "$\cup$" should be interpreted in the sense of gluing extended surfaces, defined above.) Define the gluing of (M, L, n) by (f, m) to be

$$(M, L, n)_{(f,m)} \stackrel{\text{def}}{=} (M_f, J', n + m + \sigma(K, L_1 \oplus L_2, \Delta^-)),$$

where M_f denotes the (piecewise-linear) gluing of M by f, J' is the image of J under the quotient map $Z \to \partial M_f$, Δ^- is the antidiagonal

$$\{(x, -f_*(x)) \,|\, x \in H_1(Y_1)\} \subset H_1(Y_1) \oplus H_1(Y_2),$$

$K = i_*^{-1}(j_*(J))$, and $i : Y_1 \cup Y_2 \to \partial M$ and $j : Z \to \partial M$ are the inclusions.

Contemplating the following picture may reduce the opacity of the above definition. Cap off ∂M with a 3-manifold $M^\dagger$ such that $L = \ker(H_1(\partial M) \to H_1(M^\dagger))$ and $\partial Y_1 \cup \partial Y_2$ bounds a collection of disjoint disks in $M^\dagger$. Let $\hat{Y_1}$ and $\hat{Y_2}$ denote the closed surfaces obtained by adding these disks to Y_1 and Y_2. Let $I_{\hat{f}}$ denote the mapping cylinder of the induced map $\hat{f} : \hat{Y}_1 \to \hat{Y}_2$. Assume that $\hat{Y}_1$ and $\hat{Y}_2$ bound disjoint submanifolds N_1 and N_2 of $M^\dagger$. Note that $L_i = \ker(H_1(Y_i) \to H_1(N_i))$. Let W_a be a 4-manifold of signature n bounded by $M \cup M^\dagger$. Let W_b be a 4-manifold of signature m bounded by $(-N_1) \cup I_{\hat{f}} \cup (-N_2)$. Let $W \stackrel{\text{def}}{=} W_a \cup_{N_1 \cup N_2} W_b$. ∂W can be decomposed into two pieces, one of which is $M \cup I_f \cong M_f$. Denote the other piece by $M_f^\dagger$. It is easy to see that

$$J' = \ker(H_1(\partial M_f) \to H_1(M_f^\dagger)).$$

We also have

$$\sigma(W) = n + m + \sigma(A, B, C),$$

where $A, B, C \subset H_1(\hat{Y}_1 \cup \hat{Y}_2) = H_1(\hat{Y}_1) \oplus H_1(\hat{Y}_2)$ are the kernals of the inclusion maps from $\hat{Y}_1 \cup \hat{Y}_2$ into the three 3-manifolds which it bounds in this picture. It is not hard to see that A, B and C, when pulled back to $H_1(Y_1) \oplus H_1(Y_2)$, are K, $L_1 \oplus L_2$ and Δ^-. Shifting things from $H_1(\hat{Y}_1 \cup \hat{Y}_2)$ to $H_1(Y_1) \oplus H_1(Y_2)$ does not affect the computation of $\sigma(A, B, C)$. Hence

$$\sigma(W) = n + m + \sigma(K, L_1 \oplus L_2, \Delta^-).$$

If $(M, L, n) = (M_1, J_1, n_1) \coprod (M_2, J_2, n_2)$ and $(Y_1, L_1) \subset (M_1, J_1, n_1)$, $(Y_2, L_2) \subset (M_2, J_2, n_2)$, then we will denote $(M, L, n)_{(f,m)}$ by $(M_1, J_1, n_1) \cup_{(f,m)} (M_2, J_2, n_2)$. If the map f is canonical or obvious, we denote $(M, L, n)_{(f,0)}$ by $(M_1, J_1, n_1) \cup_{(Y_1, L_1)} (M_2, J_2, n_2)$ or simply $(M_1, J_1, n_1) \cup (M_2, J_2, n_2)$. For example, if (g_1, m_1) and (g_2, m_2) are composable morphisms one can check that

$$I_{(g_2,m_2)(g_1,m_1)} \cong I_{(g_2,m_2)} \cup I_{(g_1,m_1)}.$$

Notation: In the rest of this paper, we will usually be concerned with extended objects (rather than unextended objects), so we will use capital roman letters (e.g. M, Y) to denote them. If we need to refer to the constituent pieces of extended objects, we will do so as follows. For M an extended 3-manifold, let $M^\flat$ denote the underlying unextended 3-manifold, let L_M denote the lagrangian subspace of $H_1(\partial M^\flat)$, and let n_M denote the associated integer. That is,

$$M = (M^\flat, L_M, n_M).$$

Similarly, if Y is an extended surface, then

$$Y = (Y^\flat, L_Y).$$

The prefix "e-" will be used as a shorthand for "extended" (e.g. "e-3-manifold", "e-surface"). The integers associated to e-3-manifolds and e-morphisms will be called "framing numbers". If the framing number of an e-3-manifold or e-morphism is not specified, it is assumed to be zero.

2 The definition of a TQFT

In this section we give the axioms of a topological quantum field theory (TQFT) and note some of their consequences. (The objects defined here should perhaps be called *modular* $2+1$-*dimensional TQFTs*, instead of simply TQFTs.)

Define a *label set* to be a finite set $\mathcal{L}$ equiped with an involution $a \leftrightarrow \hat{a}$ and a distinguished "trivial label", denoted by 1, with the property that $1 = \hat{1}$. $\hat{a}$ is called the *dual* of a. (In practice $\mathcal{L}$ will usually be a finite set of representations of something (e.g. a modular Hopf algebra), the involution will correspond to taking the dual representation, and 1 will be the trivial representation.)

Given a label set $\mathcal{L}$, one can define the category of *labeled, extended surfaces* (le-surfaces). The objects are e-surfaces with an element of $\mathcal{L}$ assigned to each boundary component. The morphisms are morphisms of e-surfaces which preserve the labeling. le-surfaces will be denoted (Y, l), where Y is an e-surface and l is a function from the boundary components of Y to $\mathcal{L}$. Such a function is called a *labeling* of ∂Y (or simply a labeling of Y). We will sometimes denote (Y, l) by Y, l being understood. For C any collection of parameterized circles, let $\mathcal{L}(C)$ denote the set of all labelings of C. Note that a closed le-surface is just an e-surface.

Definition. A *topological quantum field theory* based on a label set $\mathcal{L}$ consists of

- A functor V from the category of le-surfaces to the category of finite dimensional complex vector spaces and linear isomorphisms.
- An assignment $M \mapsto Z(M) \in V(\partial M)$ for each e-3-manifold M.

In addition, V and Z are required to satisfy (2.1) through (2.10) below.

(2.1) *Disjoint union axiom for V.* The theory provides an identification

$$V(Y_1 \textstyle\coprod Y_2, l_1 \textstyle\coprod l_2) = V(Y_1, l_1) \otimes V(Y_2, l_2)$$

for each pair of le-surfaces $V(Y_1, l_1)$ and $V(Y_2, l_2)$. These identifications are compatible with the actions of the mapping class groupoids (i.e. $V(f_1 \coprod f_2) = V(f_1) \otimes V(f_2)$ for le-morphisms f_1 and f_2). The identifications are also associative in the obvious sense.

(2.2) *Gluing axiom for V.* Let Y be an e-surface. Let $g : C \to -C'$ be a parameterization reflecting homeomorphism of closed disjoint submanifolds C and C' of ∂Y. Let Y_g denote Y glued by g. For l a labeling of Y_g and x a labeling of C, let $(l, x, \hat{x})$ denote the corresponding labeling of Y (i.e. g takes labels to their duals). Then the theory provides an identification

$$V(Y_g, l) = \bigoplus_{x \in \mathcal{L}(C)} V(Y, (l, x, \hat{x}))$$

which is compatible with the actions of the mapping class groupoids. The identifications are also associative (i.e. it doesn't matter what order one glues things in).

(2.3) *Duality axiom.* The theory provides an identification

$$V(Y, l) = V(-Y, \hat{l})^*$$

for each le-surface (Y, l), where $V(-Y, \hat{l})^*$ denotes the space of complex-linear maps $V(-Y, \hat{l}) \to \mathbf{C}$. These identifications are compatible with orientation reversal, the actions of the mapping class groupoids, (2.1) and (2.2) as follows.

- The identifications

  $$\begin{aligned} V(Y) &= V(-Y)^* \\ V(-Y) &= V(Y)^* \end{aligned}$$

  are mutually adjoint.

- Let $f = (f^\flat, n) : (Y_1, l_1) \to (Y_2, l_2)$ be an le-morphism and let $f^- \stackrel{\text{def}}{=} (f^\flat, -n) : (-Y_1, \hat{l}_1) \to (-Y_2, \hat{l}_2)$. Then

  $$\langle x, y \rangle = \langle V(f)x, V(f^-)y \rangle$$

  for all $x \in V(Y_1, l_1)$, $y \in V(-Y_1, \hat{l}_1)$. In other words, $V(f)$ is the adjoint inverse of $V(f^-)$.

- Let

  $$\begin{aligned} \alpha_1 \otimes \alpha_2 &\in V(Y_1 \textstyle\coprod Y_2) = V(Y_1) \otimes V(Y_2) \\ \beta_1 \otimes \beta_2 &\in V(-Y_1 \textstyle\coprod -Y_2) = V(-Y_1) \otimes V(-Y_2). \end{aligned}$$

  Then

  $$\langle \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2 \rangle = \langle \alpha_1, \beta_1 \rangle \langle \alpha_2, \beta_2 \rangle.$$

- Let

  $$\begin{aligned} \bigoplus_x \alpha_x &\in V(Y_g, l) = \bigoplus_x V(Y, (l, x, \hat{x})) \\ \bigoplus_x \beta_x &\in V(-Y_g, \hat{l}) = \bigoplus_x V(-Y, (\hat{l}, \hat{x}, x)) \end{aligned}$$

(same notation as in (2.2)). Then

$$\langle \bigoplus_x \alpha_x, \bigoplus_x \beta_x \rangle = \sum_x S(x) \langle \alpha_x, \beta_x \rangle.$$

Here $S(x) = S(x_1) \cdots S(x_n)$, where $x = (x_1, \ldots, x_n)$ and $S : \mathcal{L} \to \mathbf{C}^\times$ is a certain function which is determined by V (see (4.4))*. For the time being, the reader may wish to simply regard S as being part of the data of the theory.

(2.4) *Empty surface axiom.* Let $\emptyset$ denote the empty le-surface. Then

$$V(\emptyset) \cong \mathbf{C}.$$

(2.5) *Disk axiom.* Let D denote the (extended) disk. Then

$$V(D, a) \cong \begin{cases} \mathbf{C}, & a = 1 \\ 0, & a \neq 1. \end{cases}$$

(2.6) *Annulus axiom.* Let A denote the (extended) annulus. Then

$$V(A, (a, b)) \cong \begin{cases} \mathbf{C}, & a = \hat{b} \\ 0, & a \neq \hat{b}. \end{cases}$$

(2.7) *Disjoint union axiom for Z.* Let M_1 and M_2 be e-3-manifolds. Then

$$Z(M_1 \coprod M_2) = Z(M_1) \otimes Z(M_2).$$

(This makes sense in view of (2.1).)

(2.8) *Naturality axiom.* Let $M_1 = (M_1^\flat, L_1, n)$ and $M_2 = (M_2^\flat, L_2, n)$ be e-3-manifolds and let $f^\flat : M_1^\flat \to M_2^\flat$ be an orientation preserving homeomorphism such that

$$(f^\flat|_{\partial M_1^\flat})_*(L_1) = L_2.$$

Then

$$V(f^\flat|_{\partial M_1^\flat}, 0) Z(M_1) = Z(M_2).$$

*Note to experts: $S(a) = S_{1a}$.

Some preliminary remarks will be needed before stating the next axiom. Let M be an e-3-manifold and let $Y_1, Y_2 \subset \partial M$ be disjoint e-surfaces. Then, by (2.2), we have

$$V(\partial M) = \bigoplus_{l_1,l_2} V(Y_1, l_1) \otimes V(Y_2, l_2) \otimes V(\partial M \setminus (Y_1 \cup Y_2), (\hat{l}_1, \hat{l}_2)),$$

where l_i runs through all labelings of Y_i. Hence we can write

$$Z(M) = \bigoplus_{l_1,l_2} \sum_j \alpha^j_{l_1} \otimes \beta^j_{l_2} \otimes \gamma^j_{\hat{l}_1\hat{l}_2},$$

where $\alpha^j_{l_1} \in V(Y_1, l_1)$, $\beta^j_{l_2} \in V(Y_2, l_2)$ and $\gamma^j_{\hat{l}_1\hat{l}_2} \in V(\partial M \setminus (Y_1 \cup Y_2), (\hat{l}_1, \hat{l}_2))$. Let $f : Y_1 \to -Y_2$ be an e-morphism. Then, as in (1.12), we can form the glued e-3-manifold M_f. By (2.2) we have

$$V(\partial M_f) = \bigoplus_l V(\partial M \setminus (Y_1 \cup Y_2), (l, \hat{l})).$$

(2.9) Gluing axiom for Z. For all M and f as above,

$$Z(M_f) = \bigoplus_l \sum_j \langle V(f)\alpha^j_l, \beta^j_{\hat{l}} \rangle \gamma^j_{\hat{l}l}.$$

(2.10) *Mapping cylinder axiom.* Let I_{id} be the mapping cylinder of $\mathrm{id} : Y \to Y$ (see (1.5)). By (2.2) and (2.3) we have

$$V(\partial I_{\mathrm{id}}) = \bigoplus_{l \in \mathcal{L}(Y)} V(Y, l) \otimes V(Y, l)^*.$$

Let id_l be the identity in $V(Y, l) \otimes V(Y, l)^*$. Then

$$Z(I_{\mathrm{id}}) = \bigoplus_{l \in \mathcal{L}(Y)} \mathrm{id}_l.$$

Remarks.

$V(f)$ will usually be denoted by f_* (for f an le-morphism).

A functor V satisfying (2.1) through (2.6) is a PL version of what Segal calls a *modular functor* (see [S1]). In section 16 we will show that a modular functor as

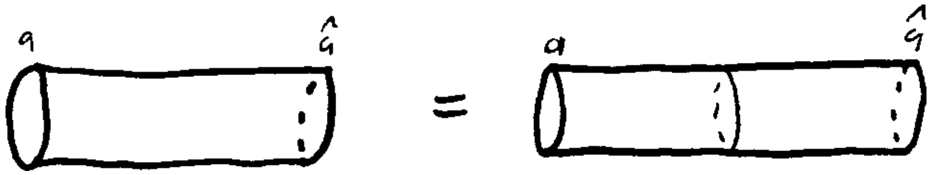

Figure 3: Two annuli are not really much better than one.

defined by Segal leads to a modular functor as defined here. According to physics jargon, Z is called the *partition function*[†].

In view of the disjoint union axiom, (2.4) is clearly a nontriviality axiom. It is also a consequence of the disjoint union axiom that $V(\emptyset)$ has a canonical identification with $\mathbf{C}$.

(2.6) is also a nontriviality axiom. For let A be an annulus and let $A \cup A$ denote the annulus obtained from gluing two copies of A together. Then, by (2.2),

$$V(A, (a, \hat{a})) \cong V(A \cup A, (a, \hat{a})) \cong \bigoplus_{x \in \mathcal{L}} V(A, (a, x)) \otimes V(A, (\hat{x}, \hat{a}))$$

(see Figure 3). This is possible only if $V(A, (a, x)) \cong 0$ for $x \neq \hat{a}$ and $V(A, (a, \hat{a}))$ is 0 or 1 dimensional. In the former case a second application of (2.2) implies that $V(Y, l) = 0$ for any le-surface (Y, l) with a boundary component labeled by a. If we drop such labels from the label set the other axioms continue to hold and we obtain a theory which satisfies (2.6). In making this modification we lose no useful information, so we might as well assume that (2.6) holds.

(2.10) is similarly a nontriviality axiom. Using (2.9) it is easy to verify that

$$Z(I_{\mathrm{id}}) = \bigoplus_{l \in \mathcal{L}(Y)} P_l,$$

where $P_l : V(Y, l) \to V(Y, l)$ is an idempotent. Furthermore, if we replace $V(Y, l)$ with $V'(Y, l) \stackrel{\mathrm{def}}{=} P_l(V(Y, l)) \subset V(Y, l)$ for all (Y, l), then the other axioms continue to hold and (2.10) is satisfied. (In particular, $Z(M) \in V'(\partial M)$ for all M.) This modification preserves all of the useful information in the theory (at least for e-3-manifolds), so we might as well assume (2.10).

(2.9) and (2.10) imply the following generalization of (2.10).

(2.11) *Mapping cylinder axiom (strong form).* Let $f : Y_1 \to Y_2$ be a morphism of e-surfaces. For $l \in \mathcal{L}(Y_1)$ let $f_l : (Y_1, l) \to (Y_2, l)$ denote the corresponding morphism

[†]Actually, the term "partition function" is usually reserved for the case where M is closed.

of le-surfaces. Then

$$Z(I_f) = \bigoplus_{l \in \mathcal{L}(Y_1)} V(f_l).$$

Using (2.9) and (2.8) one can prove the following extension of (2.8).

(2.12) *Naturality axiom (strong form).* Let $M_1 = (M_1^\flat, L_1, n_1)$ and $M_2 = (M_2^\flat, L_2, n_2)$ be e-3-manifolds and let $f^\flat : M_1^\flat \to M_2^\flat$ be an orientation preserving homeomorphism. Let $K = \ker(H_1(\partial M_2^\flat) \to H_1(M_2^\flat))$. Then

$$V(f^\flat|_{\partial M_1^\flat}, n_2 - n_1 - \sigma(K, (f^\flat|_{\partial M_1^\flat})_*(L_1), L_2)) \,:\, Z(M_1) \mapsto Z(M_2).$$

The above relation between $Z(M_1)$ and $Z(M_2)$ will be abbreviated

$$Z(M_1) \equiv Z(M_2).$$

It is worth noting some special cases of (2.9). If $Y_1, Y_2 \subset \partial M$ are closed then

$$V(\partial M) = V(Y_1) \otimes V(Y_2) \otimes V(\partial M \setminus (Y_1 \cup Y_2)),$$

and so we can write

$$Z(M) = \sum_j \alpha^j \otimes \beta^j \otimes \gamma^j.$$

For $f : Y_1 \to -Y_2$ (2.9) implies

$$Z(M_f) = \sum_j \langle \alpha^j, \beta^j \rangle \gamma^j.$$

If $\partial M_1 = (-Y_1) \coprod Y_2$ and $\partial M_2 = (-Y_2) \coprod Y_3$ (that is, M_1 is a bordism from Y_1 to Y_2 and M_2 is a bordism from Y_2 to Y_3), then

$$\begin{aligned} Z(M_1) &\in V(Y_1)^* \otimes V(Y_2) = \mathrm{Hom}(V(Y_1), V(Y_2)) \\ Z(M_2) &\in V(Y_2)^* \otimes V(Y_3) = \mathrm{Hom}(V(Y_2), V(Y_3)), \end{aligned}$$

and (2.9) implies

$$Z(M_1 \cup M_2) = Z(M_2) Z(M_1).$$

In particular, if $Y_1 = Y_3 = \emptyset$, so that $M_1 \cup M_2$ is closed, then

$$Z(M_1 \cup M_2) = \langle Z(M_2), Z(M_1) \rangle.$$

(2.13) *An alternative definition.* For $a \in \mathcal{L}$, choose $k(a) \in \mathbf{C}$ such that $k(a)^2 = S(a)$. Define $k(a_1 \dots a_n) \stackrel{\text{def}}{=} k(a_1) \cdots k(a_n)$. Define alternative pairings

$$\langle \cdot\, , \,\cdot \rangle^\natural : V(Y, l) \otimes V(-Y, \hat{l}) \to \mathbf{C}$$

by

$$\langle \cdot\, , \,\cdot \rangle^\natural = k(l) \langle \cdot\, , \,\cdot \rangle.$$

With these pairings, the compatibility condition with the gluing axiom becomes more natural:

$$\langle \bigoplus_x \alpha_x, \bigoplus_x \beta_x \rangle^\natural = \sum_x \langle \alpha_x, \beta_x \rangle^\natural.$$

But the gluing axiom for Z becomes less natural:

$$Z(M_f) = \bigoplus_l \sum_j k(l)^{-1} \langle V(f) \alpha_l^j, \beta_{\hat{l}}^j \rangle^\natural \gamma_{\hat{l}l}^j.$$

(2.14) *Unitary TQFTs.* A *unitary* modular functor is a modular functor such that each $V(Y)$ is equiped with a nonsingular hermetian pairing

$$\langle \cdot\, , \,\cdot \rangle_h : V(Y) \otimes \overline{V(Y)} \to \mathbf{C}$$

and each morphism is unitary. The hermetian structures are required to satisfy compatibility conditions similar to the ones in (2.3). In particular,

$$\langle \bigoplus_x \alpha_x, \bigoplus_x \beta_x \rangle_h = \sum_x S(x) \langle \alpha_x, \beta_x \rangle_h.$$

(The notation is similar to that of (2.3).) Note tht this implies that $S(a)$ is real and positive for all $a \in \mathcal{L}$. Also, we require that the following diagram commutes for all Y.

$$\begin{array}{ccc} V(Y) & \leftrightarrow & V(-Y)^* \\ \updownarrow & & \updownarrow \\ \overline{V(Y)^*} & \leftrightarrow & \overline{V(-Y)} \end{array}$$

(The horizontal isomorphisms come from (2.3). The vertical ones come from the hermetian structures on $V(Y)$ and $V(-Y)$.)

A *unitary* TQFT is a TQFT whose modular functor is unitary and whose partition function satisfies

$$Z(-M) = \overline{Z(M)}.$$

(The hermetian structure induces an identification $V(\partial(-M)) = \overline{V(\partial M)}$.)

3 Basic data

A modular functor contains a large amount of data satisfying some strong axioms. This raises the question of what is the minimal amount of data ("basic data") which, by virtue of the axioms, determines the entire modular functor. In this section we introduce the basic data of a modular functor. In Section 5 we show how to reconstruct a modular functor from its basic data. In Section 6 we determine what relations putative basic data must satisfy in order for the reconstruction procedure to give consistent answers.

Roughly, the idea is as follows. Any le-surface can be decomposed into a disjoint union of simple le-surfaces (disks, annuli, pairs of pants). Knowing V for these simple le-surfaces allows one, using (2.1) and (2.2), to determine V for any le-surface. A given le-surface Y can be sliced up many different ways, and one needs identifications between the corresponding descriptions of $V(Y)$. Using (2.1) and (2.2), one can express a general identification in terms of identifications corresponding to some simple decompositions of some simple le-surfaces (i.e. an annulus, a four-punctured sphere, a torus and a once punctured torus).

The material in this section, Section 5 and Section 6 is essentially a reworking of results of Moore and Seiberg [MS]. Basic data which satisfies the relations of Section 6 is called a "modular tensor category" by some authors.

First we introduce notation for some simple e- and le-surfaces. Let D denote a fixed (for the rest of this paper) extended disk, A denote a fixed annulus, and P denote a fixed pair of pants (thrice punctured sphere). (Note that if X is an extended punctured sphere, we must have $L_X = H_1(X)$, since the intersection pairing on $H_1(X)$ is zero.)

Fix a bijective correspondence of $\mathbf{Z}_3 = \{1, 2, 3\}$ with the boundary components of P. (Such a correspondence will be called a numbering, since "labeling" has been reserved for another meaning.) Fix three disjoint properly embedded arcs in P joining the point $e^{i\epsilon}$ on the j^{th} boundary component to the point $e^{-i\epsilon}$ on the $j+1^{\text{st}}$ boundary component, where $0 < \epsilon < \pi$ is fixed. Such arcs will be called seams. Similarly equip A and D with seams and numberings of their boundary components (see Figure 4). Note that if X is any other pair of pants (or annulus or disk) with numbered boundary and seams, then there is a unique homeomorphism from P (or A or D) to X which preserves the numbering and the isotopy classes of the seams.

This is as good a place as any to introduce some notation for the mapping class groups of P, A and D. Let $T_1^\flat$, $B_{23}^\flat$, $R^\flat \in \mathcal{M}(P^\flat)$ be as shown in Figure 5. Define $T_2^\flat$, $T_3^\flat$, $B_{31}^\flat$, $B_{12}^\flat \in \mathcal{M}(P^\flat)$ similarly (e.g. $T_2^\flat = (R^\flat)^{-1} T_1^\flat R^\flat$). Let $T_1 \stackrel{\text{def}}{=} (T_1^\flat, 0) \in \mathcal{M}(P)$. Define T_2, T_3, B_{23}, B_{31}, B_{12}, $R \in \mathcal{M}(P)$ similarly. Let $C \stackrel{\text{def}}{=} (\text{id}, 1) \in \mathcal{M}(P)$. $\mathcal{M}(P)$ is

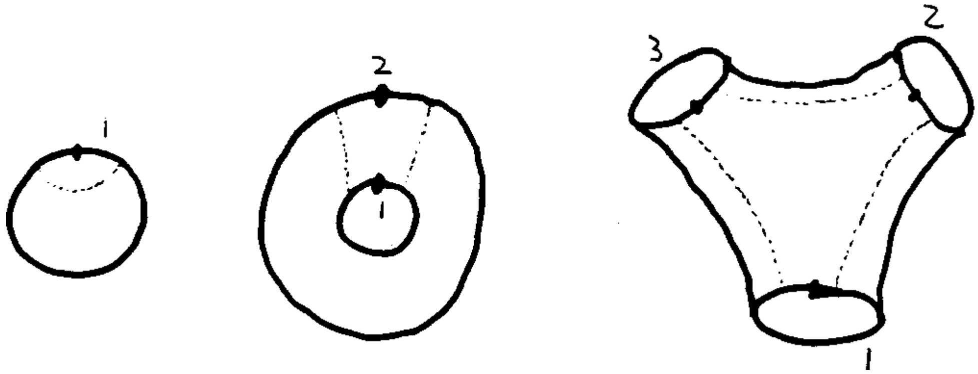

Figure 4: D, A and P.

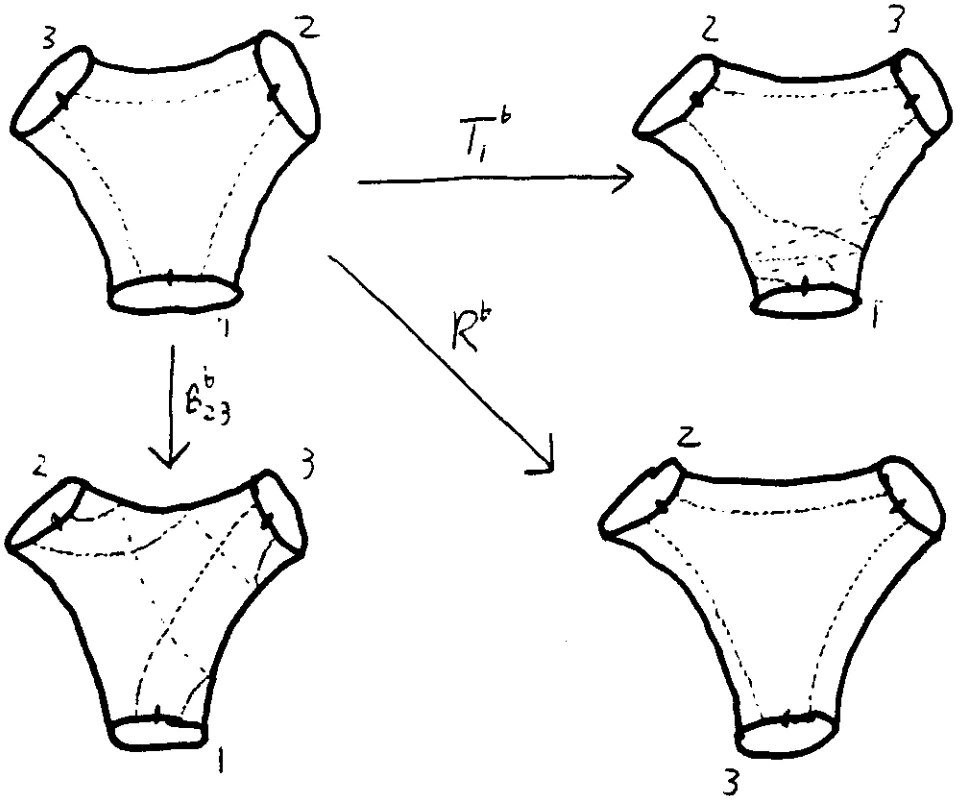

Figure 5: Elements of $\mathcal{M}(P^\flat)$

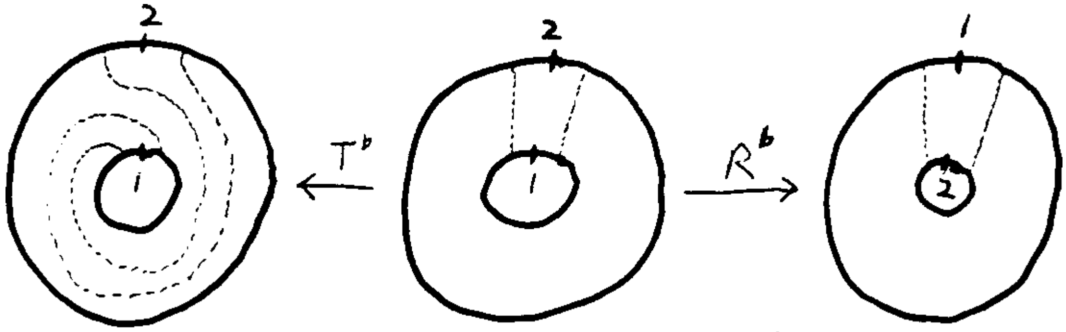

Figure 6: Elements of $\mathcal{M}(A^\flat)$

generated by T_1, T_2, T_3, B_{23}, B_{12} and C subject to the relations

$$\begin{gathered}
[C,T_i] = [C,B_{ij}] = [T_i,T_j] = 1 \\
B_{23}B_{12}B_{23} = B_{12}B_{23}B_{12} \\
B_{23}^2 = T_2T_3T_1^{-1}\ ,\ B_{12}^2 = T_1T_2T_3^{-1} \\
T_1B_{23} = B_{23}T_1\ ,\ T_2B_{23} = B_{23}T_3\ ,\ T_3B_{23} = B_{23}T_2 \\
T_1B_{12} = B_{12}T_2\ ,\ T_2B_{12} = B_{12}T_1\ ,\ T_3B_{12} = B_{12}T_3.
\end{gathered}$$

Let $T^\flat$, $R^\flat \in \mathcal{M}(A^\flat)$ be as shown in Figure 6. Let $T \stackrel{\text{def}}{=} (T^\flat, 0)$, $R \stackrel{\text{def}}{=} (R^\flat, 0)$ $C \stackrel{\text{def}}{=} (\text{id}, 1) \in \mathcal{M}(A)$. $\mathcal{M}(A)$ is generated by T, R and C subject to the relations

$$\begin{gathered}
R^2 = 1 \\
[T,R] = [R,C] = [C,T] = 1.
\end{gathered}$$

Let $C \stackrel{\text{def}}{=} (\text{id}, 1) \in \mathcal{M}(D)$. $\mathcal{M}(D)$ is freely generated by C. (The multiple definitions of R and C should not lead to confusion. In fact, for any e-surface Y define $C \stackrel{\text{def}}{=} (\text{id}, 1) \in \mathcal{M}(y)$.)

Now for the corresponding le-surfaces. For $a, b, c \in \mathcal{L}$, let P_{abc} denote P with boundary components 1, 2 and 3 labeled by a, b and c, respectively. Elements of $\mathcal{M}(P)$ give rise to elements of the mapping class *groupoid*

$$\mathcal{M}(\{P_{abc}, P_{acb}, P_{cab}, P_{cba}, P_{bca}, P_{bac}\}).$$

We will usually use the same symbols to denote the induced maps. For example, $T_1 : P_{abc} \to P_{abc}$, $B_{12} : P_{abc} \to P_{bac}$, $R : P_{abc} \to P_{cab}$. Define A_{ab} and D_a (for $a, b \in \mathcal{L}$) similarly.

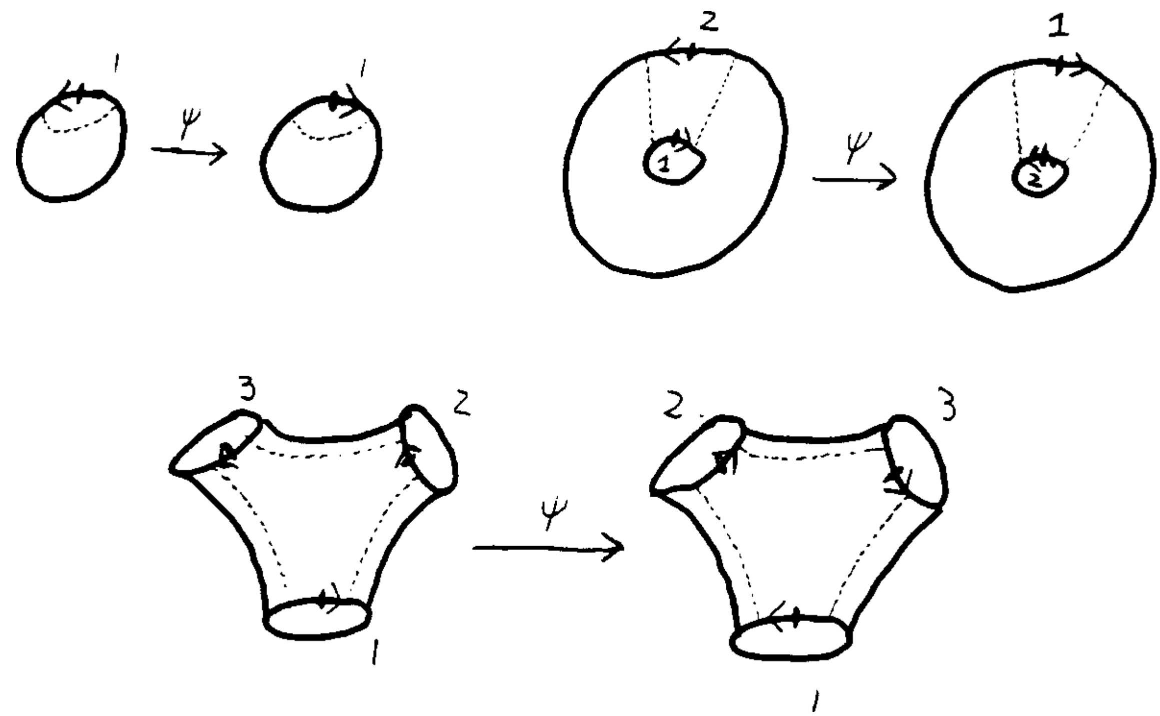

Figure 7: The standard orientation reversing maps.

For $a, b, c \in \mathcal{L}$, let

$$
\begin{aligned}
V_{abc} &\stackrel{\text{def}}{=} V(P_{abc}) \\
V_{ab} &\stackrel{\text{def}}{=} V(A_{ab}) \\
V_a &\stackrel{\text{def}}{=} V(D_a).
\end{aligned}
$$

(Note that V_{ab} is nonzero only if $a = \hat{b}$ and V_a is nonzero only if $a = 1$.) The mapping class groupoid of the le-surfaces $\{P_{abc}\}$ [or $\{A_{ab}\}$ or $\{D_a\}$] acts on the vector spaces $\{V_{abc}\}$ [or $\{V_{ab}\}$ or $\{V_a\}$], and we will use the same symbols to denote this action. So, for example, we have linear isomorphisms $B_{12} : V_{abc} \to V_{bac}$ and $R : V_{ab} \to V_{ba}$.

Since $\dim(V_{a\hat{a}}) = 1$, the action of T on $V_{a\hat{a}}$ is multiplication by a scalar $T_{(a)} \in \mathbf{C}$. The compatibility of the gluing axiom with the actions of mapping class groupoids implies the following:

(3.1) *The action of a left-handed Dehn twist along a boundary component of D, A or P on V_1, $V_{a\hat{a}}$ or V_{abc} is multiplication by a scalar $T_{(x)}$, where x is the label of the boundary component. Furthermore, $T_{(x)} = T_{(\hat{x})}$ and $T_{(1)} = 1$.*

Define the *standard orientation reversing maps* on D, A and P as in Figure 7. Each

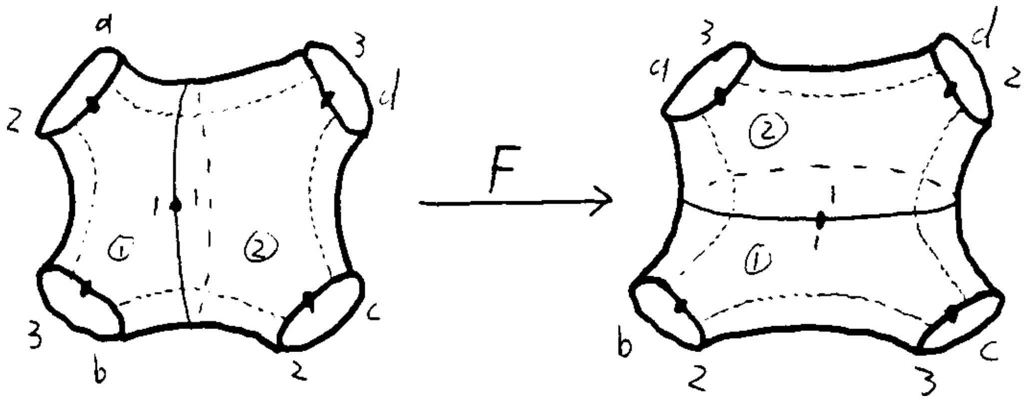

Figure 8: The le-morphism F.

of these maps will be denoted by ψ. Note that $\psi^2 = \mathrm{id}$. ψ induces identifications

$$\begin{aligned} V_{abc} &= V^*_{\hat{a}\hat{c}\hat{b}} \\ V_{a\hat{a}} &= V^*_{a\hat{a}} \\ V_1 &= V^*_1 \end{aligned}$$

for all $a, b, c \in \mathcal{L}$. The corresponding pairings wil be denoted by $\langle \cdot\, , \, \cdot \rangle$. The analog of "compatibility with the mapping class groupoids" (see (2.3)) for these pairings is

$$\langle x, y \rangle = \langle f_* x, (\psi f \psi^{-1})_* y \rangle. \tag{3.2}$$

An action of the mapping class groupoids of P (or A or D) together with a pairing which satisfies (3.2) will be called an action of the *unoriented* mapping class groupoid.

Next we introduce the simple identifications mentioned above. Let $F = F_{abcd}$ be the le-morphism between two labeled gluings of $P \coprod P$ shown in Figure 8. (The map depicted in Figure 8 is the one corresponding to horizontal translation of the drawings. The maps between $P \coprod P$ and the drawings are indicated by seams and numberings. The circled numbers are used to distinguish the first and second copies of P.) By (2.2), F induces an isomorphism

$$F : \bigoplus_{x \in \mathcal{L}} V_{xab} \otimes V_{\hat{x}cd} \to \bigoplus_{y \in \mathcal{L}} V_{ybc} \otimes V_{\hat{y}da}.$$

Let $S = S_a$ be the le-morphism from a labeled gluing of P to itself shown in Figure 9. By (2.2), S induces an isomorphism

$$S : \bigoplus_{x \in \mathcal{L}} V_{ax\hat{x}} \to \bigoplus_{y \in \mathcal{L}} V_{ay\hat{y}}.$$

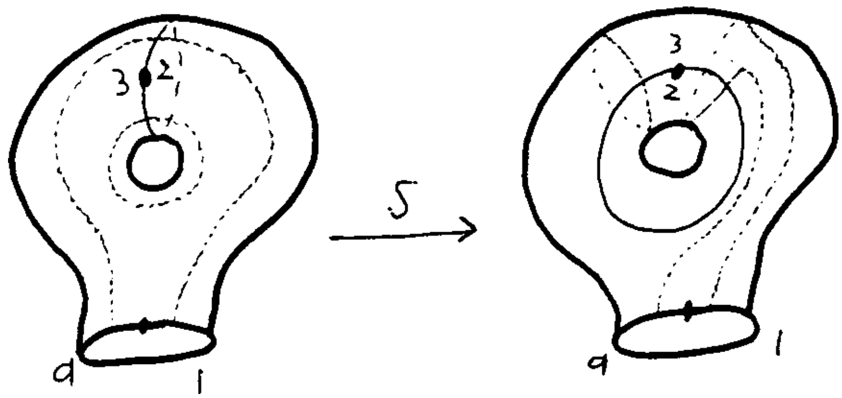

Figure 9: The le-morphism S_a.

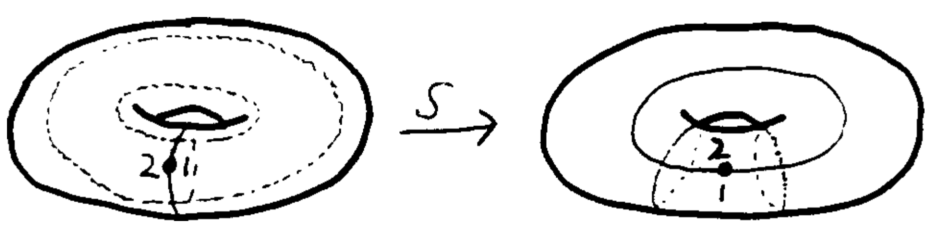

Figure 10: The le-morphism S.

