LOCALIZATION OF SPACES WITH RESPECT TO A CLASS OF MAPS

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§1. - INTRODUCTION

Let $\mathcal{C}$ be the category of pointed CW-complexes. If h_* is a generalized homology theory, denote by $\mathcal{W}$ the class of maps $X \to Y$ inducing an isomorphism $h_*(X) \xrightarrow{\sim} h_*(Y)$. Bousfield has show (2) that there exists a $\mathcal{W}$-localization functor, i.e. a functor E from the category $\mathcal{C}$ to itself endowed with a morphism $\eta : 1 \to E$ satisfying the following properties :

i) for any $X \in \mathcal{C}$, $\eta_X : X \to EX$ ~~belongs~~ lies in $\mathcal{W}$

ii) for any $X \in \mathcal{C}$ and any map $Y \to Z$ in $\mathcal{W}$, the map $[Z,EX] \to [Y,EX]$ is bijective.

My purpose is to show that this result holds in the more general following situation : $\mathcal{C}$ is the category of CW-complexes over a given topological space B and $\mathcal{W}$ is any class of maps in $\mathcal{C}$ satisfying some axiomatic conditions. For example these conditions are satisfied in the following cases :

i) $\mathcal{W}$ is the class of maps $X \to Y \to B$ inducing an isomorphism $h_*(X) \xrightarrow{\sim} h_*(Y)$ for any $* < n$ (and an epimorphism for $*=n+1$) where n is a given integer $\leq \infty$ and h_* a given twisted homology theory equalizing any homotopic maps

ii) $\mathcal{W}$ is the class of maps $X \to Y \to B$ such that $X \to Y$ is n-connected.

In this last example the $\mathcal{W}$-localized of $X \to B$ is $X' \to B$ where X' is the $n-1$ Postinov stage of X along the fiber of $X \to B$ (i.e. the homotopy fiber of $X' \to B$ is the $n-1$ Postinov stage of the homotopy fiber of $X \to B$).

Now consider the following classes of maps in $\mathcal{C}$:

i) B is a point and $\mathcal{W}$ is the class of maps $X \to Y$ inducing an isomorphism $h^*(Y) \xrightarrow{\sim} h^*(X)$, where h^* is a given generalized cohomology theory.

ii) $\mathcal{W}$ is the class of maps $X \to Y \to B$ inducing an isomorphism and a monomorphism for $* = n+1$

$h^*(Y) \to h^*(X)$ for any $* \leqslant n$, where n is a given integer $\leqslant \infty$ and h^* is a given twisted cohomology theory equalizing any homotopic maps.

iii) $\mathcal{W}$ is the class of maps $X \to Y \to B$ inducing a weak homotopy equivalence $\mathcal{S}(Y \to B) \xrightarrow{\sim} \mathcal{S}(X \to B)$, where $\mathcal{S}(Z \to B)$ is the space of maps from $Z \to B$ to $E \to B$, $E \to B$ being a given Serre fibration.

In all that cases the class $\mathcal{W}$ satisfies the axiomatic conditions except one, and one can define for any infinite cardinal number c a subclass $\mathcal{W}_c$ of $\mathcal{W}$ satisfying the following conditions :

i) $\mathcal{W}_c \subset \mathcal{W}_{c'}$ if $c \leqslant c'$

ii) any map $X \to Y \to B$ in $\mathcal{W}$ belongs ~~in~~ to $\mathcal{W}_c$ if X and Y are CW-complexes of cardinal less than c.

iii) for any c there exists a $\mathcal{W}_c$-localization functor.

A big part of this paper is devoted to the study of $\mathcal{W}$-localization where B is path-connected with fundamental group π and $\mathcal{W}$ is the class of maps $f \in \mathcal{C}$ such that the relative $\mathbb{Z}[\pi]$-chain complex $C_*(f)$ belongs in a given class $\mathcal{W}_*$ of graded differential free $\mathbb{Z}[\pi]$-modules. For example the class of maps in $\mathcal{C}$ inducing an isomorphism in homology (or cohomology) with given twisted coefficients.

In this case, I prove the following fact : if $X \to B$ is a $\mathcal{W}$-local B-space (i.e. $X \to B$ has the homotopy type of its localization), the relative homotopy groups $\pi_*(B,X)$ are local in a certain sense and the converse is true if $\mathcal{W}_*$ satisfies some splitting conditions.

The paper is organized as follows. §2 contains some definitions and the statement of the main theorem ; in §3 We give many examples of localization functors ; §4 is devoted to the localization with respect to a class $\mathcal{W}_*$ of graded differential free $\mathbb{Z}[\pi]$-modules and some theorems in this section are proved in §5 and §6 and §8 ; the last section (§9) contains the proof of the main theorem : i.e. the construction of the localization functor. The section 7 is the proof of a technical proposition of §2.

§2. - SOME DEFINITIONS AND STATEMENT OF THE MAIN RESULT

First I will give some notations about the category of CW-complexes over a space B.

2.1. - Let B be a topological space. The category of CW-spaces over B will be denoted by B-CW. An object of B-CW is a map $f : X \to B$ where X is a space having the homotopy type of a CW-complex, and will be called a B-space. A morphism of B-CW from $f : X \to B$ to $g : Y \to B$ is a map $\varphi : X \to Y$ such that : $f = g \circ \varphi$, and will be called a B-map.

The product of a B-space by the interval I is well defined and this permits to define B-homotopy, B-fibration, B-cofibration, B-homotopy equivalence etc ... . We can also define homotopy, fibration, cofibration, homotopy-equivalence ..., by using the canonical functor : $B\text{-CW} \to \text{CW}$. For example two B-maps are called homotopic if there are homotopic in the category of CW-complexes.

If $X \to B$ and $Y \to B$ are two B-spaces, the B-homotopy classes of B-maps from $X \to B$ to $Y \to B$ will be denoted by $[X \to B\ ,\ Y \to B]$ or by $[X,Y]_B$.

DEFINITION 2.2. - Let $\mathcal{W}$ be a class of B-maps. A B-space $X \to B$ is called $\mathcal{W}$-local if, for any B-maps $Y \to Z \to B$ in $\mathcal{W}$, the inducing map $[Z,X]_B \to [Y,X]_B$ is an isomorphism.

DEFINITION 2.3. - Let $\mathcal{W}$ be a class of B-maps. A $\mathcal{W}$-localization functor is a functor E from B-CW to itself endowed with a morphism $\eta : 1 \to E$ satisfying the following properties

i) for any B-space $X \to B$ the B-map η from $X \to B$ to $E(X \to B)$ belongs in $\mathcal{W}$

ii) for any B-space $X \to B$, $E(X \to B)$ is $\mathcal{W}$-local

Now I will give a list of conditions for $\mathcal{W}$ which are sufficient to have a $\mathcal{W}$-localization functor.

DEFINITIONS 2.4. - Let $\mathcal{W}$ be a class of B-maps. The class $\mathcal{W}$ is called a localizing class if it satisfies the following conditions :

L1 : $\mathcal{W}$ is closed under finite compositions

L2 : for any B-map $X \xrightarrow{f} Y \to B$ in $\mathcal{W}$, the map $X \to M(f) \to B$ ~~belongs~~ lies in $\mathcal{W}$ ($M(f)$ denotes the mapping cylinder of f)

L3 : any B-map which is a homotopy equivalence (not only a B-homotopy equivalence) ~~belongs~~ lies in $\mathcal{W}$

L4 : if $X_i \to Y_i \to B$ is a set ot B-maps in $\mathcal{W}$, the disjoint union $\coprod_i X_i \to \coprod_i Y_i \to B$ belongs in $\mathcal{W}$

L5 : if $X \to Y \to B$ is a cofibration in $\mathcal{W}$ and $X \to Z \to B$ a B-map, the B-map $Z \to Z \cup_X Y \to B$ belongs in $\mathcal{W}$

L6 : if $X \to Y \to B$ is a cofibration in $\mathcal{W}$ the B-map :
$X \times I \cup Y \times \partial I \longrightarrow Y \times I \longrightarrow B$ belongs in $\mathcal{W}$.

Let $\mathcal{W}$ be a class of B-maps. The class $\mathcal{W}$ is said not too big if there exists a cardinal number c with the following property :

L7 : for any B-map $X \to Y \to B$ in $\mathcal{W}$ and any commutative diagram :

$$\begin{tikzcd} \partial D \arrow[r, hook] \arrow[d] & D \arrow[d] & \\ X \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

where D is a disk and ∂D its boundary, there exists a factorization :

$$\begin{tikzcd} \partial D \arrow[r] \arrow[d] & D \arrow[d] & \\ X' \arrow[r, dashed] \arrow[d] & Y' \arrow[d] & \\ X \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

such that X' and Y' are CW-complexes with at most c cells, and the B-map $X' \to Y' \to B$ ~~belongs~~ lies in $\mathcal{W}$.

THEOREM 2.5. - <u>Let $\mathcal{W}$ be a not too big localizing class. Then there exists a $\mathcal{W}$-localization functor.</u>

Now suppose $\mathcal{W}$ in only a localization class.

NOTATION 2.6. - Let $\mathcal{W}$ be a localizing class of B-maps and c a infinite cardinal number. Denote by $\mathcal{W}'_c$ the class of B-maps in $\mathcal{W}$: $X \to Y \to B$, X and Y being CW-complexes of cardinal $< c$.

The smallest localizing class of B-maps containing $\mathcal{W}'_c$ will be denoted by $\mathcal{W}_c$.

PROPOSITION 2.7. - The class $\mathcal{W}_c$ is the class of B-maps in $\mathcal{W}$: $X \to Y \to B$ such that any morphism (in the category of B-maps) from a B-map $K \to L \to B$, with $\operatorname{card} K + \operatorname{card} L < c$, to $X \to Y \to B$ factors through a B-map in $\mathcal{W}'_c$.

This proposition will be proved in §7.

The classes $\mathcal{W}_c$ are not too big localizing classes and by 2.5 there exist $\mathcal{W}_c$-localization functors. These functors can be used as approximation of a virtual $\mathcal{W}$-localization functor. Actually we have a more precise theorem (proved in §9) :

THEOREM 2.8. - Let $\mathcal{W}$ be a localizing class of B-maps. Then there exists a family of functors E_c indexed by the cardinal numbers and satisfying the following properties :

i) for any B-space $X \to B$ and any c, $E_c(X \to B)$ is a B-space containing $X \to B$

ii) for any $c < c'$ and any $X \to B$, $E_c(X \to B)$ is contained in $E_{c'}(X \to B)$

iii) for any $X \to B$ and any c, $E_c(X \to B)$ is $\mathcal{W}_c$-local

iv) for any $X \to B$ and any c the B-inclusion from $X \to B$ to $E_c(X \to B)$ belongs in $\mathcal{W}_c$.

§3. - EXAMPLES OF LOCALIZING CLASSES AND LOCALIZATION FUNCTORS

It is difficult to construct not too big localizing classes of B-maps. Then I will first give examples of localizing classes.

EXAMPLE 3.1. - Let h_* (resp. h^*) be a generalized twisted homology (resp. cohomology) theory equalizing any two homotopic B-maps. Then the class of B-maps inducing isomorphisms on h_* (resp. h^*) is a localizing class.

EXAMPLE 3.2. - Let h_* (resp. h^*) be a generalized twisted homology (resp. cohomology) theory equalizing any two homotopic B-maps and n an integer. Then the class of B-maps inducing is isomorphisms on the h_* (resp. h^*) for all $* \leq n$ is a localizing class.
and an epimorphism (resp. a monomorphism) for $* = n+1$

EXAMPLE 3.3. - Let q be an integer. Then the class of B-maps $X \to Y \to B$ such that $X \to Y$ is q-connected is a localizing class.

EXAMPLE 3.4. - Let $E \to B$ be a Serre fibration, and denote by $\mathcal{S}(X \to B)$ the space of cross sections of $X \to B$ through E. Then the class of B-maps $X \to Y \to B$ inducing a weak homotopy equivalence $\mathcal{S}(Y \to B) \xrightarrow{\approx} \mathcal{S}(X \to B)$ is a localizing class.

These examples are very easy to check and we can construct a lot of other examples by using the following proposition.

PROPOSITION 3.5. - *Let $\mathcal{W}_i$ be a family of localizing classes of B-maps. Then the intersection of the classes $\mathcal{W}_i$ is a localizing class.*

THEOREM 3.6. - *Let h_* be a generalized twisted homology theory egalizing any two homotopic B-maps and n be an integer $\leq \infty$. Then the class $\mathcal{W}$ of B-maps $X \to Y \to B$ such that $h_*(Y,X)$ is zero for $* \leq n$ is a not too big localizing class and there exists a $\mathcal{W}$-localization functor.*

Proof : Clearly $\mathcal{W}$ is a localizing class.

Now let c be a cardinal number greater than the cardinal of $h_*(pt)$, for any B-map $pt \to B$.

It is easy to see that the cardinal of $h_*(X)$ is less than c for every B-map $X \to B$, X being a CW-complex of cardinal less than c.

Suppose we have a commutative diagram :

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L \arrow[d] & \\ X \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

the B-map $X \to Y \to B$ inducing an isomorphism on h_* and K being a subcomplex of the finite complex L . I will construct by induction a commutative diagram :

$$\begin{tikzcd} K = K_o \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & K_2 \cdots \arrow[r] \arrow[d] & K_p \arrow[r] \arrow[d] & \cdots \arrow[r] & X \arrow[d] & \\ L = L_o \arrow[r] & L_1 \arrow[r] & L_2 \cdots \arrow[r] & L_p \arrow[r] & \cdots \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

where K_p is a subcomplex of K_{p+1} and L_p a subcomplex of L_{p+1} and K_p a subcomplex of L_p and card L_p are less than c and the map : $h_*(L_p,K_p) \to h_*(L_{p+1},K_{p+1})$ is zero for $* \leqslant n$.

Suppose we have construct the K_i and the L_i for any $i \leqslant p$. Since the map $h_*(L_p,K_p) \to h_*(Y,X)$ is zero for $* \leqslant n$, we can attach a finite number of cells in K_p and L_p in order to kill any element in $h_*(L_p,K_p)$, $* \leqslant n$. Then we can attach less than c cells in K_p and L_p and we have construct K_{p+1} and L_{p+1} with the desired property.

