

# NONCOMMUTATIVE MASLOV INDEX AND $\eta$ -FORMS RECONSIDERED

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ABSTRACT. Invariants for Lagrangians of symplectic vector spaces, such as the Maslov index for paths and the Maslov triple index, have many applications in symplectic geometry and index theory. Here we study the properties of their generalizations for modules over  $C^*$ -algebras and correct an error in our earlier work on the subject.

## 1. INTRODUCTION

The Maslov triple index  $\tau(L_0, L_1, L_2)$ , also called Leray–Kashiwara index, is an invariant associated to a triple of Lagrangian subspaces of a finite dimensional symplectic vector space. It is related to the Maslov index for a pair of paths of Lagrangians  $\mu(L_1(t), L_2(t))$  through the formula

$$(1.1) \quad \tau(L_0(1), L_1(1), L_2(1)) - \tau(L_0(0), L_1(0), L_2(0)) = \\ 2(\mu(L_0(t), L_1(t)) + \mu(L_1(t), L_2(t)) + \mu(L_2(t), L_0(t))) .$$

Furthermore, one may associate  $\eta$ -invariants to a pair of Lagrangians and prove the following cocycle formula for the Maslov triple index

$$(1.2) \quad \tau(L_1, L_2, L_3) = \eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0) .$$

These invariants have found applications in symplectic geometry, the theory of quantization and in index theory, among others. For example, in a classical result of Wall, the Maslov triple index appears as a correction term in an additivity formula for signatures for manifolds with boundary [Wa]. Formula (1.2) arises in the context of cut-and-paste results for  $\eta$ -invariants [Bu]. For a survey on Maslov indices and references we refer to [CLM] and, more up-to-date, the webpage on the Maslov index maintained by Ranicki [R].

Using the Atiyah–Patodi–Singer index theorem for families, Bunke and Koch defined and studied (1.2) for families of Lagrangians [BK]. In previous work, we generalized (1) and (2) to a noncommutative context, considering  $C^*$ -modules instead of vector spaces [W1][W2]. The original motivation came from higher index theory. For example one may ask for a generalization of Wall’s non-additivity result to higher signatures for manifolds with boundary, which were introduced by Leichtnam and Piazza (see the survey [LP3]). Closely related is the question on cut-and-paste results for higher  $\eta$ -invariants.

In the meantime there has been an independent interest in generalizations of Maslov indices to a purely algebraic context, which was also reflected at the 2014 AMS-meeting on “The many facets of the Maslov index” [Bo]. Barge and Lannes defined and studied Maslov indices for modules over rings [BL][L]. These results are relevant for an algebraic analogue of Wall’s result in Ranicki’s  $L$ -theory. While

Barge and Lannes defined and proved (1.1) algebraically, it is not clear how an algebraic analogue of (1.2) should look like since the definition of the  $\eta$ -invariants is highly analytic. In this context, the  $C^*$ -algebraic context presents a test case where an analogue exists and its properties may be studied.

While the generalization of (1.1) to a  $C^*$ -algebraic context is rather direct and does not involve major technical difficulties [W2], this is not the case for the second formula. The technical prerequisites were developed in [W1]. The aim of the present paper is to give a short survey on the noncommutative analogue of (1.2) and to establish further properties of these invariants. We also correct an error in our earlier generalization of (1.2): Contrary to what is claimed in [W1], there are choices involved in the definition of the noncommutative  $\eta$ -forms (which replace the  $\eta$ -invariants) which cannot be eliminated.

A crucial technical condition in our framework, which is not present in the classical case, is that the Lagrangians are transverse (in the case of paths this concerns the endpoints). In this case the Maslov indices for triples and paths yield  $K$ -theory classes. The fact that in general the Maslov indices are less well-behaved, can be seen in the work of Barge and Lannes who do not make this assumption.

We heavily rely on results from noncommutative Atiyah–Patodi–Singer index theory. The framework used here, which builds on and extends foundational work by Lott, Wu and Leichtnam–Piazza [Lo][LP1][LP2], has been developed in [W1][W2][W3].

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## 2. LAGRANGIAN PROJECTIONS AND THE MASLOV TRIPLE INDEX

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra.

In this section we recall the definition and properties of noncommutative analogues of Lagrangian subspaces and of the Maslov triple index, relying on and slightly expanding [W1, §1.4]. For simplicity, we replace the symplectic vector space by a free finitely generated  $C^*$ -module. However, many of the following results can be adapted to projective modules.

Let  $n \in \mathbb{N}$ .

