Surgery on Closed Manifolds

by

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Theorem (0.1): $(n \geq 4)$ Let $(f : N^n \to M^n, \hat{f})$ be a surgery problem where M^n is a closed, oriented manifold with $\pi_1 M$ finite. Then,

$(f \times Id_{S^1}, \hat{f} \times Id_{\nu_{S^1}})$ is normal cobordant $\Leftrightarrow$ to a homotopy equivalence

$$\begin{cases} \text{index } N = \text{index } M, & \text{when } n \equiv 0\ (4) \\ \text{always}, & \text{when } n \equiv 1\ (4) \\ \operatorname{Arf}(f,\hat{f}) = 0, & \text{when } n \equiv 2\ (4) \\ \operatorname{Arf}_\mu(f,\hat{f}) = 0, & \text{when } n \equiv 3(4) \\ \quad \text{for all nontrivial homomorphisms } \mu : \pi \to \mathbb{Z}/2 & \end{cases}$$

$\operatorname{Arf}_\mu(f,\hat{f}) = \operatorname{Arf}(f_\mu : N_\mu^{n-1} \to M_\mu^{n-1}, \hat{f}_\mu)$, where $(f_\mu, \hat{f}_\mu)$ is the sub-surgery problem of $(f,\hat{f})$ which is induced via transversality by the map $M^n \to B\pi_1 M \xrightarrow{B_\mu} B\,\mathbb{Z}_2 = RP^\infty$.

Also, we can show

Theorem (0.2): For any closed manifold P^n with finite π_1 and index = 0,

$(f : M^8 \to S^8, \hat{f}) \times (Id_P, Id_{\nu_P})$ is normally cobordant to a homotopy equivalence - (where $(f,\hat{f})$ = Milnor surgery problem with index 8).

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Section 1:

For any closed, compact, oriented manifold M with $\pi_1 M \simeq \pi$, we have the Sullivan-Wall structure sequence

$$[\Sigma M, G/TOP] \xrightarrow{\theta} L^s_{n+1}(\mathbb{Z}\pi) \to S(M) \to [M, G/TOP] \xrightarrow{\sigma} L^s_n(\mathbb{Z}\pi)$$

There are also defined "intermediate" Wall groups $L^x_*(\mathbb{Z}\pi)$ where $x \subset \tilde{K}_0(\mathbb{Z}\pi)$ or $\{\pi\} \subset x \subset \tilde{K}_1(\mathbb{Z}\pi)$ is an involution invariant subgroup (see [R]). There is a homomorphism $L^s_*(\mathbb{Z}\pi) \to L^x_*(\mathbb{Z}\pi)$ so we get maps

$\sigma^x : [M, G/TOP] \to L^x_n(\mathbb{Z}\pi)$ and

$\theta^x : [\Sigma M, G/TOP] \to L^x_{n+1}(\mathbb{Z}\pi)$

It follows from work of Quinn-Ranicki that there is a homomorphism

$$A^x : \oplus H_{n-4i}(B\pi; \mathbb{Z}_{(2)}) \oplus H_{n-4i-2}(B\pi; \mathbb{Z}/2) \to L^x_n(\mathbb{Z}\pi)_{(2)}$$

where $(\ )_{(2)}$ denotes localization at 2, such that the 2 localizations of σ^x and θ^x are given by composing A^x with a certain characteristic class formula that we worked out in [T-W]. We wrote out the one for σ^x (formula 1.7) : to get the one for θ^x use the same formula but replace [M] by the homology suspension of the fundamental class. Indeed, given any compact, oriented manifold with bounding, W^n, we get a formula for the map $[W/\partial W, G/TOP] \to L_n(\mathbb{Z}\pi)$: replace [M] with $[W, \partial W]$ in (1.7)*

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*Care is needed in [T-W]. The Wu class referred to there is the Morgan-Sullivan Wu class, [M-S] p. 480-81. It is the inverse of the Wu class defined in Milnor-Stasheff [Mi-S] 11.14. In particular, some of the polynomials on [M-S] p. 481 are incorrect.

Recall that Wall ([W 2]) has shown that $L_n^x(\mathbb{Z}\pi) \to L_n^x(\mathbb{Z}\pi)_{(2)}$ is 1-1 for π finite.