Now we have the commutative diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \bigcup_p K_p \arrow[r] \arrow[d] & X \arrow[d] & \\ L \arrow[r] & \bigcup_p L_p \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

and $\bigcup_p K_p$ is a subcomplex of $\bigcup_p L_p$ and the cardinal of $\bigcup_p L_p$ is less than c and the B-map $\bigcup_p K_p \to \bigcup_p L_p \to B$ belongs in $\mathcal{W}$.

That proves that $\mathcal{W}$ is not too big.

THEOREM 3.7. - *Let* q *be an integer. Then the class* $\mathcal{W}$ *of B-maps* $X \to Y \to B$ *such that* $X \to Y$ *is q-connected is a not too big localizing class. Moreover the* $\mathcal{W}$*-localization of* $X \to B$ *is the* $q-1^{th}$ *Postnikov stage along the fiber of* $X \to B$.

Proof : The proof is exactly the same as the proof of 3.9 except one must replace h_* by π_* . Then $\mathcal{W}$ is not too big.

Now if $X \to B$ is a B-space, let $X' \to B$ be the $q-1^{th}$ Postnikov stage along the fiber of $X \to B$. It is easy to see that the B-map $X \to X' \to B$ belongs in $\mathcal{W}$ and $X' \to B$ is $\mathcal{W}$-local.

That proves the theorem.

Hence Postnikov decomposition can be considered as particular case of localization in my sense. With the same idea the plus construction and the Dror acyclic functor are localization functors with respect to a certain class $\mathcal{W}$.

THEOREM 3.8. - Let X be a connected CW-complex and N a perfect normal subgroup of $\pi_1(X)$. Denote by G the group $\pi_1(X)/N$.

Then the class $\mathcal{W}$ of $K(G,1)$-maps $Y \to Z \to K(G,1)$ inducing isomorphisms $H_*(Y,\mathbb{Z}[G]) \xrightarrow{\simeq} H_*(Z,\mathbb{Z}[G])$ is a not too big localizing class and the localization of $X \to K(G,1)$ with respect to $\mathcal{W}$ is the $K(G,1)$-space $X^+ \to K(G,1)$, X^+ being the plus construction of X with respect to N.

The proof is trivial.

THEOREM 3.9. - Let X be a connected CW-complex. Denote by $A(X)$ the acyclic Dror functor of X and $A(X) \to X$ the canonical map (5).

Then the class $\mathcal{W}$ of X-maps $Y \to Z \to X$ inducing isomorphisms $H_*(Y,Z) \to H_*(Z,Z)$ is a not too big localizing class and $A(X) \to X$ is the localization of $pt \to X$ with respect to $\mathcal{W}$.

The proof is trivial too.

§4. - LOCALIZATION WITH RESPECT TO A CLASS OF GRADED DIFFERENTIAL FREE MODULES

4.1. - If A is a ring denote by $\mathcal{C}_*(A)$ the category of graded differential free left A-modules with non negative degree.

In all this section I suppose that B is path-connected with fundamental group π, and I will study the $\mathcal{W}$-localization when $\mathcal{W}$ is a class of B-maps defined by a class $\mathcal{W}_*$ of objects in $\mathcal{C}_*(\mathbb{Z}[\pi])$ by the following way : a B-maps $X \to Y \to B$ belongs in $\mathcal{W}$ if the relative chain complex $C_*(Y,X)$ belongs in $\mathcal{W}_*$.

The class $\mathcal{W}$ will be denoted by $\mathcal{W}(\mathcal{W}_*)$.

For example is M is a $\mathbb{Z}[\pi]$-module the class of B-maps $X \to Y \to B$ inducing an isomorphism $H_*(X,M) \xrightarrow{\sim} H_*(Y,M)$ is such a class.

DEFINITION 4.2. - Let $\mathcal{W}_*$ be a class of complexes of $\mathcal{C}_*(A)$. The class $\mathcal{W}_*$ will be called a localizing class if it satisfies the following properties :

i) $\mathcal{W}_*$ is stable under homotopy equivalence

ii) if C_* and C'_* belong in $\mathcal{W}_*$ and $0 \to C_* \to C''_* \to C'_* \to 0$ is exact in $\mathcal{C}_*(A)$ then C''_* belongs in $\mathcal{W}_*$

iii) $\mathcal{W}_*$ is stable under direct sum

iv) $\mathcal{W}_*$ is stable under suspension.

PROPOSITION 4.3. - <u>Let $\mathcal{W}_*$ be a localizing class in $\mathcal{C}_*(\mathbb{Z}[\pi])$.</u>

<u>Then $\mathcal{W}(\mathcal{W}_*)$ is a localizing class of B-maps.</u>

<u>Moreover the class $\mathcal{W}(\mathcal{W}_*)$ is not too big if there exists a cardinal number c with the following property :</u>

<u>v) for any $C_* \in \mathcal{W}_*$, any cycle of C_* is contained in a subcomplex of C_* in $\mathcal{W}_*$ of cardinal less than c.</u>

The first part of this proposition is trivial and the second will be proved in section §8.

Now suppose that $\mathcal{W}_*$ is a localizing class in $\mathcal{C}_*(\mathbb{Z}[\pi])$.

NOTATION 4.4. - If $G \to \pi$ is a group homomorphism, the class of complexes $C_* \in \mathcal{C}_*(\mathbb{Z}[G])$ such that $\mathbb{Z}[\pi] \otimes_G C_* \in \mathcal{W}_*$ will be denoted by $\mathcal{W}_*(G)$.

NOTATION 4.5. - If $G \to \pi$ is a group homomorphism and n an integer, we denote by $W_n(G)$ the class of linear maps $C' \to C$ between two free $\mathbb{Z}[G]$-modules such that the n^{th} suspension of the complex $C \leftarrow C' \leftarrow 0 \leftarrow \dots$ belongs in $\mathcal{W}_*(G)$.

NOTATION 4.6. - If $G \to \pi$ is a group homomorphism and n an integer, we denote by $\overline{W}_n(G)$ the class of G equivariant homomorphisms $F' \to F$ between two free G-groups such that the abelianized $H_1(F') \to H_1(F)$ belongs in $W_n(G)$.

LEMMA 4.7. - <u>One has the following properties</u> :

i) $W_n(G)$ <u>(resp. $\overline{W}_n(G)$) is stable under composition and direct sum (resp. free product) and contains the isomorphisms between free $\mathbb{Z}[G]$-modules (resp. free G-groups)</u>

ii) $W_n(G)$ <u>is contained in</u> $W_{n+1}(G)$

$\overline{W}_n(G)$ <u>is contained in</u> $\overline{W}_{n+1}(G)$.

REMARK 4.8. - Sometimes the classes $W_n(G)$ are all the same, for example if $\mathcal{W}_*$ is the class of complexes C_* such that $H_*(C_*,M)$ vanishes for a given $\mathbb{Z}[\pi]$-module M.

DEFINITION 4.9. - Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a class of morphisms in $\mathcal{C}$. An object $X \in \mathcal{C}$ is called $\mathcal{W}$-semilocal (resp. $\mathcal{W}$-local) if, for any morphism $Y \to Z$ in $\mathcal{W}$ the induced map $\mathrm{Hom}\,(Z,X) \to \mathrm{Hom}\,(Y,X)$ is injective (resp. bijective).

Now let $X \to B$ be a Serre fibration. In a first case I suppose that $X \to B$ is 1-connected $(\pi_1(B,X) = 0)$ and I denote by G the fundamental group of X.

THEOREM 4.10. - <u>If the B-space $X \to B$ is $\mathcal{W}$-($\mathcal{W}_*$)-local, $\pi_2(B,X)$ is a G-group $\overline{W}_1(G)$-local and $\overline{W}_2(G)$-semilocal, and $\pi_n(B,X)$ is a $\mathbb{Z}[G]$-module $W_{n-1}(G)$-local and $W_n(G)$-semilocal for all</u> $n \geq 3$.

I don't know in general if the converse of this theorem is true. But it's the case if the class $\mathcal{W}_*$ satisfies a splitting condition :

DEFINITION 4.11. - A complex C_* in $\mathcal{W}_*$ is called n-splittable if there exist a n-dimensional complex C'_* in $\mathcal{W}_*$ and a n-1 connected map from C'_*

to C_* .

The complex C_* is called splittable if it is n-splittable for any n .

The class $\mathcal{W}_*$ is called n-splittable (resp. splittable) il any complex in $\mathcal{W}_*$ is n-splittable (resp. splittable).

Now suppose again that $X \to B$ is a 1-connected Serre fibration and denote by G the fundamental group of X .

THEOREM 4.12. - Suppose that $\mathcal{W}_*$ is splittable and, for any subgroup Γ of π , every complex $C_* \in \mathcal{W}_*(\Gamma)$ satisfying : $H_o(C_*) = Z$ (with trivial Γ-action) is 1- and 2-spittable. Then $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local if and only if $\pi_2(B,X)$ is $\overline{W}_1(G)$-local and $\overline{W}_2(G)$-semilocal and $\pi_n(B,X)$ is $W_{n-1}(G)$-local and $W_n(G)$-semilocal for any $n \geqslant 3$.

REMARK 4.13. - If $\mathcal{W}_*$ is not splittable one can consider the class $\mathcal{W}_*^s$ of all splittable complexes of $\mathcal{W}_*$. It is not difficult to see that $\mathcal{W}_*^s$ is a localizing splittable class inducing the same $W_n(G)$ and $\overline{W}_n(G)$ as $\mathcal{W}_*$.

Consider now the general case. The Serre fibration $X \to B$ is not necessary 1-connected. Denote by $\pi_1(B,X)$ the set of connected components of the fiber of $X \to B$ endowed with the canonical π-action.

NOTATION 4.14. - Denote by $\mathcal{L}$ the class of π-sets E such that for any $K(\pi,1)$-map $Y \to Z \to K(\pi,1)$ satisfying the following properties :

i) $Y \to K(\pi,1)$ is a covering space and the π-set $\pi_1(K(\pi,1),Y)$ is isomorphic to E

ii) Y is a subcomplex of the CW-complex Z and Z/Y is 2-dimensional

iii) the relative chain complex $C_*(Z,Y)$ is a splittable complex in $\mathcal{W}_*$.

there exists also a $K(\pi,1)$-map $Z \to Y \to K(\pi,1)$ extending the identity map $Y \to Y \to K(\pi,1)$.

REMARK 4.15. - It is not difficult to see that the class of n-dimensional splittable complexes in $\mathcal{W}_*$ depends only on $W_o(\pi),\dots,W_{n-1}(\pi)$.

Then the class $\mathcal{L}$ depends only on π and on $W_o(\pi)$ and $W_1(\pi)$ and will be denoted by $\mathcal{L}(W_o(\pi), W_1(\pi))$.

THEOREM 4.16. - Let $X \to B$ be a Serre fibration. Then, if $X \to B$ is

$\mathcal{W}(\mathcal{W}_*)$-local, $\pi_1(B,X)$ belongs in $\mathcal{L}(W_0(\pi), W_1(\pi))$ and for any point x of X, $\pi_2(B,X,x)$ is a $\pi_1(X,x)$-group $\overline{W}_1(\pi_1(X,x))$-local and $\overline{W}_2(\pi_1(X,x))$-semilocal, and $\pi_n(B,X,x)$ is, for any $n \geq 3$, a $\mathbb{Z}[\pi_1(X,x)]$-module $W_{n-1}(\pi_1(X,x))$-local and $W_n(\pi_1(X,x))$-semilocal.

THEOREM 4.17. - Let $X \to B$ be a Serre fibration. Suppose that the class satisfies the following conditions :

i) for every subgroup G of π, every complex $C_* \in \mathcal{W}_*(G)$ satisfying : $H_0(C_*) = \mathbb{Z}$ (with trivial G-action) is 1- and 2-splittable

ii) for every subgroup G of π containing the isotropy subgroup of a point in $\pi_1(B,X)$, the class $\mathcal{W}_*(G)$ is 2-splittable

iii) for every isotopy subgroup G of a point in $\pi_1(B,X)$, the class $\mathcal{W}_*(G)$ is splittable.

Then $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local if and only if $\pi_1(B,X)$ belongs in $\mathcal{L}(W_0(\pi), W_1(\pi))$ and, for any $x \in X$, $\pi_2(B,X,x)$ is a $\pi_1(X,x)$-group $\overline{W}_1(\pi_1(X,x))$-local and $\overline{W}_2(\pi_1(X,x))$-semilocal and, for any $x \in X$ and any $n \geq 3$, $\pi_n(B,X,x)$ is a $\mathbb{Z}[\pi_1(X,x)]$-module $W_{n-1}(\pi_1(X,x))$-local and $W_n(\pi_1(X,x))$-semilocal.

The theorems 4.10, 4.12, 4.16, 4.17 will be proved in section §5 and §6.

The splitting condition is difficult to check in general. But we have the following results :

PROPOSITION 4.18. - If A is a P.I.D., any localizing class in $\mathcal{C}_*(A)$ is splittable.

PROPOSITION 4.19. - Let $A \to B$ be a ring homomorphism such that every finite set in B is contained in a $s \operatorname{Im}(A \to B)$ where s is a unit of B. Then, if $\mathcal{W}_*$ is a n-splittable localizing class in $\mathcal{C}_*(B)$, the class $\mathcal{W}'_*$ in $\mathcal{C}_*(A)$ defined by :

$$C_* \in \mathcal{W}'_* \iff B \underset{A}{\otimes} C_* \in \mathcal{W}_*$$

is a n-splittable localizing class.

Proof of 4.18 : Let C_* be a complex in a localizing class $\mathcal{W}_*$ in

$\mathscr{C}_*(A)$. Since A is a P.I.D. the image of $C_n \to C_{n-1}$ is a free A-module and we have a decomposition :

$$C_* = C'_* \oplus C''_*$$

where C'_* is n-dimensional and C''_* is zero in dimension $< n$.

Denote by $\sum_N C_*$ a direct sum of a countable many copies of C_* . We have an exact sequence :

$$0 \to \sum_N C_* \to \sum_N C_* \to C'_* \to 0$$

and C'_* belongs in $\mathscr{W}_*$.

This proves thas $\mathscr{W}_*$ is n-splittable.

Proof of 4.19 : It is easy to see that $\mathscr{W}'_*$ is a localizing class. Now I will prove that $\mathscr{W}'_*$ is n-splittable.

Let C_* be a complex in $\mathscr{W}'_*$. Because $\mathscr{W}_*$ is n-splittable there exist a n-dimensional complex $C'_* \in \mathscr{W}_*$ and a n-1-connected map $f : C'_* \to B \otimes_A C_*$.