**Definition 2.1.** *Two selfadjoint projections  $P_1, P_2 \in M_n(\mathcal{A})$  are called TRANSVERSE if  $\text{Ran } P_0 \cap \text{Ran } P_1 = \{0\}$  and  $\text{Ran } P_1 + \text{Ran } P_2 = \mathcal{A}^n$ .*

Let  $\mathcal{A}^{2n}$  be endowed with the standard  $\mathcal{A}$ -valued scalar product and let

$$I_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} : \mathcal{A}^n \oplus \mathcal{A}^n \rightarrow \mathcal{A}^n \oplus \mathcal{A}^n .$$

**Definition 2.2.** *A LAGRANGIAN PROJECTION on  $\mathcal{A}^{2n}$  is a selfadjoint projection  $P \in M_{2n}(\mathcal{A})$  with*

$$PI_0 = I_0(1 - P) .$$

We define the standard Lagrangian projection

$$P_s := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_{2n}(\mathcal{A}) .$$

**Lemma 2.3.** (1) For every Lagrangian projection  $P$  on  $\mathcal{A}^{2n}$  there is a unitary  $p \in M_n(\mathcal{A})$  such that

$$P = \frac{1}{2} \begin{pmatrix} 1 & p^* \\ p & 1 \end{pmatrix}.$$

(2) For every Lagrangian projection  $P$  on  $\mathcal{A}^{2n}$  the unitary

$$U = \begin{pmatrix} 1 & 0 \\ 0 & p^* \end{pmatrix} \in M_{2n}(\mathcal{A})$$

with  $p$  as in (1) fulfills  $UI_0 = I_0U$  and  $UPU^* = P_s$ .

(3) A Lagrangian projection  $P$  is transverse to  $P_s$  if and only if  $1 - p$  is invertible.

If  $U = \begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix}$  is a unitary, then  $UPU^* = \frac{1}{2} \begin{pmatrix} 1 & p^*u \\ u^*p & 1 \end{pmatrix}$ .

Thus, two Lagrangian projections  $P_0, P_1 \in M_{2n}(\mathcal{A})$  are transverse to each other if and only if  $p_0^*p_1$  has a gap at 1. Here  $p_0, p_1 \in M_n(\mathcal{A})$  are the corresponding unitaries.

The unitary in (2) of the lemma is not uniquely determined by the conditions  $UI_0 = I_0U$  and  $UPU^* = P_s$  as we show now:

Assume that  $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  has these properties. Since

$$[U, I_0] = \begin{pmatrix} 0 & -2bi \\ 2ci & 0 \end{pmatrix}$$

it follows that  $b = c = 0$ .

Now

$$UPU^* = \frac{1}{2} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a^* & p^*d^* \\ pa^* & d^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & ap^*d^* \\ dpa^* & 1 \end{pmatrix}.$$

Thus the unitary has to fulfill the condition  $dpa^* = 1$ .

Clearly, we may enforce uniqueness by assuming  $a = 1$ .

Let  $(P_0, P_1, P_2)$  be a triple of pairwise transverse Lagrangian projections on  $\mathcal{A}^{2n}$ . For  $x \in \mathcal{A}^{2n}$  write  $x = x_1 + x_2$  with  $x_i \in \text{Ran } P_i$ ,  $i = 1, 2$ .

The form

$$h : \text{Ran } P_0 \times \text{Ran } P_0 \rightarrow \mathcal{A}, (x, y) \mapsto \langle x_2, I_0y_1 \rangle$$

is hermitian and non-singular. It defines an element in  $K_0(\mathcal{A})$ .

**Definition 2.4.** The MASLOV INDEX  $\tau(P_0, P_1, P_2) \in K_0(\mathcal{A})$  of a triple of pairwise transverse Lagrangian projections is the class of the hermitian form  $h$  in  $K_0(\mathcal{A})$ .

**Proposition 2.5.** Let  $p, q$  be projective projections over  $\mathcal{A}$ .

Then  $2([p] - [q]) \in K_0(\mathcal{A})$  may be expressed as the Maslov index of a triple of pairwise transverse Lagrangian projections.

*Proof.* Let  $p \in M_k(\mathcal{A})$  be a projection.

For  $m = -2, 0, 2$  let  $(P_0^m, P_1^m, P_2^m)$  be a triple of pairwise transverse Lagrangian projections in  $M_4(\mathbb{C})$  with  $\tau(P_0^m, P_1^m, P_2^m) = m$  via the identification  $K_0(\mathbb{C}) \cong \mathbb{Z}$ . (That such a choice is possible, follows from the Normalization property on [CLM, p. 163].)

We define the Lagrangian projections

$$P_i = P_i^2 \otimes p \oplus P_i^0 \otimes (1 - p) \in M_{4k}(\mathcal{A}) .$$

Then

$$\tau(P_0, P_1, P_2) = \tau(P_0^2, P_1^2, P_2^2)[p] + \tau(P_0^0, P_1^0, P_2^0)[1 - p] = 2[p] \in K_0(\mathcal{A}) .$$

By defining  $P_i = P_i^{-2} \otimes p \oplus P_i^0 \otimes (1 - p)$  one gets  $-2[p]$ .  $\square$

One may easily define the Maslov triple index in more generality by allowing that the form  $h$  has a projective radical and is non-singular on the complement.