Periodicity implies that A^x factors as

$$\bigoplus_i (H_{n-4i}(\pi, \mathbb{Z}_{(2)}) \oplus H_{n-4i-2}(\pi, \mathbb{Z}/2)) \xrightarrow{\oplus \mathcal{I}_{n-4i} \oplus \mathcal{K}_{n-4i-2}} \bigoplus_i L_{n-4i}^x(\mathbb{Z}\pi)_{(2)} \xrightarrow{+} L_n^x(\mathbb{Z}\pi)_{(2)}$$

$\mathcal{I}_*$ and $\mathcal{K}_*$ are determined by the surgery obstructions of certain very special surgery problems. To be more specific, let $M^8 \to S^8$ denote the 8-dimensional Milnor surgery problem, and let $K^3 \to L^3$ denote the twisted Kervaire problem, i.e. the generator of $L_3(\mathbb{Z}e; \mathbb{Z}/2)$. Define homomorphisms

$$\alpha_n^x : \Omega_n(B\pi) \to L_{n+8}^x(\mathbb{Z}\pi)$$

$$\beta_n^x : \Omega_n(B\pi; \mathbb{Z}/2) \to L_{n+2}^x(\mathbb{Z}\pi)$$

by $\alpha_n^x(P)$ is the surgery obstruction for $M^8 \times P \to S^8 \times P$ and $\beta_n^x(P)$ is the surgery obstruction for the surgery problem induced along the bockstein of $K^3 \otimes P \to L^3 \otimes P$. (See [M-S]).

The map $\mathcal{I}_n^x$ is determined by α_n^x and the $\mathcal{I}_r^x$ for $r < n$. The map $\mathcal{K}_n^x$ is determined by β_n^x and the $\mathcal{K}_r^x$ and $\mathcal{I}_r^x$ for $r < n$.

The precise relation between the α's and the $\mathcal{I}$'s is supplied by (1.7) in [T-W] : to wit, if $g : P \to B\pi$,

$$\alpha_n^x(P) = \Sigma\, \mathcal{I}_{n-4i}\, g_*(\mathcal{L}_P \cap [P])$$

where $\mathcal{L}$ is the Morgan-Sullivan $\mathcal{L}$ class [M-S]. With a bit more work, one can show

$$\beta_n^x(P) = \Sigma\, \mathcal{K}_{n-4i}\, g_*(V_P^2 \cap [P])$$

$$+ \Sigma\, \mathcal{J}_{n-4i-2}\, g_*(\delta(V_P\, Sq^1 V_P \cap [P])$$

where δ denotes the bockstein $0 \to \mathbb{Z}_{(2)} \to \mathbb{Z}_{(2)} \to \mathbb{Z}/2 \to 0$ and V_P denotes the total Wu class of the oriented tangent bundle to P.

Theorem (0.1) follows from Ranicki's product formula $L_{n+1}^h(\mathbb{Z}(\pi\times\mathbb{Z})) \simeq L_n^p(\mathbb{Z}\pi) \oplus L_{n+1}^h(\mathbb{Z}\pi)$ (see [RI]), plus the following result.

Theorem 1.1. Assume π is finite.

(a) $\mathcal{J}_o^p$ is 1-1

(b) For $j > 0$, $\mathcal{J}_j^{\overline{C\ell_1(\pi)}}$ is trivial, where

$C\ell_1(\pi) = \ker(\tilde{K}_1(\mathbb{Z}\pi) \to \tilde{K}_1(\hat{\mathbb{Z}}\pi) \oplus \tilde{K}_1(\mathbb{Q}\pi))$ and $\overline{C\ell_1}(\pi) = \{C\ell_1(\pi), \pm\pi\}$.

(c) $\mathcal{K}_o^p$ and $\mathcal{K}_1^p$ are 1-1.

(d) For $j > 1$, $\mathcal{K}_j^p$ is trivial.

Theorem (0.2) follows from (1.1)(b).

We can improve on 1.1 for some groups.

Theorem 1.2. Let π be a finite group whose 2 Sylow group is abelian. Then

(a) $\mathcal{J}_j^s$ is trivial for $j > 0$

(b) $\mathcal{K}_j^h$ is trivial for $j > 2$

(c) $\mathcal{K}_j^s$ is trivial for $j > 3$

Remarks: (i) The result for $\mathcal{K}^h$ is due to Morgan-Pardon, but the s result seems new.