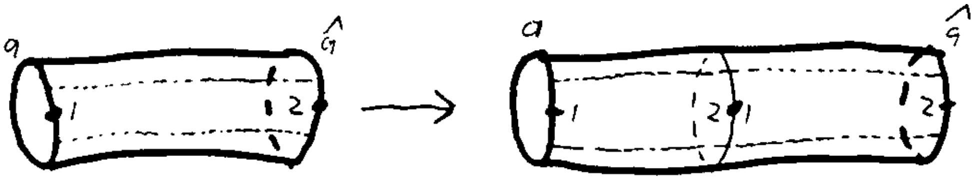

Figure 11: Labelings and gluings of A.

Let S be the le-morphism from a labeled (trivially) gluing of A to itself shown in Figure 10. By (2.2), S induces an isomorphism

$$S : \bigoplus_{x \in \mathcal{L}} V_{x\hat{x}} \to \bigoplus_{y \in \mathcal{L}} V_{y\hat{y}}.$$

Consider the le-morphism between a labeling of A and a labeled gluing of $A \coprod A$ shown in Figure 11. This induces an isomorphism

$$V_{a\hat{a}} \to V_{a\hat{a}} \otimes V_{a\hat{a}}.$$

Let $\beta_{a\hat{a}} \in V_{a\hat{a}}$ be the unique element such that

$$\beta_{a\hat{a}} \mapsto \beta_{a\hat{a}} \otimes \beta_{a\hat{a}}. \tag{3.3}$$

(Recall that $V_{a\hat{a}} \cong \mathbf{C}$.)

The elements $\beta_{a\hat{a}}$ ($a \in \mathcal{L}$) satisfy a stronger property, which will require a short digression to explain. There is an isotopy class of homeomorphisms $A^\flat \to S^1 \times I$ which takes boundary component number 1 of $A^\flat$ to $S^1 \times \{0\}$, preserves boundary parameterizations (up to conjugation of S^1), and takes seams of $A^\flat$ to arcs of the form $\{\theta\} \times I$. In other words, the seams of $A^\flat$ determine a prefered isotopy class of product structures on $A^\flat$.

Let Y be an e-surface with a distinguished boundary component. Let $Y \cup A$ denote the e-surface obtained by gluing that component to boundary component number 2 of A. There is a unique homeomorphism $f^\flat : Y^\flat \to (Y \cup A)^\flat$ which satisfies the following condition: $f^\flat$ is isotopic to $\mathrm{id} : Y^\flat \to Y^\flat \subset (Y \cup A)^\flat$ via an isotopy which takes the distinguished boundary component to $S^1 \times \{t\} \subset A^\flat$ (in a parameterization preserving fashion) at time t and which fixes the rest of $\partial Y^\flat$. Let $f = (f^\flat, 0) : Y \to Y \cup A$.

Let $l \in \mathcal{L}(Y)$. Let a be the label which l assigns to the distinguished boundary component. Then, by (2.1) and (2.2), f induces an isomorphism

$$f_* : V(Y, l) \to V(Y, l) \otimes V_{a\hat{a}}.$$

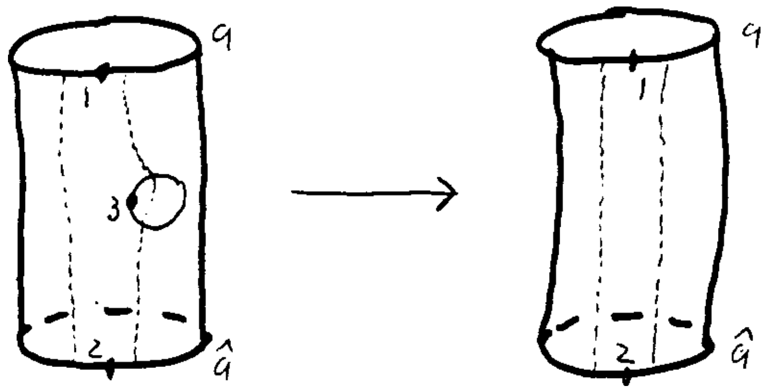

Figure 12: More le-surfaces.

I claim that for all $x \in V(Y, l)$,

$$f_* : x \mapsto x \otimes \beta_{a\hat{a}}. \tag{3.4}$$

Clearly, for some $g \in \mathrm{Aut}\,(V(Y, l))$, we have $f_*(x) = g(x) \otimes \beta_{a\hat{a}}$. But by (3.3) and the associativity of gluing (applied to $Y \cup A \cup A$), $g^2 = g$. Therefore $g = \mathrm{id}$.

There is a unique orientation reversing e-morphism from D to itself. By (2.3), this induces a nondegenerate pairing

$$V_1 \otimes V_1 \to \mathbf{C}.$$

Recall (see (2.5)) that V_1 is 1-dimensional. Fix once and for all $\beta_1 \in V_1$ such that

$$\beta_1 \otimes \beta_1 \mapsto 1.$$

(There are exactly two such elements, β_1 and $-\beta_1$.)

Consider the le-morphism between a labeled gluing of $P \coprod D$ and $A_{a\hat{a}}$ shown in Figure 12. This induces an isomorphism

$$V_{a\hat{a}1} \otimes V_1 \to V_{a\hat{a}}.$$

Let $\beta_{a\hat{a}1} \in V_{a\hat{a}1}$ be the unique element such that

$$\beta_{a\hat{a}1} \otimes \beta_1 \mapsto \beta_{a\hat{a}}. \tag{3.5}$$

Also define

$$\begin{aligned} \beta_{1a\hat{a}} &= R(\beta_{a\hat{a}1}) \in V_{1a\hat{a}} \\ \beta_{\hat{a}1a} &= R^2(\beta_{a\hat{a}1}) \in V_{\hat{a}1a}. \end{aligned}$$

(3.6) The stuff discussed above will be collectively called "basic data". In other words, basic data consists of

- vector spaces V_{abc}, $V_{a\hat{a}}$ and V_1 (for all $a, b, c \in \mathcal{L}$), together with actions of the appropriate unoriented mapping class groupoids which satisfy (3.1)
- isomorphisms

$$\begin{aligned} F : \bigoplus_{x\in\mathcal{L}} V_{xab} \otimes V_{\hat{x}cd} &\to \bigoplus_{y\in\mathcal{L}} V_{ybc} \otimes V_{\hat{y}da} \\ S : \bigoplus_{x\in\mathcal{L}} V_{ax\hat{x}} &\to \bigoplus_{y\in\mathcal{L}} V_{ay\hat{y}} \\ S : \bigoplus_{x\in\mathcal{L}} V_{x\hat{x}} &\to \bigoplus_{y\in\mathcal{L}} V_{y\hat{y}} \end{aligned}$$

  (for all $a, b, c, d \in \mathcal{L}$)

- elements $\beta_{a\hat{a}} \in V_{a\hat{a}}$, $\beta_1 \in V_1$, $\beta_{a\hat{a}1} \in V_{a\hat{a}1}$ (for all $a \in \mathcal{L}$), with $\langle \beta_1, \beta_1 \rangle = 1$.

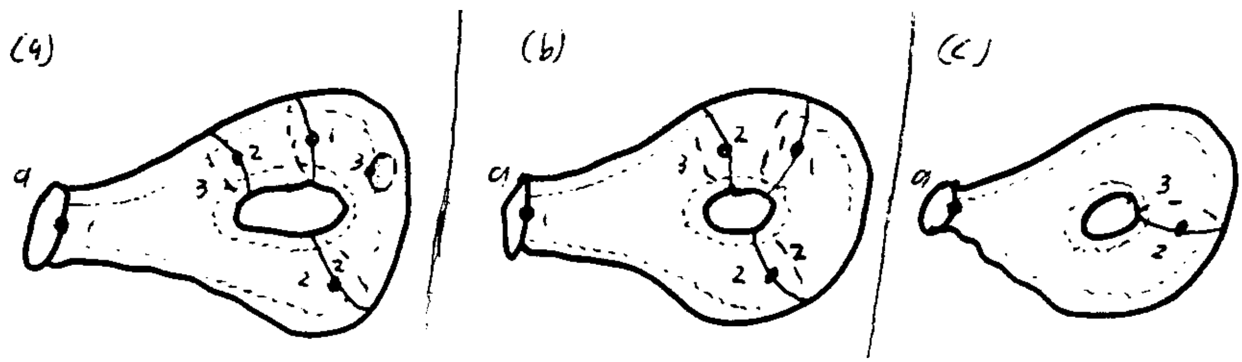

Figure 13: Some decompositions of a punctured torus.

4 Miscellaneous calculations

In this section we calculate Z for certain e-3-manifolds in terms of the basic data introduced in Section 3. We also show how TQFTs lead to invariants of labeled, framed links and give surgery formulae in terms of these invariants. This is done both to give the reader practice in applying the axioms of a TQFT and to establish certain elementary results which will be needed in later sections.

Let Y be an le-surface with a decomposition into disks, annuli and pairs of pants. Using the disjoint union and gluing axioms, V(Y) can be expressed as a direct sum of tensor products of V_1's, $V_{a\hat{a}}$'s and V_{abc}'s. For example, the decomposition shown in Figure 13(a) leads to

$$\begin{aligned} V(Y) &= \bigoplus_{x,y,z,w\in\mathcal{L}} V_{axy} \otimes V_{z\hat{x}w} \otimes V_{\hat{w}} \otimes V_{\hat{z}\hat{y}} \\ &= \bigoplus_{x} V_{ax\hat{x}} \otimes V_{x\hat{x}1} \otimes V_1 \otimes V_{\hat{x}x}. \end{aligned}$$

The decompositions shown in Figures 13(b) and 13(c) lead to

$$\begin{aligned} V(Y) &= \bigoplus_{x} V_{ax\hat{x}} \otimes V_{x\hat{x}} \otimes \otimes V_{\hat{x}x} \\ V(Y) &= \bigoplus_{x} V_{ax\hat{x}}. \end{aligned}$$

The ordering of the factors of the tensor product is not important; only their correspondence to the components of the decomposition matters. A general element $\alpha \in V(Y)$ can be written, with respect to the first decomposition, as

$$\alpha = \bigoplus_{x} \sum_{j} \alpha^{j}_{ax\hat{x}} \otimes \beta_{x\hat{x}1} \otimes \beta_1 \otimes \beta_{\hat{x}x}$$

$$
\begin{aligned}
&= \bigoplus_x \sum_j \alpha^j_{ax\hat{x}} \otimes \beta_{x\hat{x}} \otimes \otimes \beta_{\hat{x}x} \\
&= \bigoplus_x \sum_j \alpha^j_{ax\hat{x}},
\end{aligned}
$$

where $\alpha^j_{ax\hat{x}} \in V_{ax\hat{x}}$. The first and second lines are equal by (3.5). The second and third lines are equal by (3.4). In general, adding/subtracting annuli corresponds to adding/subtracting $\beta_{x\hat{x}}$'s, and moves like the one shown in Figure 12 correspond to replacing $\beta_{x\hat{x}1} \otimes \beta_1$'s with $\beta_{x\hat{x}}$'s.

In the rest of this paper, equalities such as the above will usually be written without commenting on the decompositions to which they correspond.

It is left as an exercise for the reader to show that for all $a \in \mathcal{L}$

$$
R(\beta_{a\hat{a}}) = \beta_{\hat{a}a}. \tag{4.1}
$$

(A proof is buried in Section 6.)

Let $x = x \otimes \beta_{a\hat{a}} \in V(Y, l)$ and $y = y \otimes \beta_{\hat{a}a} \in V(-Y, \hat{l})$ (see (3.4)). By (4.1), we can also write $y = y \otimes \beta_{a\hat{a}}$. By the duality axiom,

$$
\begin{aligned}
\langle x, y \rangle &= \langle x \otimes \beta_{a\hat{a}}, y \otimes \beta_{a\hat{a}} \rangle \\
&= S(a) \langle x, y \rangle \langle \beta_{a\hat{a}}, \beta_{a\hat{a}} \rangle.
\end{aligned}
$$

Hence, for all $a \in \mathcal{L}$,

$$
\langle \beta_{a\hat{a}}, \beta_{a\hat{a}} \rangle = S(a)^{-1}. \tag{4.2}
$$

(Here $\langle \,\cdot\,, \,\cdot\, \rangle$ is induced by the standard orientation reversing map on A.) Similar arguments show that for unitary modular functors

$$
\langle \beta_{a\hat{a}}, \beta_{a\hat{a}} \rangle_h = S(a)^{-1}.
$$

Let Q_n denote the standard n-punctured sphere. (So $Q_1 = D$, $Q_2 = A$ and $Q_3 = P$.) Let l be a labeling of Q_l. Define

$$
V_l \stackrel{\text{def}}{=} V(Q_n, l).
$$

Let $\bar{1} = (1, \ldots, 1)$. $V_{\bar{1}}$ is 1-dimensional and has a prefered element

$$
\beta_{\bar{1}} \in V_{\bar{1}}
$$

whose definition is similar to that of β_{111} in Section 3.

Let T^2 be a torus with a decomposition into an annulus. With respect to this decomposition,

$$V(T^2) = \bigoplus_{x \in \mathcal{L}} V_{x\hat{x}}.$$

$V_{x\hat{x}}$ is 1-dimensional and has a prefered element $\beta_{x\hat{x}}$. Thus an annulus decomposition of T^2 leads to a prefered basis of $V(T^2)$. Note that when $\beta_{x\hat{x}}$ is regarded as an element of $V(T^2)$, we have $\langle \beta_{x\hat{x}}, \beta_{x\hat{x}} \rangle = 1$. If the decomposition is changed as in Figure 10, we get an isomorphism

$$S : \bigoplus_x V_{x\hat{x}} \to \bigoplus_y V_{y\hat{y}}.$$

Let $[S_{xy}]$ $(x, y \in \mathcal{L})$ be the matrix representation of S with respect to the prefered bases. In other words,

$$\beta_{x\hat{x}} \stackrel{S}{\mapsto} \bigoplus_y S_{xy} \beta_{y\hat{y}}.$$

By (4.1) and the fact that S commutes with (a gluing of) R, we have

$$S_{xy} = S_{\hat{x}\hat{y}}.$$

Since S^2 is equal to the map $\beta_{a\hat{a}} \beta_{\hat{a}a}$,

$$(S^{-1})_{xy} = S_{\hat{x}y}.$$

Since $\psi^2 = 1$ and $\psi S \psi = S$,

$$S_{xy} = S_{\hat{y}x}.$$

Let B^3 be a 3-ball with framing number zero. B^3 is the mapping cylinder of $\mathrm{id} : D \to D$. Hence, by the mapping cylinder axiom,

$$Z(B^3) = (\mathrm{id} : V_1 \to V_1) = \beta_1 \otimes \beta_1. \tag{4.3}$$

(The second equality follows from the fact that $\langle \beta_1, \beta_1 \rangle = 1$.) Note that β_1 appears an even number of times, so it's sign ambiguity is of no concern.

Gluing two copies of B^3 together along their boundaries yields S^3 (with framing number zero). By the gluing axiom,

$$Z(S^3) = \langle \beta_1 \otimes \beta_1, \beta_1 \otimes \beta_1 \rangle = S(1) \langle \beta_1, \beta_1 \rangle \langle \beta_1, \beta_1 \rangle = S(1).$$

It follows from (4.3) that for any e-3-manifold M,

$$Z(M \setminus B^3) = S(1)^{-1} Z(M) \otimes \beta_1 \otimes \beta_1.$$

($M \setminus B^3$ denotes M with a 3-ball removed from its interior.) This implies

$$Z(M_1 \# M_2) = \frac{Z(M_1) \otimes Z(M_2)}{S(1)} = \frac{Z(M_1) \otimes Z(M_2)}{Z(S^3)}.$$

(Note that if M_1 and M_2 are closed then $Z(M_1) \otimes Z(M_2) = Z(M_1)Z(M_2)$.)

We now compute the partition function of a solid torus in two different ways. Comparing the computations will show that

$$S(a) = S_{1a} \tag{4.4}$$

for all $a \in \mathcal{L}$.

$S^1 \times D^2$ can be obtained from B^3 by gluing two disks on ∂B^3 together. If ∂B^3 is decomposed as two disks and an annulus,

$$Z(B^3) = \beta_1 \otimes \beta_1 \otimes \beta_{11}.$$

Hence, by the gluing axiom,

$$Z(S^1 \times D^2) = \langle \beta_1, \beta_1 \rangle \beta_{11} = \beta_{11} \in \bigoplus_x V_{x\hat{x}}.$$

This is with respect to an annulus decomposition of $\partial(S^1 \times D^2)$ such that the decomposing curve bounds a disk in $S^1 \times D^2$. With respect to a decomposition such that the decomposing curve is a longitude (and the seams bound a disk), we have

$$Z(S^1 \times D^2) = \bigoplus_y S_{1y} \beta_{y\hat{y}}. \tag{4.5}$$

$Z(S^1 \times D^2)$ can also be thought of as the mapping cylinder of $\mathrm{id} : A \to A$. Hence, by the mapping cylinder axiom and (4.2),

$$\begin{aligned} Z(S^1 \times D^2) &= \bigoplus_y (\mathrm{id} : V_{y\hat{y}} \to V_{y\hat{y}}) \\ &= \bigoplus_y S(y) \beta_{y\hat{y}} \otimes \beta_{y\hat{y}} \\ &= \bigoplus_y S(y) \beta_{y\hat{y}}. \end{aligned} \tag{4.6}$$

The last line is with respect to an annulus decomposition such that the decomposing curve is a longitude and the seams bound a disk. Comparing (4.5) and (4.6) yields (4.4).

Next we compute the partition function of a genus two handlebody in two different ways. Comparing the computations will yield a relation which V must satisfy. We will

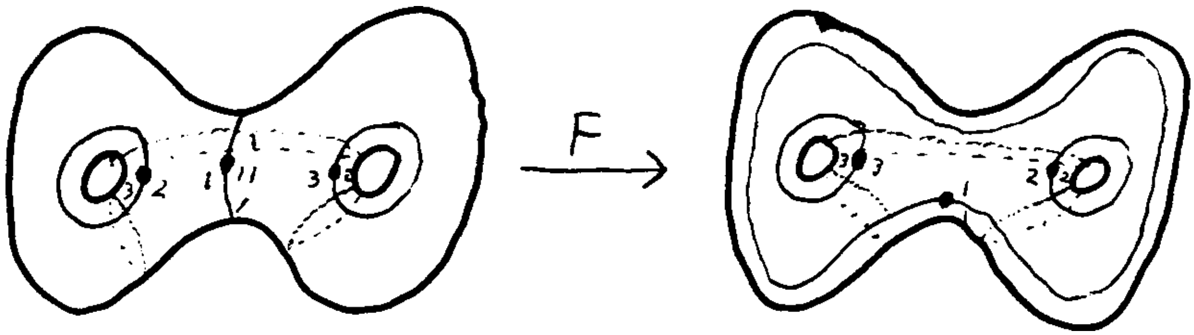

Figure 14: Two decompositions of H.

see in Section 7 that a modular functor V has a compatible partition function if and only if it satisfies this relation.

Let H denote a genus two handlebody (with framing number zero). H can be obtained by gluing two copies of $S^1 \times D^2$ together along disks in their boundaries. By (4.5) and the disjoint union axiom,

$$\begin{aligned} Z((S^1 \times D^2)\coprod(S^1 \times D^2)) &= \bigoplus_{a,b} S(a)S(b)\beta_{\hat{a}a} \otimes \beta_{b\hat{b}} \\ &= \bigoplus_{a,b} S(a)S(b)\beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}} \otimes \beta_1 \otimes \beta_1. \end{aligned}$$

Hence

$$Z(H) = \bigoplus_{a,b} S(a)S(b)\beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}}.$$

(This is with respect to the decomposition shown in the left hand side of Figure 14.)

On the other hand, H is the mapping cylinder of $\mathrm{id} : P \to P$. Thus

$$Z(H) = \bigoplus_{c,a,b} \mathrm{id}_{cab},$$

where id_{cab} denotes the identity in $V_{cab} \otimes V_{cab}^* = V_{cab} \otimes V_{\hat{c}\hat{b}\hat{a}}$. (This is with respect to the decomposition shown in the right hand side of Figure 14.) The two decompositions are related by (a gluing of) F (see Figure 14). So we must have, for all $a, b \in \mathcal{L}$,

$$S(a)S(b)F(\beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}}) = \bigoplus_{c} \mathrm{id}_{cab},$$

or

$$\beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}} \overset{F}{\mapsto} \bigoplus_{c} \frac{\mathrm{id}_{cab}}{S(a)S(b)}. \tag{4.7}$$

Let K be a framed link in a closed e-3-manifold M. (A framing of a link is a choice of homotopy class of parallel for each component of the link.) The framing of

K determines a decomposition of $\partial(M \setminus \text{nbd}(K))$ into annuli. The meridians are the decomposing curves and the framing curves are parallel to the seams. (It follows that the lagrangian in $H_1(\partial(M \setminus \text{nbd}(K)))$ is the one spanned by the meridians.) With respect to this decomposition we can write

$$Z(M \setminus \text{nbd}(K)) = \bigoplus_l J(K;l)\beta_{a_1\hat{a}_1} \otimes \cdots \otimes \beta_{a_n\hat{a}_n},$$

where $J(K;l) \in \mathbf{C}$ and $l = (a_1, \ldots, a_n)$ ranges over all labelings of the components of K. $J(K;l)$ may be regarded as an invariant of the framed, labeled link $(K;l)$. We will see in Section **??** that for certain TQFTs $J(K;l)$ is equal to a generalized Jones polynomial evaluated at a root on unity.

This construction can be extended to give invariants of labeled, properly embedded graphs in e-3-manifolds with boundary. Each boundary of a regular neighborhood of a vertex of the graph should be equiped with a homeomorphism to the standard punctured sphere with the appropriate number of punctures. (In other words, the vertices must be "pinned".) The label of a vertex is an element of V_l, where l is the labeling determined by the label of the edges incident to the vertex. The details are left to the reader. (Compare [Wi2]; see also Section 11.)

A framed link K can be regarded as representing a closed e-3-manifold $\chi(K)$ via surgery. (Glue $S^1 \times D^2$'s to $M \setminus \text{nbd}(K)$ so that the framing curves bound disks in the $S^1 \times D^2$'s.) It is left as an exercise for the reader to show that if M is a homology sphere, then the framing numbers of $\chi(K)$ and M differ by the signature of the linking matrix of K.

Using the gluing axiom and (4.6), we can express $Z(\chi(K))$ in terms of the numbers $J(K;l)$:

$$Z(\chi(K)) = \sum_l S(l)J(K;l).$$

5 Reconstructing modular functors from basic data

In this section we show how to reconstruct a modular functor V from its basic data. (In the next section we will drop the assumption that the basic data comes from a modular functor and determine what relations putative basic data must satisfy in order for this reconstruction procedure to work.)

Fix an le-surface Y. We will construct a 1-complex $\Gamma^1 = \Gamma^1(Y)$ whose 0-cells are le-morphisms from labeled gluings of copies of D, A and P to Y, and whose 1-cells are elementary transformations between the le-morphisms. (In the next section we will add 2-cells to Γ^1, obtaining a connected and simply connected 2-complex Γ. The 2-cells will determine the relations on basic data mentioned above.)

Let, for $i, j, k \geq 0$,

$$Q_{ijk} \stackrel{\text{def}}{=} \left(\coprod_1^i D\right) \amalg \left(\coprod_1^j A\right) \amalg \left(\coprod_1^k P\right).$$

Define $\mathcal{W}_{ijk}$ to be the set of all le-surfaces W such that W is a labeled gluing of Q_{ijk} and W is le-morphic to Y. Note that, in the case where Y is connected, $\mathcal{W}_{ijk}$ is nonempty exactly when $i + k = \chi(Y)$ and at least one of i, j and k is positive. Let

$$\mathcal{W} \stackrel{\text{def}}{=} \bigcup_{i,j,k\geq 0} \mathcal{W}_{ijk}.$$

Define Γ^0, the 0-skeleton of Γ^1, to be the set of all le-morphisms $f : W \to Y$ where $W \in \mathcal{W}$.

Here is a more concrete description of Γ^0. Let $\alpha \subset Y$ be a collection of disjoint parameterized circles such that the components of Y cut along α are disks, annuli and pairs of pants (see Figure 15). Call such a collection an *overmarking* of Y. (If the number of components is minimal, then it is called a *marking* (see [HT]).) An overmarking α together with

- seams on the components of Y cut along α
- numberings of the boundary components of Y cut along α
- an ordering (segregated according to topological type) of the components of Y cut along α

is called a *DAP-decomposition* of Y (see Figure 16). A DAP-decomposition determines a unique (up to isotopy) map from a labeled gluing of $Q^\flat_{ijk}$ (for appropriate i, j and k)

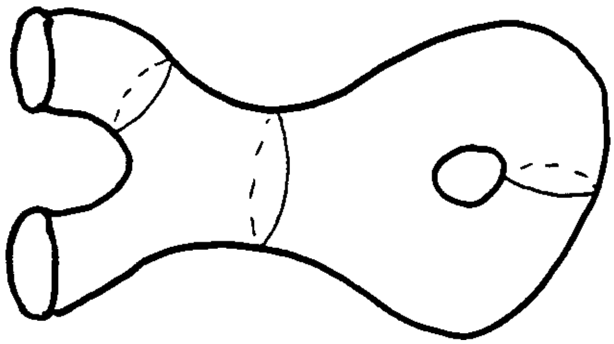

Figure 15: An overmarking.

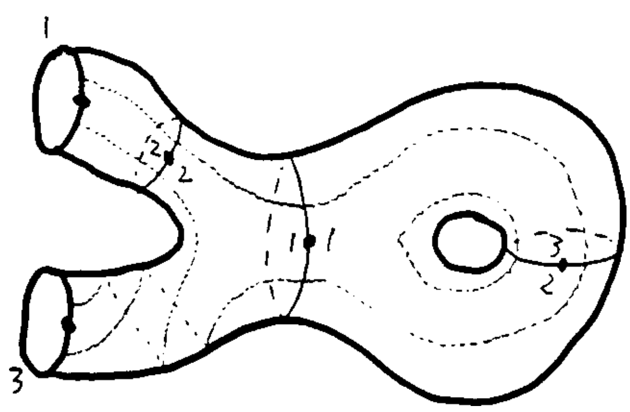

Figure 16: A DAP-decomposition.

to Y, and conversely. It follows that Γ^0 is in bijective correspondence with $\mathbf{Z}$ cross the set of isotopy classes of DAP-decompositions of Y. (The $\mathbf{Z}$ corresponds to the framing number of the morphism.)

Next we describe the 1-skeleton of Γ^1. A subset $\mathcal{E} \subset \mathcal{M}(\mathcal{W})$ will be specified below. For each $f \in \Gamma^0$ and $g \in \mathcal{E}$ with the same domain, we add a (directed) 1-cell joining f to $g^{-1}f \in \Gamma^0$. We will see in the next section that $\mathcal{E}$ generates $\mathcal{M}(\mathcal{W})$, or equivalently that Γ^1 is connected. (Homotopy theory enthusiasts will note that if $\mathcal{E} = \mathcal{M}(\mathcal{W})$ then the resulting Γ^1 is the 1-skeleton of the homotopy colimit of the functor which assigns to each $W \in \mathcal{W}$ the set of le-morphisms from W to Y.) Edges of Γ^1 can be thought of as moves between DAP-decompositions of Y.

Elements of $\mathcal{E}$ (and hence edges of Γ^1) come in five types:

Type $\mathcal{M}$. Elements of $\mathcal{M}(Q_{ijk})$ (for any i, j, k) give rise to elements of $\mathcal{M}(\mathcal{W})$. Include all of these elements in $\mathcal{E}$. (Note that $f, f' \in \Gamma^0$ are connected by a type $\mathcal{M}$ 1-cell if and only if they determine the same overmarking of Y.)

Type F. Any morphism in $\mathcal{M}(\mathcal{W})$ which is obtained by gluing the morphism F to an identity morphism is in $\mathcal{E}$. (The two copies of P on which F acts are allowed to be anywhere in the ordering.)

Type S. Any morphism in $\mathcal{M}(\mathcal{W})$ which is obtained by gluing (either version of) the morphism S to an identity morphism is in $\mathcal{E}$. (The copy of P or A on which S acts is allowed to be anywhere in the ordering.)

Type A. Let X be a component of Q_{ijk} (for any i, j, k). Let $f : X \to X \cup A$ be a morphism of the type involved in (3.4). Let $g \in \mathcal{M}(Q_{ijk}, Q_{i,j+1,k}) \subset \mathcal{M}(\mathcal{W})$ be a morphism which is a gluing of f and a morphism which corresponds to reordering the components. Include all such morphisms in $\mathcal{E}$. (In terms of DAP-decompositions, type A edges correspond to inserting a copy of A at one of the decomposing circles or at a boundary component of Y, and then reordering the components of the decomposition. A should be inserted in such a way that the seams are preserved and boundary component number two of A is not part of the boundary of Y.)

Type D. Let $f : A \to P \cup D$ be the morphism depicted in Figure 12. Let

$$g \in \mathcal{M}(Q_{ijk}, Q_{i+1,j-1,k+1}) \subset \mathcal{M}(\mathcal{W})$$

be a morphism which is a gluing of f and a morphism which corresponds to reordering the components. Include all such morphisms in $\mathcal{E}$. (In terms of DAP-decompositions, type D edges correspond to replacing an annulus component of the decomposition with a gluing of D and P (as shown in Figure 12) and reordering the components.)

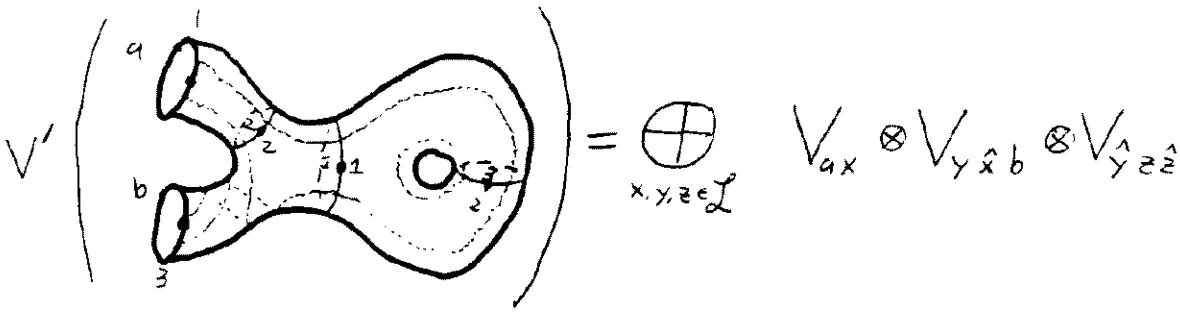

Figure 17: Illustration of $V'(W)$

It is easy to see that the assignment $Y \mapsto \Gamma^1(Y)$ is functorial. (If $h \in \mathcal{M}(Y, Y')$ and $f \in \Gamma^0(Y)$, then $h_*(f) = hf \in \Gamma^0(Y')$.) Assigning to each vertex of Γ^1 its domain and to each edge of Γ^1 its associated morphism turns Γ^1 into a big diagram of le-surfaces and morphisms. By construction, this diagram commutes.

Next we describe, using only the basic data of V, a functor V' from $\Gamma^1(\mathcal{LE})$ (where $\mathcal{LE}$ is the category of le-surfaces) to the category of complex vector spaces and isomorphisms. The composition of Γ^1 and V' will be isomorphic to V. Moreover, it is possible to recover the "modular" part of V (i.e. the identifications in the disjoint union, gluing, and duality axioms), again using only the basic data.

For $W \in \mathcal{W}$, the disjoint union and gluing axioms give an identification of $V(W)$ with a direct sum of tensor products of V_{abc}'s, $V_{a\hat{a}}$'s and V_1's. Define $V'(W)$ to be the latter space (see Figure 17). For $(f : W \to Y) \in \Gamma^0$, define $V'(f) \stackrel{\text{def}}{=} V'(W)$. (Note that, paradoxically, $V'(f)$ is a vector space while $V(f)$ is a linear map.) Let e be a 1-cell of Γ^1 with initial endpoint $f_1 : W_1 \to Y$ and final endpoint $f_2 : W_2 \to Y$. Let $m(e) \stackrel{\text{def}}{=} f_2^{-1} f_1 \in \mathcal{M}(W_1, W_2)$ be the le-morphism associated to e. It is easy to see that $V(m(e)) : V'(f_1) \to V'(f_2)$ can be expressed in terms of the basic data. Define $V'(e)$ to be this expression.

To be slightly more precise, for if e is a type $\mathcal{M}$ edge one defines $V'(e)$ in terms of the actions of $\mathcal{M}(P)$, $\mathcal{M}(A)$ and $\mathcal{M}(D)$, and well as the obvious action of the permutation group of components. For type F and S edges, ones uses the linear maps F and S. For type A edges one inserts $\beta_{a\hat{a}}$'s into the tensor products. For type D edges one replaces $\beta_{a\hat{a}}$'s with $\beta_1 \otimes \beta_{a\hat{a}1}$'s.

The assignment V' associates to Γ^1 a diagram of vector spaces and linear isomorphisms. *Because we are assuming that the basic data comes from a modular functor,* and because the underlying diagram of le-surfaces and morphisms commutes, this diagram commutes. Thus all the vector spaces $V'(f)$, for $f \in \Gamma^0$, can be unambiguously identified. Define $V'(\Gamma^1)$ to be this vector space. For any le-surface Y, define

$$V'(Y) \stackrel{\text{def}}{=} V'(\Gamma^1(Y)).$$

Let $h : Y_1 \to Y_2$ be an le-morphism. Let $f \in \Gamma^0(Y_1)$. f and $h_*(f) = hf$ share the same domain. Therefore

$$V'(h_*(f)) = V'(f).$$

Let e be an edge of $\Gamma^1(Y_1)$ connecting f_0 to f_1. Then

$$m(h_*(e)) = (hf_1)^{-1}(hf_0) = f_1^{-1}f_0 = m(e).$$

Therefore

$$V'(h_*(e)) = V'(e).$$

The above identifications induce a well defined map

$$V'(h) : V'(Y_1) \to V'(Y_2).$$

The functor V' is clearly isomorphic to the functor V.

Next we show that the basic data also determines the "modular" part of V. First consider the disjoint union axiom. Let Y_1 and Y_2 be le-surfaces. The 1-skeleton of $\Gamma^1(Y_1) \times \Gamma^1(Y_2)$ can be identified with a subcomplex of $\Gamma^1(Y_1 \coprod Y_2)$.

More precisely, $Q_{ijk} \coprod Q_{lmn}$ can be identified with $Q_{i+l,j+m,k+n}$ by putting the components of Q_{lmn} at the end of the ordering. This leads to an identification of $\Gamma^0(Y_1) \times \Gamma^0(Y_2)$ with a subset of $\Gamma^0(Y_1 \coprod Y_2)$. Let e be an edge of $\Gamma^1(Y_1)$ joining f_1 to f_1', and let $f_2 \in \Gamma^0(Y_2)$. Then there is an edge $\bar{e}$ of $\Gamma^1(Y_1 \coprod Y_2)$ joining (f_1, f_2) to (f_1', f_2) with $m(\bar{e}) = m(e) \coprod \mathrm{id}$. Similar things are true if the roles of Y_1 and Y_2 are reversed. It follows that $\Gamma^1(Y_1) \times \Gamma^1(Y_2)$ can be identified with a subcomplex of $\Gamma^1(Y_1 \coprod Y_2)$.

For $f_i \in \Gamma^0(Y_i)$ there is a natural identification

$$V'(f_1 \coprod f_2) = V'(f_1) \otimes V'(f_2).$$

It is easy to see that this gives rise to a well-defined identification

$$V'(Y_1 \coprod Y_2) = V'(Y_1) \otimes V'(Y_2).$$

It is equally easy to see that this identification is compatible with the actions of the mapping class groupoids.

Next we consider the gluing axiom (2.2), the notation of which will be used without reintroduction. For each x, $\Gamma^1(Y, (l, x, \hat{x}))$ can be identified with a subcomplex of

$\Gamma^1(Y_g, l)$. In other words, DAP-decompositions of Y, when glued, give rise to DAP-decompositions of Y_g; moves between DAP-decompositions of Y, when glued, give rise to moves between DAP-decompositions of Y_g; and this correspondence is one-to-one. Let

$$h_x : \Gamma^1(Y, (l, x, \hat{x})) \to \Gamma^1(Y_g, l)$$

denote this map. It is easy to see that for each $f \in \Gamma^0(Y_g, l)$ which lies in the image of h_x (for all/any x),

$$V'(f) = \bigoplus_x V'(h_x^{-1}(f)).$$

Also, for each edge e in the image,

$$V'(e) = \bigoplus_x V'(h_x^{-1}(e)).$$

Hence there is a well-defined identification

$$V'(Y_g, l) = \bigoplus_x V'(Y, (l, x, \hat{x})).$$

It is easy to see that this identification is compatible with the actions of the mapping class groupoids.

Now for the duality axiom. Recall the standard orientation reversing maps (Figure 7). Applying these maps to each component of a DAP-decomposition leads to a bijective correspondence $h : \Gamma^0(Y, l) \to \Gamma^0(-Y, \hat{l})$. h can be extended to an isomorphism between $\Gamma^1(Y, l)$ and $\Gamma^1(-Y, \hat{l})$. For each $f \in \Gamma^0(Y, l)$ the basic data provides an identification

$$V'(f) = V'(h(f))^*.$$

In order to see that these identifications lead to a well-defined identification

$$V'(Y, l) = V'(-Y, \hat{l})^*,$$

we must for a second time appeal to the assumption that *the basic data comes from a modular functor.* It is easy to see (without appealing to this assumption) that the above identification is compatible with the actions of the mapping class groupoids, and also with the disjoint union and gluing axioms.

This completes the reconstruction of V from its basic data.

6 Relations for basic data

In this section we establish sufficient (and necessary) conditions for basic data to come from a modular functor. (The conditions are stated in (6.4).) In the previous section we described a procedure for reconstructing a modular functor from its basic data. The assumption that the basic data was derived from a modular functor in the first place was used in only two places. The first was in showing that the diagram of vector spaces and isomorphisms associated to $\Gamma^1(Y)$ was commutative. The second was in showing that the identification $V'(Y, l) = V'(-Y, \hat{l})$ was well-defined, or equivalently that certain other diagrams commuted. Note that we were able to verify the axioms ((2.1) through (2.6)) for V' without further appeal to this assumption.

Our strategy will be to attach 2-cells to $\Gamma^1(Y)$ in such a way that the resulting 2-complex $\Gamma(Y)$ is simply connected. If the associated diagram commutes around each 2-cell, then it will commute in general. Thus the 2-cells lead to a set of relations on the basic data sufficient to guarantee that the reconstruction procedure of Section 5 produces a well-defined functor. The diagrams whose commutativity is needed for the well-definedness of the identifications $V'(Y, l) = V'(-Y, \hat{l})^*$ lead to further relations.

Throughout this section we will assume that Y is connected, leaving it to the reader to verify that the general case follows from this one.

We introduce two auxiliary 1-complexes, $\Gamma_1^1 = \Gamma_1^1(Y)$ and $\Gamma_2^1 = \Gamma_2^1(Y)$. (These are the 1-skeleta of 2-complexes Γ_1 and Γ_2 which will be defined later.) Define Γ_1^0 to be the set of all isotopy classes of overmarkings of Y (see Figure 15). There is an obvious map $\Gamma^0 \to \Gamma_1^0$; in terms of DAP-decompositions, just forget about the seams, numberings and orderings. Define Γ_1^1 to be the 1-complex obtained by adding a 1-cell connecting two points of Γ_1^0 if there exist corresponding points of Γ^0 joined by a 1-cell of type F, S, A or D.

Define Γ_2^0 to be the set of all isotopy classes of markings of Y. Since any marking is also an overmarking, $\Gamma_2^0 \subset \Gamma_1^0$. Define Γ_2^1 to be the subcomplex of Γ_1^1 spanned by Γ_2^0. Thus Γ_2^1 has 1-cells of types F and S. Each overmarking contains a marking which is unique up to isotopy. This leads to a map $\Gamma_1^0 \to \Gamma_2^0$.

The maps $\Gamma^0 \to \Gamma_1^0 \to \Gamma_2^0$ extend in an obvious fashion to maps

$$\Gamma^1 \to \Gamma_1^1 \to \Gamma_2^1.$$

Our strategy will be to use the techniques of Hatcher and Thurston [HT] to prove that Γ_2^1 is 0-connected and to find 2-cells which render it 1-connected. We will then use the structure of the above maps to obtain similar results for Γ_1^1 and finally Γ^1.