It is not difficult to find a n-dimensional complex C''_* and a homotopy equivalence $g : C''_* \to C'_*$ such that $f \circ g : C''_* \to B \otimes_A C_*$ is an isomorphism in dimension $< n - 1$ and an epimorphism with free kernel K in dimension $n - 1$.

Let Γ be a free A-module of the same dimension as K , and Γ_* be the $n-1^{th}$ suspension of the complex $\Gamma \xleftarrow{1} \Gamma \leftarrow 0 \ldots$.

Let $h : C''_* \to B \otimes_A \Gamma_*$ be a map inducing an isomorphism from K to $B \otimes_A \Gamma$.

Then the map :

$$f \circ g \oplus h : C''_* \to B \otimes_A (C_* \oplus \Gamma_*)$$

is an isomorphism in dimension $< n$.

Now let $\overline{C}_0, \ldots, \overline{C}_n$ be a sequence of free A-modules such that :

$$\overline{C}_i = C_i \oplus \Gamma_i \qquad i < n$$

$$B \otimes_A \overline{C}_n \simeq C''_n .$$

The map $C''_n \to B \otimes_A (C_n \oplus \Gamma_n)$ induces a map $\varphi : B \otimes_A \overline{C}_n \to B \otimes_A (C_n \oplus \Gamma_n)$. If $(\overline{e}_i)$ is a free basis of $\overline{C}_n$ and (e_j) is a free basis of $C_n \oplus \Gamma_n$, φ is defined by :

$$\varphi(1 \otimes \bar{e}_i) = \sum_j \beta_{ij} \otimes e_j$$

where β_{ij} belongs in B and the sum $\sum_j$ is allways finite.

By hypothesis there exist units s_i in B and elements α_{ij} in A such that for any i, j :

$$\beta_{ij} \otimes e_j = s_i \otimes \alpha_{ij} e_j .$$

Denote by ε the isomorphism from $B \underset{A}{\otimes} \bar{C}_n$ to itself defined by :

$$\varepsilon(1 \otimes \bar{e}_i) = s_i^{-1} \otimes \bar{e}_i .$$

Then the map $\varphi \circ \varepsilon$ is induced by a map $\psi : \bar{C}_n \to C_n \oplus \Gamma_n$.

The differentials of $C_* \oplus \Gamma_*$ and the composite map $\bar{C}_n \xrightarrow{\psi} C_n \oplus \Gamma_n \xrightarrow{d} C_{n-1} \oplus \Gamma_{n-1}$ define a n-dimensional complex structure on $\bar{C}_o \oplus \ldots \oplus \bar{C}_n$.

This complex $\bar{C}_*$ is canonically mapped in $C_* \oplus \Gamma_*$. It is not difficult to show that :

i) $B \underset{A}{\otimes} \bar{C}_*$ is isomorphic to C''_* then $\bar{C}_* \in \mathcal{W}'_*$

ii) $\bar{C}_*$ is n-dimensional

iii) the map $\bar{C}_* \to C_* \oplus \Gamma_* \to C_*$ is n-1 connected.

That proves the proposition.

§5. - PROOF OF THE THEOREMS 4.10 AND 4.16

Clearly the theorem 4.10 is a particular case of the theorem 4.16. Now I will prove the theorem 4.16 by a sequence of five lemmas.

LEMMA 5.1. - *Let* $X \to B$ *be a Serre fibration,* n *an integer* ≥ 3 , x *a point of* X *and* $C \leftarrow C'$ *a map in* $W_n(\pi_1(X,x))$. *Then, if* $X \to B$ *is* $\mathcal{W}(\mathcal{W}_*)$*-local, the map* $\mathrm{Hom}\,(C, \pi_n(B,X,x)) \to \mathrm{Hom}\,(C', \pi_n(B,X,x))$ *is injective.*

Proof : Let φ be a map from C to $\pi_n(B,X,x)$ such that the composite map $C' \to C \to \pi_n(B,X,x)$ is zero.

By attaching n-cells on $X \to B$ we can obtain a B-space $Y \to B$ and a B-map $X \to Y \to B$ such that $\pi_n(Y,X,x)$ is the module C and the map $\pi_n(Y,X,x) \to \pi_n(B,X,x)$ is the map φ .

Since the composite map $C' \to C \to \pi_n(B,X,x)$ is zero, we can find a commutative diagram of $\mathbb{Z}[\pi_1(X,x)]$-modules :

$$\begin{tikzcd}
\pi_{n+1}(B,Y,x) \arrow[r] & \pi_n(Y,X,x) \arrow[r] & \pi_n(B,X,x) \\
C' \arrow[u, "\varphi'"] \arrow[r] & C \arrow[u, equal] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

and we can attach n+1-cells on $Y \to B$ in order to obtain a B-space $X' \to B$ and a B-map $Y \to X' \to B$ such that $\pi_{n+1}(X',Y,x)$ is the module C' and the map $\pi_{n+1}(X',Y,x) \to \pi_{n+1}(B,Y,x)$ is the map φ' .

By construction the B-map $X \to X' \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$ because the chain complex $C_*(X',X)$ is the n^{th} suspension of the complex

$$\mathbb{Z}(\pi) \underset{\pi_1(X,x)}{\otimes} C \leftarrow Z\pi \underset{\pi_1(X,x)}{\otimes} C' \leftarrow 0 \dots .$$

Since $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local the map $X' \to B$ is homotopic rel $X \to B$ to a map $X' \to X \to B$, and this implies that the map φ in zero.

LEMMA 5.2. - *Let* $X \to B$ *be a Serre fibration,* n *an integer* ≥ 3 , x *a point of* X *and* $C \leftarrow C'$ *a map in* $W_{n-1}(\pi_1(X,x))$. *Then, if* $X \to B$ *is* $\mathcal{W}(\mathcal{W}_*)$*-local, the map* $\mathrm{Hom}\,(C, \pi_n(B,X,x)) \to \mathrm{Hom}\,(C', \pi_n(B,X,x))$ *is surjective.*

Proof : Let φ' be a map from C' to $\pi_n(B,X,x)$.
Attach $n-1$-cells on $X \to B$ by trivial attaching maps :

$$\begin{array}{ccc} S^{n-1} & \to & B^{n-1} \\ x \downarrow & & \downarrow 0 \\ X & \to & B \end{array}$$

So we can obtain a B-space $Y \to B$ and a B-map $X \to Y \to B$ such that $\pi_n(B,Y,x)$ is isomorphic to $\pi_n(B,X,x) \oplus C$. The isomorphism is given by the boundary $\pi_n(B,Y,x) \to \pi_{n-1}(Y,X,x) \simeq C$ and the standard retraction : $\pi_n(B,Y,x) \to \pi_n(B,X,x)$.

Let $\psi : C' \to \pi_n(B,Y,x)$ be the map given by the map $\varphi' : C' \to \pi_n(B,X,x)$ and the map $C' \to C$.

By the map ψ we can attach n-cells on $Y \to B$ in order to obtain a B-space $X' \to B$ and a B-map $Y \to X' \to B$ such that $\pi_n(X',Y,x)$ in the module C', and the map $\pi_n(X',Y,x) \to \pi_n(B,Y,x)$ is the map ψ.

Clearly the B-map $X \to X' \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$ and the obstruction to retract $X' \to B$, rel $X \to B$ in $X \to B$ is the cohomology class of φ' in $H^n(X',X\ ;\ \pi_n(B,X,x))$. But $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local, then this obstruction vanishes and φ' is a boundary. This proves that the map $\mathrm{Hom}\ (C,\pi_n(B,X,x)) \to \mathrm{Hom}\ (C',\pi_n(B,X,x))$ is surjective.

LEMMA 5.3. - *Let $X \to B$ be a Serre fibration, x a point of X and $F \leftarrow F'$ a map in $\overline{W}_2(\pi_1(X,x))$. Then, if $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local, the map $\mathrm{Hom}\ (F,\pi_2(B,X,x)) \to \mathrm{Hom}\ (F',\pi_2(B,X,x))$ is injective.*

Proof : Denote by $a \mapsto \overline{a}$ the map $\pi_2(B,X,x) \to \pi_1(X,x)$. We have the following properties :

i) $\overline{g(a)} = g\,\overline{a}\,g^{-1}$ for any $a \in \pi_2(B,X,x)$ and $g \in \pi_1(X,x)$

ii) $a\,b\,a^{-1} = \overline{a}(b)$ for any $a, b \in \pi_2(B,X,x)$.

Now let φ_0 and φ_1 be two maps from F to $\pi_2(B,X,x)$ giving the same map : $F' \to \pi_2(B,X,x)$.

If we choice a basis (x_i) of F, φ_0 is given by :

$$\varphi_0(x_i) = u_i \in \pi_2(B,X,x)$$

and φ_1 is given by :

$$\varphi_1(x_i) = u_i\, v_i \in \pi_2(B,X,x) .$$

I want to prove that v_i is equal to 1 .

Now choice a basis (y_j) of F' . The morphism $F' \to F$ maps y_j in a word $\omega_j \in F$.

The word ω_j is given by a finite ordered product :

$$\omega_j = \prod_\alpha g_\alpha(x_{i_\alpha})^{\varepsilon_\alpha} \qquad g_\alpha \in \pi_1(X,x) \ , \quad \varepsilon_\alpha = \pm 1 .$$

By using the properties i) and ii), it is not difficult to show the following formula :

$$\varphi_1(\omega_j) = \prod_\alpha g'_\alpha(v_{i_\alpha})^{\varepsilon_\alpha} . \varphi_0(\omega_j)$$

where g'_α is given by :

$$g'_\alpha = (\prod_{\beta<\alpha} g_\beta . \overline{u_{i_\beta}}^{\varepsilon_\beta} . g_\beta^{-1}) . g_\alpha . \overline{u_{i_\alpha}}^{(1+\varepsilon_\alpha)/2} .$$

Now let $\psi : F \to \pi_2(B,X,x)$ and $\lambda : F' \to F$ be the maps defined by :

$$\psi(x_i) = v_i$$

$$\lambda(y_j) = \prod_\alpha g'_\alpha(x_{i_\alpha})^{\varepsilon_\alpha} .$$

Since φ_0 and φ_1 are equalized by the map $F' \to F$, $\psi \circ \lambda$ is the trivial homomorphism. Moreover g_α and g'_α give the same element in π and λ and the given map $F' \to F$ induce the same map from

$$\mathbb{Z}[\pi] \underset{\pi_1(X,x)}{\otimes} H_1(F') \to \mathbb{Z}[\pi] \underset{\pi_1(X,x)}{\otimes} H_1(F) .$$

Then λ belongs in $\overline{W}_2(\pi_1(X,x))$.

Now let $Y \to B$ be the B-space obtained by attaching the 2-cells to $X \to B$ by the attaching maps v_i . This cells define a canonical $\pi_1(X,x)$-equivariant map f from F to $\pi_2(Y,X,x)$ and we have a commutative diagram of $\pi_1(X,x)$-groups

$$\begin{tikzcd}
 & \pi_3(B,Y,x) \arrow[r] & \pi_2(Y,X,x) \arrow[r] & \pi_2(B,X,x) \\
F' \arrow[rr, "\lambda"] & & F \arrow[u, "f"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Since $\psi \circ \lambda$ is the trivial homomorphism and F' is free we can find a commutative diagram :

$$\begin{tikzcd}
\pi_3(B,Y,x) \arrow[r] & \pi_2(Y,X,x) \arrow[r] & \pi_2(B,X,x) \\
F' \arrow[u, "\psi'"] \arrow[r, "\lambda"] & F \arrow[u, "f"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Now let $X' \to B$ be the B-space obtained by attaching the 3-cells to $Y \to B$ by the attaching maps $\psi'(x_j)$.

It is easy to see that $X \to X' \to B$ is a B-map in $\mathcal{W}(\mathcal{W}_*)$, then the map $X' \to B$ retracts, rel $X \to B$, to a map $X' \to X \to B$. That proves that the attaching maps v_i are trivial and, consequently, that $\varphi_0 = \varphi_1$.

LEMMA 5.4. - <u>Let $X \to B$ be a Serre fibration, x a point in X and $F \to F'$ a map in $\overline{E}_1(\pi_1(X,x))$. Then, if $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local, the map</u> $\mathrm{Hom}\,(F,\pi_2(B,X,x)) \to \mathrm{Hom}\,(F',\pi_2(B,X,x))$ <u>is surjective.</u>

<u>Proof</u> : Let φ' be a map from F' to $\pi_2(B,X,x)$.

By attaching trivial 1-cells on $X \to B$ we obtain a B-space $Y \to B$ and a B-maps $X \to Y \to B$ endowed with a bijection from a basis (x_i) of F to the set of 1-cells of Y/X .

Let L be the free group generated by the 1-cells of Y/X . The group $\pi_1(Y,x)$ is equal to $\pi_1(X,x) * L$. Let Γ be the kernel of the retraction $\pi_2(B,Y,x) \to \pi_2(B,X,x)$; $\pi_2(B,Y,x)$ is the semi direct product $\Gamma \rtimes \pi_2(B,X,x)$ and Γ is mapped isomorphicly by the boundary $\pi_2(B,Y,x) \to \pi_1(Y,x)$ to the subgroup $\mathrm{Ker}\,(\pi_1(X,x) * L \xrightarrow{1*0} \pi_1(X,x)) \subset \pi_1(Y,x)$.

It is easy to see that the group $\mathrm{Ker}\,(\pi_1(X,x) * L \to \pi_1(X,x))$ in the free $\pi_1(X,x)$-group generated by the basis of L and then Γ is isomorphic to F .

Thus $\pi_2(B,Y,x)$ is isomorphic to the semi direct product $F \rtimes \pi_2(B,X,x)$ and the action in given by the map $\pi_2(B,X,x) \to \pi_1(X,x)$ and the standard $\pi_1(X,x)$-action on F .

Now let $\psi : F' \to \pi_2(B,Y,x) = F \rtimes \pi_2(B,X,x)$ be the homomorphism defined by :

$$\psi(y_j) = (\lambda(y_j)\ ,\ \varphi'(y_j)^{-1})$$

where (y_j) is a basis of F' and λ is the given map $F' \to F$.

By attaching 2-cells on $Y \to B$ with the attaching maps $\psi(y_j)$ we obtain a B-space $X' \to B$ and a B-map $Y \to X' \to B$.