### 3. ‘SMOOTH’ LAGRANGIAN PROJECTIONS

While the  $C^*$ -algebra  $\mathcal{A}$  generalizes a compact topological space, a closed smooth manifold  $B$  is generalized by a projective system of Banach algebras, mimicking the projective system  $(C^j(B))_{j \in \mathbb{N}_0}$ . Heuristically, such an additional structure is necessary since the definition of  $\eta$ -invariants is not topological, but analytic. Practically, it is motivated by the fact that cyclic homology (or the closely related de Rham homology used here) for  $C^*$ -algebras does not contain much information.

For details on the following see [W1, §1.3].

Let  $(\mathcal{A}_j, \iota_{j+1,j} : \mathcal{A}_{j+1} \rightarrow \mathcal{A}_j)_{j \in \mathbb{N}_0}$  be a projective system of involutive Banach algebras with unit satisfying the following conditions:

- The algebra  $\mathcal{A} := \mathcal{A}_0$  is a  $C^*$ -algebra.
- For any  $j \in \mathbb{N}_0$  the map  $\iota_{j+1,j} : \mathcal{A}_{j+1} \rightarrow \mathcal{A}_j$  is injective.
- For any  $j \in \mathbb{N}_0$  the map  $\iota_j : \mathcal{A}_\infty := \varprojlim_i \mathcal{A}_i \rightarrow \mathcal{A}_j$  has dense range.
- For any  $j \in \mathbb{N}_0$  the algebra  $\mathcal{A}_j$  is stable with respect to the holomorphic functional calculus in  $\mathcal{A}$ .

The projective limit  $\mathcal{A}_\infty$  is an involutive locally  $m$ -convex Fréchet algebra with unit. Note that it is a local  $C^*$ -algebra. In particular  $K_*(\mathcal{A}_\infty) \cong K_*(\mathcal{A})$  [Bl].

One can associate to  $\mathcal{A}_\infty$  a universal differential algebra  $(\hat{\Omega}_* \mathcal{A}_\infty, d)$ . The de Rham homology  $H_*^{dR}(\mathcal{A}_\infty)$  is defined using the complex  $(\hat{\Omega}_* \mathcal{A}_\infty / [\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty], d)$ . Here by  $[\cdot, \cdot]$  we denote supercommutators. There is a Chern character  $\text{ch} : K_*(\mathcal{A}) \rightarrow H_*^{dR}(\mathcal{A}_\infty)$ .

**Proposition 3.1.** *Let  $(P_0, P_1)$  be a pair of transverse Lagrangian projections with  $P_0, P_1 \in M_{2n}(\mathcal{A}_\infty)$  and let  $(P_0^s, P_1^s)$  be a pair of (not necessarily transverse) Lagrangians with  $P_i^s \in M_{2n}(\mathbb{C})$ . Then for  $0 < x_1 < \frac{1}{2} < x_2 < 1$  there is  $U \in C^\infty([0, 1], M_{2n}(\mathcal{A}_\infty))$  such that*

- (1)  $UU^* = 1$ ,
- (2)  $UI_0 = I_0U$ ,
- (3)  $U(i)P_iU(i)^* = P_i^s$ ,
- (4)  $U$  is constant on  $[0, x_1]$  and on  $[x_2, 1]$ .

The proof is given in the following as a separate section.

**3.1. Special paths via logarithm.** Here we construct a particular path fulfilling the assumptions of the proposition. It will be useful later on.

Let  $\varphi: [0, 1] \rightarrow [0, 1]$  be an increasing function with  $\varphi(x) = 0$  on  $[0, x_1]$  and  $\varphi(x) = 1$  on  $x \in [x_2, 1]$ .

Let  $a = -i \log p_1^* p_0$ . Thus  $e^{ia} = p_1^* p_0$ . Here we assume the logarithm to be defined with respect to the gap of  $p_1^* p_0$  at 1.

Set  $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & p_0^* \end{pmatrix}$  and  $V(x) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi(x)a} \end{pmatrix}$ .

Using logarithm we easily define a path  $U_s(x) \in M_{2n}(\mathbb{C})$  such that

$$U_s(0)P_0^s U_s(0)^* = P^s \text{ and } U_s(1)P_1^s U_s(1)^* = P^s .$$

Then the path  $U^a(x) := U_s(x)^* V(x) U_0$  fulfills

$$U^a(0)P_0 U^a(0)^* = P_0^s \text{ and } U^a(1)P_1 U^a(1)^* = P_1^s .$$

Furthermore since the spectrum  $\sigma(a)$  is contained in  $(0, 2\pi)$ , there is  $\varepsilon > 0$  such that  $e^{i(2\pi-\varepsilon)} \notin \sigma(V(x))$  for all  $x \in [0, 1]$ .

The path  $U^a$  depends on the choice of  $\varphi$  and of  $U_s$ . However different choices lead to homotopic paths and thus to a homotopy of Lagrangians transverse to  $P_s$ .

We study the properties of these paths:

- *Unitary transformation.*

Let  $U = \begin{pmatrix} 1 & 0 \\ 0 & p^* \end{pmatrix}$  and define  $P'_i := U^* P_i U$  and  $p'_i = p p_i$ . Then

$a' = -i \log(p'_1)^* p'_0 = -i \log p_1^* p_0$ . It follows that  $U^{a'} = U^a U$ .