We now define Γ_2. 2-cells of Γ_2 will be specified by giving the cycles of 1-cells in Γ_2^1 to which they are attached. Figure 18 shows four classes of 2-cells for Γ_2. (A

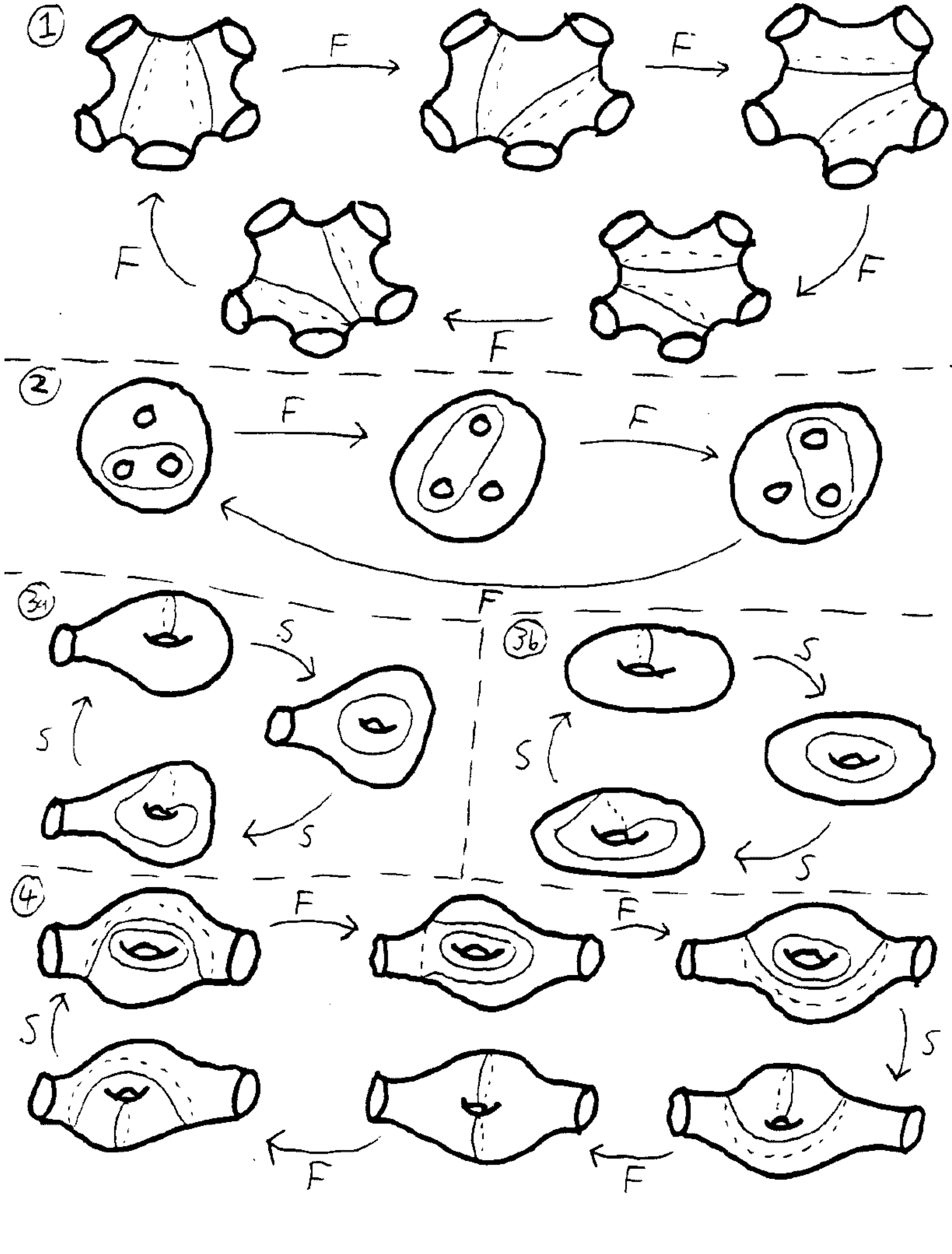

Figure 18: Type HT 2-cells.

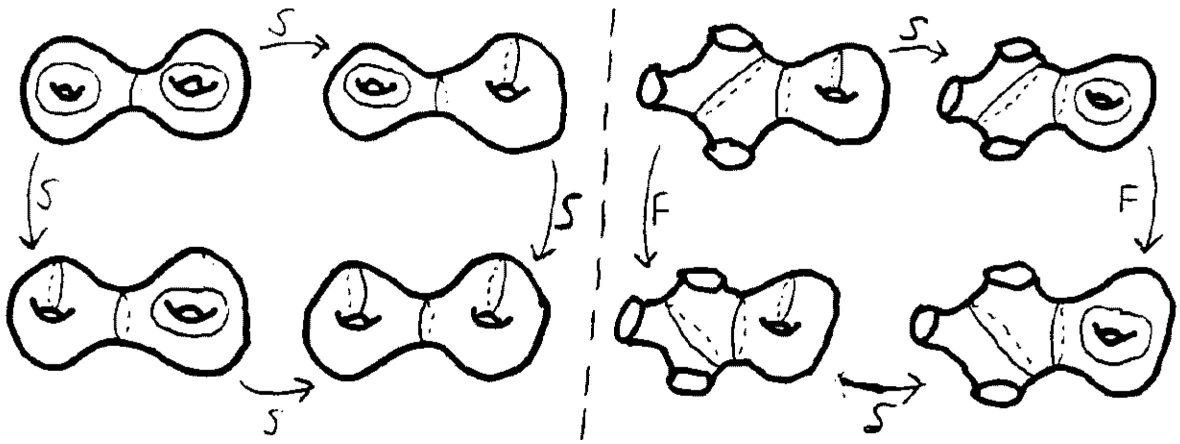

Figure 19: Type DC 2-cells.

2-cell (i.e. cycle of 1-cells) is of one of these classes if it can be obtained from one of the cycles drawn in Figure 18 by gluing on a fixed marking and/or gluing boundary components together.) We will call these classes of 2-cells collectively type HT 2-cells. Two examples of a fifth (and final) class of 2-cells are shown in Figure 19. In general, Γ_2 has 2-cells reflecting the commutativity of disjoint F or S moves on markings. We will call these 2-cells type DC 2-cells.

(6.1) **Lemma.** *Γ_2 is 0,1-connected.* □

The proof is a straight-forward application of the techniques of [HT] and will be given in Section 19.

In order to obtain similar results for Γ_1 and Γ, we need the following two lemmas.

(6.2) **Lemma.** *Let A be a 2-complex and let $f : A^1 \to B$ be a cellular map from the 1-skeleton of A to a 2-complex B such that*

- *B is 0,1-connected*
- *for each 0- or 1-cell z of B, $f^{-1}(z)$ is nonempty and 0-connected*
- *for each 0- or 1-cell z of B, the image of $\pi_1(f^{-1}(z)) \to \pi_1(A)$ is trivial*
- *for each 2-cell y of B there exists a 2-cell x of A such that $f(\partial x)$ is homotopic to ∂y.*

Then A is 0,1-connected. □

The proof is elementary and is left to the reader.

(6.3) **Lemma.** *Let A be a 0,1-connected 2-complex on which a discrete group G acts discretely. Suppose that there is a set of generators of G such that each generator has a fixed point in A. Then A/G is 0,1-connected.*

Proof: [The proof is elementary and will be included in a later version of this paper.] $\square$

(6.2) won't be applied to $\Gamma_1^1 \to \Gamma_2$ directly. (It will be applied directly to $\Gamma^1 \to \Gamma_1$.) Rather, we will apply (6.2) to

$$f : \Gamma_1^1 \to \Lambda,$$

where Λ a 2-complex with a 0-cell for each non-negative integer $\geq \chi(Y)$, a 1-cell connecting each pair of adjacent 0-cells, and no 2-cells. f sends each 0-cell x of Γ_1^0 to the 0-cell of Λ corresponding to the number of disks in the overmarking corresponding to x. f sends a 1-cell of type F, S or A to the 0-cell of Λ containing the image of its endpoints. f sends a 1-cell of type A to the 1-cell of Λ which joins the images of its endpoints.

We will attach 2-cells to Γ_1^1 (obtaining a 2-complex Γ_1) so that the third hypothesis of (6.2) is satisfied. In the course of so doing we will see that the second hypothesis is also satisfied. It will follow that Γ_1 is 0,1-connected, since the first hypothesis is easily verified and the fourth hypothesis is vacuous.

First consider $\Phi_1 \stackrel{\text{def}}{=} f^{-1}(\text{0-cell})$. This is a 1-complex whose 0-skeleton consists of overmarkings of Y with a fixed number of disk components and whose 1-skeleton has 1-cells of types F, S and A. Let $\Phi_2 \subset \Phi_1$ be the subcomplex spanned by those overmarkings with as few annulus components as possible. The 1-skeleton of Φ_2 has edges of types F and S.

Attach type HT and DC 2-cells to Γ_1^1. Redefine Φ_1 and Φ_2 to include any of these 2-cells whose boundaries they contain. Let

$$g : \Phi_1^1 \to \Phi_2$$

be the map given by erasing extra annulus components.

We will apply (6.2) to g. The fourth hypothesis of (6.2) is clearly satisfied, since $\Phi_2 \subset \Phi_1$ and g is the identity on Φ_2^1.

Next we verify the first hypothesis. Let Y' be Y with n disks removed, where n corresponds to the vertex of Λ under consideration. Let G be the kernal of $\mathcal{M}(Y') \to \mathcal{M}(Y)$. It is easy to see that

$$\Phi_2 \cong \Gamma_2(Y')/G.$$

G can be identified with the framed braid group of n points in Y. The following three types of elements generate G: twisting the framing of one of the points; sending a

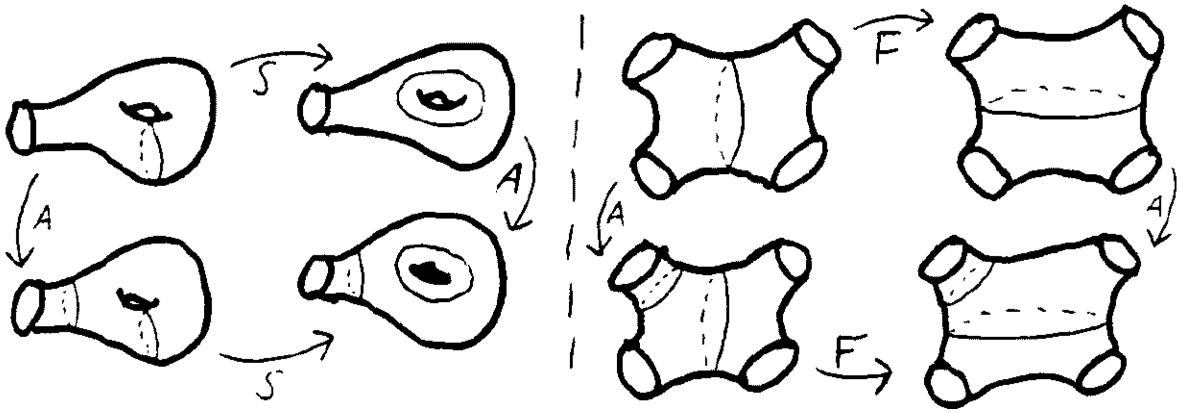

Figure 20: Examples of more type DC 2-cells

point around an embedded curve in Y; and "braiding" two points. (The last type of generator should be familiar as the usual sort of generator for the standard braid group.) If we can find a fixed point in $\Gamma_2(Y')$ for each of these elements, then (6.3) will imply the first hypothesis of (6.2).

All point of $\Gamma_2(Y')$ are fixed points of the first type of generator. Let $\gamma \in G$ send one of the points around an embedded loop $\alpha \subset Y$. Let x be a marking of Y' which contains a pair of pants with one boundary component equal to the boundary component of Y' corresponding to the point, and the other two boundary components parallel to α. Then γ fixes x. For $\gamma \in G$ which braids two points, let y be a marking of Y' which contains a pair of pants with two boundary components corresponding to the two points. Then γ fixes y.

Now we consider the second and third hypotheses of (6.2). g^{-1}(0-cell) is 0-connected by virtue of its type A 1-cells. Attach 2-cells, refered to hereafter as "type A", so that it is 1-connected. (We won't need a precise description of these 2-cells.) Also attach 2-cells reflecting the commutativity of disjoint A and F or S moves (see Figure 20). (These 2-cells will be subsumed under the DC rubric.) These 2-cells guarantee that g^{-1}(1-cell) is 1-connected. Applying (6.2) to $g : \Phi_1^1 \to \Phi_2$ shows that enough 2-cells have been added to Γ_1^1 to make Φ_1 0,1-connected.

Recall that $\Phi_1^1 = f^{-1}$(0-cell) and that our task is to add 2-cells to Γ_1^1 so that the hypotheses of (6.2) are satisfied for $f : \Gamma_1^1 \to \Lambda$. The 0-connectedness of f^{-1}(0-cell) implies that of f^{-1}(1-cell), so all that remains is to add 2-cells so that the third hypothesis holds for f^{-1}(1-cell).

Let e be a 1-cell of Λ. Let E be the set of all (type D) 1-cells of Γ_1^1 which are mapped by f to e. For $e_1, e_2 \in E$, define $e_1 \sim e_2$ to mean that e_1 and e_2 are homotopic, rel $f^{-1}(\partial e)$, in Γ_1. It is easy to see that the third hypothesis holds for $f^{-1}(e)$ if and only if $e_1 \sim e_2$ for all $e_1, e_2 \in E$.

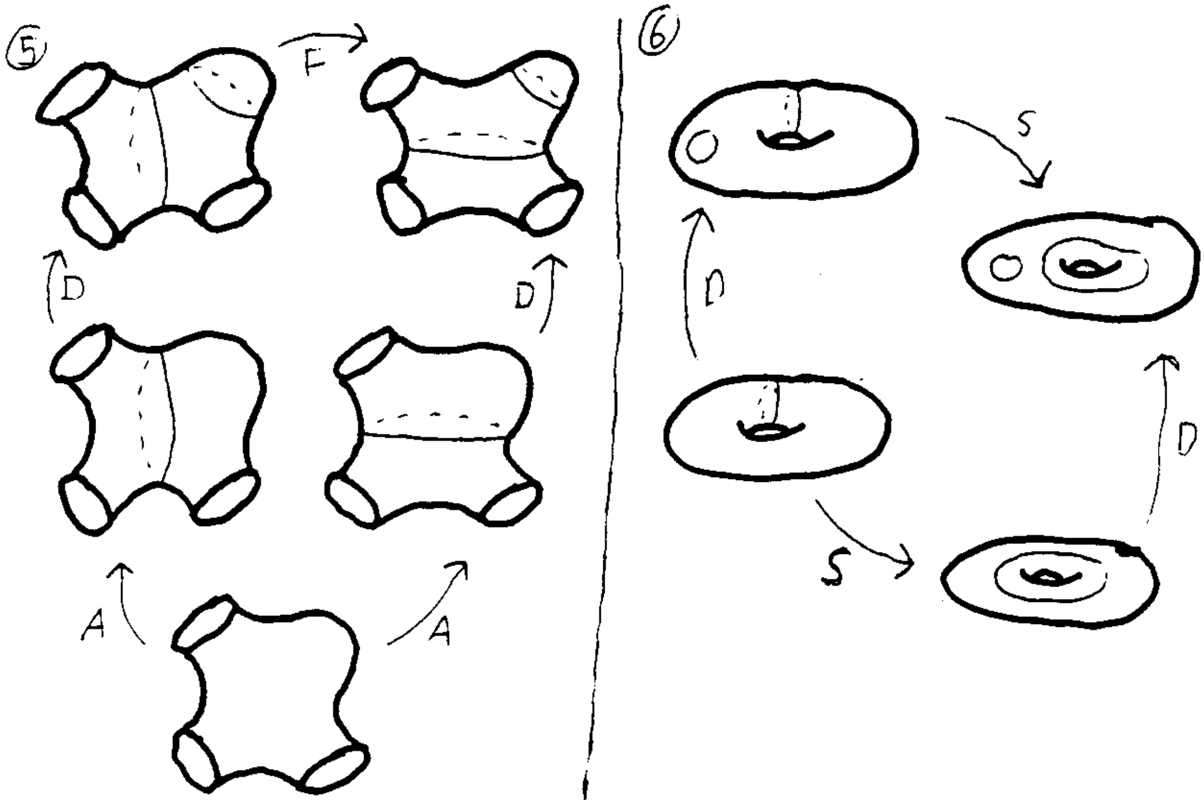

Figure 21: Type D 2-cells

Associated to e_i ($i = 1, 2$) is an overmarking m_i of Y which contains a distinguished disk and pair of pants. (This overmarking is one of the endpoints of e_i; the other endpoint is obtained by replacing the disk and pair of pants with an annulus.) Since $f^{-1}(f(m_i))$ is 0-connected, there is a sequence of overmarkings $m_1 = n_1, \ldots, n_k = m_2 \in f^{-1}(f(m_i))$ such that n_j and n_{j+1} are connected by a 1-cell of type F, S or A for all j. Corresponding to each n_j is a type D 1-cell $d_j \in E$ which corresponds to replacing the appropriate disk and pair of pants with an annulus. (Here we assume that no n_j contains an annulus which separates the appropriate disk and pair of pants, which is easily arranged.)

It suffices to attach 2-cells to Γ_1^1 so that for each j there is a 2-cell whose boundary is contained if $f^{-1}(e)$, runs over each of d_j and d_{j+1} exactly once, and runs over no other elements of E. Here are 2-cells which do the job. The first group are 2-cells (type DC) which reflect the commutativity of disjoint D and F, S or A moves. The second (and last) group, type D, is indicated in Figure 21. If the 1-cell joining n_j to n_{j+1} changes the overmarking away from the distinguished pair of pants, then a type DC 2-cell implies that $d_j \sim d_{j+1}$. If the 1-cell corresponds to a change which does involve the pair of pants, then a type D 2-cell implies that $d_j \sim d_{j+1}$.

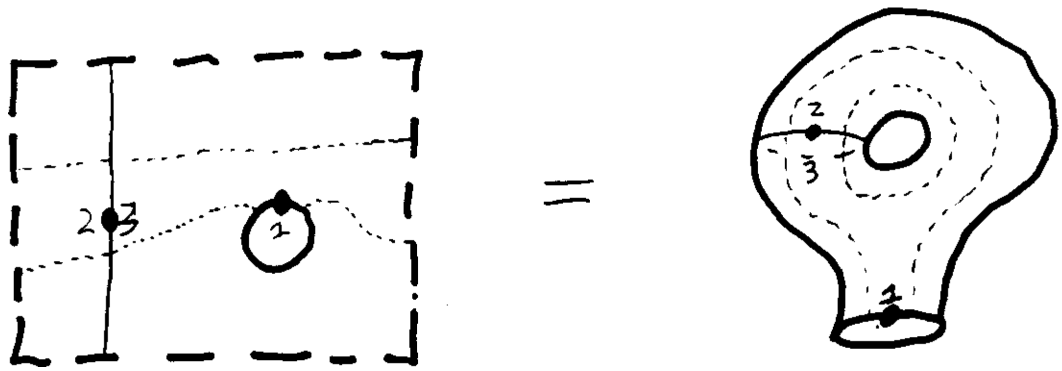

Figure 22: Convention for drawing punctured tori.

Define Γ_1 to be Γ_1^1 with 2-cells of types DC, HT, A and D attached. We have just finished verifying the hypotheses of (6.2) for $f : \Gamma_1^1 \to \Lambda$. It follows that Γ_1 is 0,1-connected.

Now we apply (6.2) to $h : \Gamma^1 \to \Gamma_1$, obtaining a collection of 2-cells for Γ^1 which render it 1-connected. We will also see that Γ^1 is 0-connected. (This fact was needed in Section 5.)

DAP-decompositions of punctured tori will sometimes be drawn as in Figure 22.

First consider h^{-1}(0-cell). All of the edges of this 1-complex are of type $\mathcal{M}$. It is easy to see that h^{-1}(0-cell) is 0-connected. Add 2-cells which make h^{-1}(0-cell) 1-connected. [It follows from (3.1) that basic data automatically satisfies the relations which these 2-cells impose. This will be proved in a later version of this paper.]

Now we consider $h^{-1}(e)$, where e is a 1-cell of Γ_1. e is of type F, S, A or D. If $h(e') = e$, then the type of e' is denoted by the same letter as the type of e. It is easy to see that $h^{-1}(e)$ is 0-connected. Add 2-cells to Γ^1 which reflect the disjoint commutativity between type $\mathcal{M}$ edges and type F, S, A and D edges, as well as the relations between the reorderings which can occur in type $\mathcal{M}$, A and D moves. Also add 2-cells (type X) as indicated in Figure 23. We have now added enough 2-cells to kill $\pi_1(h^{-1}(e))$. The proof is similar to the proof of the corresponding fact for the map $f : \Gamma_1^1 \to \Lambda$ above, and is left to the reader.

Finally, we must add, for each 2-cell of Γ_1, at least one lift to Γ^1. These 2-cells are indicated in Figures 24 through 27. We ignore the type DC and A 2-cells, since it is easy to see that basic data automatically satisfies the relations they imply. By (6.2), we have now specified enough 2-cells to make Γ 0,1-connected. These 2-cells determine a sufficient (and necessary) set of relations basic data must satisfy in order to determine a functor V.

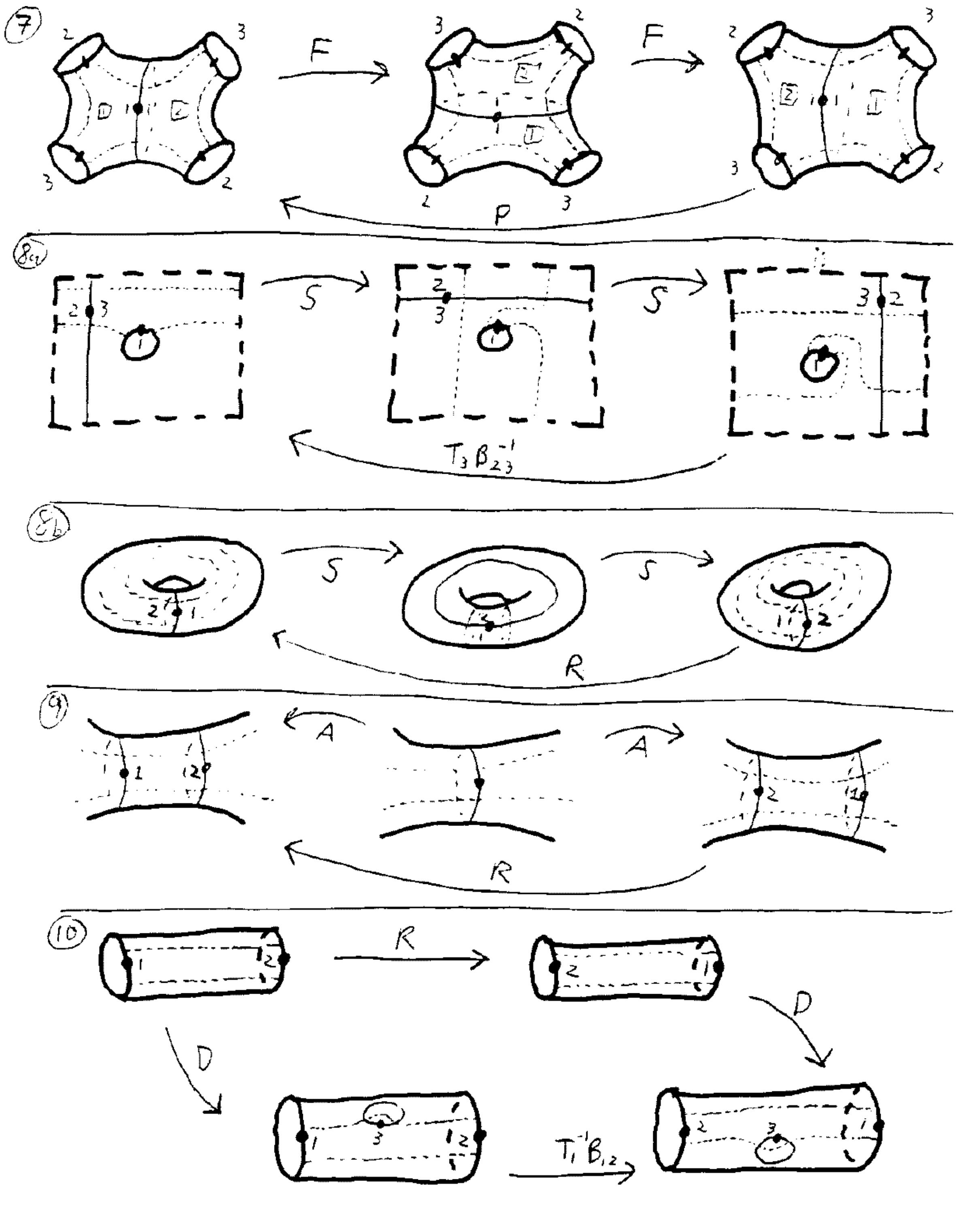

Figure 23: Type X 2-cells

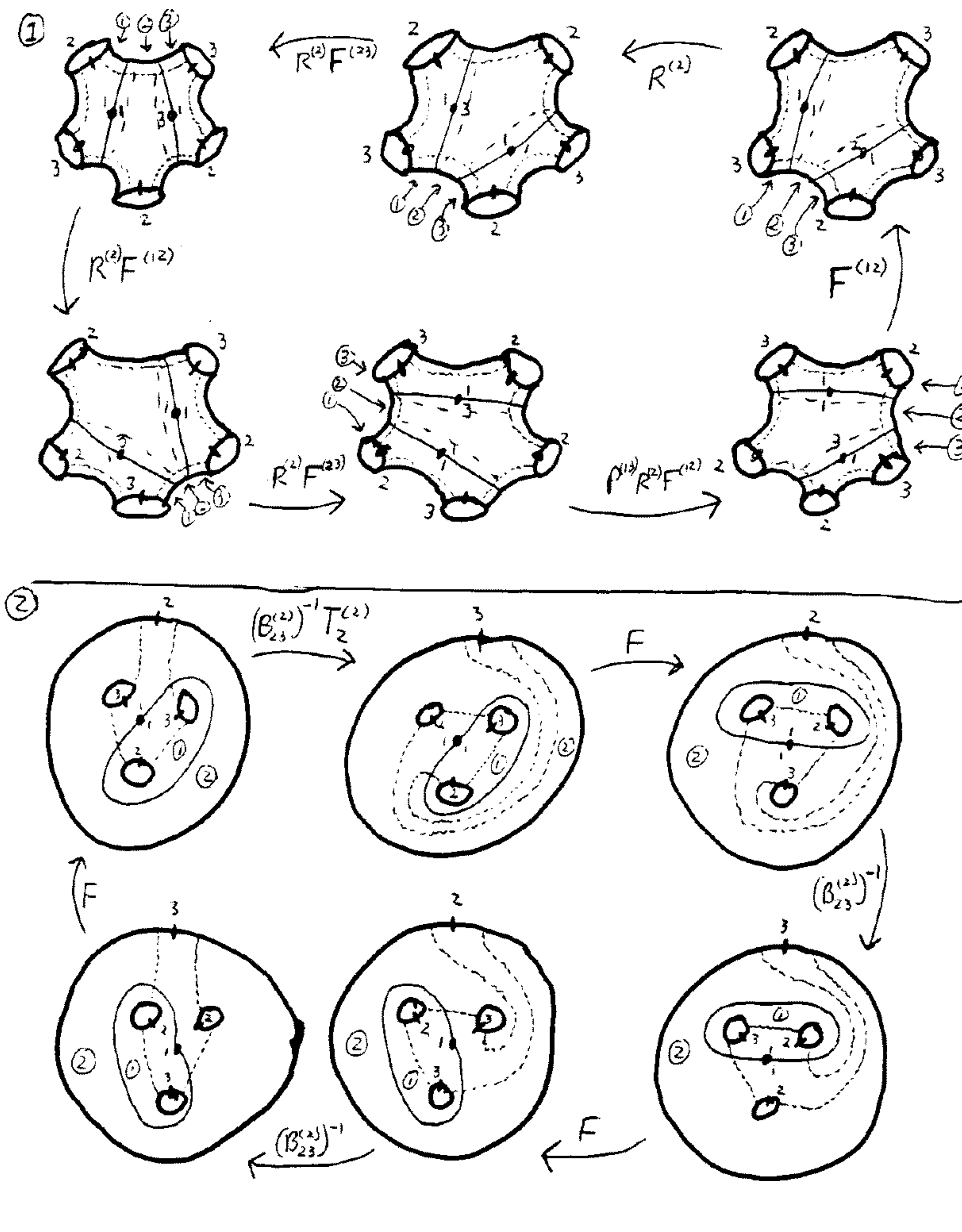

Figure 24: Some type HT 2-cells for Γ.

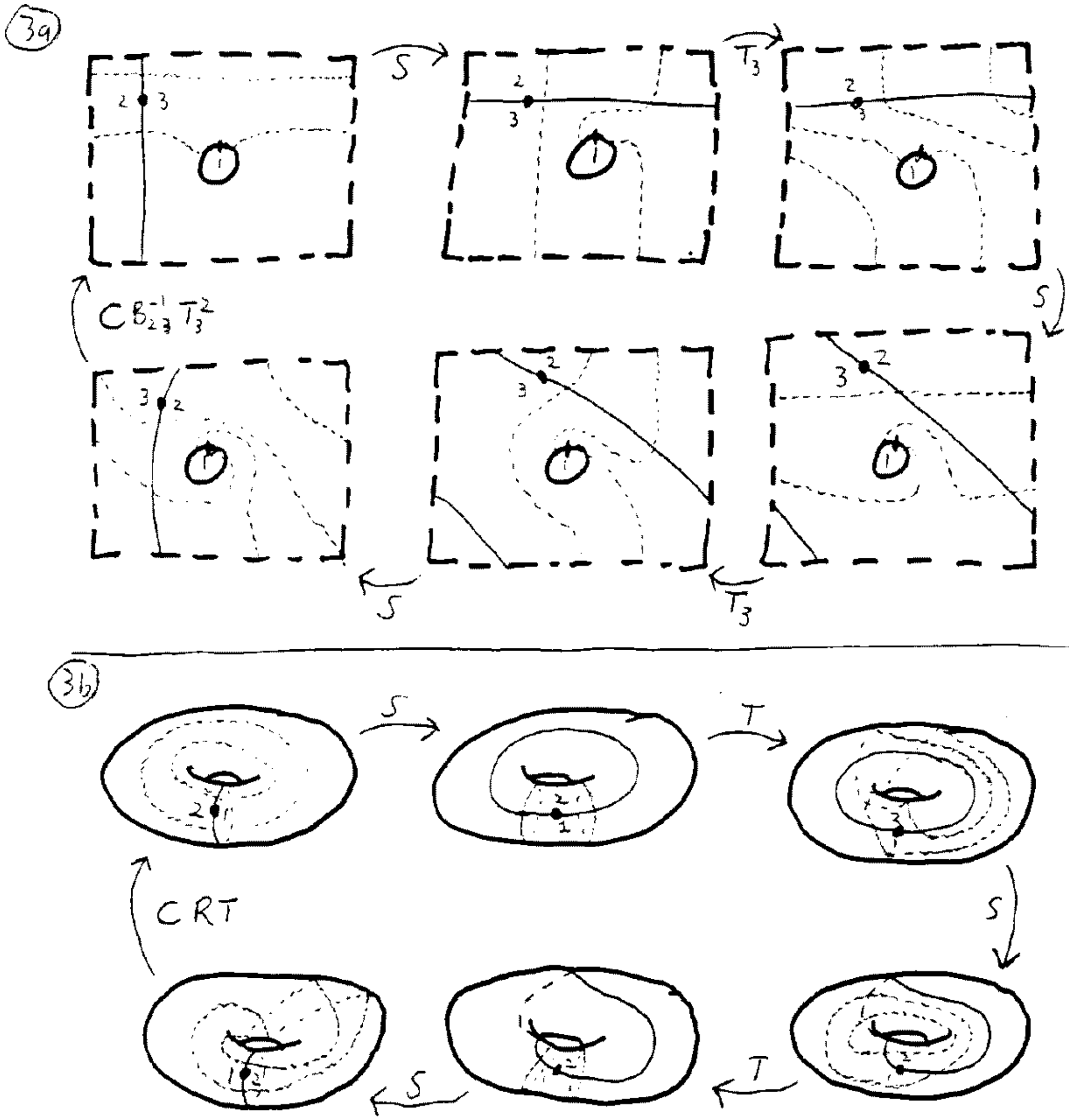

Figure 25: More type HT 2-cells for Γ.

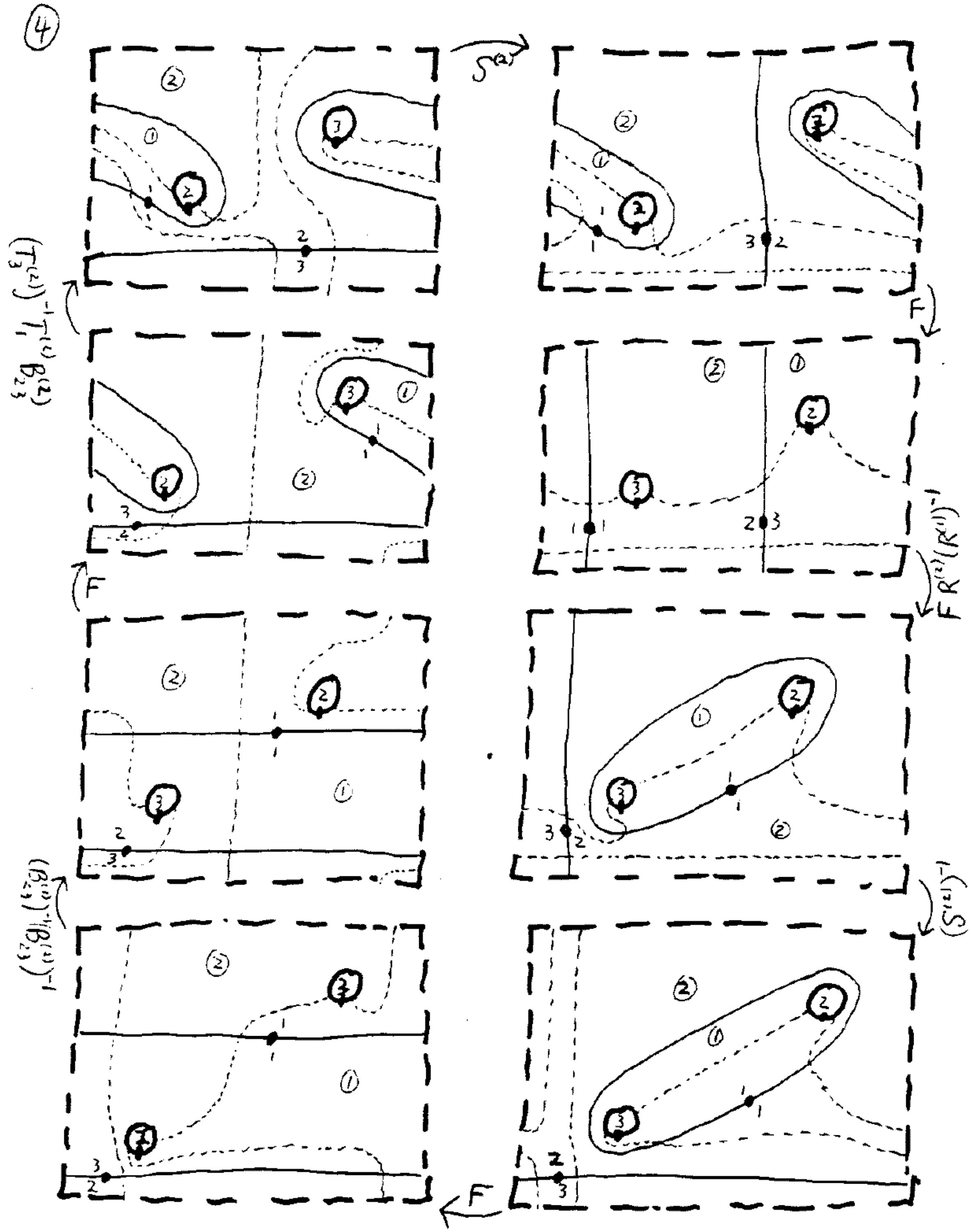

Figure 26: One more type HT 2-cell for Γ.

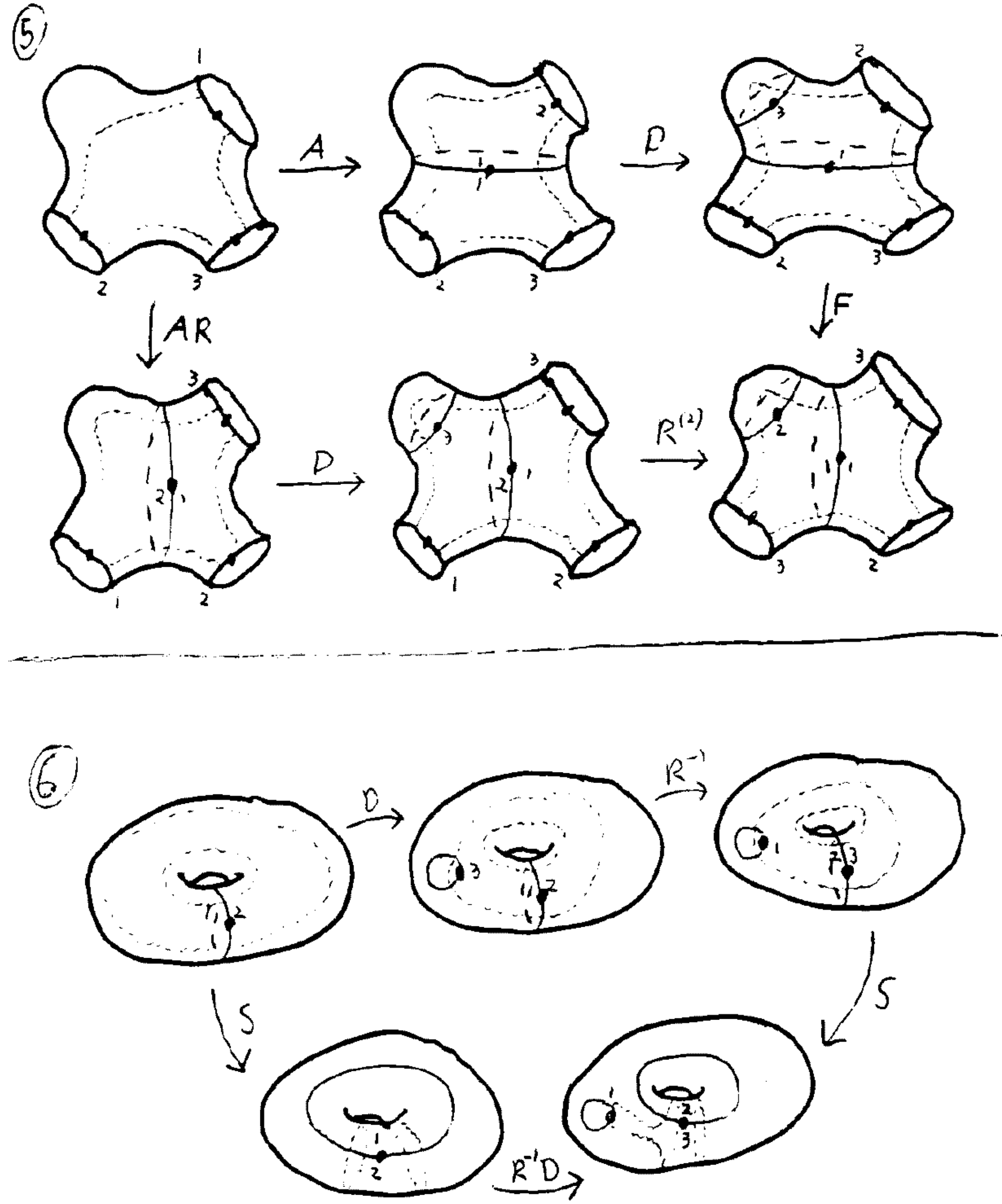

Figure 27: Some type D 2-cells for Γ.