By construction, the chain complex $C_*(X',X)$ is isomorphic to :

$$0 \leftarrow \mathbb{Z}[\pi] \underset{\pi_1(X,x)}{\otimes} H_1(F) \leftarrow \mathbb{Z}[\pi] \underset{\pi_1(X,x)}{\otimes} H_1(F') \leftarrow 0 \leftarrow \dots$$

and then $X \to X' \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$.

Since $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local $X' \to B$ retracts rel $X \to B$ to $X \to B$. This retraction maps the 1-cells of $Y - X$ in the fiber of $X \to B$ and defines elements u_i in $\pi_2(B,X,x)$.

Denote by $\varphi : F \to \pi_2(B,X,x)$ the homomorphism mapping x_i in u_i.

The retraction induces a map from $\pi_2(B,Y,x)$ to $\pi_2(B,X,x)$ and the composite map $F' \to \pi_2(B,Y,x) \to \pi_2(B,X,x)$ is trivial.

By construction this last homomorphism maps y_j in $\varphi(\lambda y_j)\,\varphi'(y_j)^{-1}$ and thus φ' is equal to $\varphi \circ \lambda$.

LEMME 5.5. - *Let $X \to B$ be a Serre fibration. Then, if $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$ local, $\pi_1(B,X)$ belongs in $\mathcal{L}(W_0(\pi), W_1(\pi))$.*

Proof : Let $Y \to Z \to K(\pi,1)$ be a $K(\pi,1)$-map, the $K(\pi,1)$-space $Y \to K(\pi,1)$ being a covering space, and suppose that $\pi_1(K(\pi,1),Y)$ is isomorphic to $\pi_1(B,X)$ and that $C_*(Z,Y)$ is a splittable 2-dimensional complex in $\mathcal{W}_*$.

The isomorphism between $\pi_1(B,X)$ and $\pi_1(K(\pi,1),Y)$ induces a commutative diagram :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ B \arrow[r] & K(\pi,1) \end{tikzcd}$$

and we will suppose that $B \to K(\pi,1)$ is a Serre fibration.

This square is 2-connected (i.e. for any point $x \to X$ the map $\pi_i(B,X,x) \to \pi_i(K(\pi,1),Y,x)$ is injective for $i = 1$ and surjective for $i \leqslant 2$), then we can attach 0, 1 and 2-cells to X in order to obtain a B-space $X' \to B$ and a B-map $X \to X' \to B$ and a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd] & \\ & X' \arrow[rr, dashed] \arrow[dl] & & Z \arrow[dl] \\ B \arrow[rr] & & K(\pi,1) & \end{tikzcd}$$

such that Z is isomorphic to $X' \bigcup_X Y$.

Since $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local and $X \to X' \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$, $X' \to B$ retracts, rel $X \to B$ to $X \to B$, and thus $Z \to K(\pi,1)$ retracts, rel $Y \to K(\pi,1)$ to $Y \to K(\pi,1)$.

Then $\pi_1(B,X)$ belongs in $\mathcal{L}(W_0(\pi), W_1\pi)$ and the lemma is proved, so are theorems 4.10 and 4.16.

§6. - PROOF OF THEOREMS 4.12 AND 4.17

Let $X \to B$ be a Serre fibration and suppose we have the following properties :

i) for every subgroup G of π, every complex $C_* \in \mathcal{W}_*(G)$ satisfying $H_o(C_*) = \mathbb{Z}$ (with trivial G-action) is 1- and 2-splittable

ii) for every subgroup G of π containing the isotopy subgroup of a point in $\pi_1(B,X)$, the class $\mathcal{W}_*(G)$ is 2-splittable

iii) for every isotopy subgroup G of a point in $\pi_1(B,X)$, the class $\mathcal{W}_*(G)$ is splittable

iv) $\pi_1(B,X)$ belongs in $\mathcal{L}(W_o(\pi), W_1(\pi))$

v) for every $x \in X$, $\pi_2(B,X,x)$ is $\overline{W}_i(\pi_1(X,x))$-local and $\overline{W}_2(\pi_1(X,x))$-semilocal

vi) for every $x \in X$ and every $n \geqslant 3$, $\pi_n(B,X,x)$ is $W_{n-1}(\pi_1(X,x))$-local and $W_n(\pi_1(X,x))$-semi-local.

I want to show that $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local.

LEMMA 6.1. - <u>Every B-map</u> $X \to Y \to B$ <u>in</u> $\mathcal{W}(\mathcal{W}_*)$, <u>where</u> Y <u>is the union of</u> X <u>and</u> n- <u>and</u> n+1-<u>cells</u> $(n \geqslant 3)$, <u>has a</u> B-<u>retraction</u>.

<u>Proof</u> : Let (X_i) (resp. (Y_i)) be the connected components of X (resp. Y) and choice a point x_i in each X_i . We have

$$C_*(Y,X) = \bigoplus_i C_*(Y_i,X_i)$$

and all the complexes $C_*(Y_i,X_i)$ belong in $\mathcal{W}_*$.

The obstructions to have a B-retraction for the B-map $X_i \to Y_i \to B$ belong in the groups $H^p(Y_i,X_i\,;\,\pi_p(B,X_i)) = H^p(Y_i,X_i,\pi_p(B,X,x_i))$.

The complex $C_*(Y_i,X_i,\mathbb{Z}\pi_1(x_i))$ is the complex

$$\leftarrow 0 \leftarrow C_n \xleftarrow{d} C_{n+1} \leftarrow 0 \leftarrow \ldots$$

and the map d belongs in $W_n(\pi_1(X,x_i))$.

Then, by using the property vi), the obstruction groups $H^p(Y_i,X_i,\pi_p(B,X_i))$ vanish and each B-map $X_i \to Y_i \to B$ has a B-retraction.

That proves the lemma.

LEMMA 6.2. - <u>Every B-map</u> $X \to Y \to B$ <u>in</u> $\mathcal{W}(\mathcal{W}_*)$, <u>where</u> Y <u>is the union of</u> X <u>and</u> 2- <u>and</u> 3-<u>cells</u>, <u>has a</u> B-<u>retraction</u>.

Proof : Define the spaces X_i and Y_i and the points x_i as above, and denote by X'_i the union of X_i and the 2-cells of Y_i.

For each i we have the following exact sequence of $\pi_1(X,x_i)$-groups :

$$\pi_3(B,X'_i) \to \pi_2(X'_i,X_i) \to \pi_2(B,X_i) .$$

Let F_i (resp. F'_i) be the free $\pi_1(X,x_i)$-groups generated by the 2-cells (resp. 3-cells) of $Y_i - X_i$. We have cannonical maps

$$F_i \to \pi_2(X'_i,X_i) \qquad F'_i \to \pi_3(B'_i,X'_i) .$$

Since F'_i is free there exists a homomorphism $f_i : F'_i \to F_i$ such that the following diagram in commutative :

$$\begin{tikzcd}
\pi_3(B,X_i) \arrow[r] & \pi_2(X'_i,X_i) \arrow[r] & \pi_2(B,X_i) \\
F'_i \arrow[u] \arrow[r, "f_i"] & F_i \arrow[u] \arrow[ur] &
\end{tikzcd}$$

It is easy to see that the B-map $X_i \to Y_i \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$, and then the map $f : F'_i \to F_i$ belongs in $\overline{W}_2(\pi_1(X,x_i))$. One deduces that the map $F_i \to \pi_2(B,X_i)$ is trivial and the B-map $X_i \to X'_i \to B$ has a B-retraction.

Now the only obstruction to have a B-retraction of $X_i \to Y_i \to B$ belongs in $H^3(Y_i,X_i,\pi_3(B,X,x_i))$. By using the property vi) this group vanishes and $X_i \to Y_i \to B$ has a B-retraction, and the lemma is proved.

LEMMA 6.3. - <u>Every B-map</u> $X \to Y \to B$ <u>in</u> $\mathcal{W}(\mathcal{W}_*)$, <u>where</u> Y <u>is the union of</u> X <u>and</u> 1- <u>and</u> 2-<u>cells</u>, <u>has a B-retraction.</u>

<u>Proof</u> : Let K be the Eilenberg-McLane space of type $K(\pi,1)$ and $f : B \to K$ the standard map. Let $\tilde{K} \to K$ be the covering space define by the π-set $\pi_1(B,X)$. We have a commutative diagram :

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr] \arrow[dd] & & \tilde{K} \arrow[dd] \arrow[dr] & \\
& Y \arrow[rr] & & Z \arrow[dl] \\
B \arrow[ur] \arrow[rr] & & K &
\end{tikzcd}$$

where Z is the push-out $Y \underset{X}{\cup} \tilde{K}$.

Clearly Z is the union of $\tilde{K}$ and 1- and 2-cells, $C_*(Z,\tilde{K})$ is a splittable complex in $\mathcal{W}_*$ and $\pi_1(K,\tilde{K})$ belongs in $\mathcal{L}(W_o(\pi),W_1(\pi))$. Then the B-map $\tilde{K} \to Z \to K$ has a K-retraction. This implies that $\pi_o(\tilde{K}) \to \pi_o(Z)$ is bijective and for every $x \in \tilde{K}$, $\pi_1(\tilde{K},x)$ and $\pi_1(Z,x)$ have the same image in $\pi_1(K,x)$. One deduces that $\pi_o(X) \to \pi_o(Y)$ is bijective and, for every $x \in X$, $\pi_1(X,x)$ and $\pi_1(Y,x)$ have the same image in $\pi_1(B,x) = \pi$.

Thus, if X' denotes the union of X and the 1-cells of $Y - X$, the B-map $X \to X' \to B$ has a B-retraction.

Define the spaces X_i, $X_i^!$, Y_i and the points x_i as above, and choice B-retractions f_i of $X_i \to X_i^! \to B$. Let F_i (resp. $F_i^!$) be the free $\pi_1(X_i)$-group generated by the 1-cells (resp. 2-cells) of $Y_i - X_i$. The obstruction to extend f_i to a B-retraction $Y_i \to X_i \to B$ is a map $\psi_i : F_i^! \to \pi_2(B,X_i)$.

Since $X_i^! \to B$ has the homotopy type of the union of $X_i \to B$ with trivial 1-cells $\pi_2(B,X_i^!)$ is isomorphic to the semi direct product $F_i \rtimes \pi_2(B,X_i)$ (see the proof of lemma 5.4) and the B-retraction f_i induces the second projection on the π_2. Moreover the attaching maps of the 2-cells induce a map : $g_i : F_i^! \to F_i \rtimes \pi_2(B,X_i)$. If $\mathcal{B}_i^!$ is the basis of $F_i^!$, g_i is given by :

$$\forall x \in \mathcal{B}_i^! \qquad g_i(x) = (h_i(x), \psi_i(x))$$

and $h_i : F_i^! \to F_i$ belongs on $\overline{W}_1(\pi_1(X_i))$.

By the property v) there exists a map $\varphi_i : F_i \to \pi_2(B,X_i)$ such that :

$$\forall x \in \mathcal{B}_i^! \qquad \varphi_i \circ h_i(x) = \psi_i(x)^{-1} .$$

Now change the B-retraction f_i by the map φ_i . This new B-retraction induces in the π_2 the morphism $F_i \rtimes \pi_2(B,X_i) \to \pi_2(B,X_i)$ mapping (u,v) to $\varphi_i(u)v$. Thus the new composite map $F_i^! \to F_i \rtimes \pi_2(B,X_i) \to \pi_2(B,X_i)$ is zero and the new B-retraction extends to $Y_i \to B$.

That proves the lemma.

LEMMA 6.4. - Every B-map $X \to Y \to B$ in $\mathcal{W}(\mathcal{W}_*)$, where Y is the disjoint union of X and a 2-complex, has a B-retraction.

Proof : Let K the complex $Y - X$. The problem is to construct a B-map from $K \to B$ to $X \to B$, or, equivalently, to construct for each connected component K' of K , a B-map from $K' \to B$ to $X \to B$.

Now let K' be a connected component of K , and G the fondamental group of K' . The complex $C_*(K' ; \mathbb{Z}G)$ is a two dimensional complex in $\mathcal{W}_*(G)$

such that $H_0(C_*(K',\mathbb{Z}G)) = \mathbb{Z}$ (with trivial G-action). Then by the condition i) $C_*(K', \mathbb{Z}[G])$ is splittable in $\mathcal{W}_*(G)$, and $C_*(K',\mathbb{Z}\pi)$ is a two dimensional splittable complex in $\mathcal{W}_*$.

Let $\widetilde{K(\pi,1)}$ be the covering space of $K(\pi,1)$ defined by $\pi_1(B,X)$. We have a commutative diagram :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{K(\pi,1)} \arrow[d] \\ B \arrow[r] & K(\pi,1) \end{tikzcd} \qquad K' \to B$$

Thus, by using the condition : $\pi_1(B,X) \in \mathcal{L}(W_0(\pi), W_1(\pi))$, the $K(\pi,1)$-map $\widetilde{K(\pi,1)} \amalg K' \to K(\pi,1)$ has a $K(\pi,1)$-retraction, and there is a $K(\pi,1)$-map : $K' \to \widetilde{K(\pi,1)} \to K(\pi,1)$.

Let $\tilde{B}$ be the pull-back $B \underset{K(\pi,1)}{\times} \widetilde{K(\pi,1)}$. The $K(\pi,1)$-map $K' \to \widetilde{K(\pi,1)} \to K(\pi,1)$ and the map $K' \to B$ induce a B-map $K' \to \tilde{B} \to B$.

Now it suffices to construct a $\tilde{B}$-map from $K' \to \tilde{B}$ to $X \to \tilde{B}$.

Choice a base point $a \in K'$ and a point $x \in X$ over the image of a in $\tilde{B}$. Denote by G the group $\pi_1(K',a)$ and by $\tilde{\pi}$ the group $\pi_1(\tilde{B},x)$; $\tilde{\pi}$ is the isotopy subgroup of a point in $\pi_1(B,X)$.

Let K_1' be the 1-skeleton of K'. Since the fibers of $X \to \tilde{B}$ are connected, the $\tilde{B}$-map $X \to X \vee K_1' \to \tilde{B}$ has a $\tilde{B}$-retraction $X \vee K_1' \to X \to \tilde{B}$. Let F be the kernel $\mathrm{Ker}\ (\pi_2(\tilde{B}, X \vee K_1', x) \to \pi_2(\tilde{B},X,x))$. By the boundary $\pi_2(\tilde{B}, X \vee K_1', x) \to \pi_1(X \vee K_1', x)$, F maps isomorphicaly on the kernel $\mathrm{Ker}\ (\pi_1(X \vee K_1', x) \to \pi_1(X,x))$, and F is a free $\pi_1(X,x)$-group (see the proof of lemma 5.4). Moreover $\pi_2(\tilde{B}, X \vee K_1', x)$ is the semi-direct product $F \rtimes \pi_2(\tilde{B},X,x)$.