- *Transposition.*

Let  $(P'_0, P'_1) = (P_1, P_0)$  and assume that  $P_i^s = P_s$ . Furthermore we assume that  $1 - \varphi(x) = \varphi(1 - x)$ .

Then  $a' = -i \log p_1^* p'_0 = -i \log p_0^* p_1 = 2\pi - a$ .

Thus

$$\begin{aligned} U^{a'}(x) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi(x)(2\pi-a)+ia} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_0^* p_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_1^* \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \varphi(x)} \end{pmatrix} U^a(1-x) . \end{aligned}$$

#### 4. THE $\eta$ -FORM ASSOCIATED TO A PAIR OF LAGRANGIANS

We recall the definition of the  $\eta$ -form from [W1, §4.1]. As in family index theory, its construction is based on the Quillen superconnection formalism.

Let  $(P_0, P_1)$  be a pair of transverse Lagrangians in  $M_{2n}(\mathcal{A}_\infty)$  and  $U$  a path of unitaries as in Prop. 3.1 with  $U(i)P_i U(i)^* = P_i^s \in M_{2n}(\mathbb{C})$ . In contrast to [W1] we do not assume here that the pair  $(P_0^s, P_1^s)$  is transverse. Indeed, we will restrict to the case  $P_i^s = P_s$  below.

Let  $C_1$  be the  $\mathbb{Z}/2$ -graded unital algebra generated by an odd element  $\sigma$  with  $\sigma^2 = 1$ . Let  $\text{tr}_\sigma(a + \sigma b) = \text{tr}(a)$ .

We write  $D_I$  for the closure of the operator  $I_0 \partial$  on  $L^2([0, 1], \mathcal{A}^{2n})$  with boundary conditions given by  $P_i((I_0 \partial)^l f)(i) = (I_0 \partial)^l f(i)$ .

We define the superconnection

$$A_I := U^* dU + \sigma D_I$$

and for  $t \geq 0$  the rescaled superconnection

$$A_t^I := U^* dU + \sqrt{t} \sigma D_I .$$

We cannot just use  $d$  instead of  $U^* dU$  since  $d$  does not commute with the projection  $P_i$  and therefore does not preserve the boundary conditions.

Note furthermore that without changing the superconnection we may change the projections  $P_i^s$ . This is seen as follows. We choose a path of unitaries  $U_s: [0, 1] \rightarrow M_{2n}(\mathbb{C})$  as in Prop. 3.1 with  $U_s(i)P_i^s U_s(i)^* = P_s$ . Then  $U_s(i)U(i)P_i U(i)^* U_s(i)^* = P_i^s$ . Since it holds that  $U_s^* dU_s = d$  we end up with the same superconnection and thus the same  $\eta$ -form.

The curvature is  $(A^I)^2 = D_I^2 + \sigma R$  with  $R = -U^* d(U I_0(\partial U^*))U$ .

For  $t \geq 0$  by Volterra development

$$\begin{aligned} e^{-tA_t^I} &= \sum_{n=0}^{\infty} (-1)^n t^n \int_{\Delta^n} e^{-u_0 t D_I^2} \sigma R e^{-u_1 t D_I^2} \sigma R \dots e^{-u_n t D_I^2} du_0 \dots du_n \\ &= \sum_{n=0}^{\infty} \sigma^n (-1)^{\frac{(n+1)n}{2}} t^n I_n(t) \end{aligned}$$

with

$$I_n(t) := \int_{\Delta^n} e^{-u_0 t D_I^2} R e^{-u_1 t D_I^2} R \dots e^{-u_n t D_I^2} du_0 \dots du_n .$$

It follows that

$$e^{-(A_t^I)^2} = \sum_{n=0}^{\infty} \sigma^n (-1)^{\frac{(n+1)n}{2}} t^{n/2} I_n(t) .$$

For every  $n \in \mathbb{N}_0$  and  $t > 0$  the operator  $I_n(t)$  is an integral operator. Let  $p_t(x, y)^n$  be its integral kernel and let  $(D_I p_t)(x, y)^n$  denote the integral kernel of  $D_I I_n(t)$ .

The following statement follows from heat kernel estimates:

**Lemma 4.1.** *For every  $n \in \mathbb{N}_0$  the integral*

$$\int_0^\infty t^{\frac{n-1}{2}} \int_0^1 \text{tr}_\sigma(D_I p_t)(x, x)^n dx dt$$

*is well-defined in  $\widehat{\Omega}_* \mathcal{A}_\infty / [\widehat{\Omega}_* \mathcal{A}_\infty, \widehat{\Omega}_* \mathcal{A}_\infty]$ .*

Thus we can define:

**Definition 4.2.** *The  $\eta$ -FORM of the superconnection  $A_I$  is*

$$\eta(A_I) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \text{Tr}_\sigma D_I e^{-(A_t^I)^2} dt \in \widehat{\Omega}_* \mathcal{A}_\infty / [\widehat{\Omega}_* \mathcal{A}_\infty, \widehat{\Omega}_* \mathcal{A}_\infty] + d \widehat{\Omega}_* \mathcal{A}_\infty .$$

Here  $\text{Tr}_\sigma$  denotes the operator trace associated to  $\text{tr}_\sigma$ .