The basic data must satisfy further relations in order for the identifications $V'(Y,l) = V'(-Y,\hat{l})^*$ to be well-defined. Let e be an edge of $\Gamma^1(Y,l)$ connecting f_1 to f_2, and let $h(e)$, $h(f_1)$ and $h(f_2)$ be the corresponding edge and vertices of $\Gamma^1(-Y,\hat{l})$ (see the end of Section 5). Associated to this setup is a diagram

$$\begin{array}{ccc} V'(f_1) & \stackrel{V'(e)}{\rightarrow} & V'(f_2) \\ \updownarrow & & \updownarrow \\ V'(h(f_1))^* & \stackrel{V'(h(e))^*}{\leftarrow} & V'(h(f_2))^*. \end{array}$$

(The vertical arrows come from the standard orientation reversing maps.) In order for the identification $V'(Y,l) = V'(-Y,\hat{l})$ to be well-defined, the diagram must commute for all edges e. If e is of type $\mathcal{M}$, the diagram commutes automatically. If e is of type F, we obtain the relation

$$F_{abcd} = F^{\dagger}_{\hat{c}\hat{b}\hat{a}\hat{d}}\,.$$

(The $\dagger$ denotes the adjoint, and the identification

$$V_{xab} \otimes V_{\hat{x}cd} = (V_{\hat{x}\hat{b}\hat{a}} \otimes V_{x\hat{d}\hat{c}})^*$$

comes from the standard orientation reversing maps. (But don't forget the correction factor of $S(x)$; see (2.3))) For type S edges, we obtain

$$S_a = S_a^{\dagger}$$

and

$$S = S^{\dagger}.$$

For type A edges we obtain

$$\langle \beta_{a\hat{a}}, \beta_{a\hat{a}} \rangle = S(a)^{-1}.$$

For type D edges we obtain

$$S(1)\langle \beta_{1a\hat{a}}, \beta_{1a\hat{a}} \rangle \langle \beta_1, \beta_1 \rangle = \langle \beta_{1a\hat{a}} \otimes \beta_1, \beta_{1a\hat{a}} \otimes \beta_1 \rangle = \langle \beta_{a\hat{a}}, \beta_{a\hat{a}} \rangle = S(a)^{-1}.$$

Since $\langle \beta_1, \beta_1 \rangle = 1$ by assumption, we have

$$\langle \beta_{1a\hat{a}}, \beta_{1a\hat{a}} \rangle = S(1)^{-1} S(a)^{-1}.$$

We have now, at long last, found a complete set of relations for basic data. It is easy to see that these relations are necessary as well as sufficient. Summarizing, we have

(6.4) **Theorem (after Moore and Seiberg).** *Basic data (see Section 3) determines (see Section 5) a modular functor (see Section 2) if and only if, for all $a, b, c, d \in \mathcal{L}$,*

1. $P^{(13)}R^{(2)}F^{(12)}R^{(2)}F^{(23)}R^{(2)}F^{(12)}R^{(2)}F^{(23)}R^{(2)}F^{(12)} = 1$

2. $F(B_{23}^{(2)})^{-1}F(B_{23}^{(2)})^{-1}F(B_{23}^{(2)})^{-1}T_2^{(2)} = 1$

3. $(T_3^{(2)})^{-1}T_1^{(2)}B_{23}^{(2)}F(B_{23}^{(1)})^{-1}(B_{23}^{(2)})^{-1}F(S^{(2)})^{-1}FR^{(2)}(R^{(1)})^{-1}FS^{(2)} = 1$

4. $CB_{23}^{-1}T_3^2ST_3ST_3S = 1$; $CRTSTSTS = 1$

5. $F(R^{-1}(x) \otimes \beta_{\hat{c}c1}) = x \otimes \beta_{\hat{a}1a}$ *for all* $x \in V_{abc}$

6. $S = \varphi^{-1}S_1\varphi$, *where* $\varphi : \beta_{x\hat{x}} \mapsto \beta_{1x\hat{x}}$

7. $F^2P = 1$

8. $T_3B_{23}^{-1}S^2 = 1$; $RS^2 = 1$

9. $R(\beta_{a\hat{a}}) = \beta_{\hat{a}a}$

10. $R = \varphi^{-1}(T_1^{-1}B_{12})\varphi$, *where* $\varphi : \beta_{x\hat{x}} \mapsto \beta_{x\hat{x}1}$

11. $F_{abcd} = F^{\dagger}_{\hat{c}\hat{b}\hat{a}\hat{d}}$

12. $S = S^{\dagger}$

13. $\langle \beta_{a\hat{a}}, \beta_{a\hat{a}} \rangle = S(a)^{-1}$

14. $\langle \beta_{1a\hat{a}}, \beta_{1a\hat{a}} \rangle = S(1)^{-1}S(a)^{-1}$.

$\square$

(Relations 1 through 10 correspond to Figures 23 through 27. Subscripts have been omitted from F's, S's, etc. whenever this has seemed unlikely to cause confusion.)

7 Constructing partition functions from modular functors

In this section we establish necessary and sufficient conditions for a given modular functor V to have a compatible partition function Z. The result is that V admits such a Z if and only if V satisfies one simple relation. [I suspect that this relation is satisfied automatically for any modular functor, but I do not at present have a proof of this.]

The proof is straightforward. One first notes that in view of the axioms ((2.7) through (2.10)), Z is determined by V. One next defines a class of decompositions of e-3-manifolds ("slicings"). Given an e-3-manifold M with slicing s, one can calculate (using the axioms) an element $Z'(M,s) \in V(\partial M)$ in terms of V and . Requiring that $Z'(M,s)$ be independent of s imposes a relation on V. If this condition is satisfied, one defines $Z(M) \stackrel{\text{def}}{=} Z'(M,s)$. Slicings are designed so that when Z is defined this way, it (almost) obviously satisfies the axioms of a partition function ((2.7) through (2.10)). (This last point is the major advantage slicings have over other ways of representing 3-manifolds, e.g. surgery descriptions and Heegaard splittings.)

For the next few paragraphs we will be dealing exclusively with unextended 2- and 3-manifolds, and so will depart from our usual notation conventions and denote these objects with unadorned capital letters rather than capital letters with superscript "$\flat$".

Let M be a (piecewise-linear) 3-manifold. A *Morse function* on M is defined to be a function f from a smoothing of M to $\mathbf{R}$ such that $f|_{\partial M}$ and $f|_{\mathrm{int}(M)}$ are Morse functions in the usual sense and all critical point have distinct levels. Critical points of a Morse function come in 10 types. These types will be denoted by a letter and an integer, the integer indicating the index of the critical point and the letter indicating whether the critical point occurs in the interior of M ("I"), on ∂M with grad(f) pointing out ("T"), or on ∂M with grad(f) pointing in ("B"). Thus the possible types are I0, I1, I2, I3, T0, T1, T2, B0, B1 and B2.

A *slicing function* for a compact 3-manifold M is defined to be a Morse function f on M together with real numbers $t_0 < \cdots < t_k$ such that each t_i is a regular value of f, at most one critical point occurs between t_i and t_{i+1} $(0 \le i \le k-1)$, all critical points lying below t_0 are of type B0 or B1, and all critical points lying above t_k are of type T2 or T1. Note that this implies that $M \cong f^{-1}([t_0, t_k])$.

A *slice* is defined to be a compact 3-manifold N together with a decomposition $\partial N = \partial_0 N \cup \partial_v N \cup \partial_1 N$ such that there exists $f : N \to [0,1]$ with $\partial_0 N = f^{-1}(0)$, $\partial_1 N = f^{-1}(1)$, and f restricted to $\mathrm{int}(M) \cup \partial_v(M)$ is a Morse function with at most one critical point. Slices come in 11 types, corresponding to the 10 types of critical points and the case of no critical point. The latter type of slice is homeomorphic to a compact surface cross I and is called a trivial slice.

Note that any slice can be constructed as follows. Start with the mapping cylinder of id : $Y \to Y$ (where Y is some surface), add a 0-, 1-, 2- or 3-handle (except for a trivial slice or types B1, B2, T1 or T0), and then decompose the boundary appropriately.

A *slicing*, s, of a compact 3-manifold is a decomposition

$$M = N_1 \cup \cdots \cup N_k,$$

where each N_i is a slice and $N_i \cap N_{i+1} = \partial_1 N_i = \partial_0 N_{i+1}$ for $1 \le i \le k-1$. (All other intersections of N_i s are, of course, empty.) Define $\partial_0(M, s) \stackrel{\text{def}}{=} \partial_0 N_1$, $\partial_1(M, s) \stackrel{\text{def}}{=} \partial_1 N_k$, and $\partial_v(M, s) \stackrel{\text{def}}{=} \cup_{i=1}^k \partial_v N_i$.

The proof of the following lemma is elementary and is left to the reader.

(7.1) **Lemma.** *Let $(f, t_0, \ldots, t_k)$ be a slicing function for a compact 3-manifold M. Then*

$$f^{-1}([t_0, t_1]) \cup \cdots \cup f^{-1}([t_{k-1}, t_k])$$

is a slicing of $f^{-1}([t_0, t_k]) \cong M$. Furthermore, every slicing of M (up to appropriately defined equivalence) arises from a slicing function in this manner. □

Using the above correspondence between slicing functions and slicings, it is easy to prove

(7.2) **Proposition.** *Let M be a compact 3-manifold and let Y_0 and Y_1 be disjoint codimension zero submanifolds of ∂M. The there exists a slicing s of M such that $\partial_0(M, s) = Y_0$ and $\partial_1(M, s) = Y_1$.* □

The statements of the next lemma and proposition, while accurate, are not as precise as they might be. They are also not as cumbersome as they might be. I hope that the latter fact makes up for the former one.

(7.3) **Lemma.** *Let M be a compact 3-manifold and let f_0 and f_1 be two Morse functions on M. Then f_0 and f_1 can be joined by a 1-parameter family of functions $\{f_t\}_{0 \le t \le 1}$ such that f_t is a Morse function for all but finitely many values of t. At each of the exceptional values of t, the critical point structure changes by one of the following "moves" (or one of their inverses).*

1. *The ordering (induced from $\mathbf{R}$) of two critical points is exchanged.*

2. *Two critical points whose indices differ by one cancel. (The critical points are necessarily both of type I, both of type B, or both of type T.)*

3. *A critical point of type Bi is replaced by two critical points of types Ii and Ti ($i = 0, 1, 2$).*

4. *A critical point of type Ti is replaced by two critical points of types Bi and I($i+1$) ($i = 0, 1, 2$).*

$\square$

The proof will be given in Section 19.

It might be helpful to give examples of the last two moves. Let $H \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbf{R}^3 \mid z \geq 0\}$. Define

$$\begin{aligned} g_t : H &\to \mathbf{R} \\ (x, y, z) &\mapsto x^2 - y^2 - (z - t)^2. \end{aligned}$$

Then for $t < 0$, g_t has a single critical point of type T1, while for $t > 0$ g_t has two critical points of types B1 and I2. Changing the signs of the terms in the definition of g_t produces examples of all the other sorts of behavior covered by the last two moves.

Using (7.3), it is easy to prove

(7.4) **Proposition.** *Any two slicings of a compact 3-manifold are related by a finite sequence of the following moves and their inverses.*

1. *Insert a trivial slice.*

2. *Replace two adjacent slices with two other slices in a manner consistent with (7.3.1).*

3. *Replace adjacent slices of types I0 and I1 with a trivial slice (when appropriate).*

4. *Same as above, but with I1, I2 $\rightsquigarrow$ trivial.*

5. *I2, I3 $\rightsquigarrow$ trivial.*

6. *B0, B1 $\rightsquigarrow$ trivial.*

7. *B1, B2 $\rightsquigarrow$ trivial.*

8. *T0, T1 $\rightsquigarrow$ trivial.*

9. *T1, T2 $\rightsquigarrow$ trivial.*

10. *Replace a slice of type B0 with slices of types I0 and T0.*

11. *Same as above, but with B1 $\rightsquigarrow$ I1, T1.*

12. *B2 $\rightsquigarrow$ I2, T2.*

13. *T0 $\rightsquigarrow$ B0, I1.*

14. *T1 $\rightsquigarrow$ B1, I2.*

15. *T2 $\rightsquigarrow$ B2, I3.*

16. *Add a slice of type B0 to the bottom of the slicing.*

17. *Add a slice of type B1 to the bottom of the slicing.*

18. *Add a slice of type T0 to the top of the slicing.*

19. *Add a slice of type T2 to the top of the slicing.*

$\square$

We now revert to our usual notation convention (e.g. M denotes an e-3-manifold).

An *e-slice* is defined as follows. Start with the mapping cylinder of $\mathrm{id} : Y \to Y$ (where Y is some e-surface), add an extended 0-, 1-, 2- or 3-handle with framing number 0 (except for a trivial e-slice or types B1, B2, T1 or T0), and then decompose the boundary appropriately. (For the case of a trivial e-slice, we will also allow the case where id is replaced with an arbitrary e-morphism. This has certain technical advantages.) Note that this is the same as the alternative description of an unextended slice, except that the mapping cylinder and handle attachment are extended. Another way of describing an e-slice is that it is an e-3-manifold $(N^\flat, L, n)$ where $N^\flat$ is a slice, $n = 0$, and L satisfies certain conditions (roughly, that the "∂_0-part" of L is as similar as possible to the "∂_1-part" of L (except for trivial e-slices)).

Define a slicing of an e-3-manifold M to be a decomposition

$$M = N_1 \cup \cdots \cup N_k,$$

where each N_i is an e-slice and $N_i \cap N_{i+1} = \partial_1 N_i = \partial_0 N_{i+1}$ for $1 \leq i \leq k-1$. (All other intersections are empty.) "$\cup$" should be interpreted in the sense of (1.12). If s is a slicing of M, $s^\flat$ will denote the underlying slicing of $M^\flat$.

Our next task is to define $Z'(N) \in V(\partial N)$, where N is an e-slice. Since an e-slice is a gluing of a mapping cylinder and (possibly) a 3-ball (i.e. a handle), and a 3-ball is just the mapping cylinder of $\mathrm{id} : D \to D$, the axioms ((2.7) through (2.10)) leave us no choice in this matter.

First, a couple of preliminary definitions. Let A be the standard extended annulus (see Section 3). Let

$$A^n \stackrel{\text{def}}{=} A\coprod \cdots \coprod A \quad (n \text{ times}).$$

Let $l = (a_1, \ldots, a_n)$ be a sequence of labels. Define

$$A_l^n \stackrel{\text{def}}{=} (A, (a_1, \hat{a}_1))\coprod \cdots \coprod (A, (a_n, \hat{a}_n)).$$

Also define

$$W_l \stackrel{\text{def}}{=} V(A_l^n) = V_{a_1\hat{a}_1} \otimes \cdots \otimes V_{a_n\hat{a}_n}$$

and

$$\beta_l \stackrel{\text{def}}{=} \beta_{a_1\hat{a}_1} \otimes \cdots \otimes \beta_{a_n\hat{a}_n} \in W_l,$$

where $\beta_{a\hat{a}} \in V_{a\hat{a}}$ is the canonical element (see Section 3).

Trivial e-slices. A trivial e-slice N is the mapping cylinder of an e-morphism $f : Y_0 \to Y_1$ with $\partial_0 N = Y_0$, $\partial_v N = \partial Y_0 \times I = A^n$, and $\partial_1 N = Y_1$. Hence

$$\begin{aligned} V(\partial N) &= \bigoplus_{l \in \mathcal{L}(Y_0)} V(Y_0, l)^* \otimes W_l \otimes V(Y_1, l) \\ &= \bigoplus_{l \in \mathcal{L}(Y_0)} \mathrm{Hom}(V(Y_0, l), W_l \otimes V(Y_1, l)). \end{aligned}$$

In view of the mapping cylinder axiom and (3.4), we define $Z'(N)$ by

$$\bigoplus_l x_l \mapsto \bigoplus_l [\beta_l] \otimes f_*(x_l),$$

where $x_l \in V(Y_0, l)$. The brackets ("[$\cdots$]") are used to distinguish $V(\partial_v N)$ from $V(\partial_1 N)$, a convention which will be used throughout this section.

Type I0. A type I0 e-slice N is the mapping cylinder of $\mathrm{id} : Y \to Y$ disjoint union an extended 3-ball B with framing number zero (i.e. a 0-handle). Also, $\partial_0 N = Y \times \{0\}$, $\partial_v N = \partial Y \times I = A^n$, and $\partial_1 N = Y \times \{1\}\coprod \partial B$. B is is equivalent to the mapping cylinder of $\mathrm{id} : D \to D$. In view of the disjoint union axiom and the definition of β_1 we define $Z'(N)$ by

$$\bigoplus_l x_l \mapsto \bigoplus_l [\beta_l] \otimes x_l \otimes \beta_1 \otimes \beta_1,$$

where $x_l \in V(Y, l)$ and $\beta_1 \otimes \beta_1 \in V(D \cup D) = V(\partial B)$. (Note that since β_1 appears twice, its sign ambiguity is of no concern.)

Type I1. A type I1 e-slice N is the mapping cylinder of $\mathrm{id} : Y \to Y$ union a 1-handle B (with framing number zero). Let $Y = Y' \cup D \cup D$, where B is attached to

$(D\cup D)\times\{1\}$. Decompose ∂B as $D\cup D\cup A$, where $D\cup D$ is the attaching region. We have $\partial_0 N = Y\times\{0\}$, $\partial_v N = \partial Y\times I = A^n$, and $\partial_1 N = Y'\times\{1\}\cup A$. Define $Z'(N)$ by

$$\bigoplus_l x_l\otimes\beta_1\otimes\beta_1 \mapsto \bigoplus_l [\beta_l]\otimes x_l\otimes\beta_{11},$$

where $x_l\in V(Y',(l,1,1))$, $\beta_1\otimes\beta_1\in V(D\cup D,(1,1)) = V_1\otimes V_1$, and $\beta_{11}\in V(A,(1,1)) = V_{11}$.

Type I2. A type I2 e-slice N is the mapping cylinder of $\mathrm{id}: Y\to Y$ union a 2-handle B. Let $Y = Y'\cup A$, where B is attached to $A\times\{1\}$. Decompose ∂B as $D\cup D\cup A$, where A is the attaching region. We have $\partial_0 N = Y\times\{0\}$, $\partial_v N = \partial Y\times I = A^n$, and $\partial_1 N = Y'\times\{1\}\cup D\cup D$. Define $Z'(N)$ by

$$\bigoplus_{l,a} x_{la\hat{a}}\otimes\beta_{a\hat{a}} \mapsto \bigoplus_l S(1)^{-1}[\beta_l]\otimes x_{l11}\otimes\beta_1\otimes\beta_1,$$

where $x_{la\hat{a}}\in V(Y',(l,a,\hat{a}))$.

Type I3. A type I3 e-slice N is the mapping cylinder of $\mathrm{id}: Y\to Y$ with a 3-handle attached. Decompose Y as $Y'\cup D\cup D$, where $D\cup D$ is the 2-sphere to which the 3-handle is attached. We have $\partial_0 N = Y\times\{0\}$, $\partial_v N = \partial Y\times I = A^n$, and $\partial_1 N = Y'\times\{1\}$. Define $Z'(N)$ by

$$\bigoplus_l x_l\otimes\beta_1\otimes\beta_1 \mapsto \bigoplus_l S(1)[\beta_l]\otimes x_l.$$

Type B0. A type B0 e-slice N is the mapping cylinder of $\mathrm{id}: Y\to Y$ union a 0-handle B. Decompose ∂B as $D\cup D$. We have $\partial_0 N = Y\times\{0\}$, $\partial_v N = (\partial Y\times I)\coprod D = A^n\coprod D$, and $\partial_1 N = (Y\times\{1\})\coprod D$. Define $Z'(N)$ by

$$\bigoplus_l x_l \mapsto \bigoplus_l [\beta_l\otimes\beta_1]\otimes x_l\otimes\beta_1.$$

(Note that this differs from $Z'(\mathrm{I0})$ only in that one of the β_1's has been moved from $V(\partial_1 N)$ to $V(\partial_v N)$.)

Type B1. A type B1 e-slice N is the mapping cylinder of $\mathrm{id}: Y\to Y$ with boundary decomposed as follows. Let $Y = Y'\cup P$, with $Y'\cap P$ consisting of either one (case 1) or two (case 2) components. Absorb the $(\partial P\cap\partial Y)\times I$ part of ∂N into $P\times\{0\}$. Let n be the number of components of $\partial Y'\cap\partial Y$. Let $\partial_0 N = Y'\times\{0\}$, $\partial_v N = ((\partial Y'\cap\partial Y)\times I)\coprod(P\times\{0\}) = A^n\coprod P$, and $\partial_1 N = Y\times\{1\}$ (see Figure 28). For $a,b,c\in\mathcal{L}$, let $\{\gamma^j_{abc}\}$ be a basis of V_{abc} and let $\{\delta^j_{\hat{a}\hat{c}\hat{b}}\}$ be the dual basis of $V_{\hat{a}\hat{c}\hat{b}}$. ($V_{\hat{a}\hat{c}\hat{b}}$ is

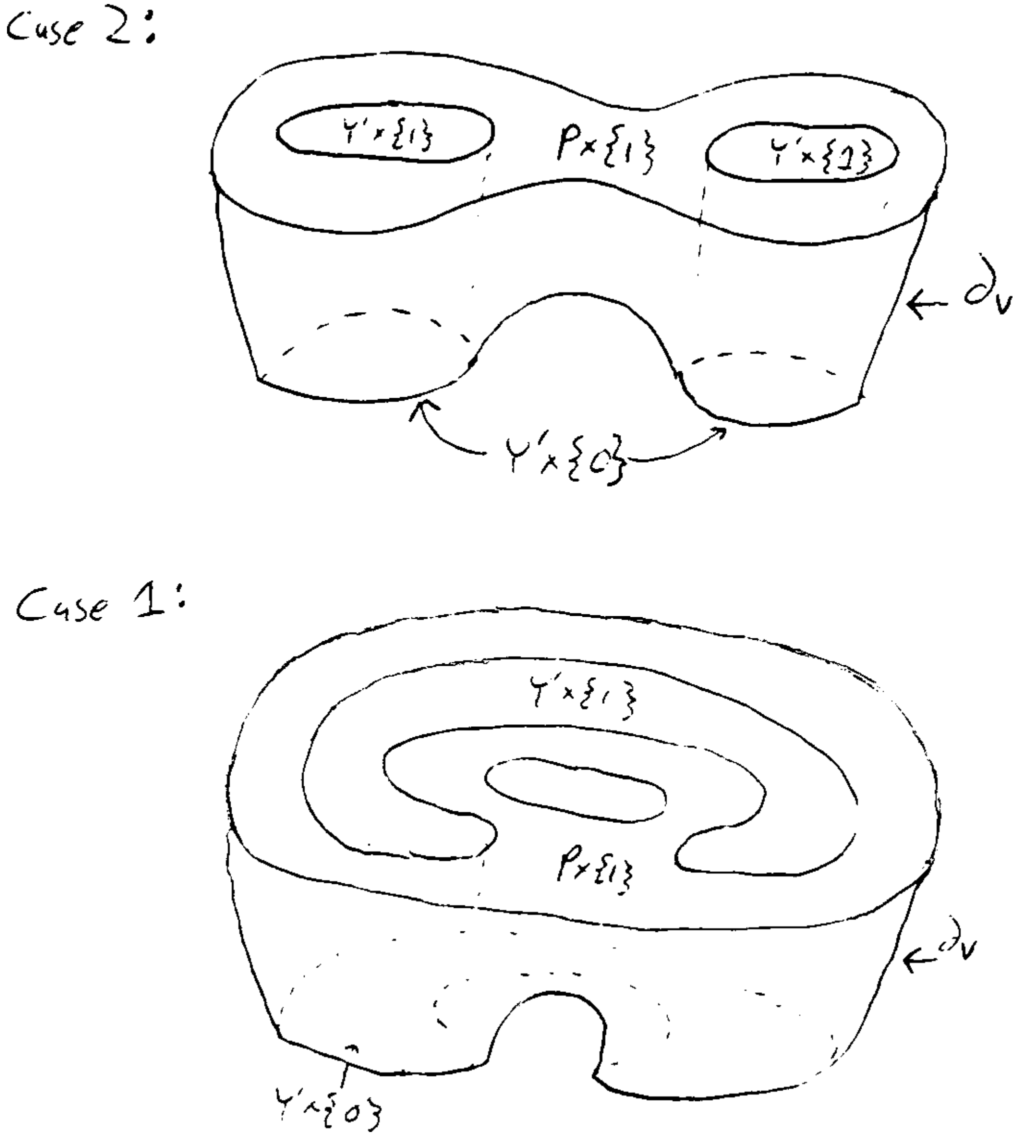

Figure 28: Type B1 e-slices.

identified with V^*_{abc} via the standard orientation reversing map (see Figure 7).) In case 1 $Z'(N)$ is defined to be

$$\bigoplus_{l,a} x_{la} \mapsto \bigoplus_{l,a,b,c} \sum_j S(a)^{-1}[\beta_l \otimes \gamma^j_{abc}] \otimes \delta^j_{\hat{a}\hat{c}\hat{b}} \otimes x_{la},$$

where a labels $Y' \cap P$, b and c label the other two components of ∂P, l labels the other n components of $\partial Y'$, and $x_{la} \in V(Y', (l, a))$. Similarly, in case two we define $Z'(N)$ by

$$\bigoplus_{l,b,c} x_{lbc} \mapsto \bigoplus_{l,a,b,c} \sum_j S(b)^{-1} S(c)^{-1}[\beta_l \otimes \gamma^j_{abc}] \otimes \delta^j_{\hat{a}\hat{c}\hat{b}} \otimes x_{lbc}.$$

Type B2. A type B2 e-slice N can be thought of as the mapping cylinder of id : $Y \cup D \to Y \cup D$, with the bottom copy of D belonging to $\partial_v N$ rather than $\partial_0 N$. Define $Z'(N)$ by

$$\bigoplus_{l,a} x_{la} \mapsto \bigoplus_{l} S(1)^{-1}[\beta_l \otimes \beta_1] \otimes x_{l1} \otimes \beta_1.$$

An e-slice of type Ti is just an e-slice of type B$(i-2)$ turned upside-down, and so requires little additional explanation.

Type T0. Define $Z'(N)$ by

$$\bigoplus_{l} x_{l1} \otimes \beta_1 \mapsto \bigoplus_{l} [\beta_l \otimes \beta_1] \otimes x_{l1}.$$

Type T1. In case one define $Z'(N)$ by

$$\bigoplus_{l,a,b,c} \sum_j x^j_{l\hat{a}} \otimes \gamma^j_{abc} \mapsto \bigoplus_{l,a,b,c} \sum_j [\beta_l \otimes \gamma^j_{abc}] \otimes x^j_{l\hat{a}}.$$

In case two define $Z'(N)$ by

$$\bigoplus_{l,a,b,c} \sum_j x^j_{l\hat{b}\hat{c}} \otimes \gamma^j_{abc} \mapsto \bigoplus_{l,a,b,c} \sum_j [\beta_l \otimes \gamma^j_{abc}] \otimes x^j_{l\hat{b}\hat{c}}.$$

Type T2. Define $Z'(N)$ by

$$\bigoplus_{l} x_l \otimes \beta_1 \mapsto \bigoplus_{l} [\beta_l \otimes \beta_1] \otimes x_l.$$

(7.5) *Remark.* There is some redundancy in the above definitions. Namely, Ii e-slices are $I(i-3)$ e-slices turned upside-down, and Bi e-slices are $T(i-2)$ e-slices turned upside-down. It is left to the reader to check that this redundancy is consistent.

For s an e-slicing define $Z'(s) \in V(\partial(M(s)))$ by applying (2.9) to $Z'(\{\text{slices}\})$ in the obvious way. We must show that if s_1 and s_2 are e-slicings with $M(s_1)^\flat \cong M(s_2)^\flat$, then

$$Z'(s_1) \equiv Z'(s_2). \tag{7.6}$$

This will be done in two steps. The first is to verify (7.6) in the case where $s_1^\flat = s_2^\flat$. The second is to show that if $s_1^\flat$ and $s_2^\flat$ are slicings which differ by one of the moves of (7.4), then there exist extensions s_j of $s_j^\flat$ which satisfy (7.6). It will follow (by (7.2)) that for any e-3-manifold M we can unambiguously define $Z(M) \equiv Z'(s)$, where s is an e-slicing such that $M^\flat \cong M(s)^\flat$.

To show that $Z'(s_1) \equiv Z'(s_2)$ if $s_1^\flat = s_2^\flat$, it suffices to consider the case where s_1 and s_2 coincide except for a single e-slice. It is not hard to see that changing a single e-slice (within its homeomorphism class) has the same effect on Z' as inserting appropriate trivial e-slices before and after the slice in question. Thus it suffices to show that the insertion of a trivial slice does not affect Z'.

Suppose that the trivial slice corresponding to

$$(\mathrm{id}, m) : (\partial_1 N_i^\flat, L_1') \to (\partial_0 N_{i+1}^\flat, L_0')$$

is inserted between the e-slices N_i and N_{i+1} of an e-slicing s. Let $\partial_1 N_i = (\partial_1 N_i^\flat, L_1)$ and $\partial_0 N_{i+1} = (\partial_0 N_{i+1}^\flat, L_0)$. By (1.12), the framing number of $M(s)$ changes by

$$m + \sigma(K, L_1, L_1') + \sigma(K, L_1', L_0') + \sigma(K, L_0', L_0) - \sigma(K, L_1, L_0), \tag{7.7}$$

where K is the kernal of the appropriate inclusion-induced map. Also, the lagrangian of $M(s)$ does not change. It is easy to see that $Z'(M)$ changes by C raised to the power

$$m + \sigma(L_1, L_1', L_0') + \sigma(L_1, L_0', L_0). \tag{7.8}$$

So what must be shown is that (7.7) and (7.8) are equal. This follows from two applications of (18.6).

Next we must show that if $s_1^\flat$ and $s_2^\flat$ are two slicings which differ by one of the 19 moves of (7.4), then there are extensions s_1 and s_2 of $s_1^\flat$ and $s_2^\flat$ such that $Z'(s_1) \equiv Z'(s_2)$. We will see that the only relation this imposes on V is (4.7).

Move 1. (Insertion of a trivial slice.) Invariance under extensions of this move was shown above.

Move 2. This move corresponds to exchanging the order of attachment of two *disjoint* handles. Except in the case where one of the corresponding slices is of type B1 and the other is of type T1, it is easy to see that this does not affect Z'. For the B1-T1 case, the T1 slice can be converted into B1 and I2 slices using move 14, the B1 slice can be moved past these two slices, and then the inverse of move 14 will put things back on track.

Move 3. (I0-I1 cancellation.) Let N_i be the type I0 e-slice and N_{i+1} be the type I1 e-slice. Since N_i and N_{i+1} cancel, one of the attaching disks of the 1-handle associated to N_{i+1} lies in the boundary of the 0-handle part of N_i and the other attaching disk does not. Let $\partial_0 N_i = Y \cup D$, where D corresponds to one of the attaching disks. A general element of $V(\partial_0 N_i, l)$ can be written as $x_l \otimes \beta_1$, where $x_l \in V(Y, (l, 1))$ and $\beta_1 \in V(D, 1) = V_1$. $Z'(N_i)$ is

$$\bigoplus_l x_l \otimes \beta_1 \mapsto \bigoplus_l [\beta_l] \otimes x_l \otimes \beta_1 \otimes \beta_1 \otimes \beta_1,$$

where the second β_1 on the right hand side corresponds to the attaching disk on the boundary of the 0-handle and the third β_1 corresponds to its complement. $Z'(N_{i+1})$ is

$$\bigoplus_l x_l \otimes \beta_1 \otimes \beta_1 \otimes \beta_1 \mapsto \bigoplus_l [\beta_l] \otimes x_l \otimes \beta_{11} \otimes \beta_1.$$

Hence $Z'(N_i \cup N_{i+1})$ is

$$\bigoplus_l x_l \otimes \beta_1 \mapsto \bigoplus_l [\beta_l \otimes \beta_l] \otimes x_l \otimes \beta_{11} \otimes \beta_1.$$

By (3.4), the left hand side is equivalent to

$$\bigoplus_l [\beta_l] \otimes x_l \otimes \beta_1.$$

Hence $Z'(N_i \cup N_{i+1}) = Z'(\text{a trivial slice})$.

It seems unwise to treat each of the remaining 16 moves in as much detail as above, so from now on the explanations will be more terse. As an illustration, we repeat the above argument in the terse format:

$$\begin{aligned} \bigoplus_l x_l \otimes \beta_1 &\overset{I0}{\mapsto} \bigoplus_l [\beta_l] \otimes x_l \otimes \beta_1 \otimes \beta_1 \otimes \beta_1 \\ &\overset{I1}{\mapsto} \bigoplus_l [\beta_l \otimes \beta_l] \otimes x_l \otimes \beta_{11} \otimes \beta_1 \\ &= \bigoplus_l [\beta_l] \otimes x_l \otimes \beta_1. \end{aligned}$$

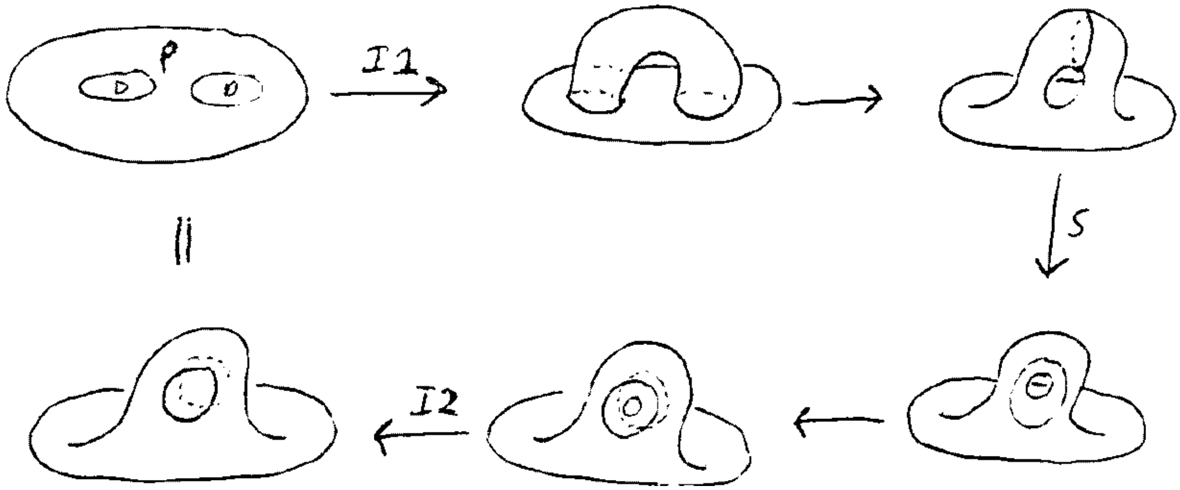

Figure 29: Illustration of I1-I2 cancellation.

Move 4. (I1-I2 cancellation.) Decompose $\partial_0 N_i$ as $Y \cup P \cup D \cup D$, where P is a pair of pants surrounding the attaching disks $D \cup D$. We have

$$\begin{aligned}
\bigoplus_l x_l \otimes \beta_{111} \otimes \beta_1 \otimes \beta_1 \quad & \overset{I1}{\mapsto} \quad \bigoplus_l [\beta_l] \otimes x_l \otimes \beta_{111} \otimes \beta_{11} \\
& = \quad \bigoplus_l [\beta_l] \otimes x_l \otimes \beta_{111} \\
& \approx \quad \bigoplus_l \sum_a S_{1a}[\beta_l] \otimes x_l \otimes \beta_{1a\hat{a}} \\
& = \quad \bigoplus_l \sum_a S_{1a}[\beta_l] \otimes x_l \otimes \beta_{1a\hat{a}} \otimes \beta_{a\hat{a}} \\
& \overset{I2}{\mapsto} \quad \bigoplus_l S_{11} S(1)^{-1} [\beta_l \otimes \beta_l] \otimes x_l \otimes \beta_{111} \otimes \beta_1 \otimes \beta_1 \\
& = \quad \bigoplus_l [\beta_l] \otimes x_l \otimes \beta_{111} \otimes \beta_1 \otimes \beta_1,
\end{aligned}$$

where "$\approx$" indicates the natural map from $V(\partial_1 N_i, l)$ to $V(\partial_0 N_{i+1}, l)$ (see Figure 29). This is equivalent to a trivial e-slice. (Recall that $S(a) = S_{1a}$.)

Move 5. (I2-I3 cancellation.) This is just move 3 turned upside-down. By (7.5), move 5 invariance is equivalent to move 3 invariance.

Move 6. (B0-B1 cancellation.) Note that the B1 e-slice must be of subtype 2. Recall the notation from the definition of $Z'(\mathrm{B1})$. By (6.4.14),

$$S(1)S(a)\beta_{a1\hat{a}} \otimes \beta_{\hat{a}a1} = \mathrm{id} \in V_{a1\hat{a}} \otimes V_{\hat{a}a1},$$

so we may assume that $\gamma_{a1\hat{a}} = S(1)S(a)\beta_{a1\hat{a}}$ and $\delta_{\hat{a}a1} = \beta_{\hat{a}a1}$. We have

$$\begin{aligned}
\bigoplus_{l,a} x_{la} &\overset{B0}{\mapsto} \bigoplus_{l,a} [\beta_l \otimes \beta_{a\hat{a}} \otimes \beta_1] \otimes x_{la} \otimes \beta_1 \\
&\overset{B1}{\mapsto} \bigoplus_{l,a} [\beta_l \otimes \beta_l \otimes \beta_{a\hat{a}} \otimes \beta_1 \otimes \beta_{a1\hat{a}}] \otimes \beta_{\hat{a}a1} \otimes x_{la} \otimes \beta_1 \\
&= \bigoplus_{l,a} [\beta_l \otimes \beta_{a\hat{a}}] \otimes x_{la}.
\end{aligned}$$

Move 7. (B1-B2 cancellation.) Note that the B1 e-slice must be of subtype 1. We have

$$\begin{aligned}
\bigoplus_{l,a} x_{la} &\overset{B1}{\mapsto} \bigoplus_{l,a,b,c} \sum_j S(a)^{-1} [\beta_l \otimes \gamma^j_{abc}] \otimes \delta^j_{\hat{a}\hat{c}\hat{b}} \otimes x_{la} \\
&\overset{B2}{\mapsto} \bigoplus_{l,a} [\beta_l \otimes \beta_l \otimes \beta_{a\hat{a}1} \otimes \beta_1] \otimes \beta_{\hat{a}1a} \otimes \beta_1 \otimes x_{la} \\
&= \bigoplus_{l,a} [\beta_l \otimes \beta_{a\hat{a}}] \otimes x_{la}.
\end{aligned}$$

Move 8. (T0-T1 cancellation.) This is just move 7 turned upside-down. By (7.5), move 8 invariance is equivalent to move 7 invariance.

Move 9. (T1-T2 cancellation.) This is just move 6 turned upside-down.

Move 10. (B0 $\rightsquigarrow$ I0, T0.) We have

$$\begin{aligned}
\bigoplus_{l} x_l &\overset{I0}{\mapsto} \bigoplus_{l} [\beta_l] \otimes x_l \otimes \beta_1 \otimes \beta_1 \\
&\overset{T0}{\mapsto} \bigoplus_{l} [\beta_l \otimes \beta_l \otimes \beta_1] \otimes x_l \otimes \beta_1 \\
&= \bigoplus_{l} [\beta_l \otimes \beta_1] \otimes x_l \otimes \beta_1.
\end{aligned}$$

This is equivalent to a B0 e-slice.

Move 11. (B1 $\rightsquigarrow$ I1, T1.) For $a, b, c \in \mathcal{L}$, define $\{\eta^j_{cab}\}$ and $\{\theta^j_{\hat{c}\hat{b}\hat{a}}\}$ by

$$F(\beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}}) = \bigoplus_c \sum_j \eta^j_{cab} \otimes \theta^j_{\hat{c}\hat{b}\hat{a}} \in \bigoplus_c V_{cab} \otimes V_{\hat{c}\hat{b}\hat{a}}.$$

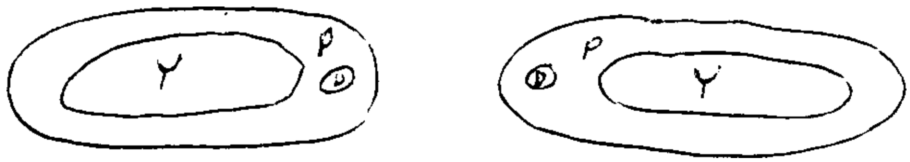

Figure 30: Second case of move 11.

There are two cases, depending on whether the B1 and T1 e-slices are of subtypes 1 or 2. In the second case, decompose $\partial_0(\mathrm{I1})$ as $Y \cup P \cup P \cup D \cup D$, where $D \cup D$ is the attaching region for the 1-handle and each copy of P surrounds one of the D's and contains one of the two boundary components of $\partial_0(\mathrm{I1})$ which will be joined by the T1 e-slice (see Figure 30). We have

$$
\begin{aligned}
&\bigoplus_{l,a,b} x_{lab} \otimes \beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}} \otimes \beta_1 \otimes \beta_1 \\
&\quad \overset{I1}{\mapsto} \bigoplus_{l,a,b} [\beta_l \otimes \beta_{\hat{a}a} \otimes \beta_{b\hat{b}}] \otimes x_{lab} \otimes \beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}} \otimes \beta_{11} \\
&\quad = \bigoplus_{l,a,b} [\beta_l \otimes \beta_{\hat{a}a} \otimes \beta_{b\hat{b}}] \otimes x_{lab} \otimes \beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}} \\
&\quad = \bigoplus_{l,a,b,c} \sum_j [\beta_l \otimes \beta_{\hat{a}a} \otimes \beta_{b\hat{b}}] \otimes x_{lab} \otimes \eta^j_{cab} \otimes \theta^j_{\hat{c}\hat{b}\hat{a}} \\
&\quad \overset{T1}{\mapsto} \bigoplus_{l,a,b,c} \sum_j [\beta_l \otimes \beta_l \otimes \beta_{\hat{a}a} \otimes \beta_{b\hat{b}} \otimes \eta^j_{cab}] \otimes x_{lab} \otimes \theta^j_{\hat{c}\hat{b}\hat{a}} \\
&\quad = \bigoplus_{l,a,b,c} \sum_j [\beta_l \otimes \eta^j_{cab}] \otimes x_{lab} \otimes \theta^j_{\hat{c}\hat{b}\hat{a}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\bigoplus_{l,a,b} x_{lab} \otimes \beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}} \otimes \beta_1 \otimes \beta_1 &= \bigoplus_{l,a,b} x_{lab} \\
&\overset{B1}{\mapsto} \bigoplus_{l,a,b,c} \sum_j S(a)^{-1} S(b)^{-1} [\beta_l \otimes \gamma^j_{cab}] \otimes \delta^j_{\hat{c}\hat{b}\hat{a}} \otimes x_{lab}.
\end{aligned}
$$

Thus the I1 and T1 e-slices are equivalent to the B1 e-slice if, for all $a, b \in \mathcal{L}$,

$$
\bigoplus_c \sum_j \eta^j_{cab} \otimes \theta^j_{\hat{c}\hat{b}\hat{a}} = \bigoplus_c \sum_j S(a)^{-1} S(b)^{-1} \gamma^j_{cab} \otimes \delta^j_{\hat{c}\hat{b}\hat{a}}
$$

or

$$
F(\beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}}) = \bigoplus_c \frac{\mathrm{id}_{cab}}{S(a)S(b)}.
$$

where id_{cab} is the identity in $V_{cab} \otimes V_{\hat{c}\hat{b}\hat{a}}$. This is just (4.7).