Let F' be the free $\pi_1(X,x)$- group generated by the 2-cells of K'. The attaching maps define a morphism $g : F' \to F \rtimes \pi_2(\tilde{B},X,x)$.

This map is defined on the standard basis of F' by :

$$g(x) = (h(x), \psi(x))$$

where $h : F' \to F$ is a $\pi_1(X,x)$-equivariant map and $\psi : F' \to \pi_2(\tilde{B},X,x)$ is the obstruction to extend the $\tilde{B}$-retraction $X \vee K_1' \to X \to \tilde{B}$ to $X \vee K' \to \tilde{B}$.

Let $\varphi : F \to \pi_2(\tilde{B},X,x)$ be a $\pi_1(X,x)$-equivariant map. We can change the $\tilde{B}$-retraction $X \vee K_1' \to X \to \tilde{B}$ in order to have a new morphism $F \rtimes \pi_2(\tilde{B},X,x) \to \pi_2(\tilde{B},X,x)$ mapping (u,v) to $\varphi(u)v$. Then the new obstruction map from F' to $\pi_2(\tilde{B},X,x)$ is given on the basis of F' by :

$$g'(x) = \varphi(h(x)\, \psi(x) .$$

Now let ψ' be the map : $F' \to \pi_2(\tilde{B},X,x)$ equal to ψ^{-1} on the basis of F' . Then, in order to prove the lemma, it suffices to find a map $\varphi : F \to \pi_2(\tilde{B},X,x)$ such that :

$$\psi' = \varphi \circ h .$$

SUBLEMMA 1. - <u>There exists a 2-dimensional complex in</u> $\mathcal{W}_*(\tilde{\pi})$:

$$\mathbb{Z}[\tilde{\pi}] \leftarrow \mathbb{Z}[\tilde{\pi}] \underset{\pi_1(X,x)}{\otimes} H_1(F) \xleftarrow{h'} \mathbb{Z}[\tilde{\pi}] \underset{\pi_1(X,x)}{\otimes} H_1(F') \leftarrow 0 \ldots$$

<u>where</u> h' <u>is the morphism induced by</u> h .

<u>Proof</u> : Denote by (x_i) (resp. (y_j)) the 1-cells (resp. 2-cells) of K' . The group $\pi_1(K_1',x)$ is the free group L generated by the x_i 's, and F' in free $\pi_1(X,x)$-group generated by the y_j 's.

Let $\alpha_*: \pi_1(X,x) * L \to \pi_1(X,x)$ be the map defined by the $\tilde{B}$-retraction : $X \vee K_1' \to X \to \tilde{B}$. Then F is the free $\pi_1(X,x)$-group generated by the $x_i\alpha_*(x_i^{-1})$'s.

It is not difficult to see that the map g is defined by :

$$g(y_j) = (\rho_j\alpha_*(\rho_j^{-1}),\, z_i)$$

where ρ_j is the relation in L defined by the cell y_j and z_j is the element in $\pi_2(\tilde{B},X,x)$ induced by y_j .

Let k be the map from L to the group $F = \text{Ker}\,(\pi_1(X,x) * L \to \pi_1(X,x))$ defined by :

$$k(u) = u\alpha_*(u^{-1}) .$$

The map h is also defined by :

$$h(y_j) = k(\rho_j) .$$

Now, consider the following complex :

$$\mathbb{Z}[\tilde{\pi}] \xleftarrow{\partial} \mathbb{Z}[\tilde{\pi}] \underset{\pi_1(X,x)}{\otimes} H_1(F) \xleftarrow{h'} \mathbb{Z}[\tilde{\pi}] \underset{\pi_1(X,x)}{\otimes} H_1(F')$$

where ∂ maps $x_i\alpha_*(x_i^{-1})$ to $\bar{x}_i - 1$ ($\bar{x}_i$ is the image of x_i by the map $L \to \tilde{\pi}$) and h' is induced by h.

It is not dificult to see that h' is induced by the Fox derivative (4) $\partial : C_2(K_1', \mathbb{Z}\pi_1(K_1',x)) \to C_1(K_1', \mathbb{Z}\pi_1(K_1',x))$ and, thus, the above complex is isomorphic to :

$$\mathbb{Z}[\tilde{\pi}] \underset{\pi_1(K_1',x)}{\otimes} C_*(K_1', \mathbb{Z}[\pi_1(K_1',x)])$$

and belongs in $\mathcal{W}_*(\pi)$.

SUBLEMMA 2. - <u>There exist two free</u> $\pi_1(X,x)$-<u>groups</u> F_1 <u>and</u> F_1' <u>and a map</u> $\bar{h} : F' * F_1' \to F * F_1$ <u>such that</u> :

i) $\bar{h}$ <u>restricts to</u> h <u>over</u> F'

ii) $\bar{h} \in \overline{W}_1(\pi_1(X,x))$.

<u>Proof</u> : Denote by : $C_0 \xleftarrow{\partial} C_1 \xleftarrow{h'} C_2$ the complex ;

$$\mathbb{Z}[\tilde{\pi}] \xleftarrow{\partial} \mathbb{Z}[\tilde{\pi}] \underset{\pi_1(X,x)}{\otimes} H_1(F) \xleftarrow{h'} \mathbb{Z}[\tilde{\pi}] \underset{\pi_1(X,x)}{\otimes} H_1(F')$$

The group $\tilde{\pi}$ is an isotopy subgroup of a point of $\pi_1(B,X)$. Then, by using the property iii), $C_0 \leftarrow C_1 \leftarrow C_2$ is splittable, and there exist a map $d \in W_0(\tilde{\pi})$ and a 0-connected map λ from $C_0' \xleftarrow{d} C_1'$ to $C_0 \leftarrow C_1 \leftarrow C_2$. One can suppose that the map $C_0' \to C_0$ is surjective with free kernel $\bar{C}_0$, and the relative complex of λ is homotopy equivalent to the following complex :

$$0 \leftarrow C_1 \oplus \bar{C}_0 \xleftarrow{\bar{h}'} C_2 \oplus C_1' \leftarrow 0 \ldots$$

One can see that $\bar{h}'$ restricts to h' over C_2 and $\bar{h}'$ belongs on $W_1(\tilde{\pi})$.

Now let F_1 (resp. F_1') be the free $\pi_1(X,x)$-group generated by a basis of $\bar{C}_0$ (resp. C_1'). Since $\pi_1(X,x) \to \tilde{\pi}$ is surjective the map $F * F_1 \to C_1 \oplus \bar{C}_0$ is surjective and we can lift $\bar{h}'$ in a $\pi_1(X,x)$-equivariant map $h' : F' * F_1' \to F * F_1$ extending h.

Furthermore, h' belongs in $\overline{W}_1(\pi_1(X,x))$, because $\bar{h}'$ belongs in $W_1(\tilde{\pi})$, and the sublemma is proved.

Now we can prove the lemma.

The map $\psi' : F' \to \pi_2(\tilde{B},X,x)$ extends to a map $\psi_1' : F' * F_1' \to \pi_2(\tilde{B},X,x)$, which factors through $F * F_1$ because h' belongs in $\overline{W}_1(\pi_1(X,x))$, and $\pi_2(B,X,x)$ is $\overline{W}_1(\pi_1(X,x))$-local. This factorization give a factorization of ψ' through F and the lemma is proved.

LEMMA 6.5. - <u>Let $X \to Y \to B$ be a B-map in $\mathcal{W}(\mathcal{W}_*)$. Then there exists a commutative diagram :</u>

$$\begin{tikzcd} & Y_o \arrow[dr] & \\ X \arrow[ur] \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

<u>such that $X \to Y_o \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$ and Y_o is the disjoint union of X and a 2-complex and $Y_o \to Y$ is 0-connected.</u>

<u>Proof</u> : Let K be a connedted component of Y not intersecting X. Denote by π' the group $\pi_1(K)$ and by G the image of π' in $\pi = \pi_1(B)$. Clearly the $\mathbb{Z}[G]$-complex $C_*(K, \mathbb{Z}[G])$ belongs in $\mathcal{W}_*(G)$ and, by i), is 2-splittable. Moreover the map $\pi' \to G$ is epic, and then the complex $C_*(K, \mathbb{Z}[\pi'])$ is a 2-splittable complex of $\mathcal{W}_*(\pi')$.

Let C'_* be a 2-dimensional complex of $\mathcal{W}_*(G)$ and f a 1-connected map from C'_* to the complex $C_* = C_*(K, \mathbb{Z}[\pi'])$.

One can suppose that f is an isomorphism in dimension 0 and an epimorphism with free kernel Γ in dimension 1, and one can construct a map f' from C'_* to $C_* \oplus (0 \leftarrow \Gamma \xleftarrow{1} \Gamma \leftarrow 0 \leftarrow \dots)$ inducing an isomorphism in dimension 0 and 1.

Add 1- and 2-cells to K, and we get a complex K' and a homotopy equivalence $K' \to K$, such that $C_*(K', Z\pi')$ is isomorphic to $C_* \oplus (0 \leftarrow \Gamma \leftarrow \Gamma \leftarrow 0 \dots)$.

Now let K'' be the union of the 1-skeleton of K' and 2-cells corresponding to a basis of C'_2, mapping in K' by a map induced by f'. The complex K'' is 2-dimensional and $C_*(K'' ; \mathbb{Z}[\pi])$ belongs in $\mathcal{W}_*$.

Then, the space Y_o is the disjoint union of X and all the K'' defined as above, and corresponding to the various connected components of Y.

One has a commutative diagram :

$$\begin{tikzcd} & Y_o \arrow[dr] & \\ X \arrow[ur] \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

Y_o is the union of X and a 2-complex, $X \to Y_o \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$ and $Y_o \to Y$ is 0-connected.

LEMMA 6.6. - Let $X \to Y \to B$ be a B-map in $\mathcal{W}(\mathcal{W}_*)$, $X \to Y$ being 0-connected. Then there exists a commutative diagram :

$$\begin{tikzcd} & Y_1 \arrow[dr] & \\ X \arrow[ur] \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

such that :

i) $X \to Y_1 \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$

ii) Y_1 is the union of X and 1-cells and 2-cells.

iii) $Y_1 \to Y$ is 1-connected.

Proof : Let K be a connected component of Y and $\overline{K}$ the inverse image of K by the map $X \to Y$, and $\overline{K} \to K' \to K$ a factorization of $\overline{K} \to K$ where K' is the union of $\overline{K}$ and non-zero dimensional cells and $K' \to K$ is a homotopy equivalence.

Let π' be the group $\pi_1(K')$ and G the image of π' in π . Clearly G contains the isotopy subgroup of a point in $\pi_1(B,X)$, and $C_*(K',\overline{K}\ ;\ \mathbb{Z}[G])$ is a 2-splittable complex of $\mathcal{W}_*(G)$. Actually $C_*(K',\overline{K}\ ;\ \mathbb{Z}[\pi'])$ is a 2-splittable complex of $\mathcal{W}_*(\pi')$ because $\pi' \to G$ is epic.

Denote by C_* the complex $C_*(K',\overline{K}\ ;\ \mathbb{Z}[\pi'])$. Since C_o vanishes and C_* is 2-splittable, there exists a map $C_2' \to C_1'$ in $W_1(\pi')$ and a commutative diagram :

$$\begin{tikzcd} 0 & C_1 \arrow[l] & C_2 \arrow[l] & C_3 \arrow[l] & \dots \arrow[l] \\ & C_1' \arrow[u] & C_2' \arrow[l] \arrow[u] & & \end{tikzcd}$$

such that the map $C_1' \to C_1$ is epic with free kernel.

Then one can modify the number of 1- and 2-cells of K' in order to have an isomorphism $C_1' \to C_1$.

Let K_1 be the union of $\overline{K}$ and the 1-cells of K' .

The map $\pi_2(K',K_1) \to H_2(K',K_1,\ \mathbb{Z}[\pi'])$ is epic. Then if (e_α) is a basis of C_2' , one can choice for any α an element u_α in $\pi_2(K',K_1)$ such that e_α and u_α have the same image in $H_2(K',K_1,\mathbb{Z}[\pi']) = C_2/C_3$. Let K_2 be the union of K_1 and the 2-cells attached by the elements $\partial u_\alpha \in \pi_1(K_1)$. The u_α 's induce a map $K_2 \to K'$, and one can show that $\overline{K} \to K_2 \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$ and $K_2 \to K$ is 1-connected.

Denote by Y_1 the union of all K_2 and one has a diagram :

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right] & Y_1 \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

satisfying the condition i) ii, iii) of the lemma.

LEMMA 6.7. - Let $X \to Y \to B$ be a B-map in $\mathcal{W}(\mathcal{W}_*)$, $X \to Y$ being 1-connected. Then there exists a commutative diagram :

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d, equal] & Y_2 \arrow[r] & Y_3 \arrow[r] & \dots \arrow[r] & Y' \arrow[dr] & \\ X \arrow[rrrrr] & & & & & Y \to B \end{tikzcd}$$

such that :

i) $Y_n \to Y_{n+1} \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$ for any $n \geq 1$

ii) Y_n is the union of Y_{n-1} and n-cells and n+1-cells

iii) Y'. is the union of the Y_n 's

iv) $Y' \to Y$ is a homotopy equivalence.

Proof : Suppose we have construct the following diagram :

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d, equal] & \dots \arrow[r] & Y_n \arrow[dr] & \\ X \arrow[rrr] & & & Y \to B \end{tikzcd}$$

where $Y_i \to Y_{i+1} \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$, Y_i is the union of Y_{i-1} and i- and i+1-cells, and $Y_n \to Y$ is n-connected.

I will construct Y_{n+1} in the case Y connected.

Since $Y_n \to Y$ is n-connected, one can find a factorization $Y_n \to Y' \to Y$ of $Y_n \to Y$ such that $Y' \to Y$ is a homotopy equivalence and Y' is the union of Y_n and cells of dimension $> n$. The complex $C_* = C_*(Y',Y_n ; \mathbb{Z}[\pi_1 Y])$ is zero in dimension $\leqslant n$, and $\mathbb{Z}[\pi] \underset{\pi_1(Y)}{\otimes} C_*$ belongs in $\mathcal{W}_*$.