In degree zero

$$\eta^0(A_I) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \text{Tr}_\sigma D_I e^{-t(D_I)^2} dt ,$$

which depends only on the pair  $(P_0, P_1)$  and not on the choice of  $U$ .

But in higher degree the form depends on the path  $U$ . Unfortunately, the proof of [W1, Prop. 4.1.7], whose aim is to construct an  $\eta$ -form independent of  $U$ , has a gap since it may not be allowed to interchange integration and the limit process

used there. Indeed, we will see in §7 that the  $\eta$ -form depends nontrivially on  $U$ . In the following section we will study this dependence in more detail.

Clearly, instead of starting with a pair of Lagrangians we may start with a path of unitaries  $U: [0, 1] \rightarrow M_{2n}(\mathcal{A}_\infty)$  as in the following definition.

**Definition 4.3.** *For a smooth path  $U: [0, 1] \rightarrow M_{2n}(\mathcal{A}_\infty)$  of unitaires such that*

- (1)  $UI_0 = I_0U$ ,
- (2) *the pair of Lagrangians  $(P_0, P_1) := (U(0)^*P_sU(0), U(1)^*P_sU(1))$  is transverse,*
- (3)  *$U$  is constant near 0 and 1*

*we define the form  $\eta(U)$  as the  $\eta$ -form  $\eta(A_I)$  constructed above.*

For a path  $U$  we define the path  $\bar{U}$  by setting  $\bar{U}(x) := U(1-x)$ .

## 5. INVARIANCE PROPERTIES OF THE $\eta$ -FORM

The following invariance properties are immediate:

- For a unitary  $V \in M_{2n}(\mathcal{A}_\infty)$  with  $[V, I_0] = 0$  we have  $\eta(UV) = \eta(U)$ .
- For a smooth path of unitaries  $V: [0, 1] \rightarrow M_{2n}(\mathbb{C})$  with  $V(x) = 1$  for  $x$  near 0, 1 and  $[V, I_0] = 0$  it holds that  $\eta(VU) = \eta(U)$ .
- The degree zero component  $\eta^0(U)$  depends only on the pair  $(P_0, P_1) = (U(0)^*P_sU(0), U(1)^*P_sU(1))$  and not on the path  $U$ .
- $\eta(\bar{U}) = -\eta(U)$ .

Recall that the Maslov index of a pair of paths of Lagrangian projections

$$\mu(P_0(t), P_1(t)) \in K_0(\mathcal{A})$$

may be defined as the spectral flow of the operator  $D_I$  with boundary conditions given by the path of pairs  $(P_0(t), P_1(t))$  [W2, §7]. For the definition it is needed that  $P_0(t), P_1(t)$  are transverse at  $t = 0, 1$ . The Maslov index for pairs fulfills an analogue of (1.1).

**Proposition 5.1.** *We have the following homotopy invariance properties:*

- (1) *If  $U, U'$  are homotopic through paths of unitaries commuting with  $I_0$ , then  $\eta^m(U) - \eta^m(U')$ ,  $m > 0$ , is closed and can be calculated via the Chern character of the Maslov index for paths of pairs of Lagrangians.*
- (2) *If  $U, U'$  are homotopic through paths of unitaries which commute with  $I_0$  and whose endpoints define transverse Lagrangians, then  $\eta^m(U) = \eta^m(U')$  for  $m > 0$ .*

*Proof.* Let  $W: \mathbb{R} \times [0, 1] \rightarrow U(\mathcal{A}_\infty^{2n})$  be smooth with  $W(x_1, x_2) = U(x_2)$  for  $x_1 \leq 0$ ,  $W(x_1, x_2) = U'(x_2)$  for  $x_1 \geq 1$ , and  $[W, I_0] = 0$  and such that  $W(x_1, x_2)$  is independent of  $x_2$  in a small neighborhood of  $x_2 = 0$  resp.  $x_2 = 1$ .

Define the paths  $P_i(x_1) := W(x_1, i)^*P_sW(x_1, i)$  for  $i = 0, 1$ .

We set

$$\tilde{W} := \text{diag}(W, W) \in M_{4d}(\mathcal{A}_\infty).$$

Consider the Dirac operator  $\tilde{\partial}_Z = c(dx_1)\partial_1 + c(dx_2)\partial_2$  on  $C_c^\infty(\mathbb{R} \times [0, 1], \mathcal{A}_\infty^{4d})$  with boundary conditions given at  $(x_1, 0)$  by  $\text{diag}(P_0(x_1), P_0(x_1))$  and at  $(x_1, 1)$  by  $(P_0(x_1), P_1(x_1))$ . Via the unitary transformation  $\tilde{W}$  we transform this operator to

$$\tilde{W}\tilde{\partial}_Z\tilde{W}^* = \tilde{\partial}_Z + \tilde{W}c(dx_1)(\partial_1\tilde{W}^*) + \tilde{W}c(dx_2)(\partial_2\tilde{W}^*)$$

with constant boundary conditions given by  $\text{diag}(P_s, P_s)$  at  $x_2 = 0$  and  $x_2 = 1$ .