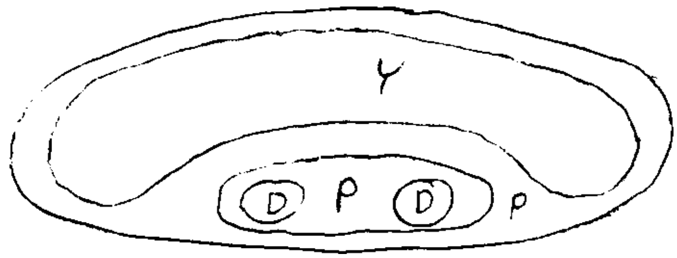

Figure 31: First case of move 11.

For the first case, decompose $\partial_0(\mathrm{I1})$ as $Y \cup P \cup P \cup D \cup D$, where $D \cup D$ is the attaching region for the 1-handle, as shown in Figure 31. We have

$$\begin{aligned}
&\bigoplus_{l,a} x_{la} \otimes \beta_{1\hat{a}a} \otimes \beta_{111} \otimes \beta_1 \otimes \beta_1 \\
&\quad \stackrel{I1}{\mapsto} \bigoplus_{l,a} [\beta_l \otimes \beta_{a\hat{a}}] \otimes x_{la} \otimes \beta_{1\hat{a}a} \otimes \beta_{111} \otimes \beta_{11} \\
&\quad = \bigoplus_{l,a} [\beta_l \otimes \beta_{a\hat{a}}] \otimes x_{la} \otimes \beta_{1\hat{a}a} \otimes \beta_{111} \\
&\quad \approx \bigoplus_{l,a,b} S_{1b} [\beta_l \otimes \beta_{a\hat{a}}] \otimes x_{la} \otimes \beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}} \\
&\quad = \bigoplus_{l,a,b,c} \sum_j S(b) [\beta_l \otimes \beta_{a\hat{a}}] \otimes x_{la} \otimes \eta^j_{cab} \otimes \theta^j_{\hat{c}\hat{b}\hat{a}} \\
&\quad \stackrel{T1}{\mapsto} \bigoplus_{l,a,b,c} \sum_j S(b) [\beta_l \otimes \beta_l \otimes \beta_{a\hat{a}} \otimes \eta^j_{cab}] \otimes x_{la} \otimes \theta^j_{\hat{c}\hat{b}\hat{a}} \\
&\quad = \bigoplus_{l,a,b,c} \sum_j S(b) [\beta_l \otimes \eta^j_{cab}] \otimes x_{la} \otimes \theta^j_{\hat{c}\hat{b}\hat{a}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\bigoplus_{l,a} x_{la} \otimes \beta_{1\hat{a}a} \otimes \beta_{111} \otimes \beta_1 \otimes \beta_1 &= \bigoplus_{l,a} x_{la} \\
&\stackrel{B1}{\mapsto} \bigoplus_{l,a,b,c} \sum_j S(a)^{-1} [\beta_l \otimes \gamma^j_{cab}] \otimes \delta^j_{\hat{c}\hat{b}\hat{a}} \otimes x_{la}.
\end{aligned}$$

The above two expressions are equivalent if, for all $a, b \in \mathcal{L}$,

$$F(\beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}}) = \bigoplus_c \frac{\mathrm{id}_{cab}}{S(a)S(b)}.$$

Again, this is just (4.7).

Move 12. (B2 $\rightsquigarrow$ I2, T2.) We have

$$\begin{aligned}\bigoplus_{l,a} x_{la}\otimes\beta_{a\hat{a}} &\stackrel{I2}{\mapsto} \bigoplus_{l} S(1)^{-1}[\beta_l]\otimes x_{l1}\otimes\beta_1\otimes\beta_1 \\ &\stackrel{T2}{\mapsto} \bigoplus_{l} S(1)^{-1}[\beta_l\otimes\beta_l\otimes\beta_1]\otimes x_{l1}\otimes\beta_1 \\ &= \bigoplus_{l} S(1)^{-1}[\beta_l\otimes\beta_1]\otimes x_{l1}\otimes\beta_1.\end{aligned}$$

This is equivalent to $Z'(\mathrm{B2})$.

Move 13. (T0 $\rightsquigarrow$ B0, I1.) This is move 12 upside-down.

Move 14. (T1 $\rightsquigarrow$ B1, I2.) This is move 11 upside-down.

Move 15. (T2 $\rightsquigarrow$ B2, I3.) This is move 10 upside-down.

Move 16. (Add B0 to bottom.) Easy.

Move 17. (Add B1 to bottom.) Easy.

Move 18. (Add T1 to top.) Easy.

Move 19. (Add T2 to top.) Easy.

We have just shown that given a modular functor V which satisfies (4.7), one can unambiguously define an element $Z(M)\in V(\partial M)$ for each e-3-manifold M. (To repeat, $Z(M)$ is defined by $Z(M)\equiv Z'(s)$, where s is any slicing such that $M(s)^\flat\cong M^\flat$.)

Next we show that Z, as defined above, satisfies the axioms (2.7) through (2.10). By the nature of its definition it is obvious that Z satisfies (2.7), (2.8) and (2.10). For (2.9), first assume that M can be decomposed as $M_1\coprod M_2$, with $Y_j\subset\partial M_j$. Choose slicings s_1 and s_2 such that $M(s_j)^\flat = M_j$, $\partial_1(M(s_1)) = Y_1$, and $\partial_0(M(s_2)) = Y_2$. (This is possible by (7.2).) Using the fact that s_1 and s_2 can be glued together to give a slicing of $M(s_1)\cup_f M(s_2)$, it is easy to see that (2.9) holds for the gluing

$$M(s_1)\coprod M(s_2)\rightsquigarrow M(s_1)\cup_f M(s_2).$$

It is also easy to see that this implies that (2.9) holds for the gluing

$$M_1 \coprod M_2 \rightsquigarrow M_1 \cup_f M_2.$$

Now assume that Y_1 and Y_2 lie in the same component of M. Let I_f be the mapping cylinder of f. Let g be the "identity" map from $Y_1 \coprod Y_2 \subset \partial M$ to $Y_1 \coprod Y_2 \subset \partial I_f$. Using (2.12) (which follows from the definition of Z), it is easy to see that the truth of (2.9) for the gluing

$$M \rightsquigarrow M_f$$

is equivalent to the truth of (2.9) for the gluing

$$M \coprod I_f \rightsquigarrow M \cup_g I_f.$$

But we have already shown that (2.9) hold for the latter gluing. This completes the proof that Z satisfies the axioms.

The results of this section may be summarized as follows.

(7.9) **Theorem.** *A modular functor V admits a compatible partition function Z if and only if*

$$F(\beta_{1\hat{a}a} \otimes \beta_{1b\hat{b}}) = \bigoplus_c \left(\frac{1}{S(a)S(b)} \right) \mathrm{id}_{cab}$$

for all $a, b \in \mathcal{L}$, in which case Z is unique. □

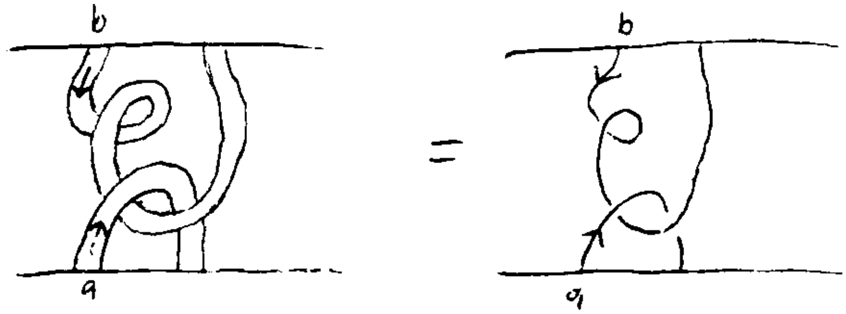

Figure 32: Convention for drawing ribbon tangles.

8 Review of [RT2]

In this section we present a slightly modified version of the work of Reshetihkin and Turaev on modular Hopf algebras and 3-manifold invariants [RT2]. These results will be used in the next section to construct TQFTs starting from modular Hopf algebras (as defined in [RT2] or below). The proof of the key lemma, (8.21), was obtained during a conversation with V. Turaev. (This lemma might not seem very important to the reader, but once it is proved the results of the next section are more or less obvious, assuming one is well versed in the results of [RT2].) It is assumed that the reader has some familiarity with the results of [RT1]. (These are reviewed in [RT2] and, very briefly, here.)

First we give a brief review of the results of [RT1]. (See [RT1] of [RT2] for details and definitions.) Let U be a ribbon Hopf algebra (RHA). Let $\operatorname{Rep} U$ denote the category of finite dimensional representations of U. (The objects are representations and the morphisms are U-linear maps between representations. The dual of a representation a will be denoted $\check{a}$.) Let $\operatorname{Tang} U$ denote the category of (isotopy classes of oriented, homogeneous, directed) ribbon tangles in $\mathbf{R}^2 \times I$ labeled by objects of $\operatorname{Rep} U$. 'Oriented' means that the ribbons have a prefered side. 'Homogeneous' means that at the ends of the ribbons the prefered side faces frontward. 'Directed' means that the cores of the ribbons are oriented. The objects of $\operatorname{Tang} U$ are labeled sequences of directed ribbon ends. The morphisms are tangles. $t_1 t_2$ is given by placing t_2 above t_1. $t_1 \otimes t_2$ is given by placing t_1 and t_2 side by side.

In the figures, ribbons will be drawn as arcs or circles. The ribbons can be reconstructed by using the "blackboard framing" (see Figure 32.)

Reshetihkin and Turaev construct a functor $[\,\cdot\,] : \operatorname{Tang} U \to \operatorname{Rep} U$. This functor has the following properties (see Figures 33 and 34):

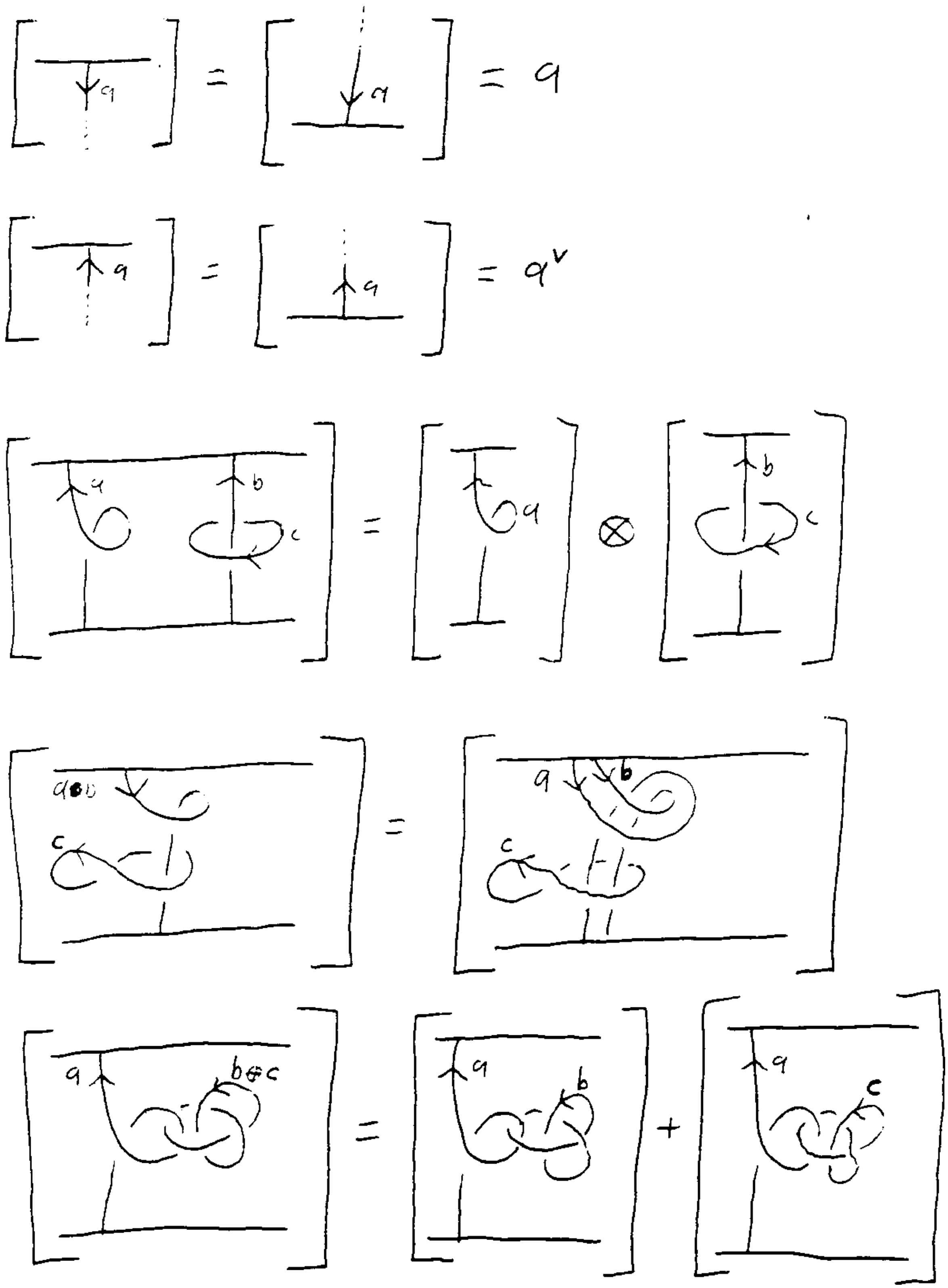

Figure 33: Properties of the functor $[\,\cdot\,]$

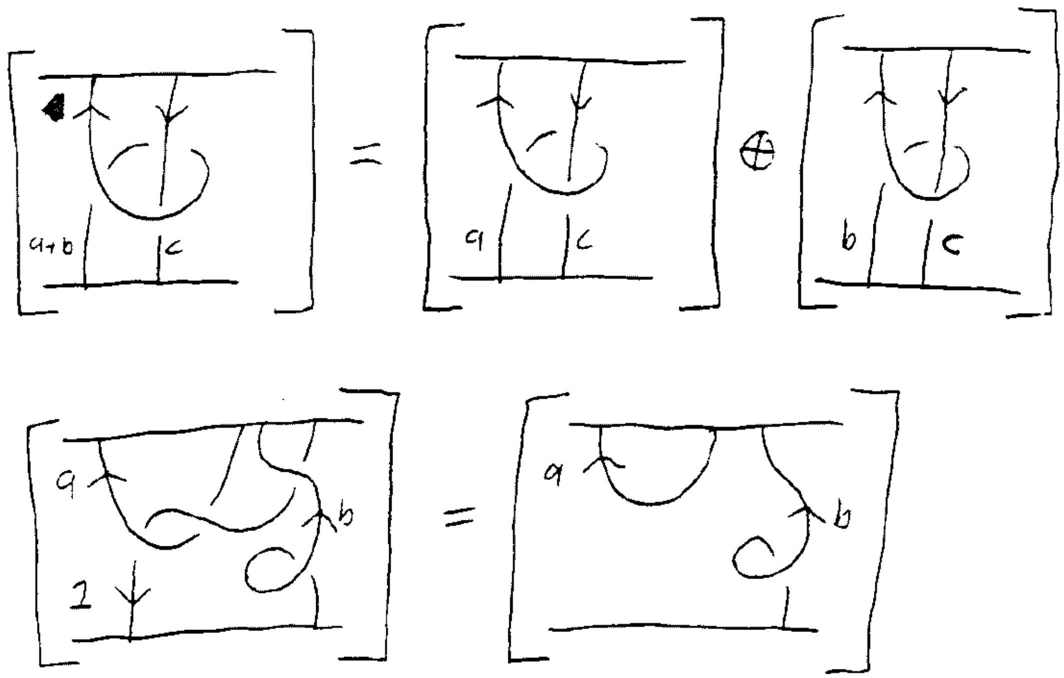

Figure 34: More properties of the functor $[\,\cdot\,]$

(8.1) Let x be an obect of Tang U consisting of a single ribbon end labeled by the representation a. If x is directed downward, then $[x] = a$. If x is directed upward, then $[x] = a\check{}$. Also, $[\emptyset] = 1$, where $\emptyset$ is the empty object of Tang U and 1 is the trivial representation of U.

(8.2) $[\,\cdot\,]$ preserves tensor products.

(8.3) Let t be a tangle which contains a ribbon r labeled by $a \otimes b$. Let t' be the tangle obtained by replacing r with two parallel copies of r labeled by a and b. Then $[t] = [t']$.

(8.4) Let t be a tangle and let r be a closed ribbon (annulus) of t labeled by $a \oplus b$. Let t_1 and t_2 be the tangles obtained by changing the label of r to a and b, respectively. Then $[t] = [t_1] + [t_2]$. Similarly, if r connects the top and bottom of the tangle, then $[t] = [t_1] \oplus [t_2]$.

(8.5) Let t be a tangle with a ribbon r labeled by the trivial representation. Let t' be t with r deleted. Then $[t] = [t']$.

We won't really be concerned with the RHA U, but rather with the functor $[\,\cdot\,]$. Such functors will be called *tangle functors*.

It will be useful to add morphisms to Tang U, obtaining the category Graph U of

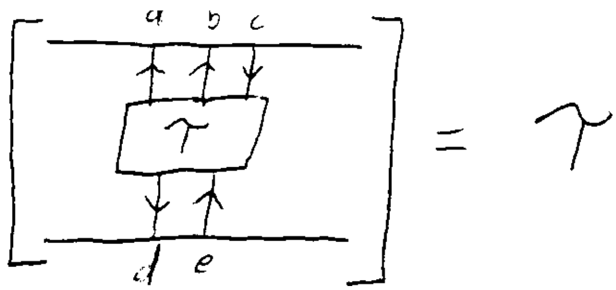

Figure 35: The definition of $[\,\cdot\,]$ on coupons.

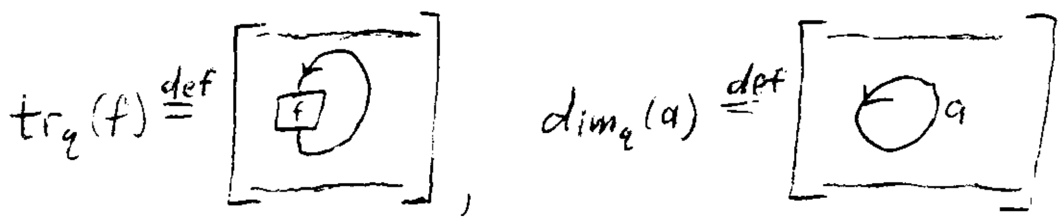

Figure 36: The definition of tr_q and $\dim_q$.

labeled ribbon graphs. A ribbon graph contains, in addition to ribbons, "coupons": rectangles with ribbons incident to the top and the bottom. Coupons are labeled with appropriate morphisms in $\mathrm{Rep}\,U$. The functor $[\,\cdot\,]$ extends to $\mathrm{Graph}\,U$, as illustrated in Figure 35.

The *quantum trace* of a morphism $f : a \to a$ of $\mathrm{Rep}\,U$, $\mathrm{tr}_q f$, is defined in Figure 36. The *quantum dimension* of a representation a, $\dim_q a$, is defined to be the quantum trace of the identity morphism of a.

Let $R(U)$ denote the semiring of representations of U. For $R \subset R(U)$ a subsemiring, define R_B, the *bad part of* R, to be the set of all $x \in R$ such that $\mathrm{tr}_q f = 0$ for all $f \in \mathrm{End}(x)$. Define R_G, the *good part of* R, to be the subsemigroup of R generated by all irreducible representations $a \in R$ such that $\dim_q a \neq 0$. (A representation a of U is irreducible if it contains no U-invariant subspace. This this a stronger condition than indecomposability. a is indecomposable if it cannot be written (U-invariantly) as $a_1 \oplus a_2$.)

Note that if $a \in R$ is irreducible, then $a \in R_B$ if and only if $\dim_q a = 0$. (This follows from Schur's lemma.) Note also that for $x \in R_B$ and $a \in R$, $x \otimes a, a \otimes x \in R_B$. In other words,

(8.6) $$R \otimes R_B = R_B \otimes R = R_B.$$

For a proof, see Figure 37.

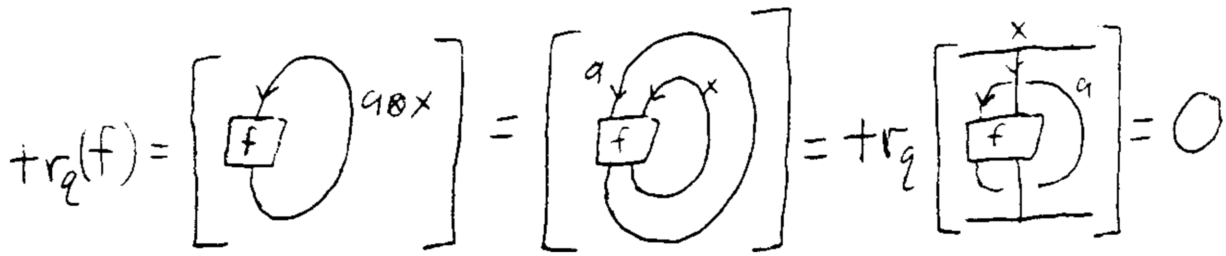

Figure 37: Proof of (8.6)

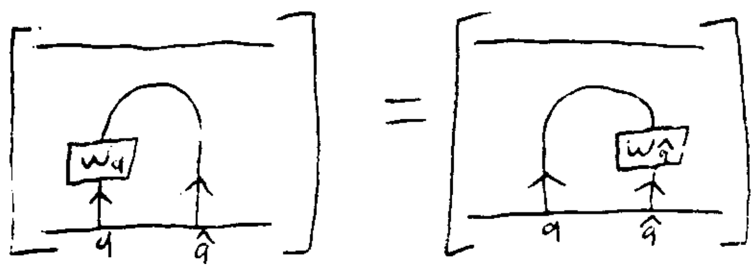

Figure 38: Part of (8.7)

We are now ready to define a modular Hopf algebra (MHA). The definition given here differs slightly from the one given in [RT2], but by the end of this section it should be apparent that the two definitions are equivalent. This fact is left to the reader to verify.

Definition. A *modular Hopf algebra* consists of an RHA U and a finite set $\mathcal{L}$ of finite dimensional, irreducible, mutually non-isomorphic representations of U. It is assumed that $1 \in \mathcal{L}$ and that $\mathcal{L}$ is equiped with an involution $a \leftrightarrow \hat{a}$ such that $1 = \hat{1}$. U and $\mathcal{L}$ are required to satisfy axioms (8.7) through (8.11) below.

(8.7) There exist U-linear isomorphisms

$$w_a : a\check{} \to \hat{a}$$

(for all $a \in \mathcal{L}$) such that $w_1 = \mathrm{id}$ and the identity in Figure 38 holds. (This allows us to identify a ribbon end labeled by a with an oppositely directed ribbon end labeled by $\hat{a}$. Figure 38 guarantees that this identification extends to tangles.)

(8.8) Let $R(\mathcal{L})$ be the subsemiring of $R(U)$ generated by $\mathcal{L}$. Then $R(\mathcal{L})_G$ is equal to the subsemigroup of $R(\mathcal{L})$ generated by $\mathcal{L}$. Furthermore, any $x \in R(\mathcal{L})$ can be written uniquely (up to isomorphism) as

$$x = x_G \oplus x_B,$$

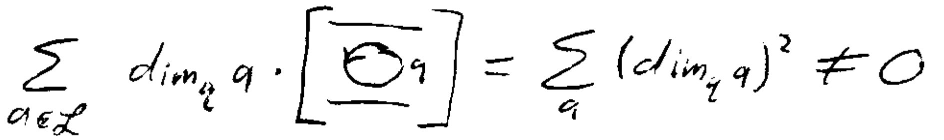

Figure 39: An axiom for MHAs.

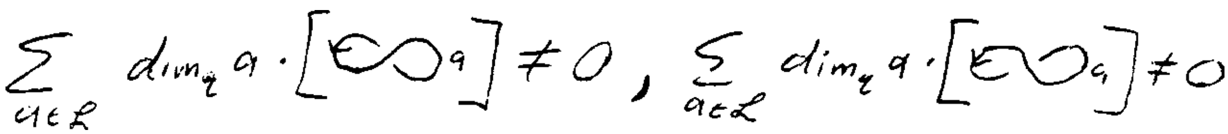

Figure 40: Another axiom for MHAs.

with $x_G \in R(\mathcal{L})_G$ and $x_B \in R(\mathcal{L})_B$. More concretely, any $x \in R(\mathcal{L})$ can be written uniquely as

$$\begin{aligned} x &= \left(\bigoplus_{a\in\mathcal{L}} x_a\right) \oplus x_B \\ x_a &\cong N_x^a a, \end{aligned}$$

where $x_B \in R(\mathcal{L})_B$, $N_x^a \in \mathbf{Z}_{\geq 0}$, and $N_x^a a$ denotes the direct sum of N_x^a copies of a. (Note that this axiom implies that $\dim_q a \neq 0$ for all $a \in \mathcal{L}$.)

The last three axioms are nondegeneracy assumptions.

(8.9) [See Figure 39.]

(8.10) [See Figure 40.]

(We will see below that one half of (8.10) implies the other.)

(8.11) For all $a \in \mathcal{L}$, $a \neq 1$, there exist tangles t and t' such that (i) each of t and t' have a closed ribbon labeled by a, (ii) t and t' are equal if this component is deleted, and (iii) $[t] \neq [t']$.

It is easy to see that for $x, y \in R(\mathcal{L})$ and $f : x \to y$ a morphism, $f(x_G) \subset y_G$. Let f_G denote the restriction of f to x_G. Similarly, for $a \in \mathcal{L}$, $f(x_a) \subset y_a$. Let f_a denote the restriction of f to x_a. Note that for morphisms f and g,

(8.12) $$(fg)_G = f_G g_G$$

(8.13) $$(f \otimes g)_G = (f_G \otimes g_G)_G$$

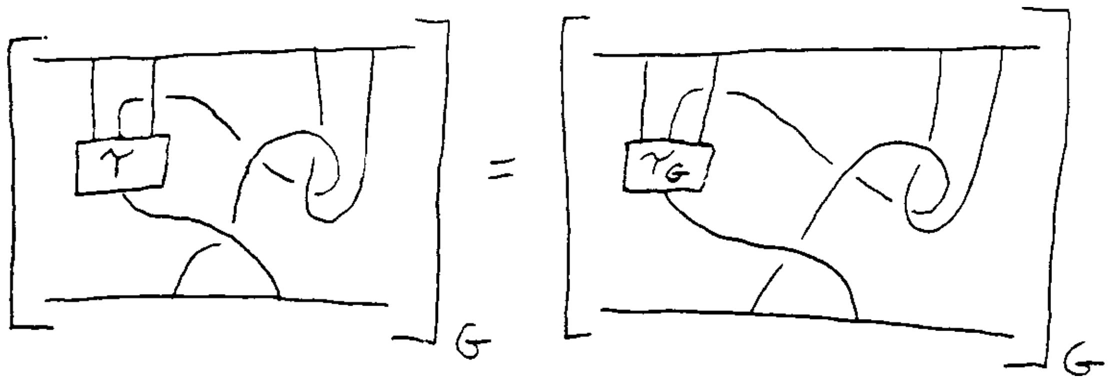

Figure 41: The good part of the whole is equal to the tensor product and composition of the good parts of its parts.

(The second equation follows from (8.6).) (8.12) and (8.13) imply the substitution property illustrated in Figure 41

We won't really be concerned with the MHA U, but rather with the functor $[\,\cdot\,]_G$. Call such functors *modular tangle functors.*

We are now ready to define invariants of labeled ribbon graphs in the presence of surgery diagrams. Define a *extended surgered ribbon graph* (ESRG) to be a morphism of Graph U together with a disjoint collection of ribbons (the "surgery ribbons") in the complement of the graph. The surgery ribbons are neither labeled nor directed. The labels of (the ribbon part of) the graph are required to be in $R(\mathcal{L}) \subset R(U)$. By surgering the surgery ribbons, an ESRG leads to a ribbon graph in an oriented 3-manifold. Two ESRGs are defined to be equivalent if the corresponding pairs of (3-manifold, ribbon graph) are equivalent and if the signatures of the linking matrices of the surgery ribbons are equal. (The signature should be thought of as the framing number of an extension of the 3-manifold. See Section 1.) By Kirby's theorem [K], two ESRGs are equivalent if and only if they differ by a sequence of handle slides and balanced [de]stabilizations. (A balanced [de]stabilization is the addition [deletion] of an unlinked pair of surgery ribbons with framings 1 and -1. See Figure 42 for an illustration. A dot on a component indicates that it is a surgery ribbon.)

Choose $X \in \mathbf{C}^\times$ such that

$$X^2 = \sum_{a\in\mathcal{L}} (\dim_q a)^2.$$

For $a \in \mathcal{L}$, define

$$s(a) \stackrel{\text{def}}{=} \frac{\dim_q a}{X}.$$

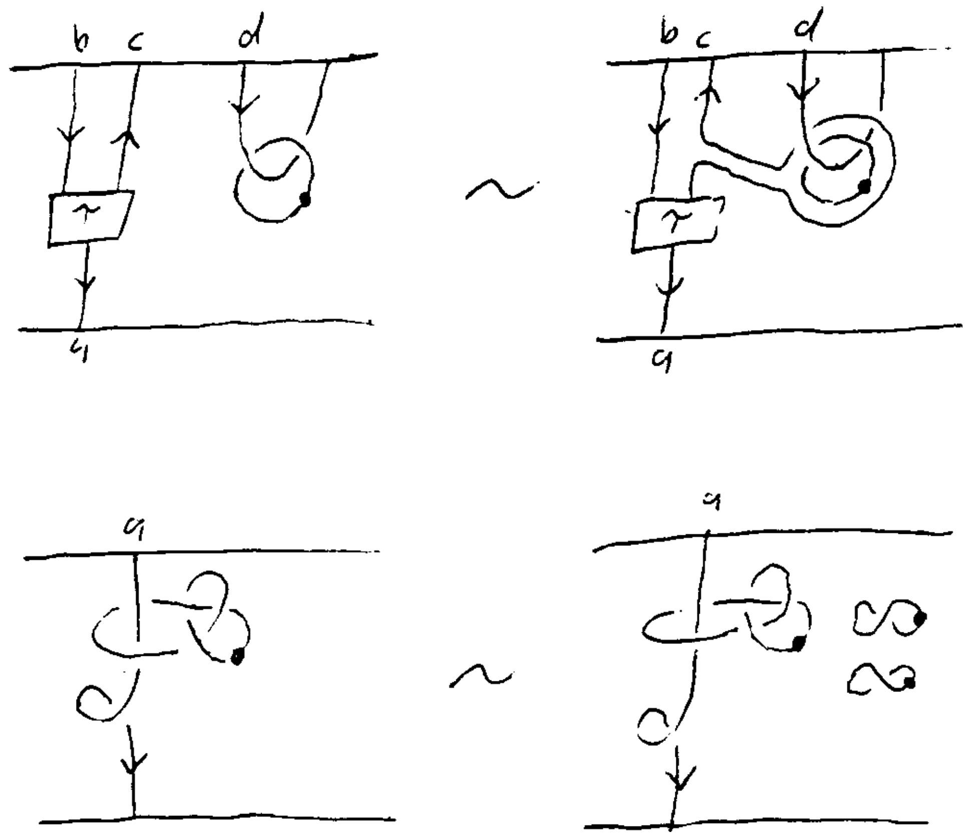

Figure 42: Moves for ESRGs.

For $l = (a_1, \dots, a_n)$ a sequence of elements of $\mathcal{L}$, define

$$s(l) \stackrel{\text{def}}{=} \prod_{i=1}^{n} s(a_i).$$

The following lemma is a consequence of (8.7) and is proved in [RT2].

(8.14) **Lemma.** *Let t be a ribbon graph with a closed ribbon labeled by $a \in \mathcal{L}$ Let t' be the ribbon graph obtained by reversing the direction of this ribbon and changing its label to $\hat{a}$. Then $[t] = [t']$.* □

It follows that

$$s(a) = s(\hat{a}) \tag{8.15}$$

for all $a \in \mathcal{L}$.

For t an ESRG and l a labeling of the surgery ribbons by elements of $\mathcal{L}$, let $(t; l)$ denote the corresponding ribbon graph. This entails choosing directions for the surgery ribbons. Finally, define, for t an ESRG,

$$[t] = \sum_{l} s(l)[(t; l)],$$

where the sum is taken over all labelings of the surgery ribbons (by elements of $\mathcal{L}$). It follows from (8.14) and (8.15) that $[t]$ does not depend on the choice of directions for the surgery ribbons.

We will show that $[t]_G$ (the good part of $[t]$) depends only on the equivalence class of the ESRG t.

(8.16) **Lemma.** *Let y be a closed ESRG. Let t be the ESRG obtained by placing y beside the identity tangle with label $b \in R(\mathcal{L})$. Let t' be the ESRG obtained by sliding the b ribbon over a surgery ribbon of y. Then $[t]_G = [t']_G$. (See Figure 43.)*

Proof: By (8.4), we may assume that $b \in \mathcal{L}$, and hence that $[t] = [t]_G$, $[t'] = [t']_G$. Since b is irreducible, $[t] = [t']$ if and only if $\mathrm{tr}_q[t] = \mathrm{tr}_q[t']$. Let (t;a) denote the ESRG obtained by changing the surgery ribbon on which the handle slide occurs to a ribbon labeled by $a \in R(\mathcal{L})$. It follows from Figure 44 that

$$\begin{aligned}
\mathrm{tr}_q[t] &= \sum_{a \in \mathcal{L}} s(a)[(t; a)]\mathrm{dim}_q b \\
\mathrm{tr}_q[t'] &= \sum_{a \in \mathcal{L}} s(a)[(t; a \otimes b)] && \text{(by (8.2))} \\
&= \sum_{a \in \mathcal{L}} s(a) \left(\textstyle\sum_{c \in \mathcal{L}}[(t; (a \otimes b)_c)] + [(t; (a \otimes b)_B)]\right) && \text{(by (8.4))} \\
&= \sum_{a,c \in \mathcal{L}} s(a) N^c_{a \otimes b}[(t; c)] && \text{(by (8.4))}
\end{aligned}$$

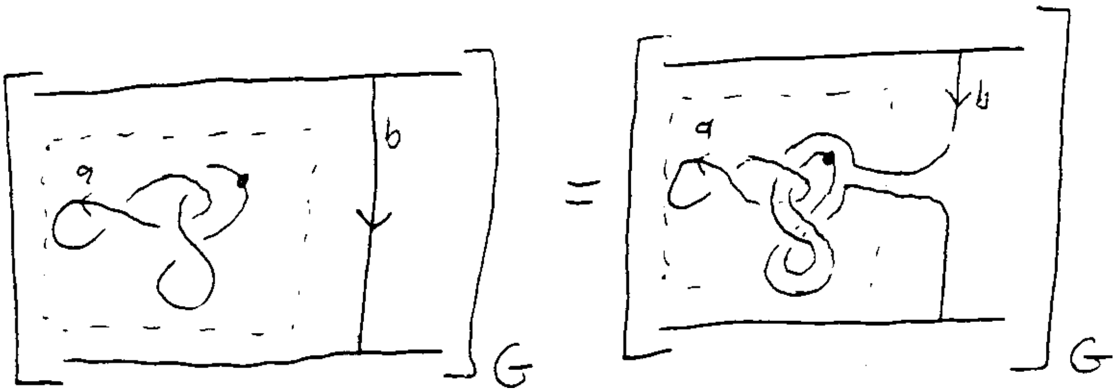

Figure 43: Picture for (8.16)

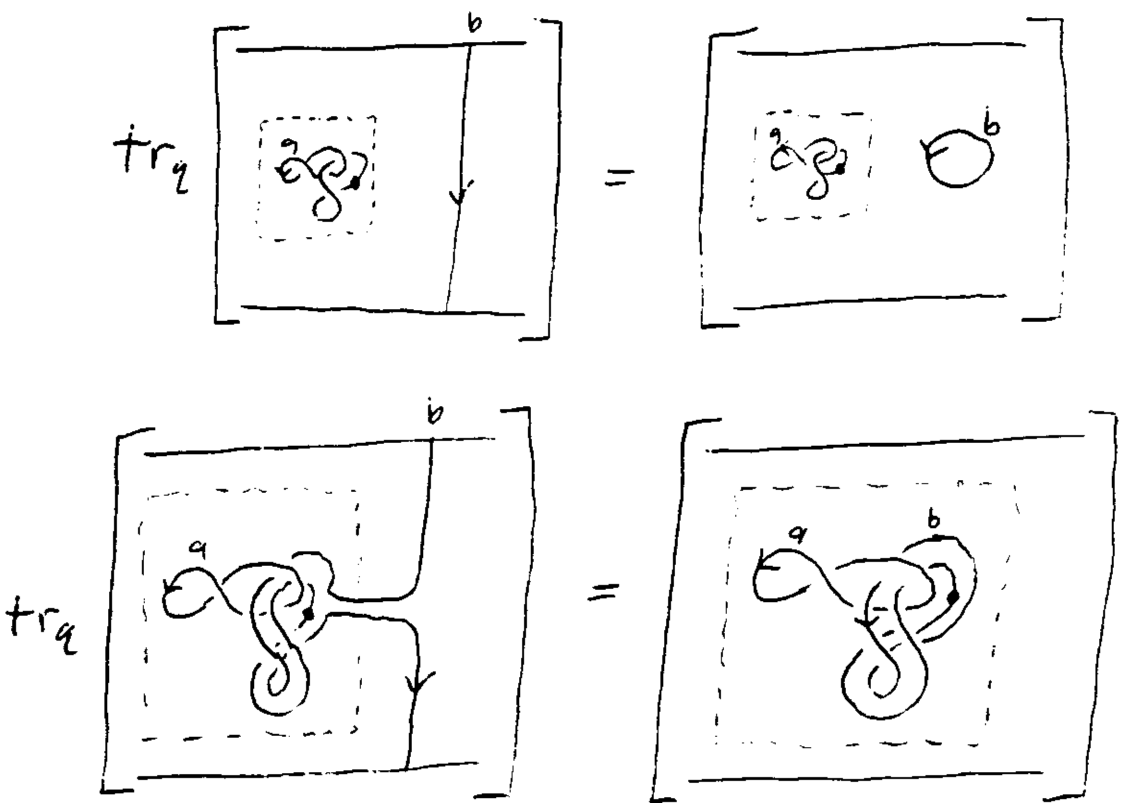

Figure 44: Part of the proof of (8.16).

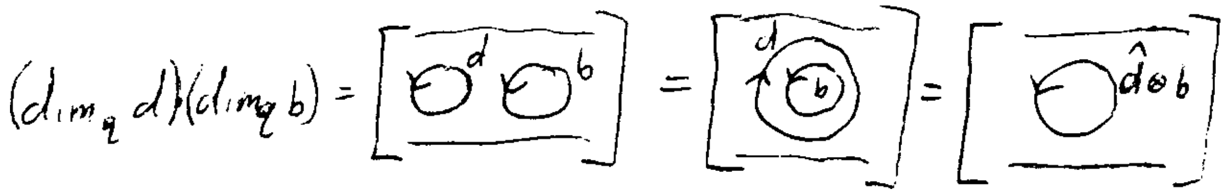

Figure 45: Part of the proof of (8.16).