Since $\pi_1(X)$ and $\pi_1(Y)$ have the same image in π , the image G of $\pi_1(Y) \to \pi$ is the isotropy subgroup of a point in $\pi_1(B,X)$ and $\mathbb{Z}[G] \underset{\pi_1(Y)}{\otimes} C_*$ is splittable in $\mathcal{W}_*(G)$. Since $\pi_1(Y) \to G$ is epic, C_* is a splittable complex in $\mathcal{W}_*(\pi_1 Y)$ and there exist a map $C'_{n+2} \to C'_{n+1}$ in $W_{n+1}(\pi_1 Y)$ and a commutative diagram :

$$\begin{tikzcd} \leftarrow 0 & C_{n+1} \arrow[l] & C_{n+2} \arrow[l] & \dots \arrow[l] \\ & C'_{n+1} \arrow[u] & C'_{n+2} \arrow[l] \arrow[u] & \end{tikzcd}$$

such that $C'_{n+1} \to C_{n+1}$ is epic with free kernel Γ.

Then there exists a diagram :

$$\begin{array}{ccccccc} \cdots \leftarrow 0 \leftarrow & C_{n+1} \oplus \Gamma & \leftarrow & C_{n+2} \oplus \Gamma & \leftarrow C_{n+3} \leftarrow \cdots \\ & \uparrow\wr & & \uparrow & \\ & C'_{n+1} & \longleftarrow & C'_{n+2} & \end{array}$$

Let Y'' be the union of Y_n, and the n+1-cells of Y' and n+1-cells corresponding to a basis of Γ. The space Y'' maps to Y' by a map trivial on the extra n+1-cells. The map

$$\pi_{n+2}(Y',Y'') \to H_{n+2}(Y',Y'' ; \mathbb{Z}(\pi_1 Y)) = C_{n+2}/C_{n+3} \oplus \Gamma$$

is epic, and one can attach n+2-cells in order to get a space Y_{n+1} and a map $Y_{n+1} \to Y'$ such that $C_*(Y_{n+1}, Y_n ; \mathbb{Z}(\pi_1 Y))$ is isomorphic to C'_* and $Y_{n+1} \to Y'$ is n+1-connected.

If Y is not connected, one can do the same thing for all connected components of Y, and in all the cases, one can construct the Y_n by inducing with the desired properties.

LEMMA 6.8. - <u>For any B-map $Y \to Z \to B$ in $\mathcal{W}(\mathcal{W}_*)$ the map $[Z \to B , X \to B] \to [Y \to B , X \to B]$ is epic.</u>

<u>Proof</u> : If suffices to prove the lemma in the case $Y \to Z$ is a cofibration.

Let $Y \to X \to B$ be a B-map and X' be the space $X \underset{Y}{\cup} Z$. One has the following diagram :

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \arrow[ddr] & \\ X \arrow[r] \arrow[drr] & X' \arrow[dr] & \\ & & B \end{tikzcd}$$

and $X \to X' \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$.

By 6.5 there exists a space X_o, disjoint union of X and a 2-complex and a commutative diagram :

$$\begin{tikzcd} & X_o \arrow[dr] & & \\ X \arrow[ur] \arrow[rr] & & X' \arrow[r] & B \end{tikzcd}$$

such that $X \to X_o \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$ and $X_o \to X'$ is 0-connected.

By 6.4 the B-map $X \to X_o \to B$ has a B-retraction $X_o \to X \to B$. Let X'' be the push-out of X and the mapping cylinder of $X_o \to X'$ over X_o. One has a B-homotopy commutative diagram :

$$\begin{tikzcd} & X_o \arrow[r] \arrow[d] & X' \arrow[d] \\ X \arrow[ur] \arrow[r, "1"] & X \arrow[r] & X'' \arrow[r] & B \end{tikzcd}$$

and $X \to X''$ is 0-connected and $X \to X'' \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$.

By 6.6 there exist a space X_1 union of X and 1- and 2-cells and a commutative diagram :

$$\begin{tikzcd} & X_1 \arrow[d] \\ X \arrow[ur] \arrow[r] & X'' \arrow[r] & B \end{tikzcd}$$

such that $X \to X_1 \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$ and $X_1 \to X''$ is 1-connected.

By 6.3 the B-map $X \to X_1 \to B$ has a B-retraction $X_1 \to X \to B$. Let X''' be the push-out $X \cup_{X_1} M(X_1 \to X'')$. One has a B-homotopy commutative diagram :

$$\begin{tikzcd} & X_1 \arrow[r] \arrow[d] & X'' \arrow[d] \\ X \arrow[ur] \arrow[r, "1"] & X \arrow[r] & X''' \arrow[r] & B \end{tikzcd}$$

and $X \to X'''$ is 1-connected and $X \to X''' \to B$ belongs in $\mathcal{W}(\mathcal{W}_*)$.

By 6.7 there exist spaces X_n, $n \geqslant 2$ and a commutative diagram :

$$\begin{tikzcd} & X_2 \arrow[r] & X_3 \arrow[r] & \dots \arrow[r] & \overline{X} \arrow[dr] \\ X \arrow[ur] \arrow[rrrrr] & & & & & X''' \arrow[r] & B \end{tikzcd}$$

such that $X \to X_2 \to B$, and the $X_n \to X_{n+1} \to B$ belong in $\mathcal{W}(\mathcal{W}_*)$, X_2 is the union of X and 2- and 3-cells, X_n is the union of X_{n-1} and n- and n+1-cells, $\overline{X}$ is the union of the X_n 's and $\overline{X} \to X'''$ is a homotopy equivalence.

By using 6.1 and 6.2 one can construct, by induction B-maps α_n from $X_n \to B$ to $X \to B$ such that α_n extends α_{n-1}. These B-maps induce a B-retraction $\overline{X} \to X \to B$ of $X \to \overline{X} \to B$.

Since $X \to B$ is a Serre fibration the B-maps $X \to X''' \to B$ has a B-retraction, and $X \to X'' \to B$ and $X \to X' \to B$ have a B-retraction too.

Thus the B-map $Y \to X \to B$ extends to $Z \to B$ and the lemma is proved.

LEMMA 6.9. - $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local.

Proof : Let $Y \to Z \to B$ be a B-map in $\mathcal{W}(\mathcal{W}_*)$, $Y \to Z$ being a cofibration.

By the last lemma, the map $[Z \to B\ ,\ X \to B] \to [Y \to B\ ,\ X \to B]$ is epic. In the same way the map :

$$[Z \times I \to B\ ,\ X \to B] \to [Y \times I \cup Z \times \partial I \to B\ ,\ X \to B]$$

is epic.

That implies the injectivity of $[Z \to B\ ,\ X \to B] \to [Y \to B\ ,\ X \to B]$ and $X \to B$ is $\mathcal{W}(\mathcal{W}_*)$-local.

That proves the lemma and theorems 4.12 and 4.17.

§7. - PROOF OF THE PROPOSITION 2.7

Denote by $\overline{\mathcal{W}}_c$ the class of B-maps in $\mathcal{W}$: $X \to Y \to B$ such that any morphism from a B-map $K \to L \to B$, with card K + card L < c , to $X \to Y \to B$ factors through a B-map in $\mathcal{W}'_c$.

It is easy to see that $\overline{\mathcal{W}}_c$ satisfies the conditions L_2, L_3, L_4 of 2.4.

LEMMA 7.1. - <u>The class $\overline{\mathcal{W}}_c$ is closed under finite compositions.</u>

<u>Proof</u> : Consider the following commutative diagram :

$$\begin{tikzcd}
K \arrow[rr] \arrow[d] & & L \arrow[d] & \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & B
\end{tikzcd}$$

where K and L are CW-complexes, of cardinal < c , and $X \to Y \to B$ and $Y \to Z \to B$ belong ~~in~~ to $\overline{\mathcal{W}}_c$.

Since $Y \to Z \to B$ belongs ~~in~~ to $\overline{\mathcal{W}}_c$ the morphism from $K \to L \to B$ to $Y \to Z \to B$ factors through a B-map $K' \to L' \to B$ in $\mathcal{W}'_c$. The morphism from $K \to K' \to B$ to $X \to Y \to B$ factors through a B-map $K_o \to K'_o \to B$ in $\mathcal{W}'_c$.

The B-maps $K_o \to K'_o \to B$ and $K'_o \to K'_o \underset{K'}{\cup} L' \to B$ belong in $\mathcal{W}'_c$ and the morphism from $K \to L \to B$ to $X \to Z \to B$ factors through the B-map in $\mathcal{W}'_c$: $K_o \to K'_o \underset{K'}{\cup} L' \to B$

$$\begin{tikzcd}
K \arrow[r] \arrow[dr] \arrow[dd] & L \arrow[dr] & & \\
& K' \arrow[r] \arrow[dd] & L' \arrow[dd] & \\
K_o \arrow[dr] \arrow[dd] & & & \\
& K'_o \arrow[r] \arrow[d] & K'_o \underset{K'}{\cup} L' \arrow[d] & \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & B
\end{tikzcd}$$

This implies that $X \to Z \to B$ belongs ~~in~~ to $\mathcal{W}'_c$.

LEMMA 7.2. - <u>If</u> $X \to Y \to B$ <u>is a cofibration in</u> $\overline{\mathcal{W}}_c$ <u>and</u> $X \to Z \to B$ <u>is a B-map, the B-map</u> $Z \to Z \underset{X}{\cup} Y \to B$ <u>belongs ~~in~~ to</u> $\overline{\mathcal{W}}_c$.

<u>Proof</u> : Let $K \to L \to B$ be a B-map, K being a subcomplex of L and card L being less than c , and α a morphism from $K \to L \to B$ to $Z \to Z \underset{X}{\cup} Y \to B$. The morphism α factors through a B-map $K'' \to K'' \underset{K'}{\cup} L' \to B$,

where $K' \to L' \to B$ is a cofibration and $K' \to K'' \to B$ is a B-map and K', L', K'' are of cardinal less than c, and the morphism from $K'' \to K' \underset{K'}{\cup} L' \to B$ to $Z \to Z \underset{X}{\cup} Y \to B$ is induced by morphisms from $K' \to K'' \to B$ to $X \to Z \to B$ and from $K' \to L' \to B$ to $X \to Y \to B$.

Since $X \to Y \to B$ belongs in $\overline{\mathcal{W}}_c$ the morphism from $K' \to L' \to B$ to $X \to Y \to B$ factors through a B-map $K'_o \to L'_o \to B$ in $\mathcal{W}'_c$, the map $K' \to K'_o$ being a cofibration.

Then the morphism from the diagram :

$$\begin{array}{ccc} K' & \to & L' \\ \downarrow & & \searrow \\ K'' & \longrightarrow & B \end{array}$$

to the diagram :

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & & \searrow \\ Z & \longrightarrow & B \end{array}$$

factors through the diagram :

$$\begin{array}{ccc} K'_o & \longrightarrow & L'_o \\ \downarrow & & \searrow \\ K'' \underset{K'}{\cup} K'_o & \longrightarrow & B \end{array}$$

Denote by $K'_1 \to L'_1 \to B$ the B-map $K'' \underset{K'}{\cup} K'_o \to K'' \underset{K'_o}{\cup} L'_o \to B$. This B-map belongs in $\mathcal{W}'_c$ and the morphism from $K \to L \to B$ to $Z \to Z \underset{X}{\cup} Y \to B$ factors through $K'_1 \to L'_1 \to B$. That proves the lemma.

LEMMA 7.3. - <u>If</u> $X \to Y \to B$ <u>is a cofibration in</u> $\overline{\mathcal{W}}_c$ <u>the B-map</u> : $X \times I \cup Y \times \partial I \to Y \times I \to B$ <u>belongs</u> ~~in~~ to $\overline{\mathcal{W}}_c$.

<u>Proof</u> : Suppose that X is a subcomplex of the CW-complex Y.

Let $K \to L \to B$ be a B-map, K and L being CW-complex of cardinal less than c, and α a morphism from $K \to L \to B$ to $X \times I \cup Y \times \partial I \to Y \times I \to B$.

Denote by L' the smallest subcomplex of Y containing the image of $L \to Y \times I \to Y$ and by K' the complex $L' \cap X$.

Since $X \to Y \to B$ belongs in $\overline{\mathcal{W}}_c$, the morphism from $K' \to L' \to B$ to $X \to Y \to B$ factors through a B-map $K'' \to L'' \to B$ in $\mathcal{W}_c'$. Then the morphism α factors through $K'' \to L'' \to B$ and the lemma is proved.

Thus $\overline{\mathcal{W}}_c$ is a localizing class and contains $\mathcal{W}_c$.

LEMMA 7.4. - Let γ be a cardinal number $\geqslant c$. Then any morphism from a B-map $K \to L \to B$ satisfying : card K + card $L < \gamma$, to a B-map in $\overline{\mathcal{W}}_c$ factors through a B-map $K' \to L' \to B$ in $\mathcal{W}_c$ where K' is a subcomplex of L' and L' a CW-complex of cardinal $< \gamma$.

Proof : This lemma will be proved by induction on γ .

Let γ be a cardinal number $> c$ and suppose that the lemma is proved for any $\gamma' < \gamma$.

If γ is not a successor it is easy to prove the lemma for this γ .

Now suppose that γ is equal to $\gamma_0 + 1$ $(\gamma_0 \geqslant c)$ and let $K \to L \to B$ be a B-map where card K + card $L \leqslant \gamma_0$ and $X \to Y \to B$ a B-map in $\overline{\mathcal{W}}_c$ and α a morphism from $K \to L \to B$ to $X \to Y \to B$; I will suppose that K is a subcomplex of L .

Choice subcomplexes L_i of L indexed by a well ordered set I such that :

$$i \leqslant j \Rightarrow L_i \subsetneq L_j \quad \text{and} \quad L = \bigcup_i L_i \quad \text{and} \quad \text{card } L_i < \gamma_0 .$$

Denote by K_i the complex $L_i \cap K$.

Suppose we have construct for any $i < i_0$ a B-map $X_i \to Y_i \to B$ with the following properties :

$i \leqslant j \Rightarrow X_i$ is a subcomplex of X_j

$i \leqslant j \Rightarrow Y_i$ is a subcomplex of Y_j

card $Y_i < \gamma_0$

and $X_i \to Y_i \to B$ belongs to $\mathcal{W}_c$.

Suppose furthermore we have a factorization of the morphism α restricted to $K_i \to L_i \to B$ through $X_i \to Y_i \to B$, and all that factorizations are compatible.