The index of this operator equals the spectral flow of the operator  $D_I$  with boundary conditions defined by the path of pairs  $(P_0(x_1), P_1(x_1))$ , thus the Maslov index  $\mu(P_0, P_1)$ . In situation (2) this operator is invertible for any  $x_1$  and therefore the spectral flow vanishes. (Alternatively one may use homotopy invariance to show directly that the index vanishes: The path of boundary conditions is homotopic to constant boundary conditions.)

Since  $\tilde{W}c(dx_1)(\partial_1\tilde{W}^*)$  is compactly supported, the index of  $\tilde{W}\tilde{\partial}_Z\tilde{W}^*$  equals the index of  $\tilde{\partial}_Z + \tilde{W}c(dx_2)(\partial_2\tilde{W}^*)$ , which is the Dirac operator associated to the connection  $dx_1\partial_1 + dx_2\partial_2 + dx_2\tilde{W}(\partial_2\tilde{W}^*)$ . Hence we can use the superconnection  $d + c(dx_1)\partial_1 + c(dx_2)\partial_2 + c(dx_2)\tilde{W}(\partial_2\tilde{W}^*)$  for the index theorem.

The methods from [W1] imply an Atiyah–Patodi–Singer index theorem in this situation, since the support of  $\tilde{W}(\partial_2\tilde{W}^*)$  does not meet the boundary.

As usual, the local term is calculated from the Chern character of the connection

$$\nabla := d + dx_1\partial_1 + dx_2\partial_2 + dx_2\tilde{W}(\partial_2\tilde{W}^*) .$$

Its curvature equals

$$\nabla^2 = dx_1dx_2(\partial_1\tilde{W})(\partial_2\tilde{W}^*) - dx_2d(\tilde{W}(\partial_2\tilde{W}^*)) .$$

Hence

$$\int_{[0,1] \times \mathbb{R}} \text{ch}(\nabla) = \int dx_1dx_2 \text{tr}((\partial_1\tilde{W})(\partial_2\tilde{W}^*)) .$$

In particular, the local term does not have any components in degree  $m > 0$ .

Hence for  $m > 0$

$$(5.1) \quad \eta^m(U') - \eta^m(U) = 2 \text{ch}^m \mu(P_0, P_1) .$$

In situation (2) the Maslov index  $\mu(P_0, P_1)$  vanishes, as mentioned above.  $\square$

The proposition allows us to define  $\eta(U)$  for general paths of unitaries  $U \in C([0, 1], U_{2n}(\mathcal{A}))$  with  $[U, I_0] = 0$  since such a path can always be approximated by a path in  $C^\infty([0, 1], U_{2n}(\mathcal{A}_\infty))$  and any two approximations are homotopic through a homotopy  $W$  as in the proof. This follows from the fact that  $\mathcal{A}_\infty$  is a local  $C^*$ -algebra [Bl, 3.1.7].

**Corollary 5.2.** (1) For a path  $U^a$  defined as in §3.1 it holds that  $\eta(U^a)^m = 0$  for  $m > 0$ .

(2) In general,  $\eta(U)^m$  is closed for  $m > 0$  and can be expressed via the Maslov index for paths of Lagrangians.

*Proof.* (1) Let  $U^a(x) = V(x)U_0$  as in §3.1 with  $V(x) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi(x)a} \end{pmatrix}$  and  $\sigma(a) \in (0, 2\pi)$ . Set  $a_t := (1 - \varphi(t))a + \varphi(t)\varepsilon$ . Then  $a_0 = a$  and  $a_1 = \varepsilon$ . For  $\varepsilon > 0$  small enough  $\sigma(a_t) \in (0, 2\pi)$  for all  $t \in [0, 1]$ . Set  $W(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi(x_2)a_{x_1}} \end{pmatrix}$ .

Since  $1 \notin \sigma(e^{i\varphi(1)a_{x_1}})$  for all  $x_1 \in \mathbb{R}$ , the projections  $W(x_1, 0)^*P_sW(x_1, 0) = P_s$  and  $W(x_1, 1)^*P_sW(x_1, 1)$  are transverse to each other for all  $x_1$ . It holds that  $\eta(U^a) = \eta(W(0, \cdot)U_0)$ . For  $m > 0$  by homotopy invariance  $\eta^m(W(0, \cdot)U_0) = \eta^m(W(1, \cdot)U_0)$ . Furthermore  $\eta(W(1, \cdot)U_0) = \eta(W(1, \cdot)) \in \mathbb{C}$  since  $W(1, x_2) \in U_{2n}(\mathbb{C})$  for all  $x_2$ .