So the lemma will be proved if it can be shown, for all $d \in \mathcal{L}$, that

$$s(d)\dim_q b = \sum_{a\in\mathcal{L}} s(a) N^d_{a\otimes b}.$$

This is equivalent to

(8.17) $$(\dim_q d)(\dim_q b) = \sum_{a\in\mathcal{L}} (\dim_q a) N^d_{a\otimes b}.$$

It is easy to see that

$$N^d_{a\otimes b} = N^1_{a\otimes b\otimes \hat{d}} = N^{\hat{a}}_{b\otimes\hat{d}}.$$

Keeping in mind that $\dim_q a = \dim_q \hat{a}$, we see that the right hand side of (8.17) can be rewritten as

$$\sum_{a\in\mathcal{L}} (\dim_q \hat{a}) N^{\hat{a}}_{b\otimes\hat{d}} = \dim_q(b\otimes\hat{d}).$$

(8.17) now follows from Figure 45 . □

(8.18) **Corollary.** *[See Figure 46.]*

Proof: [See Figure 47.] □

(8.19) **Corollary.** *Let t and t' be two ESRGs which differ by a handle slide. Then $[t] = [t']$.*

Proof: Define $C_+, C_- \in \mathbf{C}$ as in Figure 48. It follows from (8.10) that $C_\pm \neq 0$. This being the case, it suffices prove the result after t and t' have been stabilized with equal numbers of unlinked, ± 1-framed surgery ribbons. (Both $[t]$ and $[t']$ change by $C^p_+ C^q_- \neq 0$.) The corollary now follows from (8.18), (8.12), (8.13), and a well known result of Fenn and Rourke [FR]. □

(8.20) **Lemma.** *Let t_b be the ESRG shown in Figure 49 ($b \in R(\mathcal{L})$). Then $[t_b]_G$ is equal to X time the projection onto b_1 (the trivial part of b).*

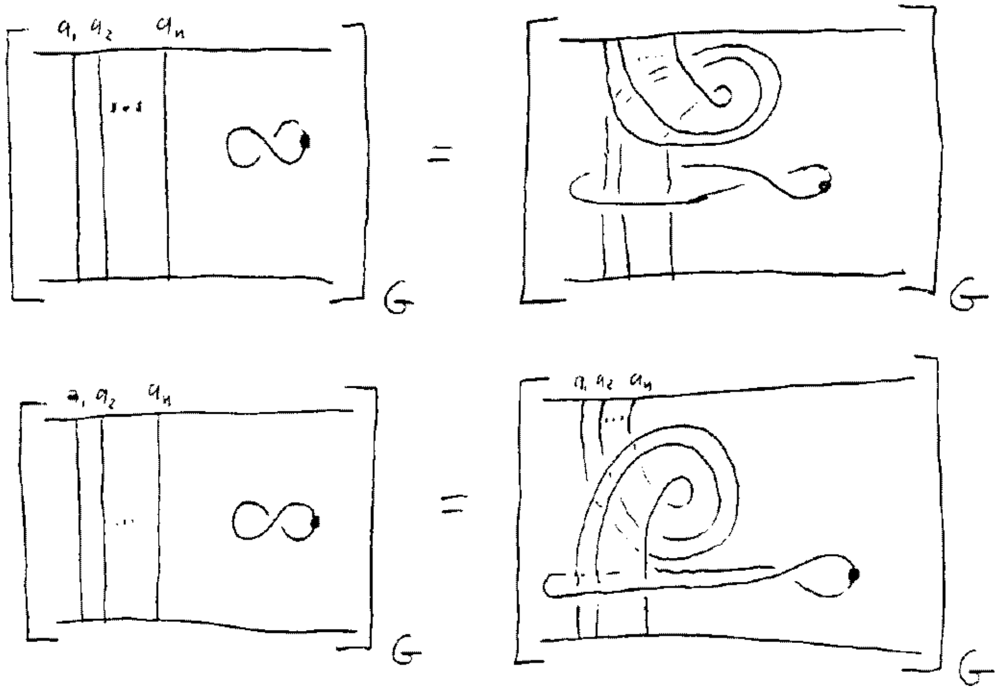

Figure 46: Statement of (8.18).

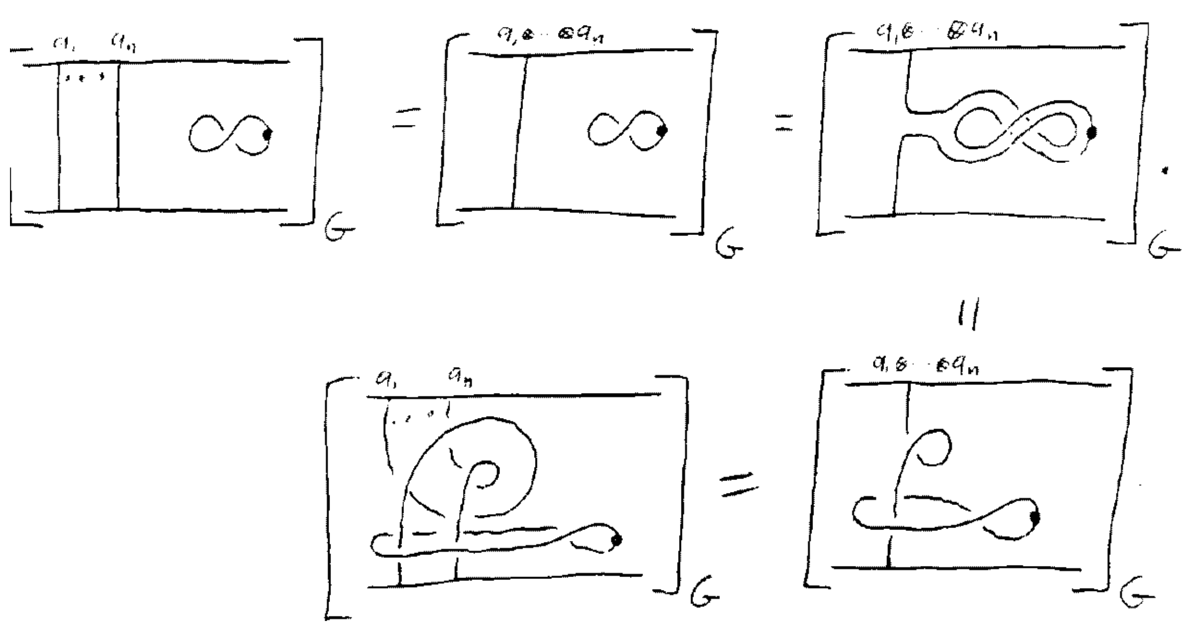

Figure 47: Proof of (8.18).

Figure 48: Definition of $C_\pm$

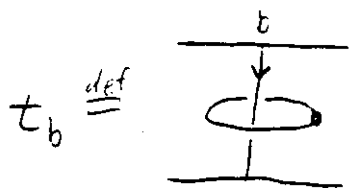

Figure 49: Definition of t_b.

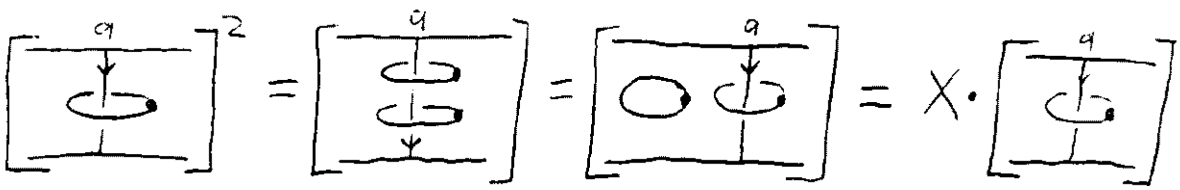

Figure 50: $[t_a]^2 = X[t_a]$.

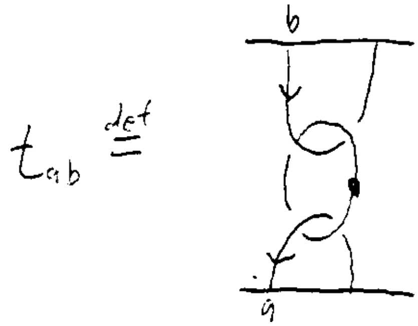

Figure 51: Definition of t_{ab}.

Proof: It suffices to show that $[t_1] = X \cdot \mathrm{id}$ and that $[t_a] = 0$ for $a \in \mathcal{L}$, $a \neq 1$. The first equality follows from (8.5). If $[t_a] \neq 0$, then Figure 50 and the irreducibility of a imply that $[t_a] = X \cdot \mathrm{id}$.

Let y be a tangle with a closed component c labeled by a. Let y' be y with a small meridian linking surgery ribbon around c. Then $[y'] = X[y]$. By (8.19), $[y']$ does not change if crossings with c are changed. Since $X \neq 0$, the same is true of y. This contradicts (8.11). □

(8.21) **Corollary.** *Let t_{ab} be the ESRG shown in Figure 51 ($a, b \in \mathcal{L}$). Then*

$$[t_{ab}]_G = \begin{cases} 0, & a \neq b \\ X(\dim_q a)^{-1}\mathrm{id}_G, & a = b. \end{cases}$$

Proof: [See Figure 52.] □

(8.22) **Corollary.** $C_+C_- = 1$.

Proof: [See Figure 53.] □

Note: The only instances of (8.19) used in proving (8.22) and (8.20) were sliding ribbons over unknotted surgery ribbons of framing zero. The reader may verify that these handle slides can be effected using only $+1$ [or only -1] stabilizations and the $+1$ [-1] version of (8.18). (More generally, the same is true for handle slides over knotted surgery ribbons of framing ≤ 1 [≥ -1].) It follows that half of (8.10) (i.e. the assumption that $C_- \neq 0$ [$C_+ \neq 0$]) is unnecessary.

It follows from (8.19) and (8.22) that $[t]_G$ depends only on the equivalence class of the ESRG t.

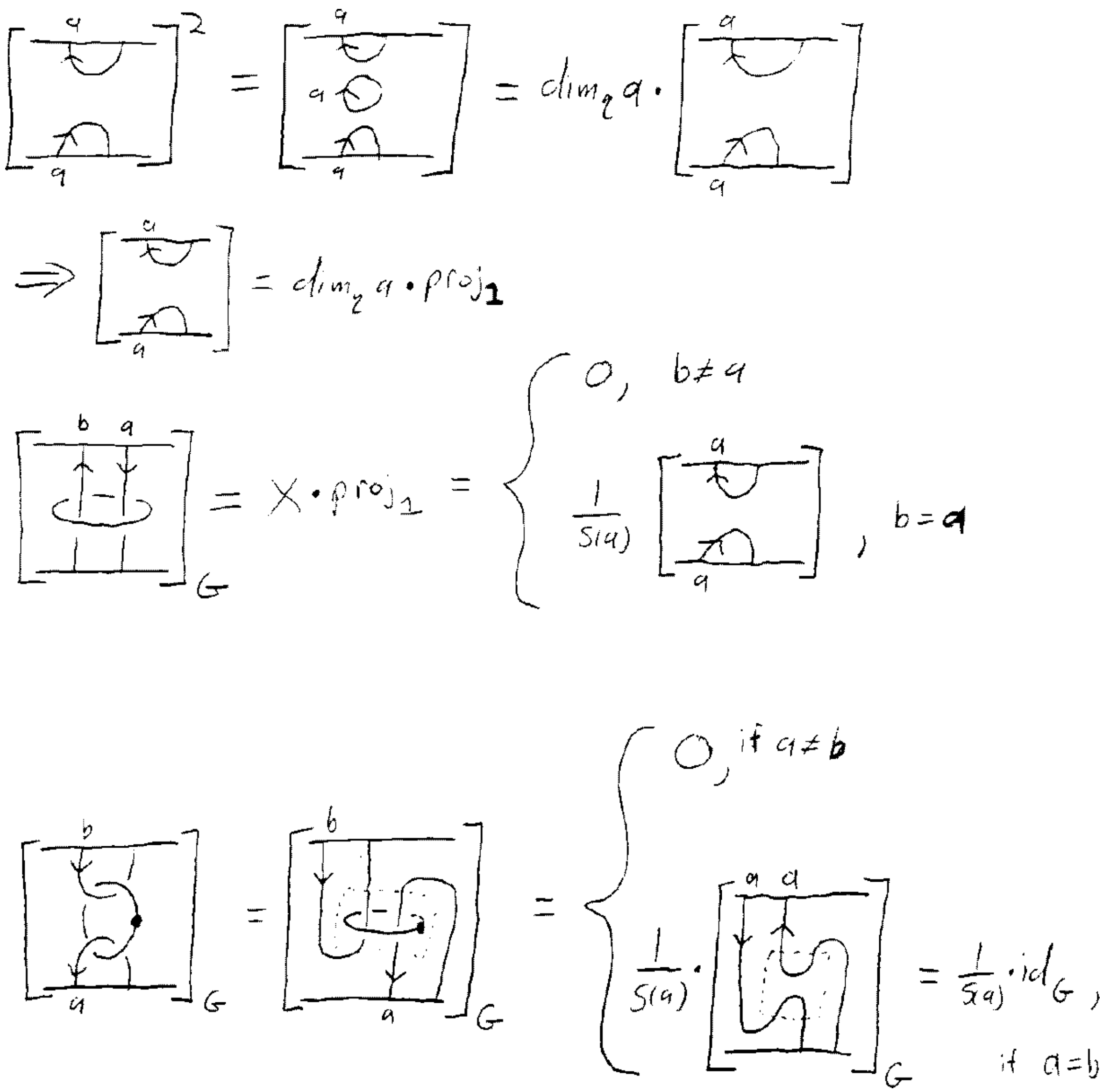

Figure 52: Proof of (8.21).

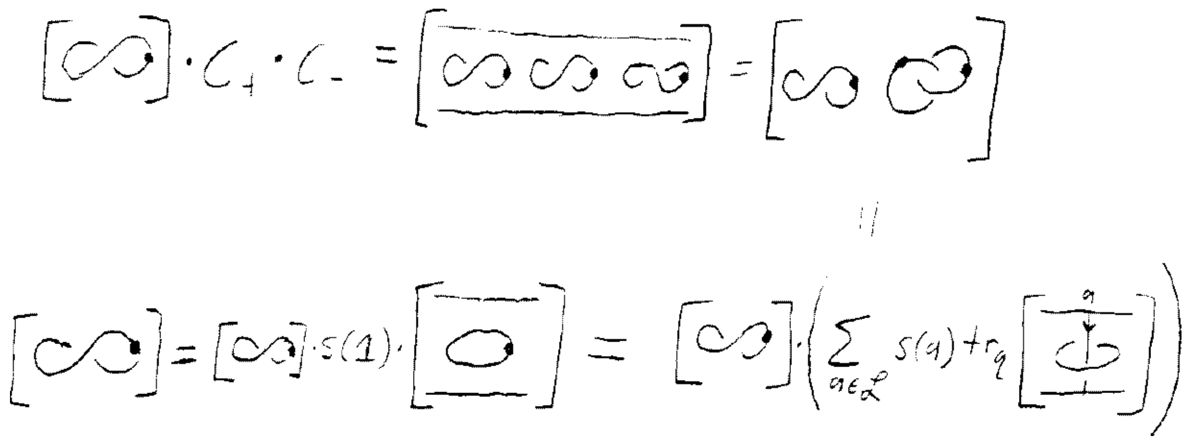

Figure 53: Proof of (8.22).

9 TQFTs from modular Hopf algebras

In this section we use the results of [RT2], as described in the previous section, to show how to construct a TQFT from a modular Hopf algebra. (Actually, we need only the corresponding modular tangle functor.) More precisely, we will derive basic data (Section 3) and gluing coefficients from a modular Hopf algebra and show that it satisfies the relations of (6.4) and (7.9). The verification of the relations is accomplished by using the fact that the functor $[\,\cdot\,]_G$ depends only on the equivalence class of the extended surgered ribbon graph; one is free to perform isotopies, handle slides, and balanced (de)stabilizations.

(A more direct approach to achieving the goals of this section is given in [RT2], but at present I don't see how it can be refined to produce a TQFT which satisfies the "gluing with corners" axiom. (The projectors which appear in [RT2] can be shown to be the identity using (8.21). The phase ambiguities can be resolved by working in the "extended" category.))

Fix a MHA $(U, \mathcal{L})$. $\mathcal{L}$ will be our label set, and the trivial representation will be the the trivial label. Define, for $a, b, c \in \mathcal{L}$,

$$\begin{aligned} V_1 &\stackrel{\text{def}}{=} 1 \cong \mathbf{C} \\ V_{a\hat{a}} &\stackrel{\text{def}}{=} (a \otimes \hat{a})_1 \cong \mathbf{C} \\ V_{abc} &\stackrel{\text{def}}{=} (a \otimes b \otimes c)_1. \end{aligned}$$

(Note that 1, the trivial representation of U, is canonically identified with $\mathbf{C}$.) Elements of V_{abc} can be thought of as U-linear maps from 1 to $a \otimes b \otimes c$, so can be represented as in Figure 54. Figures (tangles) with unlabeled coupons will represent the map from the space of coupon labels to the space associated to the top of the tangle.

Next we define the actions of the mapping class groupoids. The actions of generators of $\mathcal{M}(P)$, $\mathcal{M}(A)$ and $\mathcal{M}(D)$ are shown in Figure 55. (The action of the central element C is defined to be multiplication by $C_+ = C_-^{-1} \in \mathbf{C}$.) That these generators define

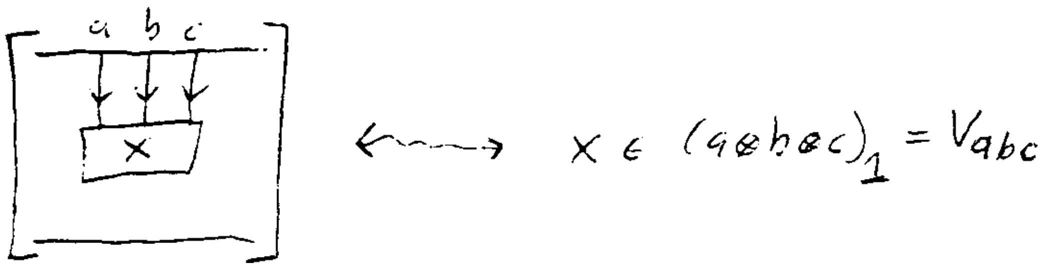

Figure 54: Representing elements of V_{abc}.

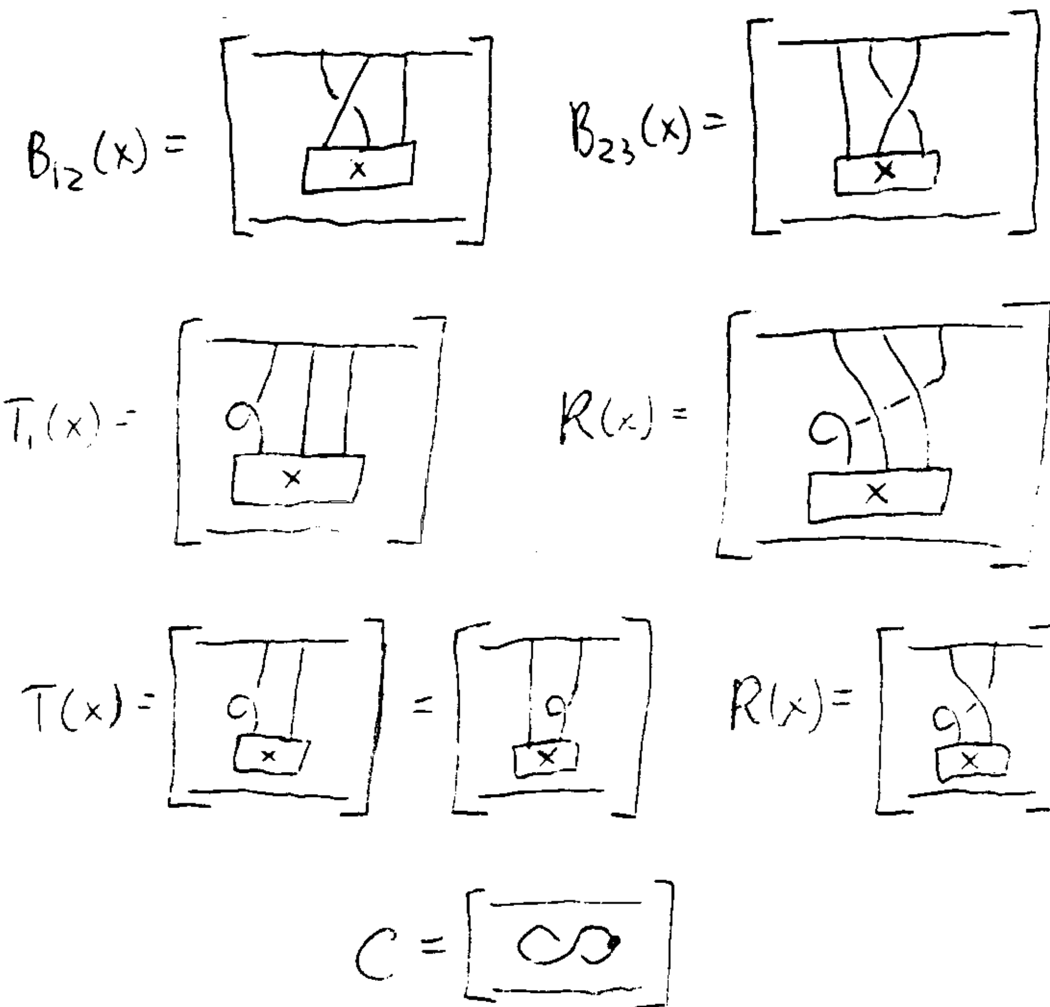

Figure 55: Generators of the actions of the mapping class groupoids.

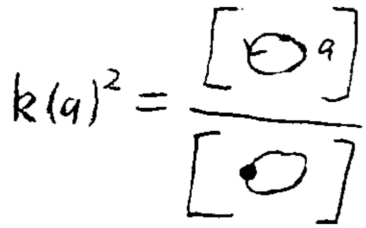

Figure 56: Definition of $k(a)$

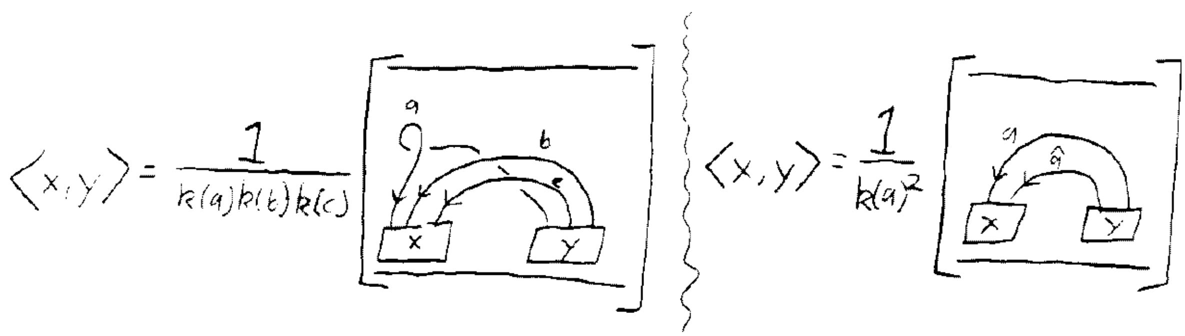

Figure 57: The standard pairings.

representations of the mapping class groupoids follows from the fact that isotopy classes of braided ribbons connecting the top of $\mathbf{R}^2 \times I$ to the top of a coupon are isomorphic to the mapping class groupoid of a punctured sphere. Alternatively, one could check the relations for the mapping class groupoids (see Section 3).

Choose, for each $a \in \mathcal{L}$, $k(a) \in \mathbf{C}^\times$ such that

$$k(a)^2 = s(a) = \frac{\dim_q a}{X}$$

(see Figure 56.)

The pairings corresponding to the standard orientation reversing maps are shown in Figure 57. The trivial representation of U (i.e. V_1) can be identified with $\mathbf{C}$, and we define $\langle x, y\rangle \overset{\text{def}}{=} k(1)^{-1}xy$ for $x, y \in V_1$. The proof that these pairings satisfy (3.2) is indicated in Figure 58. (The figure illustrates the fact that if a braid representing $\psi f \psi^{-1}$ is pushed through the tangle corresponding to the pairing, it becomes a braid representing f^{-1}.)

The definition of $\beta_{a\hat{a}} \in V_{a\hat{a}}$ is shown in Figure 59. Let $\beta_1 = k(1)^{1/2} \in \mathbf{C} = V_1$.

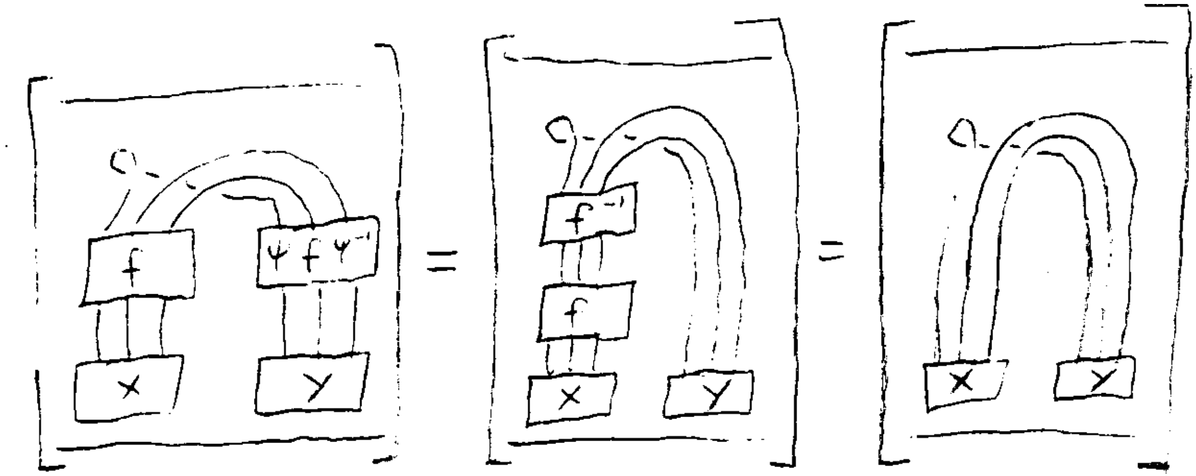

Figure 58: Compatibility of pairings with actions.

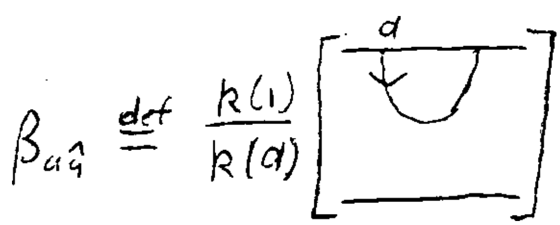

Figure 59: Definition of $\beta_{a\hat{a}}$.

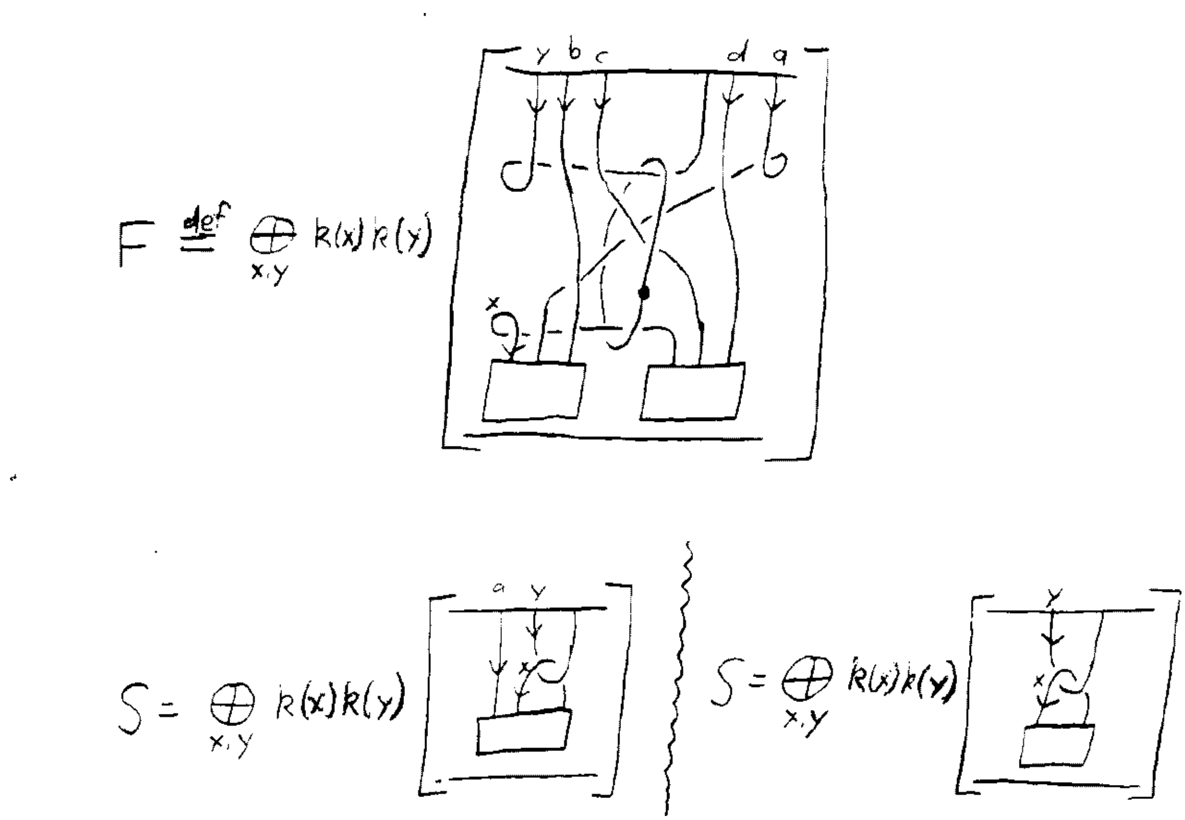

Figure 60: Definitions of F and S.

$V_{1a\hat{a}} = (1 \otimes a \otimes \hat{a})_1$ is canonically identified with $(a \otimes \hat{a})_1 = V_{a\hat{a}}$. Let $\beta_{1a\hat{a}}$ be the element corresponding to $k(1)^{-1/2}\beta_{a\hat{a}}$. Define $\beta_{\hat{a}1a}$ and $\beta_{a\hat{a}1}$ similarly.

The definitions of F and S are shown in Figure 60. This completes the specification of the basic data.

Next we verify the relations of (6.4) and (7.9). This is done in Figures 61 through 9. (The proofs rely heavily on (8.21), (8.19) (and (8.18)), (8.22), and isotopy invariance. For relations 2 and 4a, the "$\bigoplus k(\cdot)k(\cdot)$" is omitted. The proofs of relations 6, 10 and 11 are easy and are left to the reader. The proofs of 4b and 12b are similar to the proofs of 4a and 12a. The proof of 14 is similar to the proof of 13. [The proofs of relations 1 and 3 are omitted from this draft; they will be included in a later one. (I'm tired of drawing figures, and the proofs of these two relations are similar to the others.)])

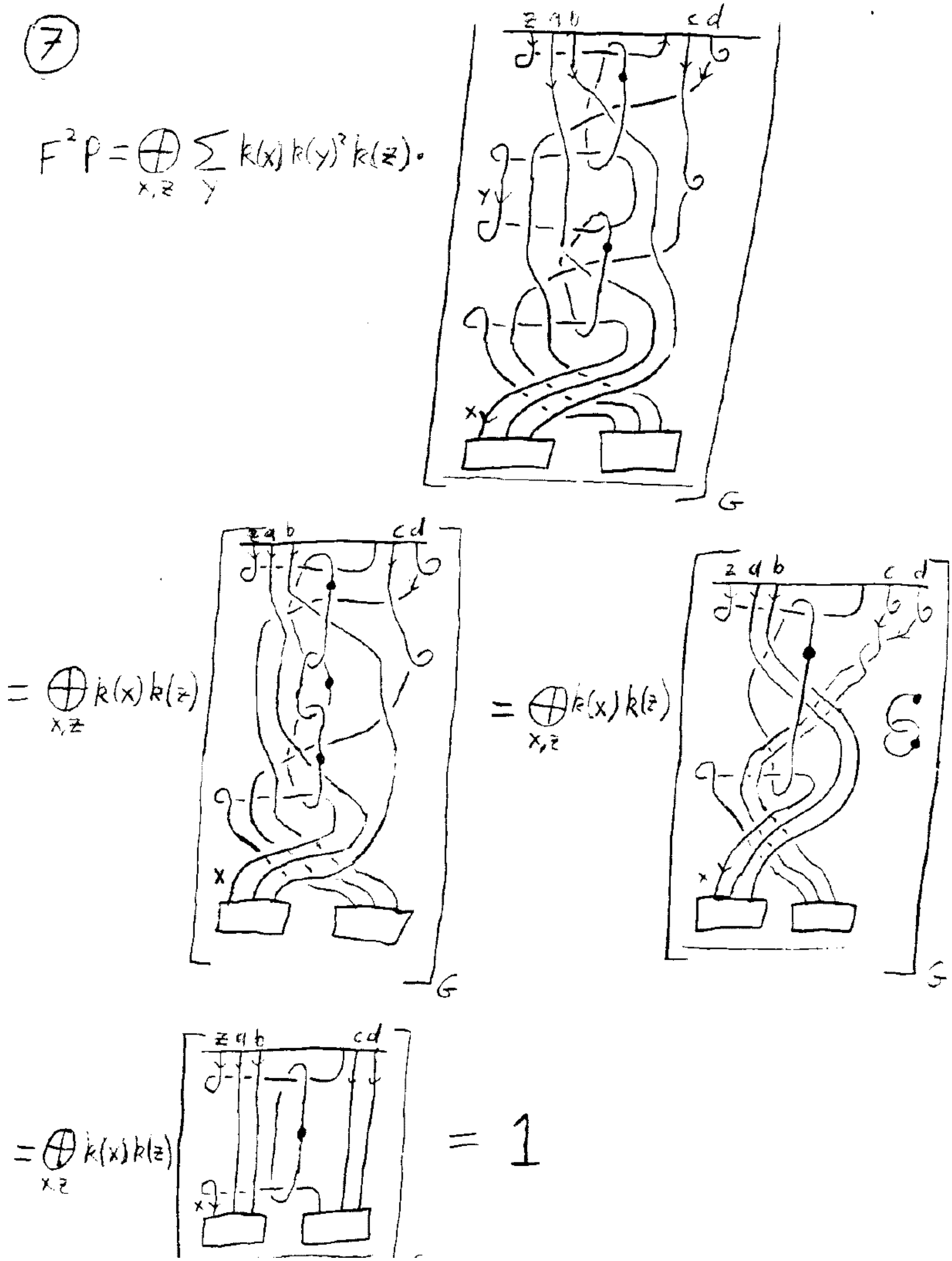

Figure 61: Proof of relation 7.

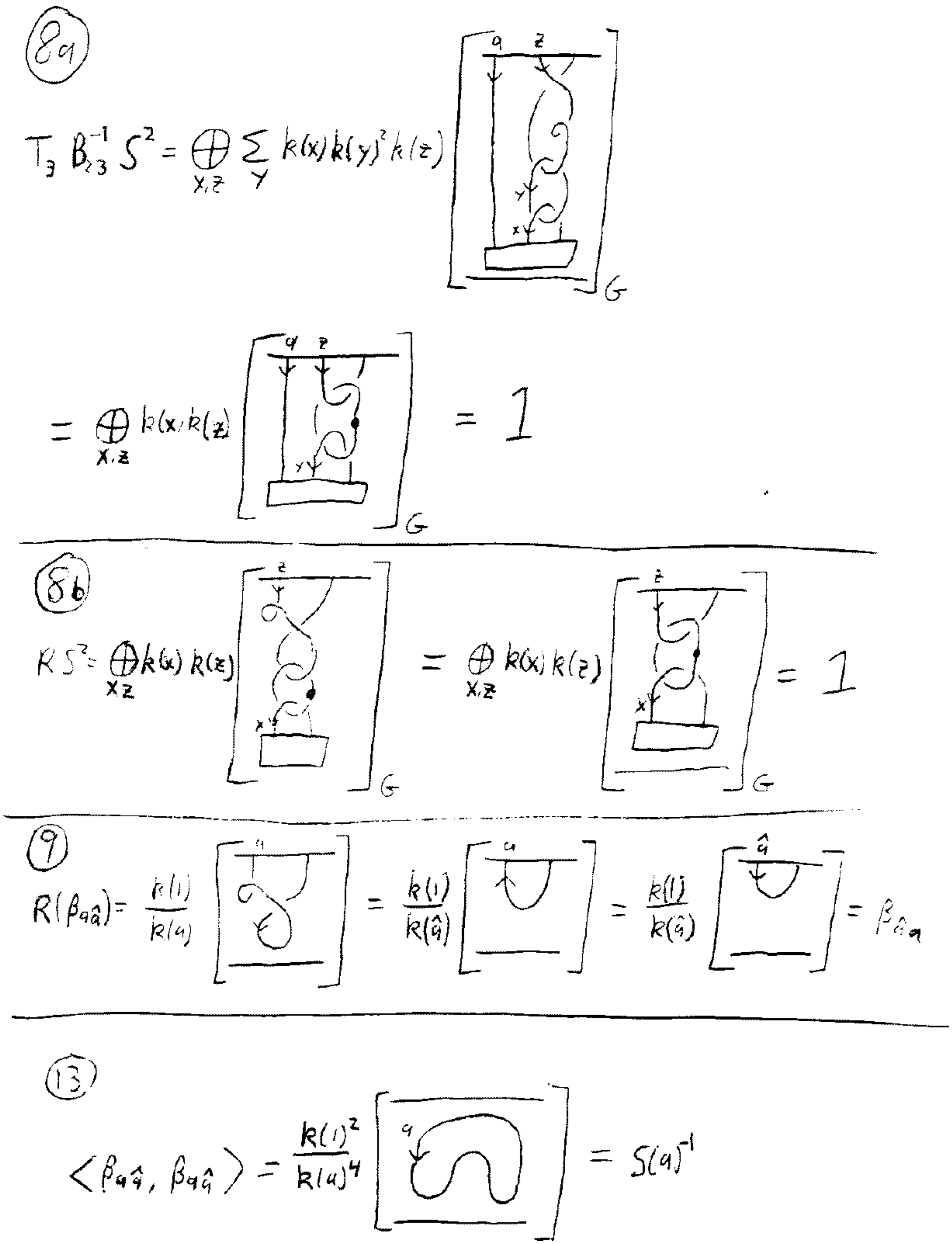

Figure 62: Proof of relations 8, 9 and 13.

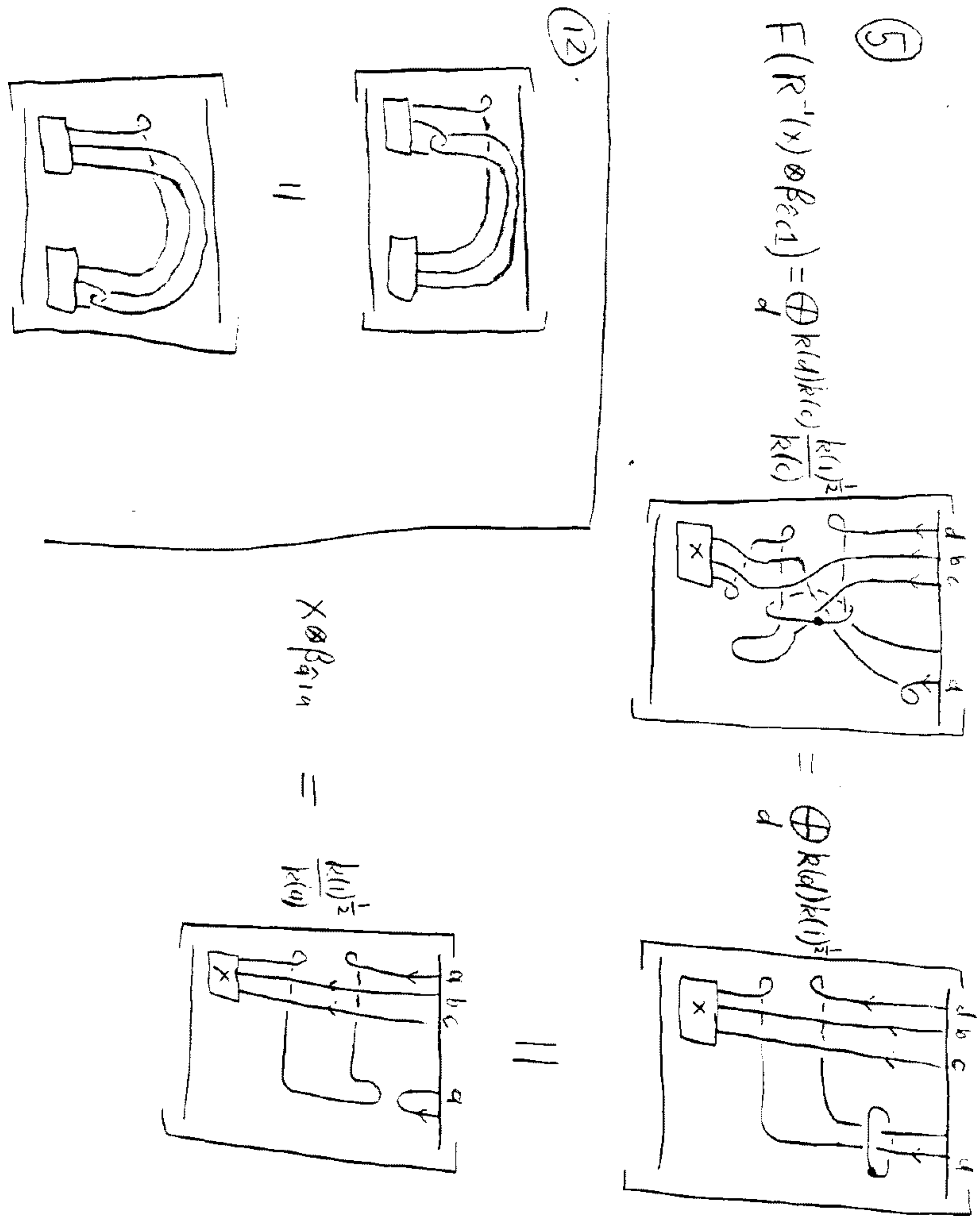

Figure 63: Proof of relations 5 and 12a.