If i_0 is equal to $i_1 + 1$, the morphism from :

$$K_{i_0} \underset{K_{i_1}}{\cup} X_{i_1} \to L_{i_0} \underset{L_{i_1}}{\cup} Y_{i_1} \to B$$

to $X \to Y \to B$ factors through a B-map $X_{i_0} \to Y_{i_0} \to B$ is $\mathcal{W}_c$ such that X_{i_0}

is a subcomplex of Y_{i_0} and card Y_{i_0} is less than γ_0, because the lemma is true for γ_0. We can suppose that the maps $X_{i_1} \to X_{i_0}$ and $Y_{i_0} \to Y_{i_1}$ are inclusions of complexes and we have construct $X_{i_0} \to Y_{i_0} \to B$ with all the desired properties.

If i_0 is not equal to $i_1 + 1$, let $X_{i_0} = \bigcup_{i<i_0} X_i$ and $Y_{i_0} = \bigcup_{i<i_0} Y_i$. The only thing to prove is that the B-map $X_{i_0} \to Y_{i_0} \to B$ belongs to $\mathcal{W}_c$.

Let M_i be the mapping cylinder of $X_i \to Y_i$. The B-maps $X_{i_0} \to M_i \bigcup_{X_i} X_{i_0} \to B$ belong to $\mathcal{W}_c$, then, by using the next lemma, the B-map $X_{i_0} \to M_{i_0} \to B$ belongs to $\mathcal{W}_c$ too.

Let K' be the union of the X_i and L' the union of the Y_i. The B-map $K' \to L' \to B$ belongs to $\mathcal{W}_c$, card K' + card L' is less than γ, and the morphism α factors through $K' \to L' \to B$.

LEMMA 7.5. - *Let $\mathcal{W}$ be a localizing class and $Y \to B$ a B-space and Y_i subspaces of Y containing a subspace X of Y. Suppose we have the following properties* :

i) *for each i the inclusion $Y_i \hookrightarrow Y$ is a cofibration*

ii) *Y is the union of the Y_i's with the limite topology*

iii) *for any i and j $Y_i \cup Y_j$ is contained in a Y_k*

iv) *the B-map $X \to Y_i \to B$ belongs to $\mathcal{W}$*

Then the B-map $X \to Y \to B$ belongs to $\mathcal{W}$.

PROOF OF PROPOSITION 2.7. - By the lemma 7.4, for any B-map $X \to Y \to B$ in $\overline{\mathcal{W}}_c$ the identity morphism from $X \to Y \to B$ to itself factors through a B-map $X' \to Y' \to B$ in $\mathcal{W}_c$.

We have two commutative diagrams :

$$\begin{tikzcd} X \arrow[r] & Y \arrow[dr] & \\ & & B \\ X' \arrow[uu, "\alpha"] \arrow[r] & Y' \arrow[uu, "\beta"'] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[dd, "\alpha'"'] & Y \arrow[dr] \arrow[dd, "\beta'"] & \\ & & B \\ X' \arrow[r] & Y' \arrow[ur] & \end{tikzcd}$$

and we have $\alpha\alpha' = 1$, $\beta\beta' = 1$.

Let M be the telescope of the system

$$X' \xrightarrow{\alpha'\alpha} X' \xrightarrow{\alpha'\alpha} X' \xrightarrow{\alpha'\alpha} \dots$$

and N the telescope of : $Y' \xrightarrow{\beta'\beta} Y' \xrightarrow{\beta'\beta} \dots$

By using the lemma 7.5 the B-map $M \to N \to B$ belongs in $\mathcal{W}_c$ and $X \to Y \to B$

belongs in $\mathcal{W}_c$ and $X \to Y \to B$ belongs in $\mathcal{W}_c$ too. The proposition is proved.

PROOF OF THE LEMMA 7.5. -

i) first case

I suppose in this case one has a sequence $Y_o \subset Y_1 \subset Y_2 \subset \ldots$ of subspaces of Y.

Let Z be the subspace of $Y \times [0,\infty)$ defined by :

$$Z = \bigcup_{n=o}^{\infty} Y_n \times [n,\ n+1].$$

It is easy to see that the maps :

$$Z \to Y \times [0,\infty) \to Y$$

are homotopy equivalences.

By hypothesis the B-map $X \to Y_n \to B$ belongs in $\mathcal{W}$ for any n. Then by using the conditions L4 and L5 one proves that the B-map :

$$X \times [0,\infty) \to X \times [0,\infty) \cup \bigcup_{n=1}^{\infty} Y_n \times \{n\} \to B$$

belongs in $\mathcal{W}$.

Let $K \to L \to B$ be this last B-map.

Let $U_n \to V_n \to B$ be the following B-map :

$$X \times [n,\ n+1] \cup Y_n \times \{n\} \cup Y_n \times \{n+1\} \to Y_n \times [n,\ n+1] \to B.$$

By L6 this B-map belongs in $\mathcal{W}$.

But $Z \to B$ is the push-out of $L \to B$ and $\coprod_n V_n \to B$ over $\coprod_n U_n \to B$. Then by L4 and L5 the B-map :

$$X \times [0,\infty) \to Z \to B$$

belongs in $\mathcal{W}$.

Moreover $X \to X \times [0,\infty)$ and $Z \to Y$ are homotopy equivalences, then by using the conditions L1 and L3 the B-map $X \to Y \to B$ belongs in $\mathcal{W}$.

ii) general case

Denote by $\mathcal{J}$ the set of all i 's. The set $\mathcal{J}$ is a filtrant ordered set. Let T be the classifying space of $\mathcal{J}$; T is a simplified complex and a simplex of T is a sequence $(i_o < i_1 < \dots < i_n)$ in $\mathfrak{I}$.

Let $Y \times T$ be the topological limite of the $Y \times K$, K finite complex of T and Z be the subspace of $Y \times T$ defined by :

$$Z = \bigcup_{i_o<\dots<i_n} Y_{i_o} \times (i_o < \dots < i_n) .$$

The space Z is filtred by the subspaces Z_n :

$$Z_n = \bigcup_{\substack{i_o<\dots<i_p \\ p\leq n}} Y_{i_o} \times (i_o < \dots < i_p) \cup X \times T .$$

Let S_n be the set of n-simplices of T. By using n times the condition L6 one can see that for any $\sigma \in S_n$ and any $i \in \mathfrak{I}$, the B-map :

$$X \times \sigma \cup Y_i \times \partial\sigma \to Y_i \times \sigma \to B$$

belongs in $\mathcal{W}$.

Then for any n, the B-map :

$$\coprod_{i_o<\dots<i_n} X \times (i_o < \dots < i_n) \cup Y_{i_o} \times \partial(i_o < \dots < i_n) \to$$

$$\to \coprod_{i_o<\dots<i_n} Y_{i_o} \times (i_o < \dots < i_n) \to B$$

belongs in $\mathcal{W}$ too.

Hence by using the condition L5 one proves that the B-map

$$Z_{n-1} \to Z_n \to B$$

belongs to $\mathcal{W}$.

This implies that for any n the B-map

$$X \times T \to Z_n \to B$$

belongs to $\mathcal{W}$.

But the first case of the lemma is proved. Then the B-map $X \times T \to Z \to B$ belongs to $\mathcal{W}$.

On the other hand one can see that T is contractible and the map $Z \to Y$ is a homotopy equivalence. Thus the B-maps $X \to X \times T \to B$ and $Z \to Y \to B$ belong in $\mathcal{W}$ and the B-map $X \to Y \to B$ belongs to $\mathcal{W}$ too.

§8. - PROOF OF THE PROPOSITION 4.3

8.1. - Let $\mathcal{W}_*$ be a localizing class in $\mathcal{C}_*(\mathbb{Z}[\pi])$ and $\mathcal{W}$ be the class of B-maps defined by :

a B-map $X \to Y \to B$ belongs to $\mathcal{W}$ if and only if the relative $\mathbb{Z}[\pi]$-chain complex $C_*(Y,X)$ belongs to $\mathcal{W}_*$. It's easy to see that, if $\mathcal{W}_*$ is a localizing class, $\mathcal{W}$ is a localizing class too. Now suppose that c is a cardinal number greater of the cardinal of π such that the condition v) of 4.3 is satisfied. I will prove that $\mathcal{W}$ is not too big.

First I prove the following lemma :

LEMMA 8.2. - <u>Let C_* be a complex in $\mathcal{C}_*(\mathbb{Z}[\pi])$ and C_*^i, $i \in I$ be a set of subcomplexes of C_* satisfying the following conditions :</u>

i) <u>for each $i, j \in I$, $C_*^i + C_*^j$ is contained in a C_*^k</u>.

ii) <u>C_* is the union of the C_*^i's</u>

iii) <u>C_*^i belongs to $\mathcal{W}_*$</u>

<u>Then C_* belongs to $\mathcal{W}_*$</u>.

<u>Proof</u> : The proof of this lemma is essentialy the same as the proof of 7.5. If I is isomorphic to $\mathbb{N}$ one has an exact sequence :

$$0 \to \bigoplus_i C_*^i \to \bigoplus_i C_*^i \to C_* \to 0 .$$

Then C_* has the homotopy type of an extension of the suspension of $\bigoplus_i C_*^i$ by $\bigoplus_i C_*^i$ and C_* belongs to $\mathcal{W}_*$. This proves the lemma in this case, or more generaly, if I is countable.

In the general case let T_* be the chain complex of the category I. The basis of T_n is the set of all sequences $i_o < \dots < i_n$ in I and the boundary is the standard map.

Let C'_* be the subcomplex of $C_* \otimes T_*$ generated by the elements $u \otimes (i_o < \dots < i_n)$, $u \in C_*^{i_o}$. It is not difficult to see that the map $C'_* \to C_*$ is an homotopy equivalence.

On the other hand, let T_{*n} be the n-skeleton of T_* and $C_*'^n$ be the complex $C'_* \cap C_* \otimes T_{*n}$. One has the following exact sequence :

$$0 \to C_*'^{n-1} \to C_*'^n \to \bigoplus_{i_o<\dots<i_n} C_*^{i_o} \to 0 .$$

By induction, that proves that all the C'^n_* belong ~~to~~ $\mathcal{W}_*$. By applying the lemma, proved in this case, to the complex C'_* filtered by the C'^n_*, one prove that C'_* and consequently C_* belong ~~to~~ $\mathcal{W}_*$.

LEMMA 8.3. - <u>Any map from a complex in $\mathcal{W}_*(\mathbb{Z}[\pi])$ of cardinal less than c to a complex in $\mathcal{W}_*$ factorizes through a complex in $\mathcal{W}_*$ of cardinal less than</u> c.

<u>Proof</u> : Let $f : C'_* \to C_*$ be a map, and suppose that C'_* is a complex of cardinal less than c and C_* belongs ~~to~~ $\mathcal{W}_*$. Choice a graded basis $\mathcal{B} = \{e_i\ ,\ i \in I\}$ of C'_* and a good order on I such that for every $i \in I$, the module generated by the e_j, $j < i$ is a differential module denoted by C'_i.

Let $I \in I$, suppose defined for any $j < i$ a complex C''_j in $\mathcal{W}_*$ of cardinal less than c and a basis $\mathcal{B}_i$ of C''_j and a commutative diagram :

$$\begin{tikzcd} C'_j \arrow[rr, "f"] \arrow[dr, "\lambda_i"'] & & C_* \\ & C''_j \arrow[ur, "\mu_j"'] & \end{tikzcd}$$

Suppose also, for any $j \leqslant k \leqslant i$, C''_j is contained in C''_k and $\mathcal{B}_j$ in $\mathcal{B}_k$ and the following in commutative :

$$\begin{tikzcd} C'_k \arrow[r, "\lambda_k"] & C''_k \arrow[dr, "\mu_k"] & \\ & & C_* \\ C'_j \arrow[uu] \arrow[r, "\lambda_i"] & C''_j \arrow[uu] \arrow[ur, "\mu_j"'] & \end{tikzcd}$$

If i is equal to $j+1$, ~~I~~ we will construct C''_i and $\mathcal{B}_i$ by the following way :

Consider the following diagram :

$$\begin{tikzcd} 0 \arrow[r] & C'_j \arrow[r] \arrow[d] & C'_i \arrow[r] \arrow[d] & C \arrow[r] & 0 \\ & C''_j \arrow[r] & C_* & & \end{tikzcd}$$

where C is a complex isomorphic to $\mathbb{Z}\pi$.

Let F be the fiber of the map $C''_j \to C_*$ (i.e the sequence $F \to C''_j \to C_*$, has the homotopy type of a short exact sequence). The conditions ii) and iv) imply that the suspension of F belongs in $\mathcal{W}_*$. Let ∂ be the boundary $C \to C'_j$. Because this boundary is homotopic to zero in C_* and by using the condition v), there exist a complex $\overline{C}$ in $\mathcal{W}_*$ of cardinal less than c and a homotopy commutative diagram :

$$\begin{array}{ccccccc} & & C & \longrightarrow & C'_j & & \\ & & \downarrow & & \downarrow & & \\ \Sigma^{-1}\overline{C} & \to & F & \to & C''_j & \to & C_* \end{array}$$

Let C'''_i be the homotopy cofiber of $\Sigma^{-1}\overline{C} \to C''_j$. We have the following homotopy commutative diagram :

$$\begin{array}{ccccc} C'_j & \to & C'_i & & \\ \downarrow & & \downarrow & & \\ C''_j & \to & C'''_i & \to & C_* \end{array}$$

The complex C'''_i belongs in $\mathcal{W}_*$. Let C''_i be a complex of cardinal less than c having the homotopy type of C'''_i endowed with a basis $\mathcal{B}_i$ and such that there exist a commutative diagram :

$$\begin{array}{ccccc} C'_j & \to & C'_i & & \\ \downarrow & & \downarrow{\scriptstyle \lambda_i} & & \\ C''_j & \to & C''_i & \xrightarrow[\mu_i]{} & C \end{array}$$

the map $C''_j \to C''_i$ being an inclusion mapping $\mathcal{B}_j$ in $\mathcal{B}_i$.

If i is not of the from $j+1$, define C''_i as the union of the C''_j, $j < i$ and $\mathcal{B}_i$ as the union of the $\mathcal{B}_j$, $j < i$. By the lemma 8.2, C''_i belongs in $\mathcal{W}_*$ and the cardinal of C''_i is less than c.

Hence one can construct the C''_i 's by induction and consequently one get a factorization of f through a complex C'' in $\mathcal{W}_*$ of cardinal less than c.