(2) Let  $\varphi$  be as before. Clearly  $\eta(U) = \eta(U \circ \varphi)$ . Define the homotopy  $W(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i x_2 \varphi(1-x_1)} \end{pmatrix} U(\varphi(x_1)\varphi(x_2))$ . Then

$$W(0, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i x_2} \end{pmatrix} U(0), \quad W(1, x_2) = U(\varphi(x_2)).$$

For  $i = 0, 1$  we define the path of Lagrangian projections  $P_i := W(x_1, i)^* P_s W(x_1, i)$ . The pairs  $(P_0(0), P_1(0))$  and  $(P_0(1), P_1(1))$  are transverse. By (5.1) for  $m > 0$

$$\eta^m(U) - \eta^m(W(0, \cdot)) = 2 \operatorname{ch}^m \mu(P_0, P_1).$$

Furthermore  $\eta^m(W(0, \cdot)) = \eta^m(W(0, \cdot)U(0)^*) = 0$  for  $m > 0$  since  $W(0, x_2)U(0)^* \in M_{2n}(\mathbb{C})$ . Thus

$$\eta^m(U) = 2 \operatorname{ch}^m \mu(P_0, P_1).$$

□

## 6. RELATION TO $\eta$ -FORMS FOR THE CIRCLE

In [W1, Prop. 4.5.1] we established a relation between  $\eta(U)$  and  $\eta$ -forms on the circle. However, the proof only works for paths of the form  $U^a$ . Since  $\eta(U)$  does depend on the path, as we will see below, we have to study the general case as well.

Let  $u \in U_n(\mathcal{A}_\infty)$ . We define the  $\mathcal{A}$ -vector bundle

$$\mathcal{L}(u) = ([0, 1] \times \mathcal{A}^n) / \sim$$

with  $(0, v) \sim (1, uv)$ . Then we can identify a smooth section of  $\mathcal{L}(u)$  with a smooth function  $f : \mathbb{R} \rightarrow \mathcal{A}_i^n$  satisfying  $f(x+1) = uf(x)$ . The trivial connection  $\nabla f := f' dx$  induces a hermitian connection on  $\mathcal{L}(u)$ . The associated Dirac operator is denoted by  $\not\partial_{\mathcal{L}(u)}$ .

Now we start with a path  $u \in C^\infty([0, 1], U_n(\mathcal{A}_\infty))$  of unitaries with is locally constant near 0, 1. We assume that  $u(0) = 1$  and that  $1 \notin \sigma(u(1))$ . Then  $\not\partial_{\mathcal{L}(u(1))}$  is invertible. We consider the associated rescaled superconnection  $A_t := u \, d u^* + \sqrt{t} \sigma \not\partial_{\mathcal{L}(u(1))}$  and the  $\eta$ -form

$$\eta_{S^1}(u) := \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}_\sigma \not\partial_{\mathcal{L}(u(1))} e^{-(A_t)^2} dt \in \hat{\Omega}_* \mathcal{A}_\infty / [\overline{\hat{\Omega}_* \mathcal{A}_\infty}, \hat{\Omega}_* \mathcal{A}_\infty] + d \hat{\Omega}_* \mathcal{A}_\infty.$$

$$\text{Set } U(x) = \begin{pmatrix} 1 & 0 \\ 0 & u(x)^* \end{pmatrix}.$$

Then the pair of Lagrangians  $(U(0)^* P_s U(0), U(1)^* P_s U(1))$  is transverse.

**Proposition 6.1.** *It holds that  $\eta_{S^1}(u) = \eta(U)$ .*

*Proof.* For  $u = e^{i\varphi a}$  with  $\sigma(a) \in (0, 2\pi)$  this was shown in [W1, Prop. 4.5.1].

Since both sides of the asserted equation do not depend on  $u$  in degree zero, it holds that  $\eta_{S^1}^0(u) = \eta^0(U)$  for general  $u$ .

For  $m > 0$  we have  $\eta^m(U) = 2 \operatorname{ch}^m \mu(P_0, P_1)$  for the pair of paths  $(P_0, P_1)$  defined as in the proof of Cor. 5.2.

Note that here  $P_0$  is the constant path  $P_s$  and  $P_1(0) \in M_{2n}(\mathbb{C})$ . Since  $P_1(1)$  is transverse to  $P_s$ , we can extend the path  $P_1$  to a path  $\tilde{P}_1 : [0, 2] \rightarrow M_{2n}(\mathcal{A}_\infty)$  such that  $\tilde{P}_1(x)$  is a Lagrangian projection transverse to  $P_s$  for all  $x \in [1, 2]$  and such that  $\tilde{P}_1(2) = \tilde{P}_1(0)$ . Thus  $\tilde{P}_1$  is now a loop of Lagrangian projections fulfilling  $\mu(P_0, \tilde{P}_1) = \mu(P_0, P_1)$ . We write  $\tilde{P}_1 = \frac{1}{2} \begin{pmatrix} 1 & \tilde{u}^* \\ \tilde{u} & 1 \end{pmatrix}$ . By the relation established