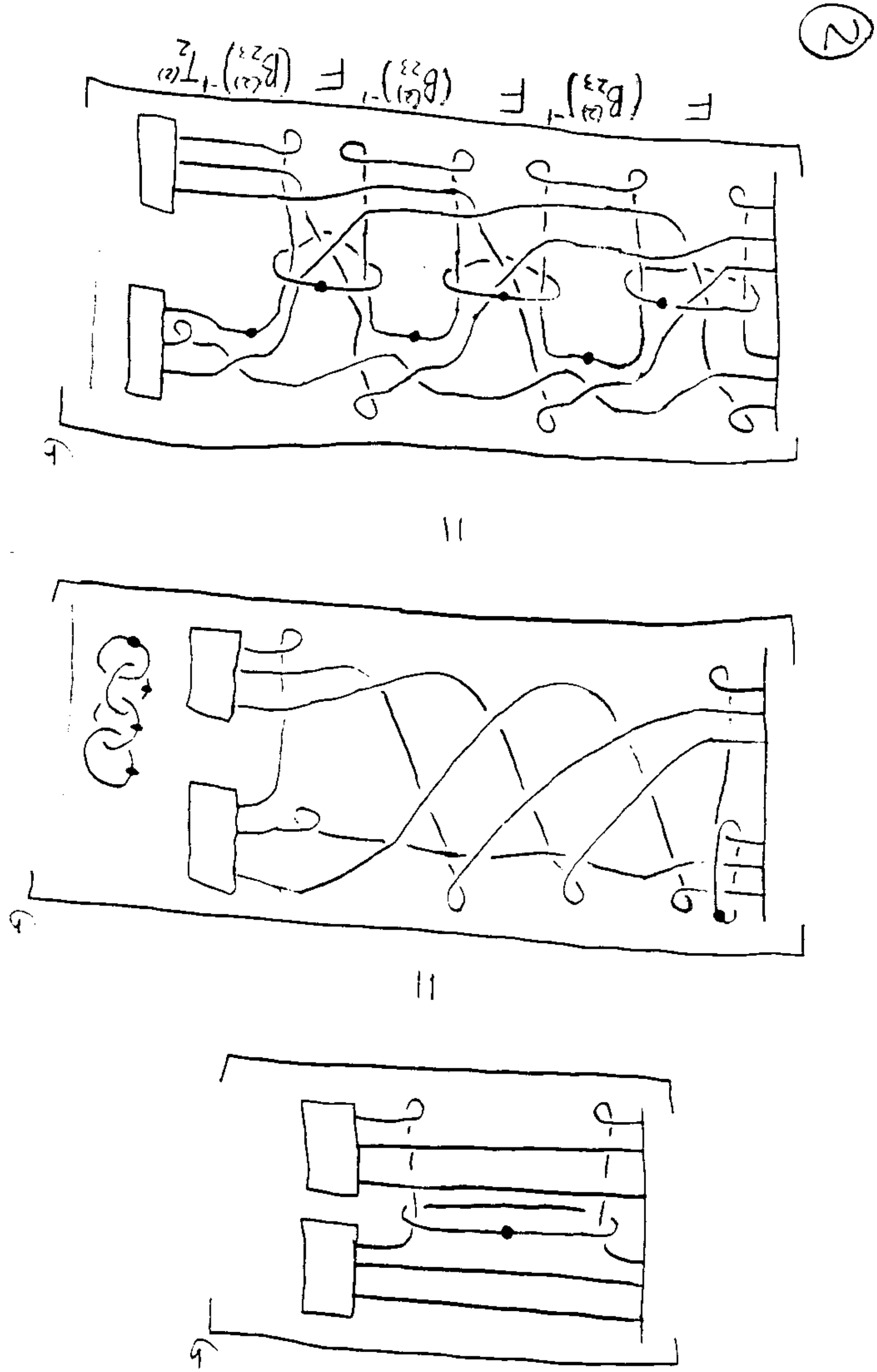

Figure 64: Proof of relation 2.

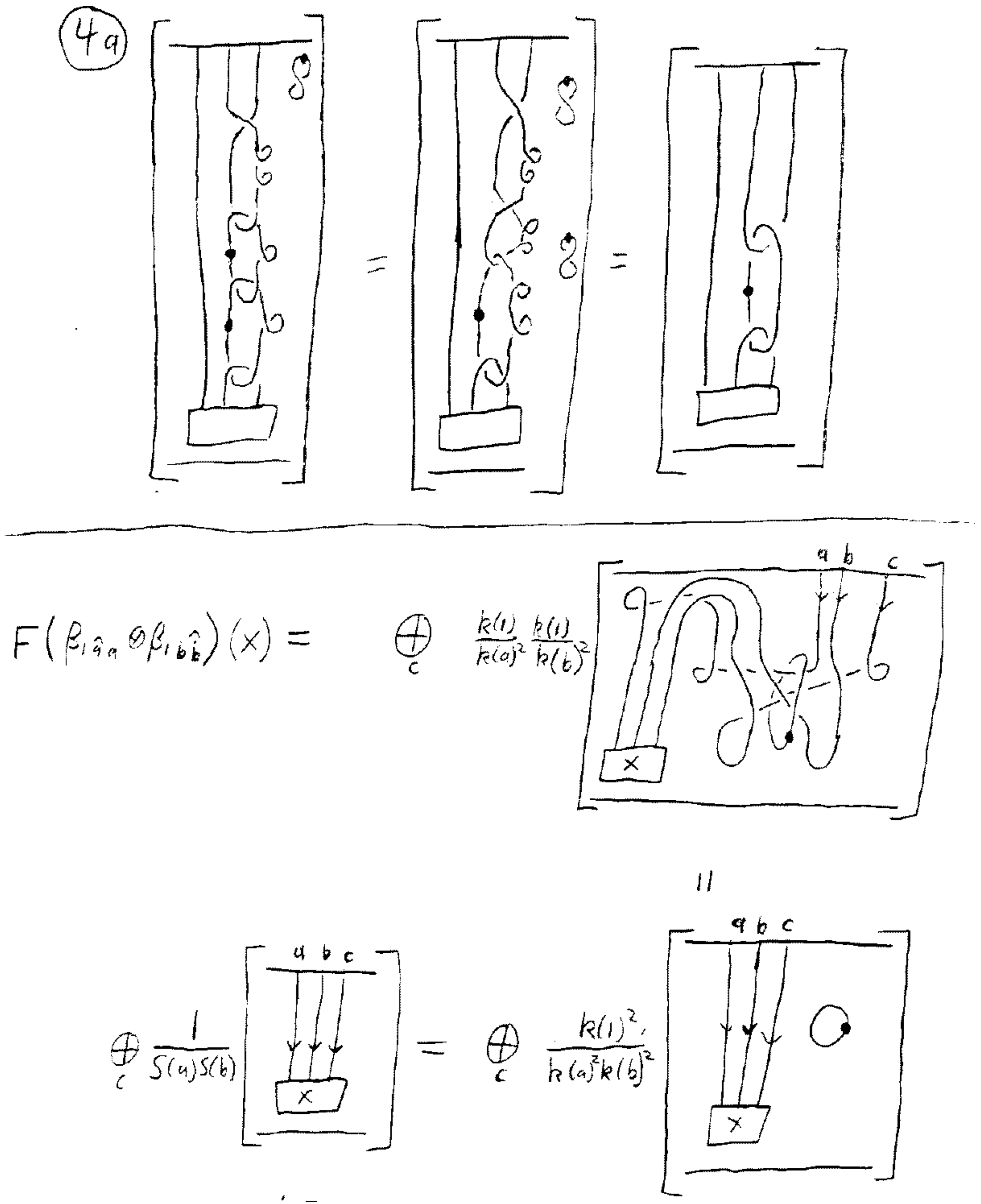

Figure 65: Proof of relation 4a and (4.7).

10 Reduced tangle functors

[Warning: This section is furthr from its final form than most of the other sections in this paper. It should be considered a very rough draft.]

In this section we axiomatize (or rather, sketch an axiomatization of) that part of a tangle functor (see Section 8) which is easily recoverable from a TQFT; in the next section we show how to recover such an object. The basic idea is to replace a representation v of U with $\oplus_x \mathrm{Hom}_U(v, x)$ (where x ranges over the irreducible representations), and to regard morphisms as lying in $\mathrm{Hom}_U(v, w)$ rather that $\mathrm{Hom}(v, w)$. I have found it somewhat tedious and cumbersome to make this precise, so some of the definitions are only sketched.

First we axiomatize some properties of the spaces $\mathrm{Hom}_U(a_1 \otimes \cdots \otimes a_n, x)$, where the a_i's and x are irreducible representations of U. Let $\mathcal{L}$ be a label set, not necessarily finite. That is, $\mathcal{L}$ is equiped with an involution $a \leftrightarrow \hat{a}$ and a trivial label $1 = \hat{1}$. (For example, $\mathcal{L}$ could be a complete set of irreducible representations of an RHA U, $\hat{a} \cong a\check{}$, and 1 the trivial representation. Also, $\mathcal{L}$ could be the distinguished set of good irreducible representations of a MHA.)

Definition. A *Hom system* for $\mathcal{L}$ consists of a finite dimensional complex vector space W_l^x for each $x \in \mathcal{L}$ and sequence $l = (l_1, \ldots, l_n)$, $l_i \in \mathcal{L}$, together with some identifications specified below. We require that for fixed l, W_l^x is nonzero for only finitely many x. We also require that the identifications satisfy some coherence conditions.

[Because I have chosen to specify a long list of identifications, the coherence conditions are rather unwieldy and are therefore omitted. Another option would be to build everything up out of the W_{ab}^x's ($x, a, b \in \mathcal{L}$). This would reduce the number of coherence conditions. (There would be the "pentagon" and two or three others.) But this approach seems unnatural to me. (On the other hand, I can't, at the moment, think of a better way to do it, so I might end up adopting this approach in a later version of this paper.)]

Here are the identifications:

- For any $x \in \mathcal{L}$ and sequences l_1, l_2, l_3,

$$W_{l_1 l_2 l_3}^x = \bigoplus_{y \in \mathcal{L}} W_{l_1 y l_3}^x \otimes W_{l_2}^y.$$

- For any $x \in \mathcal{L}$ and sequences l_1, l_2,

$$W_{l_1 l_2}^x = \bigoplus_{y,z \in \mathcal{L}} W_{yz}^x \otimes W_{l_1}^y \otimes W_{l_2}^z.$$

(This follows from the previous identification.)

- For any $x, y \in \mathcal{L}$ and sequence l,

$$W_{yl}^{x} = W_{l\hat{x}}^{\hat{y}}.$$

- For any $x \in \mathcal{L}$ and sequence l,

$$W_{l1}^{x} = W_{l}^{x}.$$

Definition. A *reduced tangle functor* based on a label set $\mathcal{L}$ consists of

- A hom system for $\mathcal{L}$.
- A linear map
$$[t]_x : W_{l_1}^{x} \to W_{l_2}^{x}$$
for each $x \in \mathcal{L}$ and each $\mathcal{L}$-labeled ribbon tangle, where l_1 $[l_2]$ is the sequence associated to the bottom [top] of t.

Define

$$\begin{aligned} W_l &\stackrel{\text{def}}{=} \bigoplus_x W_l^x \\ [t] &\stackrel{\text{def}}{=} \bigoplus_x [t]_x : W_{l_1} \to W_{l_2}. \end{aligned}$$

The $[t]_x$'s are required to satisfy the following axioms ((10.1) through (10.4)).

(10.1) *Functoriality.* The assignment (downward directed ribbon ends labeled by l) $\mapsto W_l$, $t \mapsto [t]$ is a functor from the category of ribbon tangles labeled by $\mathcal{L}$ to the category of $\mathcal{L}$-graded complex vector spaces.

(10.2) *Tensoriality.* Let t^i $(i = 1, 2)$ be a ribbon tangle with bottom [top] sequence l_1^i $[l_2^i]$. We have identifications

$$\begin{aligned} W_{l_1^1 l_1^2}^{x} &= \bigoplus_{y,z} W_{yx}^{x} \otimes W_{l_1^1}^{y} \otimes W_{l_1^2}^{z} \\ W_{l_2^1 l_2^2}^{x} &= \bigoplus_{y,z} W_{yx}^{x} \otimes W_{l_2^1}^{y} \otimes W_{l_2^2}^{z}. \end{aligned}$$

With respect to these identifications, we require (for all $x \in \mathcal{L}$)

$$[t^1 \otimes t^2]_x = \bigoplus_{y,z} \mathrm{id} \otimes [t^1]_y \otimes [t^2]_z.$$

(10.3) *Cabling.* Let t be a ribbon tangle with an unlabeled ribbon r. Let t_y denote t with r labeled by $y \in \mathcal{L}$. Let t' denote t with r replaced by an n-cable with label $l = (a_1, \dots, a_n)$.
Case (i): r is closed. Then

$$[t'] = \sum_y \dim(W_l^y)[t_y].$$

Case (ii): r goes from the bottom to the top. The domains and ranges of $[t']_x$ and $[t_y]_x$ are

$$\begin{aligned} [t']_x : W^x_{k_1^1 l k_1^2} &\to W^x_{k_2^1 l k_2^2} \\ [t_y]_x : W^x_{k_1^1 y k_1^2} &\to W^x_{k_2^1 y k_2^2} \end{aligned}$$

(for appropriate sequences k_i^j). Furthermore, we have identifications

$$\begin{aligned} W^x_{k_1^1 l k_1^2} &= \bigoplus_y W_l^y \otimes W^x_{k_1^1 y k_1^2} \\ W^x_{k_2^1 l k_2^2} &= \bigoplus_y W_l^y \otimes W^x_{k_2^1 y k_2^2}. \end{aligned}$$

With respect to these identifications, we require (for all $x \in \mathcal{L}$)

$$[t']_x = \bigoplus_y \mathrm{id}_y \otimes [t_y]_x,$$

where id_y denotes the identity on W_l^y.
Case (iii): r goes from the bottom to the bottom. $[t']_x$ and $[t_y]_x$ share the same range. The domain of $[t']_x$ is $W^x_{k_1 l k_2 l^* k_3}$ (for appropriate sequences k_i), and the domain of $[t_y]_x$ is $W^x_{k_1 y k_2 \hat{y} k_3}$. We have

$$W^x_{k_1 l k_2 l^* k_3} = \bigoplus_{y,z} W_l^y \otimes W_{l^*}^z \otimes W^x_{k_1 y k_2 z k_3},$$

which contains the subspace

$$\bigoplus_y W_l^y \otimes W_{l^*}^{\hat{y}} \otimes W^x_{k_1 y k_2 \hat{y} k_3}.$$

We require that $[t']_x$ be zero when $z \neq \hat{y}$, and

$$[t']_x = \bigoplus_y \langle \,\cdot\,,\,\cdot\, \rangle_y [t_y]_x$$

on the above subspace, where $\langle \,\cdot\,,\,\cdot\, \rangle_y$ denotes the pairing

$$W_l^y \otimes W_{l^*}^{\hat{y}} \hookrightarrow W^1_{y\hat{y}} \otimes W_l^y \otimes W_{l^*}^{\hat{y}} \to \mathbf{C}$$

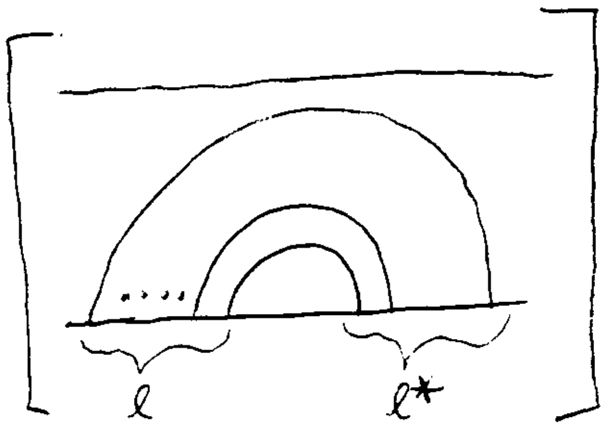

Figure 66: A pairing.

shown in Figure 66.
Case (iv): r goes from the top to the top. This is similar to the previous case. We require that

$$[t']_x(\alpha) = \bigoplus_y \mathrm{id}_y \otimes [t_y]_x(\alpha),$$

where α is in the common domain of $[t']_x$ and $[t_y]_x$, and id_y denotes the identity in $W_l^y \otimes W_{l^*}^{\hat{y}}$ (with respect to the above pairing).

(10.4) *Trivial label.* Let t be a ribbon tangle with a ribbon labeled by the trivial label, and let t' denote t with this ribbon deleted. Then $[t'] = [t]$.

It is easy to see that a tangle functor induces a reduced tangle functor. Hence ribbon Hopf algebras give rise to reduced tangle functors.

Definition. A *modular reduced tangle functor* (MRTF) is a reduced tangle functor which satisfies the following additional axioms.

(10.5) $\mathcal{L}$ is finite.

(10.6) $\dim_q(a) \neq 0$ for all $a \in \mathcal{L}$. ($\dim_q$ is defined as in Figure 36.)

(10.7) [Same as (8.9).]

(10.8) [Same as (8.10).]

(10.9) [Same as (8.11).]

It is not hard to see that modular tangle functors (and hence modular Hopf algebras) give rise to MRTFs. (Use $[\ \cdot\ ]_G$, not $[\ \cdot\ ]$.) Furthermore, the the proofs of the results in the end of Section 8 and in Section 9 can easily be modified to work for MRTFs. Hence MRTFs lead to invariants of extended surgered ribbon graphs and to TQFTs.

11 Modular reduced tangle functors from TQFTs

[Warning: This section is furthr from its final form than most of the other sections in this paper. It should be considered a very rough draft.]

In this section we show how to construct a modular reduced tangle functor from a TQFT. In other words, we describe the inverse of the construction of Section 9.

Let V be a modular functor with label set $\mathcal{L}$. Define

$$W_l^x \stackrel{\text{def}}{=} V_{\hat{x}l}.$$

(Recall that $V_{\hat{x}l}$ is the vector space associated to the standard punctured sphere labeled be $\hat{x}l$.) The identifications required to make the W_l^x's into a hom functor are easily derived from the axioms of a modular functor.

Let (Z, V) be a TQFT. Let t be an ESRG (extended surgered ribbon graph) with labels in $\mathcal{L}$. Let X_t denote the complement of t: Regard t as lying is $B^2 \times I$. Surger the surgery ribbons of t and excise regular neighborhoods of the other ribbons of t. The framing number of X_t is defined to be the signature of the linking matrix of t. The lagrangian of X_t is defined to be the one spanned by the meridians of the excised ribbons.

The boundary of X_t can be decomposed into annuli and two punctured spheres. One annulus (the "outer" one) corresponds to $\partial B^2 \times I$. Its seams are given by the product structure of $\partial B^2 \times I$. The other ("ribbon") annuli correspond to the excised ribbons, which determine their seams. The labels of t determine labels for the ribbon annuli.

Let (Y, l) be an le-surface. Define

$$\langle \,\cdot\,,\,\cdot\,\rangle^\natural : V(Y, l) \otimes V(-Y, \hat{l}) \to \mathbf{C}$$

by

$$\langle \,\cdot\,,\,\cdot\,\rangle^\natural = k(l)\langle \,\cdot\,,\,\cdot\,\rangle,$$

where $\langle \,\cdot\,,\,\cdot\,\rangle$ is the usual pairing, $k(a)^2 = S(a)$ for $a \in \mathcal{L}$, and $k(a_1 \ldots a_n) = k(a_1)\cdots k(a_n)$. (See (2.13).)

Given a labeling x of the outer annulus and l of the ribbon annuli, let $\hat{x}l_1$ [$\hat{x}l_2$] be the labeling associated to the bottom [top] punctured sphere of X_t. (Here the bottom punctured sphere given the orientation opposite of the one induced from X_t.) With respect to the above decomposition, $Z(X_t)$ can be written

$$Z(X_t) = \bigoplus_{x,l} \beta_{x\hat{x}} \otimes \beta_l \otimes \alpha_{l_2}^x \otimes \alpha_{l_1}^x,$$

where x ranges over $\mathcal{L}$, l ranges over labelings of the ribbon annuli, $\alpha_{l_1}^x \in (W_{l_1}^x)^*$ and $\alpha_{l_2}^x \in W_{l_2}^x$. Here we identify $W_{l_1^*}^{\hat{x}} = V_{xl_1^*}$ with $W_{l_1}^x)^* = V_{\hat{x}l_1}^*$ *using the modified pairing*

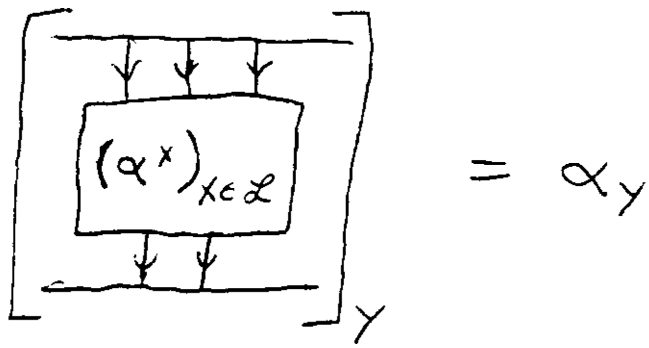

Figure 67: Extending $[\,\cdot\,]$ to coupons.

$\langle\,\cdot\,,\,\cdot\,\rangle^\natural$. (Actually, $Z(X_t)$ can be written as a *sum* of such things, but for the sake of clarity of exposition we will ignore this fact.) Let m be the actual labeling of the ribbon annuli (the one induced from t). Let $\bar{m}$ be the sublabeling of m corresponding to the non-closed ribbons. Define, for all $x \in \mathcal{L}$,

$$[t]_x = k(x)^{-1}k(\bar{m})^{-1}\alpha^x_{m_2} \otimes \alpha^x_{m_1} \in \mathrm{Hom}(W^x_{m_1}, W^x_{m_2})$$

and

$$[t] = \bigoplus_x [t]_x.$$

Using the axioms of a TQFT, it is easy to check that $[\,\cdot\,]$ satisfies the axioms of a MRTF. In particular, tensoriality (10.2) follows from (4.7)

$[\,\cdot\,]$ is extended to ribbon graphs (coupons) as follows. Let c be a coupon with bottom [top] sequence l_1 [l_2]. A label for c is a collection of elements $\alpha^x \in \mathrm{Hom}(W^x_{l_1}, W^x_{l_2})$, $x \in \mathcal{L}$. $[\,\cdot\,]$ is extended to ribbon graphs with labeled coupons as shown in Figure 67.

In Section 9 we showed how to construct a TQFT from a MRTF. Next we show that this construction is the inverse of the one described above.

Let (Z, V) be a TQFT. Let $[\,\cdot\,]$ be the MRTF derived from (Z, V) as above. Let (Z', V') be the TQFT derived from $[\,\cdot\,]$ as in Section 9. We must show that $(Z', V') = (Z, V)$. To do this it suffices to show that $V' = V$, and to do that it suffices to show that V' and V have isomorphic basic data.

.
.
.

12 The Verlinde algebra

In this section we define the Verlinde algebra and prove some well known results concerning it. The approach given here is similar to those in [Ko] and [S2].

Let $T^2 = S^1 \times S^1$ be the standard torus. Define the *meridian* to be the curve $\mu \stackrel{\text{def}}{=} S^1 \times \{1\}$ and the *longitude* to be the curve $\lambda \stackrel{\text{def}}{=} \{1\} \times S^1$. Note that $\langle \mu, \lambda \rangle = 1$.

Let (Z, V) be a TQFT. Define the *Verlinde algebra* associated to Z to be

$$\mathcal{A} \stackrel{\text{def}}{=} V(T^2).$$

(The multiplication on $\mathcal{A}$ will be defined below.) The orientation reversing involution $(\alpha, \beta) \leftrightarrow (\bar{\alpha}, \beta)$ induces an identification

$$\mathcal{A} = \mathcal{A}^*. \tag{12.1}$$

Let P be a pair of pants. Identify each boundary component of $P \times S^1$ with T^2 by sending $\partial \times \{1\}$ to μ and $pt \times S^1$ to λ. With respect to this identification we have

$$Z(P \times S^1) \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} = \mathcal{A} \otimes \mathcal{A}^* \otimes \mathcal{A}^*.$$

Thus $Z(P \times S^1)$ defines a multiplication

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

It is easy to see from the naturality and gluing axioms for Z that this multiplication is commutative and associative. Gluing $D^2 \times S^1$ to one of the boundary components of $P \times S^1$ in such a way that the meridian of the boundary component bounds a disk in $D^2 \times S^1$ results in $A \times S^1 \cong T^2 \times I$. It follows that

$$e \stackrel{\text{def}}{=} Z(D^2 \times S^1)$$

is the identity for the multiplication on $\mathcal{A}$.

Consider the two decompositions of T^2 into a copy of the standard annulus A shown in Figure 68. In the meridian decomposition boundary component number one of A goes to μ and the seams are parallel to λ. In the longitude decomposition boundary component number one of A goes to λ and the seams are parallel to μ. These two decompositions give rise to two bases of $\mathcal{A}$:

$$\begin{aligned} m_a &\stackrel{\text{def}}{=} \beta_{a\hat{a}} \in \mathcal{A} && \text{(meridian decomposition)} \\ l_a &\stackrel{\text{def}}{=} \beta_{a\hat{a}} \in \mathcal{A} && \text{(longitude decomposition)} \end{aligned}$$

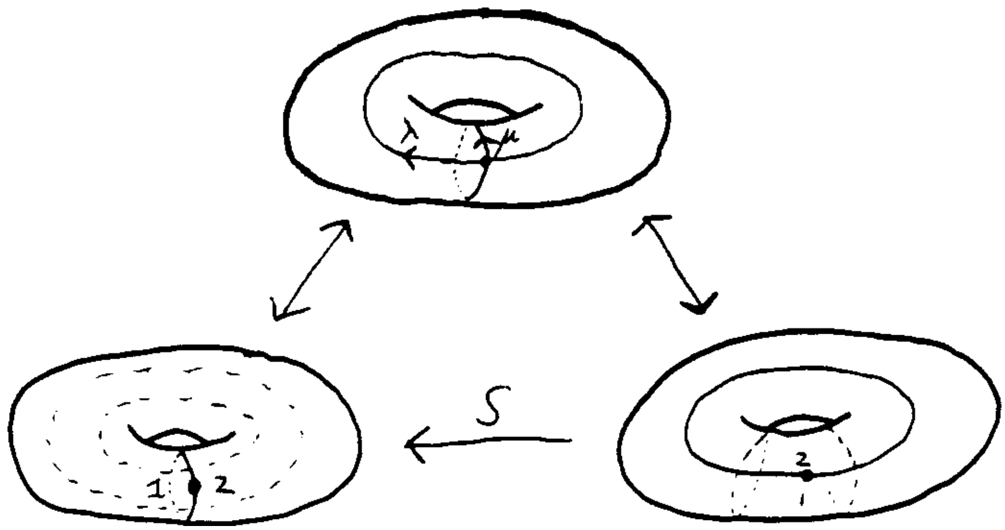

Figure 68: Two decompositions of T^2.

($a \in \mathcal{L}$). These two bases are related by S:

$$l_a = \sum_b S_{ab} m_b \tag{12.2}$$

$$m_a = \sum_b (S^{-1})_{ab} l_b = \sum_b S_{\hat{a}b} l_b. \tag{12.3}$$

With respect to (12.1) we have

$$\begin{aligned} \langle m_a, m_b \rangle &= \delta_{a\hat{b}} \\ \langle l_a, l_b \rangle &= \delta_{ab}. \end{aligned}$$

Let $N_{abc} \stackrel{\text{def}}{=} \dim(V_{abc})$ ($a, b, c \in \mathcal{L}$). With respect to the meridian decomposition on $\partial(P \times S^1)$ we have (by the mapping cylinder and gluing axioms)

$$\begin{aligned} Z(P \times S^1) &= \bigoplus_{a,b,c} \operatorname{tr}(\mathrm{id} : V_{abc} \to V_{abc}) \beta_{a\hat{a}} \otimes \beta_{b\hat{b}} \otimes \beta_{c\hat{c}} \\ &= \bigoplus_{a,b,c} N_{abc} m_a \otimes m_b \otimes m_c. \end{aligned}$$

It follows that

$$m_b m_c = \sum_a N_{a\hat{b}\hat{c}} m_a. \tag{12.4}$$

(Most authors take this as the definition of multiplication. Note that $N_{a\hat{b}\hat{c}} = N_{\hat{a}bc}$.)

$P \times S^1$ can also be constructed by starting with $D^2 \times S^1$ and gluing two pairs of "longitudinal" annuli together (see Figure 69). By (4.5) and (3.4),

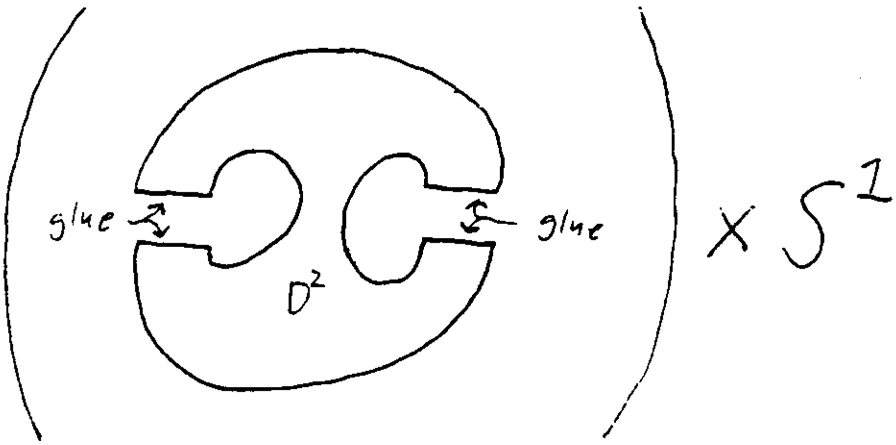

Figure 69: Constructing $P \times S^1$ from $D^2 \times S^1$.

$$Z(D^2 \times S^1) = \bigoplus_a S_{1a}\beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}}.$$

By (4.2) and the gluing axiom,

$$\begin{aligned} Z(P \times S^1) &= \bigoplus_a S_{1a}^{-1}\beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \\ &= \bigoplus_a S_{1a}^{-1}\beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \otimes \beta_{a\hat{a}} \\ &= \bigoplus_a S_{1a}^{-1} l_a \otimes l_a \otimes l_a. \end{aligned}$$

It follows that

(12.5) $$l_a l_b = \delta_{ab} S_{1a}^{-1} l_a.$$

Using (12.4), (12.5) and (12.3) we can express N_{abc} interms of the S_{xy}'s:

$$\begin{aligned} N_{abc} &= \langle m_{\hat{a}}, m_{\hat{b}} m_{\hat{c}} \rangle \\ &= \langle \sum_x S_{ax} l_x, (\sum_y S_{by} l_y)(\sum_z S_{cz} l_z) \rangle \\ &= \langle \sum_x S_{ax} l_x, \sum_y S_{by} S_{cy} S_{1y}^{-1} l_y \rangle \\ &= \sum_x \frac{S_{ax} S_{bx} S_{cx}}{S_{1x}}. \end{aligned}$$

Next we calculate $\dim(V_l)$, where V_l is the vector space associated to an n-punctured sphere with labels $l = (a_1, \ldots, a_n)$. Let Y be an (unlabeled) n-punctured sphere. Then

$$\begin{aligned} Z(Y \times S^1) &= \bigoplus_l \dim(V_l) \beta_{a_1 \hat{a}_1} \otimes \cdots \otimes \beta_{a_n \hat{a}_n} \\ &= \bigoplus_l \dim(V_l) m_{a_1} \otimes \cdots \otimes m_{a_n}. \end{aligned}$$

(l ranges over all labelings of ∂Y.) As $Z(Y \times S^1)$ can be interpreted as the tensor describing $(n-1)$-fold multiplication on $\mathcal{A}$, we have

$$\begin{aligned} \dim(V_l) &= \langle m_{\hat{a}_1}, m_{\hat{a}_2} \cdots m_{\hat{a}_n} \rangle \\ &= \langle \sum_x S_{a_1 x} l_x, \sum_y (\prod_{i=2}^n S_{a_i 1}) S_{1y}^{-n+2} l_y \rangle \\ &= \sum_x \left(\prod_{i=1}^n S_{a_i x} \right) S_{1x}^{2-n}. \end{aligned}$$

Finally, we consider the general case of a genus g surface Y with n punctures labeled by $l = (a_1, \ldots, a_n)$. Cutting Y along g well chosen curves results in an $(n+2g)$-punctured sphere. Hence

$$\begin{aligned} \dim(V(Y)) &= \sum_{y=(y_1,\ldots,y_g)} \dim(V_{ly\hat{y}}) \\ &= \sum_y \sum_{x \in \mathcal{L}} S_{1x}^{2-2g-n} \left(\prod_{i=1}^n S_{a_i x} \right) \left(\prod_{j=1}^g S_{y_i x} S_{\hat{y}_j x} \right) \\ &= \sum_x S_{1x}^{2-2g-n} \left(\prod_i S_{a_i x} \right) \left(\sum_{z \in \mathcal{L}} S_{zx} S_{\hat{z}x} \right)^g \\ &= \sum_x S_{1x}^{2-2g-n} \left(\prod_i S_{a_i x} \right) (1)^g \\ &= \sum_x \left(\prod_i S_{a_i x} \right) S_{1x}^{\chi(Y)}, \end{aligned}$$

where $\chi(Y) = 2 - 2g - n$ is the Euler characteristic of Y.

13 $\mathbf{sl}_2$ theories

In this section we describe explicitly the TQFTs arising from $U_q(\mathbf{sl}_2))$, where q is a root of unity. This material is extracted from [RT2] and [KR].

$$\vdots$$

14 Remarks on other surgery approaches

In this section we relate the approach to TQFTs given in this paper to [KM], [L] and [MSt].

$$\vdots$$

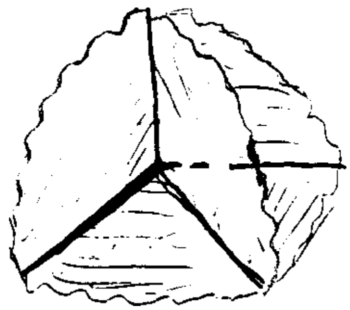

Figure 70: The local model for a generic cell decomposition.

15 State models

Let Z be a unitary TQFT. In this section we derive a “state model” for $|Z(M)|^2$, based on a generic cell decomposition of M. In the case where Z is an $\mathbf{sl}_2$ theory and M is closed, this state model is the same as the one described by Turaev and Viro [TV]. Since we will only compute $|Z(M)|^2$, we won't concern ourselves with the lagrangian or framing number of M.

Let M be an e-3-manifold equiped with a generic cell decomposition. This means that near each point of M the decomposition looks locally like some point of Figure 70. (For points of ∂M, Figure 70 should be interpreted as having one of the four components of the 3-dimensional stratum removed.) If M is closed, then the cell decomposition dual to a triangulation of M is a generic. Note that the cell decompositions considered here differ from the cell decompositions considered in [TV] at ∂M. Also, in [TV] the components of the 2-dimensional stratum are allowed to be planar surfaces; here we require that they be disks.

Let c_i ($i = 0, 1, 2, 3$) denote the number of i-cells of the decomposition. Let M_i denote a regular neighborhood of the i-skeleton. M_i may be obtained from M_{i-1} by adding c_i i-handles. We will prefer to view M_1 as obtained by gluing c_0 caltrops together (see Figure 71). Each caltrop consists of a 0-handle and half of each of the four adjacent 1-handles. ∂(caltrop) can be decomposed into four disks (the dual cores of the 1-handles) and a 4-punctured sphere. With respect to this decomposition,

$$Z(\text{caltrop}) = \beta_1 \otimes \beta_1 \otimes \beta_1 \otimes \beta_1 \otimes \beta_{1111},$$

where $\beta_{1111} \in V_{1111}$ is the canonical element (see Section 4). It follows that

$$Z(M_1) = \beta_{1111}^{\otimes c_0}.$$

This equation is with respect to the decomposition of ∂M_1 into c_0 4-punctured spheres arising from the decomposition of M_1 into c_0 caltrops. Call this decomposition the

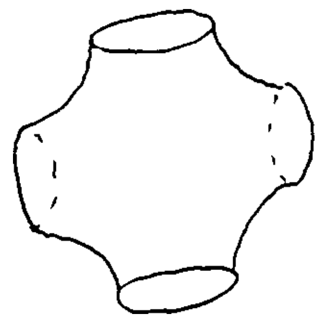

Figure 71: A caltrop.

edge decomposition. Note the its decomposing curves correspond bijectively to the edges (1-cells) of the cell decomposition of M.

There is a second decomposition of ∂M_1 given by the attaching circles of the 2-handles. Call this decomposition the *face decomposition*. With respect to this decomposition we can write

$$Z(M_1) = \bigoplus_{l_f} Z(M_1)_{l_f}, \tag{15.1}$$

where l_f ranges over all labelings of the faces (2-cells), and $Z(M_1)_{l_f}$ is the (orthogonal) projection of $Z(M_1)$ onto the l_f summand. Let p_f denote the projection onto the 1_f summand, where 1_f assigns the trivial label to each face. Let

$$\alpha \stackrel{\text{def}}{=} p_f(Z(M_1)) = Z(M_1)_{1_f}.$$

We will regard $p_f(Z(M_1))$ as lying in $V(\partial M_1)$, and regard α as lying in $V(\partial M_1$ cut along the attaching circles). Hence

$$|p_f(Z(M_1))|^2 = S(1)^{c_2}|\alpha|^2.$$

Closely related to the face decomposition is the decomposition of ∂M_1 obtained by replacing each attaching circle with an annulus. With respect to to this decomposition we have

$$p_f(Z(M_1)) = \beta_{11}^{\otimes c_2} \otimes \alpha.$$

It follows that

$$Z(M_2) = S(1)^{-c_2}(\beta_1 \otimes \beta_1)^{\otimes c_2} \otimes \alpha.$$

(The $(\beta_1 \otimes \beta_1)$'s correspond to the non-attaching part of the boundaries of the 2-handles.)

∂M_2 consists of ∂M and c_3 2-spheres. With respect to this decomposition write

$$Z(M_2) = \gamma \otimes (\beta_1 \otimes \beta_1)^{\otimes c_3}.$$

(The $(\beta_1 \otimes \beta_1)$'s lie in the $V(S^2)$'s, and $\gamma \in V(\partial M)$.) It follows that

$$Z(M) = S(1)^{c_3}\gamma.$$

If we could calculate α, it would be easy to calculate γ and hence $Z(M)$. This cannot be conveniently be done, so instead we will calculate $|p_f(Z(M_1))|^2$ (in terms of a state model), and hence $|Z(M)|^2$. Note that

$$\begin{aligned} |Z(M)|^2 &= S(1)^{2c_3}|\gamma|^2 \\ &= S(1)^{c_3}|Z(M_2)|^2 \\ &= S(1)^{c_3}|\alpha|^2 \\ &= S(1)^{c_3-c_2}|p_f(Z(M_1))|^2. \end{aligned} \tag{15.2}$$

The operation of attaching a 2-handle to a curve C in the boundary of a 3-manifold N can be factored into the following three operations. First, excise from N a regular neighborhood of a curve C', where C' is isotopic to C and contained in the interior of N. Second, Dehn surger the resulting manifold with respect to a curve on $\partial(\text{nbd}(C'))$ which is isotopic to C. Third, cut along the disk in the surgered manifold which consists of a meridinal disk of the surgery solid torus and an annulus connecting the surgery curve to C.

Let M_1^- be the manifold obtained from M_1 by excising neighborhoods of curves in the interior of M_1 isotopic to the attaching circles of the 2-handles. Let M_1^{-+} be the manifold obtained from M_1^- by doing Dehn surgery as described above. We will calculate each of $Z(M_1^-)$ and $Z(M_1^{-+})$ in two different ways. Comparing these calculations will yield a calculation of $|p_f(Z(M_1))|^2$.

Consider the decomposition of ∂M_1 obtained from the face decomposition by replacing each attaching circle with three parallel annuli. Gluing the outer two annuli together yields M_1^- (see Figure 72). It follows from (15.1) that

$$Z(M_1^-) = \bigoplus_{l_f} S(l_f)^{-1}\beta_{l_f} \otimes Z(M_1)_{l_f}. \tag{15.3}$$

(β_{l_f} denotes $\beta_{a_1\hat{a}_1} \otimes \cdots \otimes \beta_{a_n\hat{a}_n}$, where $l_f = (a_1, \ldots, a_n)$.) In the above equation, $\partial M_1 \subset \partial M_1^-$ has the face decomposition and the tori in ∂M_1^- are decomposed along the surgery curves.