8.4. - PROOF OF THE PROPOSITION

Consider the following diagram :

$$\begin{array}{ccccc} K & \to & L & & \\ \downarrow & & \downarrow & & \\ X & \to & Y & \to & B \end{array}$$

where K is a subcomplex of the finite complex L and $X \to Y \to B$ belongs in $\mathcal{W}$.

We can suppose that Y is a CW-complex and X is a subcomplex of Y.

Let Z_o be a subcomplex of Y containing X and the image of L and such that Z_o/X is finite.

By using the lemma 1, there exists a factorization :

$$C_*(Z_o,X) \to C'_{*o} \to C_*(Y,X)$$

where C'_* is a complex in $\mathcal{W}_*$ of cardinal less tan c.

Then one can construct a subcomplex Z_1 of Y containing Z_o such that the cardinal of Z_1/X is less than c and C'_{*c} maps in $C_*(Z_1,X)$. By induction one construct also a sequence $Z_o \subset Z_1 \subset Z_2 \subset \dots$ of subcomplex of Y and a diagram :

$$C_*(Z_o,X) \to C'_{*o} \to C_*(Z_1,X) \to C'_{*1} \to C_*(Z_2,X) \to C'_{*2} \to \dots$$

such that the cardinal of Z_n/X is less than c and C'_{*n} belongs to $\mathcal{W}_*$.

The limite $\varinjlim C'_{*n}$ belongs to $\mathcal{W}_*$ because one has the exact sequence :

$$0 \to \bigoplus_n C'_{*n} \to \bigoplus_n C'_{*n} \to \varinjlim C'_{*n} \to 0 .$$

Let Z be the union of the Z_n 's. The chain complex $C_*(Z,X)$ is isomorphic to $\varinjlim C'_{*n}$ and thus $X \to Z \to B$ belongs to $\mathcal{W}$ and the cardinal of Z/X is less than c.

Choose a subcomplex Y' of Y of cardinal less than c such that Z is the union of X and Y' and denote by X' the complex $X \cap Y'$. One has the commutative diagram :

$$\begin{array}{ccccc} K & \to & L & & \\ \downarrow & & \downarrow & & \\ X' & \to & Y' & & \\ \downarrow & & \downarrow & & \\ X & \to & Y & \to & B \end{array}$$

and X' and Y' are of cardinal less ~~tahn~~ than c and $X' \to Y' \to B$ belongs to $\mathcal{W}$. This proves the proposition.

9.1. - Let $\mathcal{W}$ be a not too big localizing class and c be a cardinal number such that the condition L7 is satisfied. I will construct a "good" set S contained in $\mathcal{W}$ and a system of functor Φ_u with some nice properties with respect to S. The functor E will be the limite of the Φ_u 's. By induction I prove that the canonical inclusion map from a B-space $X \to B$ to $E(X \to B)$ belongs in $\mathcal{W}$. The last thing to do is to show that $E(X \to B)$ is $\mathcal{W}$-local. In fact I prove that $E(X \to B)$ is S-local in general and if the condition L7 is satisfied $E(X \to B)$ is $\mathcal{W}$-local.

9.2. - THE SET S

Take a set Σ_o of cardinal c and let Σ be the simplicial complex generated by Σ_o (the simplices of Σ are the non trivial finite subsets of Σ_o).

Denote by S the set of B-maps $K \to L \to B$ in $\mathcal{W}$ such that L is a subcomplex of Σ and K is a subcomplex of L. It is not difficult to see the following properties :

i) every B-map $X \to Y \to B$ in $\mathcal{W}$ such that X and Y are CW-complexes of cardinal $< c$ has the B-homotopy type of a B-map in S.

ii) for any integer $n \geqslant 0$ and any map $I^{n+1} \to B$, the B-map $I^n \times 0 \hookrightarrow I^{n+1} \to B$ is isomorphic to a B-map in S.

9.3. - THE FUNCTOR Φ

For any B-space $X \to B$ denote by $\mathcal{A}(X \to B)$ the set of all commutative diagrams :

$$\begin{array}{ccc} K & \to & L \\ \downarrow & & \downarrow \\ X & \to & B \end{array}$$

such that $K \to L \to B$ belongs in S.

For most convenience, the diagrams of $\mathcal{A}(X \to B)$ will be indexed by the character α :

$$\mathcal{A}(X \to B) = \left\{ \begin{array}{ccc} K_\alpha & \to & L_\alpha \\ \downarrow^{\alpha} & & \downarrow^{\alpha} \\ X & \longrightarrow & B \end{array} \right\}$$

Denote by $Y \to Z \to B$ the following B-map :

$$\coprod_\alpha K_\alpha \to \coprod_\alpha L_\alpha \to B .$$

The maps $K_\alpha \to X$ induce a map $Y \to X$ and we have a commutative diagram :

$$\begin{array}{ccc} Y & \to & Z \\ \downarrow & & \downarrow \\ X & \to & B \end{array}$$

Now let $\phi(X \to B)$ be the B-space $X \underset{Y}{\cup} Z \to B$ and λ be the canonical inclusion $X \to X \underset{Y}{\cup} Z \to B$. Clearly Φ is a functor from the category of B-spaces to itself and λ is a morphism from 1 to Φ .

LEMMA 9.4. - <u>For any B-space</u> $X \to B$ <u>we have the following properties</u> :

i) <u>the B-map</u> λ <u>from</u> $X \to B$ <u>to</u> $\Phi(X \to B)$ <u>belongs in</u> $\mathcal{W}$

ii) <u>for any</u> $K \to L \to B$ <u>in</u> S <u>any B-map</u> $K \to X \to B$ <u>entends to</u> $L \to B$ <u>in</u> $\Phi(X \to B)$.

<u>Proof</u> : The property ii) is trivial. The property i) is an easy consequence of the conditions L4 and L5 satisfied by $\mathcal{W}$.

9.5. - THE FUNCTORS Φ_u

Let $X \to B$ be a B-space. For any ordinal number u one defines a B-space $\Phi_u(X \to B)$ by the following conditions :

i) if $u < v$, $\Phi_u(X \to B)$ is contained in $\Phi_v(X \to B)$

ii) $\Phi_o(X \to B) = X \to B$

iii) $\Phi_{u+1}(X \to B) = \Phi(\Phi_u(X \to B))$

iv) $\Phi_v(X \to B) = \varinjlim_{u<v} \Phi_{u+1}(X \to B)$

If u is less than v the inclusion $\Phi_u(X \to B) \to \Phi_v(X \to B)$ will be denoted by λ_{uv} . It is easy to see that the Φ_u 's are functors from the category of B-sapces to itself and λ_{uv} is a morphism from ϕ_u to ϕ_v .

LEMMA 9.6. - <u>For any B-space</u> $X \to B$ <u>and any ordinal number</u> u <u>the B-map</u> λ_{ou} <u>from</u> $X \to B$ <u>to</u> $\Phi_u(X \to B)$ <u>belongs in</u> $\mathcal{W}$.

<u>Proof</u> : If u is an ordinal number, denote by $X_u \to B$ the B-space $\Phi_u(X \to B)$.

If u is less than v , X_u is contained in X_v and the inclusion map $X_u \to X_v$ is a cofibration.

I will prove that $X \to X_u \to B$ belongs in $\mathcal{W}$ by induction on u .

Now let u be an ordinal number and suppose that the B-map $X \to X_v \to B$

belongs in $\mathcal{W}$ for any $v < u$.

If $u = v + 1$, $X \hookrightarrow X_v \to B$ belongs in $\mathcal{W}$ and $X_v \hookrightarrow X_u \to B$ too by 9.4. Then by using L1, $X \hookrightarrow X_u \to B$ belongs in $\mathcal{W}$.

If u is not $v + 1$, X_u is the union of the X'_v 's for $v < u$. Then $X \hookrightarrow X_u \to B$ belongs in $\mathcal{W}$ as consequence of the lemma 7.5 and the lemma 9.6 is proved.

9.7. - THE FUNCTOR E

Let ω be the first ordinal number of cardinal c, and let E be the functor Φ_ω and η the morphism $\lambda_{o\omega}$.

LEMMA 9.8. - <u>For any</u> B-<u>space</u> $X \to B$, <u>the</u> B-<u>space</u> $E(X \to B)$ <u>is</u> S-<u>local</u>.

<u>Proof</u> : Let $K \to L \to B$ be a B-map in S and f be a B-map from $K \to B$ to $E(X \to B) = \Phi_\omega(X \to B)$. By construction the image of f is contained in a $\Phi_u(X \to B)$, $u < \omega$ and by the lemma 9.4 f extends to $L \to B$ in $\Phi_{u+1}(X \to B)$.

Then the map $[L \to B,\ E(X \to B)] \to [K \to B,\ E(X \to B)]$ is surjective.

Now let f_o and f_1 be two B-maps from $L \to B$ to $E(X \to B)$, B-homotopic on $K \to B$.

Denote by $K' \to L' \to B$ the B-map $K \times I \cup L \times \partial I \to L \times I \to B$. This B-map is isomorphic to a B-map in S. On the other hand the two B-maps f_o and f_1 and a B-homotopy between the restrictions on $K \to B$ of f_o and f_1 give a B-map g from $K' \to B$ to $E(X \to B)$. The image of this B-map is contained in a $\Phi_u(X \to B)$ and g extends to $L \times I \to B$ in $\Phi_{u+1}(X \to B)$. Thus f_o and f_1 are B-homotopic and the map :

$$[L \to B\ ,\ E(X \to B)] \to [K \to B\ ,\ E(X \to B)]$$

is injective.

That proves the lemma.

LEMMA 9.9. - <u>For any</u> B-<u>space</u> $X \to B$ <u>the map</u> $E(X \to B)$ <u>is a Serre fibration</u>.

<u>Proof</u> : Consider a commutative diagram :

$$\begin{array}{ccc} I^n \times 0 & \longrightarrow & Y \\ \downarrow & & \downarrow \\ I^{n+1} & \longrightarrow & B \end{array}$$

where $Y \to B$ is the B-space $E(X \to B)$.

The B-map $I^n \times 0 \to Y \to B$ factorizes through a $\Phi_n(X \to B)$ $u < \omega$ and then extends to $I^{n+1} \to B$ in $\Phi_{u+1}(X \to B)$.

This proves that $Y \to B$ is a Serre fibration.

Now I will use the condition L7 satisfied by $\mathcal{W}$ to prove that $E(X \to B)$ is $\mathcal{W}$-local for any B-space $X \to B$, and consequently to show that (E,η) is a $\mathcal{W}$-localization functor.

LEMMA 9.10. - For any B-space $X \to B$ and any B-maps $Y \to Z \to B$ in $\mathcal{W}$, the map

$$[Z \to B\ ,\ E(X \to B)] \to [Y \to B\ ,\ E(X \to B)]$$

is surjective.

Proof : By using the condition L2 one can suppose that the map $Y \to Z$ is a cofibration.

Now take a B-map from $Y \to B$ to $E(X \to B)$. If $X_\omega \to B$ is the B-space $E(X \to B)$ one have the following diagram :

$$\begin{array}{ccc} Y & \to & Z \\ \downarrow & & \downarrow \\ X_\omega & \to & B \end{array}$$

and by using the condition L5 the B-map $X_\omega \to X_\omega \underset{Y}{\cup} Z \to B$ belongs in $\mathcal{W}$.

Let $X' \to B$ be the B-space $E(X_\omega \underset{Y}{\cup} Z \to B)$. One have the following commutative diagram :

$$\begin{array}{cccc} Y & \to & Z & \\ \downarrow & & \downarrow & \\ X_\omega & \to & X' & \\ & & & \searrow \\ & & & & B \end{array}$$

Then in order to prove the lemma, it suffices to show that the B-map

$X_\omega \to X' \to B$ is a B-homotopy equivalence. But $X_\omega \to B$ and $X' \to B$ are Serre fibrations. Hence it suffices to show that the map $X \to X'$ is a weak homotopy equivalence.

Thus consider a disk D and its boundary S and a commutative diagram :

$$\begin{array}{ccc} S & \to & D \\ \downarrow & & \downarrow \\ X_\omega & \to & X' \end{array}$$

By using the condition L7 and the definition of S one can find a factorization :

$$\begin{array}{ccccc} S & \to & D & & \\ \downarrow & & \downarrow & & \\ K & \to & L & & \\ \downarrow & & \downarrow & & \\ X_\omega & \to & X' & \to & B \end{array}$$

such that $K \to L \to B$ belongs in S . But $X_\omega \to B$ is S-local, then there exists a B-map $f : L \to X_\omega \to B$ extending the B-map $K \to X_\omega \to B$. Moreover $X' \to B$ is S-local and f is B-homotopic rel $K \to B$ to the B-map $L \to X' \to B$.

This proves that the map of pairs from $S \to D$ to $X_\omega \to X'$ is homotopic to a map in X_ω and the relative homotopy groups of $X_\omega \to X'$ vanish. Thus the map $X_\omega \to X'$ is a weak homotopy equivalence and the lemma is proved.

LEMMA 9.11. - <u>For any</u> $X \to B$ <u>the</u> B-<u>space</u> $E(X \to B)$ <u>is</u> $\mathcal{V}$-<u>local</u>.

<u>Proof</u> : Let $Y \to Z \to B$ be a B-map in $\mathcal{U}$ and consider the map

$$[Z \to B \ , \ E(X \to B)] \to [Y \to B \ , \ E(X \to B)] \ .$$

By the last lemma one know that this map is surjective.

Now I will prove the injectivity in the sufficient case where $Y \to Z$ is a cofibration.

Let f_0 and f_1 be two B-maps from $Z \to B$ to $E(X \to B)$ B-homotopic on $Y \to B$. This two B-maps and the B-homotopy give a B-map from $Y \times I \cup Z \times \partial I$ to $E(X \to B)$. By using the condition L6 and the last lemma this B-map extends to $Z \times I \to B$ and the two B-maps f_0 and f_1 are B-homotopic.

That proves the lemma and consequently the theorem 2.5.

9.12. - The functor E defined as above depends only on the choice of the set S , thus of the set Σ_o . Take as set Σ_o the set of all ordinal numbers less than the first ordinal number of cardinal c . This set defines a set Σ_o and a functor E denoted by E_c from the category of B-spaces to itself, and an inclusion morphism $1 \to E_c$. It is easy to check that for any $c \leqslant c'$ one has an inclusion morphism $E_c \to E_{c'}$, and E_c is a $\mathcal{W}_c$-localization functor.

That proves the theorem 1.8.