in [W2, §7.2] between Maslov index for loops and Bott periodicity on the one hand, and in [W2, §4] between Bott periodicity and spectral flow on the other hand, the Maslov index  $\mu(P_0, \tilde{P}_1)$  equals the spectral flow of the loop  $(\partial_{\mathcal{L}(\tilde{u}(x))})_{x \in [0,2]}$ . This, in turn, equals the spectral flow of the path  $(\partial_{\mathcal{L}(u(x))})_{x \in [0,1]}$ . Now we apply the classical equality spectral flow=index of a cylinder to this situation. Then we can apply the Atiyah–Patodi–Singer index theorem. The local term vanishes in degree  $m > 0$  by arguments as in the proof of Prop. 5.1. It follows that for  $m > 0$

$$\text{ch } \mu^m(P_0, P_1) = \eta_{S^1}^m(u) .$$

□

For a fixed unitary  $v \in U_n(\mathcal{A}_\infty)$  and  $u$  as above, we can define  $\eta_{S^1}(uv)$  as above and get  $\eta_{S^1}(uv) = \eta_{S^1}(u)$ . This implies that we can define  $\eta_{S^1}(u)$  for any path  $u \in C^\infty([0,1], U_n(\mathcal{A}_\infty))$  of unitaries with is locally constant near 0, 1 and such that  $u(1) - u(0)$  is invertible and the proposition still holds for these paths.

## 7. MASLOV INDEX AND $\eta$ -FORMS

Let  $(P_0, P_1, P_2)$  be a triple of pairwise transverse Lagrangian projections. Let  $U_{01}, U_{12}, U_{20}$  be paths of unitaries as in Prop. 3.1 corresponding to the pairs  $(P_0, P_1)$ ,  $(P_1, P_2)$ ,  $(P_2, P_0)$ , respectively. (Here we take  $P_s^i = P_s$  in Prop. 3.1.) We make the following crucial assumption:

**Assumption 7.1.** *For  $x \in [0, \frac{1}{2}]$  it holds that  $U_{ij}(1-x) = U_{jk}(x)$ .*

**Theorem 7.2.** *For a triple  $(P_0, P_1, P_2)$  of pairwise transverse Lagrangian projections*

$$\text{ch } \tau(P_0, P_1, P_2) = \eta(U_{01}) + \eta(U_{12}) + \eta(U_{20}) \in H_*^{dR}(\mathcal{A}_\infty) .$$

*Proof.* One defines the unitary  $W$  in [W1, §2.1.2] using the  $U_{ij}$ . Note that in [W1] we assumed the complex triple  $(P_0^s, P_1^s, P_2^s)$ , to which the triple  $(P_0, P_1, P_2)$  is transformed, to consist of mutually transverse projections. However, one may well take more general projections since, as in §4, the choice of the complex projections cancels out in the definition of the superconnection used in the proof of the index theorem [W1, Theorem 4.4.11]. This theorem implies the assertion. □

In §3.1 we exhibited a way to associate a canonical (up to homotopy) path of unitaries  $U^a$  to a pair of transverse Lagrangians. In Cor. 5.2, it was shown that the associated canonical form  $\eta(U^a)$  has no higher degree components. However, it follows from Prop. 2.5 that  $\text{ch } \tau(P_0, P_1, P_2)$  can have nontrivial higher degree components. Hence in general  $\eta(U)$  depends nontrivially on  $U$ . In particular it is in general not possible to choose each of the three unitaries  $U_{ij}$  in the theorem homotopic to a canonical path.

However, if we assume that all  $P_i$  are transverse to a fixed Lagrangian projection  $P_3$ , then we may fix a procedure of defining the unitaries  $U_{ij}$ : Via logarithm as in §3.1 (with  $U_s^i = 1$ ) we construct a path of unitaries  $U_i$  such that  $U_i(0)P_iU_i(0)^* = U_i(\frac{1}{2})P_3U_i(\frac{1}{2})^* = P_s$ ,  $i = 0, 1, 2$ . Then we concatenate the paths  $U_i(x)$  and  $U_j(1-x)$  at  $x = \frac{1}{2}$  to obtain  $U_{ij}$ . This construction leads to a form  $\eta_{P_3}(P_i, P_j)$ ,  $i = 0, 1, 2$ ,  $i \neq j$ , which does not depend on the choices since different choices lead to homotopic paths of unitaries.

The following invariance properties follow immediately from the invariance properties of the  $\eta$ -form and of the paths constructed via logarithm:

- For a unitary  $V \in M_{2n}(\mathcal{A}_\infty)$  with  $[V, I_0] = 0$  it holds that

$$\eta_{P_3}(P_i, P_j) = \eta_{V^*P_3V}(V^*P_iV, V^*P_jV) .$$

- It holds that  $\eta_{P_3}(P_i, P_j) = -\eta_{P_3}(P_j, P_i)$ .

The previous theorem implies:

**Corollary 7.3.** *For a quadruple  $(P_0, P_1, P_2, P_3)$  of pairwise transverse Lagrangian projections it holds that*

$$\text{ch } \tau(P_0, P_1, P_2) = \eta_{P_3}(P_0, P_1) + \eta_{P_3}(P_1, P_2) + \eta_{P_3}(P_2, P_0) \in H_*^{dR}(\mathcal{A}_\infty) .$$

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