M_1^- can also be obtained be gluing together c_0 tunneled caltrops (see Figure 73). The boundary of a tunneled caltrop can be decomposed into six annuli, four 4-punctured spheres, and another 4-punctured sphere. With respect to this decomposition write

$$Z(\text{tunneled caltrop}) = \bigoplus_{m,n} \varphi_{mn} \otimes \beta_m.$$

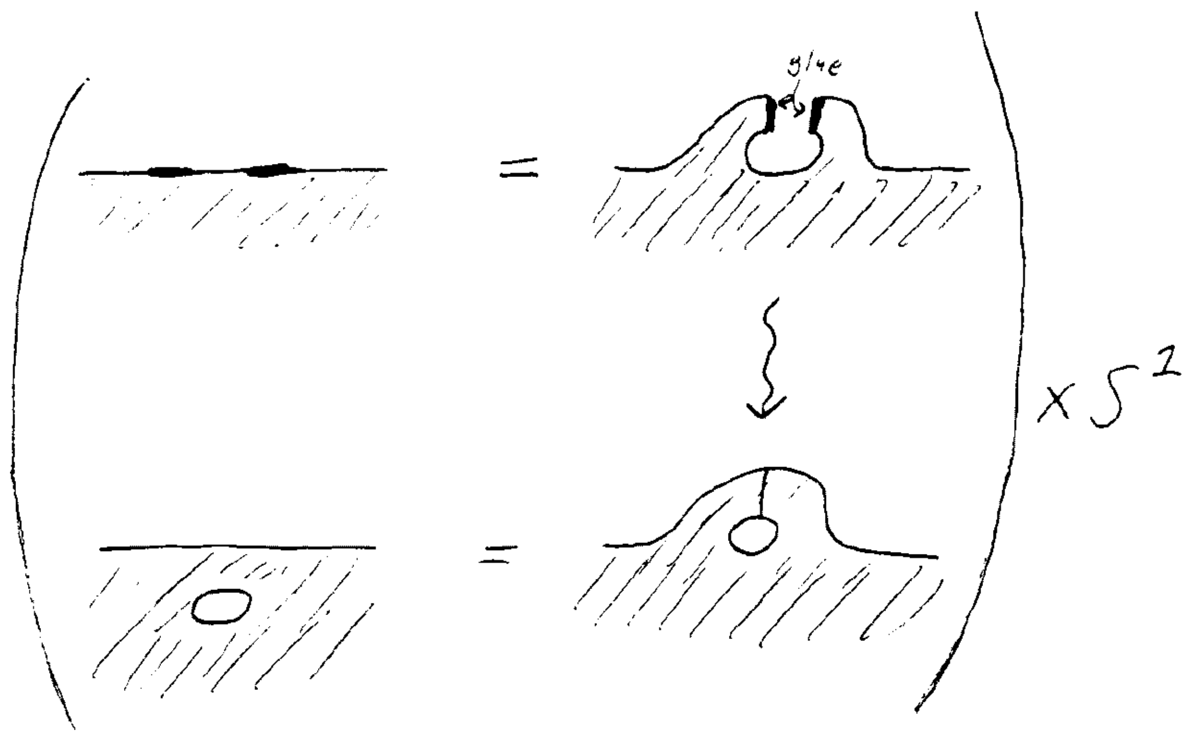

Figure 72: Achieving an excision by gluing up the boundary.

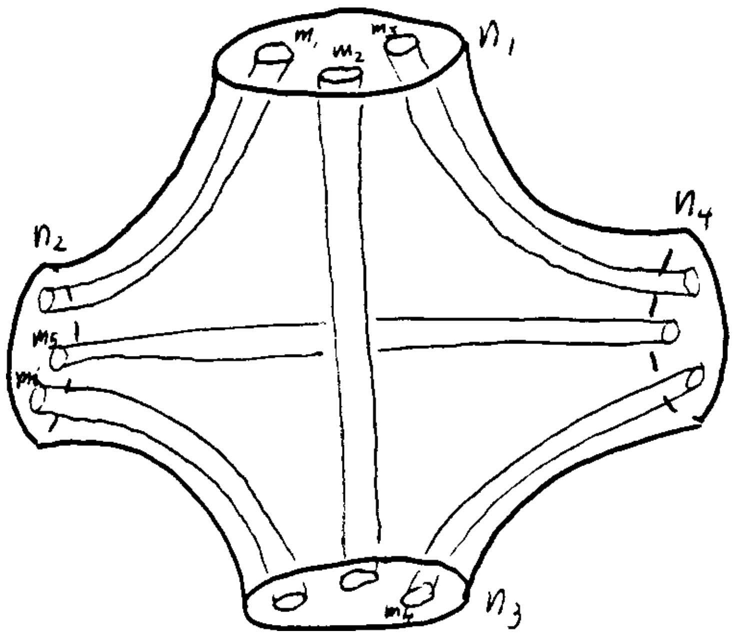

Figure 73: A tunneled caltrop.

The sum is over all labelings $m = (m_1, \ldots, m_6)$ and $n = (n_1, \ldots, n_4)$ (see Figure 73). β_m denotes $\beta_{m_1\hat{m}_1} \otimes \cdots \otimes \beta_{m_6\hat{m}_6}$, and

$$\varphi_{mn} \in V_{n_1n_2n_3n_4} \otimes V_{\hat{n}_1m_1m_2m_3} \otimes V_{\hat{n}_2\hat{m}_1m_5m_6} \otimes V_{\hat{n}_3m_4\hat{m}_2\hat{m}_6} \otimes V_{\hat{n}_4\hat{m}_4\hat{m}_5\hat{m}_3}.$$

We will think of ϕ_{mn} as a tensor with five indices.

Let l_e be a labeling of the edges and l_f be a labeling of the faces. l_e and l_f induce a labeling (m, n) for the tunneled caltrop associated to each 0-cell. Let $\mathrm{con}(l_e, l_f)$ denote the result of contracting all of the resulting φ_{mn}'s together in the obvious fashion. Then

$$Z(M_1^-) = \bigoplus_{l_e, l_f} \beta_{l_f} \otimes \mathrm{con}(l_e, l_f). \tag{15.4}$$

Here the tori in ∂M_1^- are decomposed along meridinal curves, and $\mathrm{con}(l_e, l_f)$ should be thought of as lying in the vector space associated to the edge decomposition of $\partial M_1 \subset \partial M_1^-$.

Now we use (15.3) and (15.4) to calculate $Z(M_1^{-+})$ in two different ways. For (15.3) this is easy since the tori are already decomposed along the surgery curves. The result is

$$Z(M_1^{-+}) = S(1)^{-c_2} Z(M_1)_{1_f} = S(1)^{-c_2}\alpha. \tag{15.5}$$

For (15.4) we must first apply S to the tori. The result is

$$Z(M_1^{-+}) = \bigoplus_{l_e} \sum_{l_f} S(l_f)\mathrm{con}(l_e, l_f). \tag{15.6}$$

Recall that the first equation is with respect to the face decomposition of $\partial M_1^{-+} = \partial M_1$, and the second equation is with respect to the edge decomposition.

Let 1_e denote the labeling which assigns the trivial label to each edge. Let p_e denote the projection of $V(\partial M_1)$ onto the 1_e summand (with respect to the edge decomposition). By (15.5) and (15.6),

$$p_f(Z(M_1)) = \alpha = \bigoplus_{l_e} \sum_{l_f} S(1)^{c_2} S(l_f) \mathrm{con}(l_e, l_f).$$

Hence

$$p_e(p_f(Z(M_1))) = S(1)^{c_2} \sum_{l_f} S(l_f) \mathrm{con}(1_e, l_f).$$

Keeping in mind that $p_e(Z(M_1)) = Z(M_1) = \beta_{1111}^{\otimes c_0}$, we see that

$$\begin{aligned} |p_f(Z(M_1))|^2 &= \langle p_f(Z(M_1)), p_f(Z(M_1)) \rangle_h \\ &= \langle p_f(Z(M_1)), Z(M_1) \rangle_h \\ &= \langle p_e(p_f(Z(M_1))), Z(M_1) \rangle_h \\ &= S(1)^{c_2} \sum_{l_f} S(l_f) \langle \mathrm{con}(1_e, l_f), \beta_{1111}^{\otimes c_0} \rangle_h. \end{aligned}$$

Thus, by (15.2),

$$|Z(M)|^2 = S(1)^{c_3} \sum_{l_f} S(l_f) \langle \mathrm{con}(1_e, l_f), \beta_{1111}^{\otimes c_0} \rangle_h. \tag{15.7}$$

[It is not hard to express the tensors φ_{m1} which appear in (15.7) in terms of the partition function of a labeled tetrahedron in B^3 or S^3. (Compare [Wi2].) Using the results of Sections 8 through 11 this can, in turn, be related to $6j$-symbols, in the case where Z comes from a modular Hopf algebra. Using Section 13, one can obtain the formulae of [TV]. Details will be included in a later draft.]

.
.
.

16 PL Modular Functors from Holomorphic Modular Functors

The definition of a modular functor given in Section 2 is a piecewise linear version of Segal's original definition [S1]. In this section we show how to construct such a PL modular functor starting from one of Segal's holomorphic modular functors. A similar construction is described in [S2].

[There are some details missing below. Some are missing because the corresponding details are missing from [S1]. Some are missing because the corresponding details are not missing from [S1], and there's no point in repeating them. Some are missing because this is merely the first draft of this section. And there are probably some details which are *not* missing, but are wrong. Reader's capable of detecting these errors should have no trouble correcting them.]

Define an *lh-surface* to be a smooth, compact, oriented surface whose boundary components are equiped with smooth parameterizations and are labeled by elements of some label set $\mathcal{L}$, and whose interior is equiped with a holomorphic structure. The orientations of the boundary components coming from the parameterizations should agree with the orientations induced by the orientation of the surface. lh-surfaces have well-defined gluing operations.

Define a *holomorphic modular functor* to be a functor E from the the category of lh-surfaces and structure-preserving diffeomorphisms to the category of finite dimensional complex vector spaces and isomorphisms which satisfies axioms analogous to (2.1) through (2.6) and also

(16.1) If X is a [holomorphic] family of lh-surfaces then $\{E(\Sigma)\}_{\Sigma\in X}$ has the structure of a [holomorphic] vector bundle over X.

Let $Y = (Y^\flat, L)$ be an le-surface. Define $\mathcal{C}(Y)$ to be the space of all lh-structures on Y. A point of $\mathcal{C}(Y)$ is represented by a pair (Σ, h), where Σ is an lh-surface and $h : \Sigma \to Y^\flat$ is a label and parameterization preserving homeomorphism. (Σ_1, h_1) is considered equivalent to (Σ_2, h_2) if there is a morphism $g : \Sigma_1 \to \Sigma_2$ such that the diagram

$$\begin{tikzcd} \Sigma_1 \arrow[r, "g"] \arrow[dr, "h_1"'] & \Sigma_2 \arrow[d, "h_2"] \\ & Y^\flat \end{tikzcd}$$

commutes up to homotopy. (Homotopies are required to fix the boundary.) $\mathcal{C}(Y)$ is contractable. (We will use only the fact that $\mathcal{C}(Y)$ is 0- and 1-connected.)

Let E be a holomorphic modular functor. By (16.1), there is a complex vector bundle $E(Y)$ over $\mathcal{C}(Y)$ whose fiber at (Σ, f) is $E(\Sigma)$. The operation of gluing annuli to boundary components of lh-surfaces gives rise to a projectively flat connection on $E(Y)$ whose (scalar valued) curvature is equal to $c\omega$, where $c \in \mathbf{R}$ depends only on E and ω is a certain 2-form on $\mathcal{C}(Y)$. We will assume that $c \neq 0$. There is also, over $\mathcal{C}(Y)$, the determinant line bundle $\mathrm{Det}(Y)$ which is equiped with a connection whose curvature is equal to ω. [More details in a later version of this paper.]

Let us now pause to summarize and motivate the remainder of the section. If c were an integer, then $\mathrm{Det}(Y)^{-c}$ would be a line bundle with curvature $-c\omega$, and $E(Y) \otimes \mathrm{Det}(Y)^{-c}$ would be a flat bundle. Since $\mathcal{C}(Y)$ is $0,1$-connected, the fibers of $E(Y) \otimes \mathrm{Det}(Y)^{-c}$ could be unambiguously identified as a single vector space $V(Y^\flat)$. The important thing here is that $V(Y^\flat)$ is *functorially* associated to $Y^\flat$ (not just to Y). Is is easy to see that V, so defined, would satisfy (2.1) through (2.6), since E satisfies similar axioms.

In reality, c is not in general an integer, so defining $\mathrm{Det}(Y)^{-c}$ requires more work. This extra work includes using the fiberwise universal cover of the square of the unit determinant line bundle $\mathrm{Det}^1(Y)^2$. This step means that $\mathrm{Det}(Y)^{-c}$ is not functorially associated to $Y^\flat$ (because $\mathrm{Det}(\Sigma)^{-c}$ is not functorially associated to the lh-surface Σ). This is essentially the fact that the universal cover is not a functor of topological spaces. It is, however, a functor of *pointed* topological spaces. The maximal isotropic subspace L of $Y = (Y^\flat, L)$ allows us to define a section of $\mathrm{Det}^1(Y)^2$ over $\mathcal{C}(Y)$. This section plays the role of the base point of a pointed topological space, and allows us to functorially associate the fiber of $E(Y) \otimes \mathrm{Det}(Y)^{-c}$ to the le-surface Y.

Define $\mathrm{Det}^1(Y)$, the unit determinant line bundle of Y, to be the S^1 bundle over $\mathcal{C}(Y)$ whose fiber at (Σ, f) is $\mathrm{Det}(\Sigma)/\mathbf{R}^+$. $\mathrm{Det}^1(Y)$ inherits a connection with curvature ω. Let $\mathrm{Det}^1(Y)^2$ be the square of this bundle. The curvature of $\mathrm{Det}^1(Y)^2$ is 2ω.

Let $(\Sigma, f) \in \mathcal{C}(Y)$. $\mathrm{Det}(\Sigma)$ can be identified with the maximal exterior power of $H_1(\Sigma)/H_1(\partial\Sigma)$ thought of as a complex vector space. [More needs to be said in the case where the genus of Y is zero.] Thus elements of $\mathrm{Det}(\Sigma)$ are represented by things of the form $e_1 \wedge \cdots e_g$, $e_i \in H_1(\Sigma)/H_1(\partial\Sigma)$. $f_*^{-1}(L)/H_1(\partial\Sigma)$ is a totally real subspace of $H_1(\Sigma)/H_1(\partial\Sigma)$. Hence

$$\mathrm{Det}(L)_{(\Sigma,f)} \stackrel{\text{def}}{=} a_1 \wedge \cdots \wedge a_g,$$

where (a_i) is a basis of $f_*^{-1}(L)/H_1(\partial\Sigma)$, is well defined up to non-zero real numbers. It follows that $\mathrm{Det}(L)_{(\Sigma,f)}$ is a well-defined element of $\mathrm{Det}^1(\Sigma)^2$. Let $\mathrm{Det}(L)$ denote the corresponding section of $\mathrm{Det}^1(Y)^2$.

Let $\widetilde{\mathrm{Det}}^1(Y)^2$ be the fiberwise universal cover of $\mathrm{Det}^1(Y)^2$: The fiber at $p \in \mathcal{C}(Y)$ is the set of all homotopy classes (rel boundary) of paths in $\mathrm{Det}^1(Y)^2_p$ which start at $\mathrm{Det}(L)_p \in \mathrm{Det}^1(Y)^2_p$. This is an $\mathbf{R}$ bundle over $\mathcal{C}(Y)$ with curvature 2ω. (The structure group is $\mathbf{R}$, acting by translations.) Let $\mathrm{Det}^1(Y)^{-c}$ be the S^1 bundle obtained by dividing out by the $\mathbf{Z}$ action generated by translation by $-\pi c^{-1}$ in each fiber of $\widetilde{\mathrm{Det}^1}(Y)^2$. This S^1 bundle has curvature $-c\omega$. Define $\mathrm{Det}(Y)^{-c}$ to be the corresponding complex line bundle.

$E(Y) \otimes \mathrm{Det}(Y)^{-c}$ is a flat vector bundle over the $0,1$-connected space $\mathcal{C}(Y)$. Therefore there is a well-defined identification between any two fibers. Define $V(Y)$ to be "the" fiber of $E(Y) \otimes \mathrm{Det}(Y)^{-c}$.

Let $f = (f^\flat, n)$ be a morphism from $Y_1 = (Y_1^\flat, L_1)$ to $Y_2 = (Y_2^\flat, L_2)$. We must define a map

$$V(f) : V(Y_1) \to V(Y_2).$$

To do this it suffices to define a (connection preserving) isomorphism

$$\bar{f} : E(Y_1) \otimes \mathrm{Det}(Y_1)^{-c} \to E(Y_2) \otimes \mathrm{Det}(Y_2)^{-c}.$$

$\bar{f}$ gives rise to an isomorphism $\bar{f} : V(Y_1) \to V(Y_2)$, and we define

$$V(f) \stackrel{\mathrm{def}}{=} (e^{2\pi i c})^{\frac{n}{4}} \bar{f}.$$

$f^\flat$ induces a homeomorphism $f_* : \mathcal{C}(Y_1) \to \mathcal{C}(Y_2)$. By the axioms of a holomorphic modular functor, this map is covered by a isomorphism $\bar{f}_1 : E(Y_1) \to E(Y_2)$. So all that remains is to define an isomorphism

$$\bar{f}_2 : \mathrm{Det}(Y_1)^{-c} \to \mathrm{Det}(Y_2)^{-c}.$$

We will then define

$$\bar{f} = \bar{f}_1 \otimes \bar{f}_2.$$

Before defining $\bar{f}_2$, a digression. Let K_1 and K_2 be lagrangians in a symplectic vector space. Let $e_1, \ldots, e_m$ be a basis of K_1 such that $e_1, \ldots, e_k$ is a basis of $K_1 \cap K_2$. Then there is a basis $f_1^\pm, \ldots, f_n^\pm$ of K_2 such that

$$\begin{array}{rcll} f_i^\pm & = & e_i, & i \le k \\ \Omega(e_i, f_i^\pm) & = & \pm 1, & i > k \\ \Omega(e_i, f_j^\pm) & = & 0, & i \ne k. \end{array}$$

(Ω is the symplectic form.) Define

$$P^\pm(K_1, K_2)_t \stackrel{\mathrm{def}}{=} \mathrm{span}(\{(1-t)e_i + t f_i^\pm\}_{1 \le i \le m}).$$

$P^\pm(K_1, K_2)$ is a path of lagrangians from K_1 to K_2. It is easy to see that the homotopy class of $P^\pm(K_1, K_2)$ is independent of the choices of the e_i's and f_i's. Thus $P^+(K_1, K_2)$

and $P^-(K_1, K_2)$ are a canonical pair of (homotopy classes of) paths from K_1 to K_2. For future reference we observe that $P^\pm$ is additive in the following sense:

$$P^\pm(K_1 \oplus J_1, K_2 \oplus J_2) = P^\pm(K_1, K_2) \oplus P^\pm(J_1, J_2). \tag{16.2}$$

(It is interesting to note that similar paths of lagrangians are used in [W].)

Let $p_1 \in \mathcal{C}(Y_1)$ and $p_2 = f_*(p_1) \in \mathcal{C}(Y_2)$. $f^\flat$ induces an isomorphism

$$f_* : \mathrm{Det}^1(Y_1)^2_{p_1} \to \mathrm{Det}^1(Y_2)^2_{p_2}.$$

Thus if $a \in \widetilde{\mathrm{Det}^1}(Y_1)^2_{p_1}$, then $f_*(a)$ is a homotopy class of path in $\mathrm{Det}^1(Y_2)^2_{p_2}$ which starts at $f_*(\mathrm{Det}(L_1)_{p_1}) = \mathrm{Det}(f_*(L_1))_{p_1}$. $\mathrm{Det}(P^\pm(L_2, f_*(L_1)))_{p_2}$ is a path in $\mathrm{Det}^1(Y_2)^2_{p_2}$ from $\mathrm{Det}(L_2)_{p_2}$ to $\mathrm{Det}(f_*(L_1))_{p_2}$. (Here we are letting L_j denote its image in $H_1(Y_j)/H_1(\partial Y_j)$.) Hence we have two elements

$$\mathrm{Det}(P^\pm(L_2, f_*(L_1)))_{p_2} * f_*(a) \in \widetilde{\mathrm{Det}^1}(Y_2)^2_{p_2}.$$

("$*$" denotes composition of paths.) Define $\bar{f}_2(a)$ to be the average of these two elements. The maps (for all p_1)

$$\bar{f}_2 : \widetilde{\mathrm{Det}^1}(Y_1)^2_{p_1} \to \widetilde{\mathrm{Det}^1}(Y_2)^2_{p_2}$$

induce a map

$$\bar{f}_2 : \mathrm{Det}(Y_1)^{-c} \to \mathrm{Det}(Y_2)^{-c}.$$

This is the desired isomorphism, and completes the definition of $V(f)$.

Next we must show that V is a functor. That is, we must show that

$$V(gf) = V(g)V(f). \tag{16.3}$$

Recall that if

$$\begin{aligned} f &= (f^\flat, n) : (Y_1, L_1) &\to& \ (Y_2, L_2) \\ g &= (g^\flat, m) : (Y_2, L_2) &\to& \ (Y_3, L_3), \end{aligned}$$

then

$$gf = (g^\flat f^\flat, m + n + \sigma((g^\flat f^\flat)_* L_1, g^\flat_* L_2, L_3)).$$

Let $\sigma \stackrel{\mathrm{def}}{=} \sigma((g^\flat f^\flat)_* L_1, g^\flat_* L_2, L_3)$. Since a holomorphic modular functor is a functor, $(\overline{gf})_1 = \bar{g}_1 \bar{f}_1$. So (16.3) reduces to

$$(e^{2\pi i c})^{\frac{m+n+\sigma}{4}} (\overline{gf})_2 = (e^{2\pi i c})^{\frac{m}{4}} \bar{g}_2 (e^{2\pi i c})^{\frac{n}{4}} \bar{f}_2$$

or

$$(e^{2\pi i c})^{\frac{\sigma}{4}} = \bar{g}_2 \bar{f}_2 (\overline{gf})_2^{-1}.$$

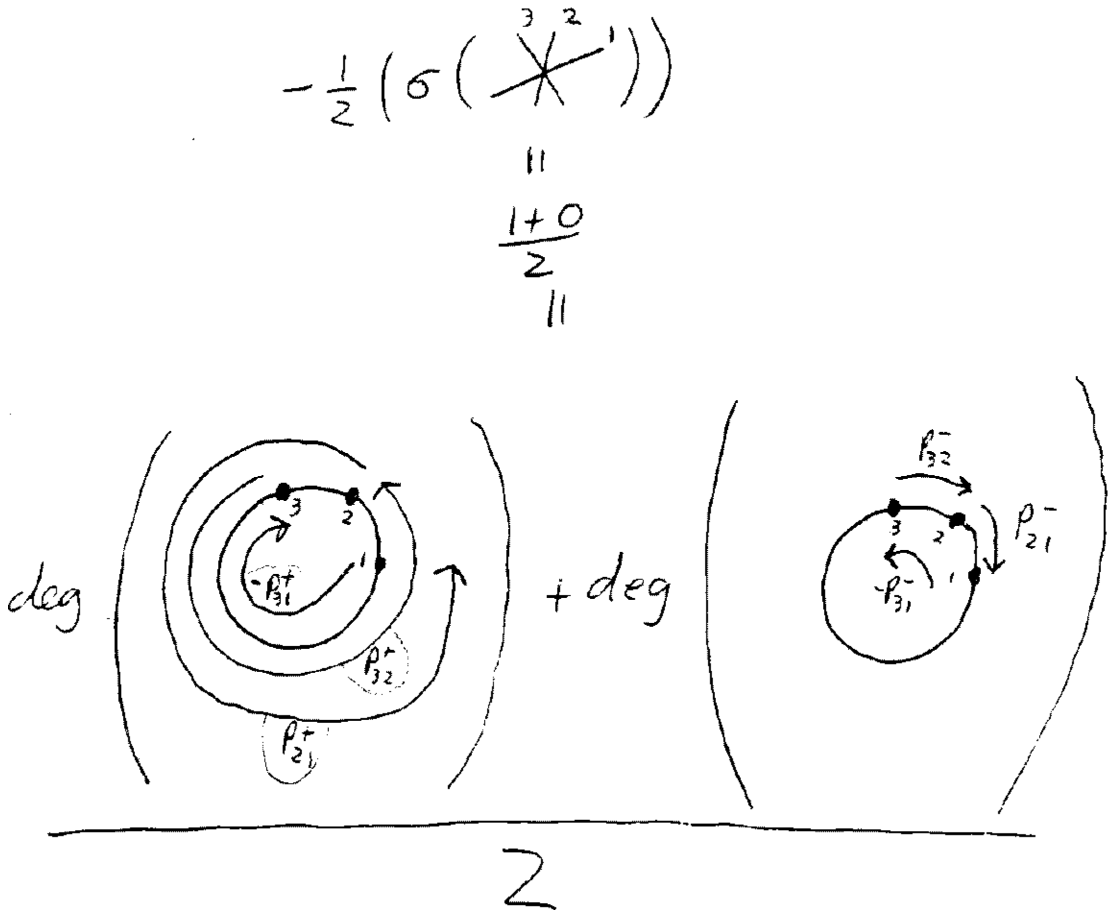

Figure 74: Part of the proof of (16.4).

This implied by the claim that, for any three lagrangians K_1, K_2, K_3,

$$(16.4)\quad -\frac{\sigma(K_1, K_2, K_3)}{2} = \deg\left(\mathrm{Det}(P^{\pm}(K_3, K_2) * P^{\pm}(K_2, K_1) * (-P^{\pm}(K_3, K_1))\right).$$

Here "$*$" denotes composition of paths and "$-$" means traverse the path in the opposite direction. The right hand side should be interpreted as the average of the degrees of the two closed paths coming from P^+ and P^-. These paths lie in $\mathrm{Det}^1(W)^2$, where W is the symplectic vector space containing the K_i's and equiped with a compatible complex structure.

By (18.7), (18.4) and (16.2), it suffices to verify (16.4) when $\dim(K_i) = 1$. This is easily done by hand. One case of the verification is shown in Figure 74.

We have now succeeded is defining a functor V from the category of le-surfaces to the category of complex vector spaces. As was pointed out above, it is easy to show that V satisfies the axioms of a PL modular functor by using the fact that E satisfies the corresponding axioms of a holomorphic modular functor.

$$\vdots$$

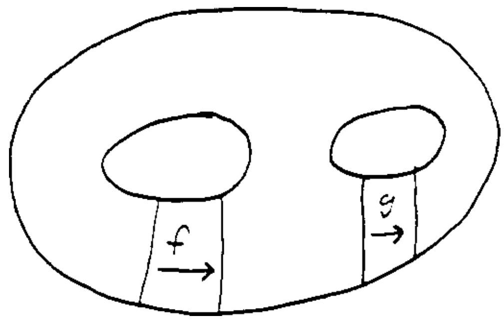

Figure 75: 4-manifold for d.

17 Central extensions of mapping class groups

In this section we remark briefly on the central extensions of mapping class groups arising from e-surfaces.

.
.
.

[Outline:

Point #1: (1.7) is the well-known Shale-Weil cocycle (see [LV]).

Point #2: (16.4) shows that this central extension is equal to -4 times the central extension arising from the action of the mapping class group on the determinant line bundle. (The right hand side corresponds to $-1/2$ times our favorite cocycle, and the left hand side corresponds to a cocycle representing 2 times the determinant line bundle extension (= the extension aring from the square of the determinant line).)

Point #3: [This was typed in a hurry; there might be some sign and ordering errors.] In [A1] (see also [A3]) Atiyah discusses the following cocycle. Let Y be a closed surface. Let $f, g \in \mathcal{M}(Y)$. Let $D(f,g)$ be the Y bundle over a pair of pants with monodromies f and g (see Figure 75). Define

$$d(f,g) \stackrel{\text{def}}{=} \sigma(D(f,g)),$$

where σ denotes the signature of a 4-manifold. Atiyah shows that d arises naturally when one considers framed 3-manifolds.

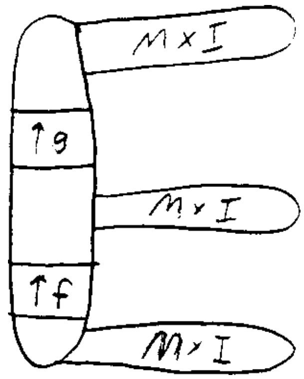

Figure 76: 4-manifold for c.

We wish to show that c (see (1.7)) and d are cohomologous. One way to do this would be to use Point #2 above and a similar fact about d (see [A1]). Another way would be to exploit the correspondence between framed 3-manifolds and 3-manifolds equiped with a bordism class of null-bordisms (i.e. e-3-manifolds), as explained in Section 1.

A third, simpler way goes as follows. Let $L \subset H_1(Y)$ be a lagrangian subspace. Let M be such that $\partial M = Y$ and $\ker(H_1(Y) \to H_1(M)) = L$. (This puts some restrictions on L.) Let $C(f,g)$ be the 4-manifold obtained by gluing three copies of $M \times I$ to $Y \times D^2$ as shown in Figure 76. Then

$$c(f,g) \stackrel{\mathrm{def}}{=} \sigma(C(f,g)).$$

Let $f \in \mathcal{M}(Y)$ Let $J(f)$ be the 4-manifold obtained by gluing $M \times I$ to the mapping torus of f cross I as shown in Figure 77. Define

$$j(f) \stackrel{\mathrm{def}}{=} \sigma(J(f)).$$

Let X be the 5-manifold obtained by gluing $M \times D^2$ to $D(f,g) \times I$ as shown in Figure 78. ∂X can be decomposed as

$$\partial X = C(f,g) \cup -D(f,g) \cup J(f) \cup J(g) \cup -J(fg).$$

This decomposition is along *closed* 3-manifolds. It follows from Novikov additivity and the fact that the signature of a boundary is zero that

$$\begin{aligned} 0 &= \sigma(\partial X) \\ &= \sigma(C(f,g)) - \sigma(D(f,g)) + \sigma(J(f)) + \sigma(J(g)) - \sigma(J(fg)) \\ &= c(f,g) - d(f,g) + j(f) + j(g) - j(fg) \end{aligned}$$

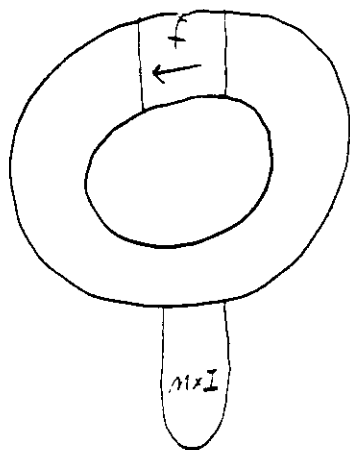

Figure 77: 4-manifold for j.

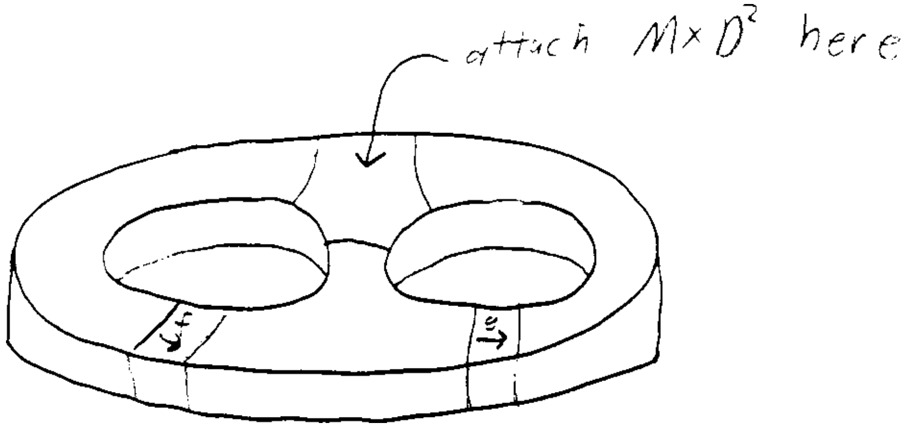

Figure 78: A 5-manifold.

In other words, $d-c$ is equal to the coboundary of the 1-cochain j.

Point #4: $H_2(\mathcal{M}(T^2);\mathbf{Q}) = H_2(SL_2(\mathbf{Z});\mathbf{Q}) = 0$. It follows that when $Y = T^2$ there is a $\mathbf{Q}$-valued 1-cochain whose coboundary is c. For the proper choice of L this cochain is precisely $\Phi/3$, where $\Phi : SL_2(\mathbf{Z}) \to \mathbf{Z}$ is Rademacher's Φ function. (For d, Φ is replaced by Ψ; see [A3].)]

18 Nonadditivity of the signature

This section contains the definition of $\sigma(\cdot,\cdot,\cdot)$ and reviews Wall's theorem on the nonadditivity of the signature. For more details see [Wa]. A decomposition theorem for triples of lagrangian subspaces is also proved.

Let A, B and C be subspaces of a real vector space W. The three place relation $a+b+c=0$ $(a \in A, b \in B, c \in C)$ induces isomorphisms

$$U \stackrel{\text{def}}{=} \frac{A \cap (B+C)}{(A \cap B) + (A \cap C)} = \frac{B \cap (C+A)}{(B \cap C) + (B \cap A)} = \frac{C \cap (A+B)}{(C \cap A) + (C \cap B)}.$$

Let ω be an antisymmetric bilinear form on W. Assume that A, B and C are isotropic with respect to ω (i.e. $\omega(A \times A) = \omega(B \times B) = \omega(C \times C) = 0$). Let $a+b+c=0=a'+b'+c'$. Then a and a' represent elements $[a],[a'] \in U$. Define

$$\psi([a],[a']) \stackrel{\text{def}}{=} \omega(a,b').$$

ψ is a well-defined, symmetric bilinear form on U. If A, B and C are maximal isotropic, then ψ in nondegenerate. Define

$$\sigma(A,B,C) \stackrel{\text{def}}{=} \operatorname{sign}(\psi).$$

($\operatorname{sign}(\psi)$ denotes the signature of ψ, i.e. the number of positive eigenvalues of ψ minus the number of negative eigenvalues of ψ.)

The proofs of the following lemmas are elementary and are left to the reader.

(18.1) **Lemma.** *If A, B and C are permuted, then $\sigma(A,B,C)$ changes by the sign of the permutation.* □

(18.2) **Lemma.** *If ω is replaced by $-\omega$, then the sign of $\sigma(A,B,C)$ changes.* □

Let $N \subset W$ be the null space of ω. Then W/N inherits a symplectic structure from ω. If A, B and C are maximal isotropic, then A/N, B/N and C/N are lagrangian. (Geometrically, this corresponds to capping off the boundary of a surface.)

(18.3) **Lemma.**

$$\sigma(A,B,C) = \sigma(A/N, B/N, C/N).$$

□

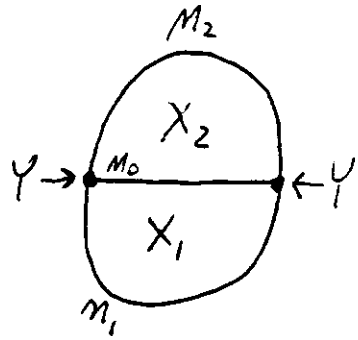

Figure 79: Two 4-manifolds glued together.

(18.4) **Lemma.** *Let A_i, B_i and C_i be isotropic subspaces of W_i with respect to ω_i ($i = 1, 2$). Then $A_1 \oplus A_2$, $B_1 \oplus B_2$ and $C_1 \oplus C_2$ are isotropic subspaces of $W_1 \oplus W_2$ with respect to $\omega_1 \oplus \omega_2$, and*

$$\sigma(A_1 \oplus A_2, B_1 \oplus B_2, C_1 \oplus C_2) = \sigma(A_1, B_1, C_1) + \sigma(A_2, B_2, C_2).$$

□

Let X be a 4-manifold decomposed along a properly embedded 3-manifold M_0 into X_1 and X_2. As *oriented* manifolds, let

$$\begin{aligned} \partial X_1 &= (-M_1) \cup M_0 \\ \partial X_2 &= (-M_0) \cup M_2 \\ Y &= \partial M_0 = \partial M_1 = \partial M_2 \end{aligned}$$

(see Figure 79). Let

$$K_i \stackrel{\text{def}}{=} \ker(H_1(Y; \mathbf{R}) \to H_1(M_i; \mathbf{R})$$

($i = 0, 1, 2$). K_i is a lagrangian subspace of $H_1(Y; \mathbf{R})$ with respect to the intersection pairing. Let $\sigma(X_*)$ denote the signature of X_*.

(18.5) **Theorem (Wall).**

$$\sigma(X) = \sigma(X_1) + \sigma(X_2) - \sigma(K_1, K_0, K_2).$$

□

Let Y be a surface and let $L \subset H_1(Y; \mathbf{R})$ be a maximal isotropic subspace. Let f and g be elements of the mapping class group of Y. Define

$$c(f,g) \stackrel{\text{def}}{=} \sigma((fg)_* L, f_* L, L).$$

As explained in Section 1, Wall's theorem implies that $c(f,g)$ is a 2-cocycle on $\mathcal{M}(Y)$. The fact that the coboundary of c is zero implies the following proposition.

(18.6) **Proposition.** *Let L_1, L_2, L_3 and L_4 be maximal isotropic subspaces. Then*

$$\sigma(L_1, L_2, L_3) + \sigma(L_1, L_3, L_4) = \sigma(L_2, L_3, L_4) + \sigma(L_1, L_2, L_4).$$

□

(Presumably this proposition can be proved algebraically without resorting to Wall's theorem, but I have not done so.)

(18.7) **Proposition.** *Let W be a $2n$-dimensional symplectic vector space and let L_1, L_2 and L_3 be lagrangian subspaces of W. Then there exist 2-dimensional symplectic subspaces $W^i \subset W$ $(i = 1, \ldots, n)$ and lagrangian subspaces $L_j^i \subset W^i$ $(j = 1, 2, 3)$ such that*

$$\begin{aligned} W &= W^1 \oplus \cdots \oplus W^n \\ L_j &= L_j^1 \oplus \cdots \oplus L_j^n. \end{aligned}$$

Proof:

.
.
.

□

19 The combinatorics of 1- and 2-parameter families of Morse functions

In this section we prove (7.3) and (6.1).

.
.
.

References

[A1] M. Atiyah, On framings of 3-manifolds, Topology **29** (1990), 1-7.

[A2] M. Atiyah, Topological quantum field theories, Publ. Math. IHES **68** (1989), 175-186.

[A3] M. Atiyah, The logarithm of the Dedekind η-function, Math. Ann. **278** (1987), 335-380.

[CLM] S. Cappell, R. Lee and E. Miller, Invariants of 3-manifolds from conformal field theory, preprint.

[Ce] Cerf

[C] L. Crane,

[FR] R. Fenn and C. Rourke, On Kirby's calculus of links, Topology **18** (1979), 1-15.

[GG] M. Golubitsky and V. Guillemin, Stable mappings and their singularities, Springer-Verlag, 1973.

[HT] A. Hatcher and W. Thurston, A presentation for the mapping class group of a closed orientable surface, Topology **19** (1980), 221-237.

[K] R. Kirby, A calculus for framed links in S^3, Invent. Math. **45** (1978), 35-56.

[KM] R. Kirby and P. Melvin,

[KR] A. Kirillov and N. Reshetikhin, Representations of the algebra $U_q(\mathbf{sl}_2)$, q-orthogonal polynomials and invariants of links, LOMI preprint E-9-88 (1988).

[Koh] T. Kohno,

[Ko] M. Kontsevich, Rational conformal field theory and invariants of 3-dimensional manifolds, preprint (1988).

[L] W. B. R. Lickorish, Three-manifolds and the Temperly-Lieb algebra, preprint (1990).

[LV]

[MS] G. Moore and N. Seiberg, Polynomial equations for rational conformal field theories, to appear in Phy. Lett.

[MSt] H. Morton and P. Strickland, Satellites and surgery invariants, preprint.

[RT1] N. Reshetikhin and V. Turaev, Ribbon graphs and their invariants derived from quantum groups, Comm. Math. Phys. **127** (1990), 1-26.

[RT2] N. Reshetikhin and V. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, to appear in Invent. Math.

[S1] G. Segal, Conformal field theories, preprint.

[S2] G. Segal, a later version of the above.

[TV] V. Turaev and O. Viro, State sum invariants of 3-manifolds and quantum $6j$-symbols, preprint.

[W] K. Walker, An extension of Casson's invariant, preprint.

[Wa] C. T. C. Wall, Non-additivity of the signature, Invent. Math. **7** (1969), 269-274.

[We] H. Wenzl, Braid tensor categories and invariants of 3-manifolds, preprint.

[Wi1] E. Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. **121** (1989), 351-399.

[Wi2] Gauge theories and integrable lattice models, Nucl. Phys. B **322** (1989), 629-697.