(ii) See Theorem 4.1 for results on generalized quaternionic and semi-dihedral groups.

(iii) Using results of Quillen [Q] and the naturality of the $\mathcal{J}$, and $\mathcal{K}$, one can prove the same result for the dihedral groups; the symmetric and alternating groups; and many others.

(iv) When we sketch the proof of 1.2 we will also determine $\mathcal{K}_2^h$.

Section 2:

Following Wall ([WI], Theorem 12) it is easy to reduce Theorem 1.1 to the result for finite 2-groups.

Relative Detection Theorem 2.1: If π is a finite 2-group, then

(a) $K_i(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \to \bigoplus_{\substack{\text{special}\\ \text{subquotients}\\ G}} K_i(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G)$ is 1-1 for all i.

(b) $L_i^{C\ell_\epsilon(\pi)\to 0}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \to \bigoplus_{\substack{\text{special}\\ \text{subquotients}\\ G}} L_i^{C\ell_\epsilon(\pi)\to 0}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi)$ is 1-1

for all i and $\epsilon = 0$ or 1.

$(C\ell_\epsilon(\pi) = \ker \tilde{K}_\epsilon(\mathbb{Z}\pi) \to \tilde{K}_\epsilon(\hat{\mathbb{Z}}\pi) \oplus \tilde{K}_\epsilon(\mathbb{Q}\pi))$

Remarks:

1. A subquotient of π is a quotient group $G = H/N$ where H = subgroup of π.

2. A 2-group G is special if all normal abelian subgroups of G are cyclic. A special group is either cyclic, generalized quaternionic, dihedral, or semi-dihedral.

3. The maps in (2.1) are compositions of restriction maps associated to subgroups $H \subset \pi$ and projection maps associated to quotients $H \to H/N = G$.

4. $L_i^{C\ell_1(\mathbb{Z}\pi) \to 0}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \simeq L_i^{\overline{C\ell_1}(\mathbb{Z}\pi) \to \{\pm\pi\}}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi)$

$\simeq L_i^{SK_1 \to SK_1}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \simeq \mathfrak{L}_i^{SK_1 \to SK_1}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi)$

where $\mathfrak{L}_i$ = the L-groups defined by Wall in [W2]. $L_i^X \neq \mathfrak{L}_i^X$ in general (see [W2] Section 5.4).

5. $L_i^{C\ell_0(\mathbb{Z}\pi) \to 0}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \simeq L_i^p(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi)$.

6. Theorem 2.1 was motivated by the calculations of Wall [W2], Section (5.2), Carlsson-Milgram [C-M], Pardon [P], Bak-Kolster [K1], [B-K], [K2], and especially Milgram-Hambleton [M-H].

Theorem 1.1 (b) is reduced to the result for special 2-groups as follows:

A is induced by a map of spectra $\underline{A}$ which fits into a commutative diagram

$$\begin{array}{ccc} \underline{L}(\mathbb{Z}) \wedge B\pi^+ & \xrightarrow{\underline{A}} & \underline{L}(\mathbb{Z}\pi) \\ \downarrow & & \downarrow \\ \underline{L}(\hat{\mathbb{Z}}_2) \wedge B\pi^+ & \xrightarrow{\underline{A}_2} & \underline{L}(\hat{\mathbb{Z}}_2\pi) \end{array}$$

If we localize at (2), then $\underline{L}(\mathbb{Z}) \to \underline{L}(\hat{\mathbb{Z}}_2)$ is equivalent to

$$\prod_i K(\mathbb{Z}_{(2)}; 4i) \times \pi K(\mathbb{Z}/2; 4i+2) \xrightarrow{\text{project}} \pi K(\mathbb{Z}/2; 4i+2) \xleftarrow{\text{include}} \pi K(\mathbb{Z}/2; 4i+2)$$

$\pi K(\mathbb{Z}/2; 4i+1)$

This implies that $\mathcal{J}^x_j$ lifts to a map

$$\tilde{\mathcal{J}}^x_j : H_j(\pi, \mathbb{Z}_{(2)}) \longrightarrow L^x_{j+1}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi)_{(2)} \text{ for all } x.$$

Apply Theorem 2.1 with $\epsilon = 1$ and $x = \overline{C\ell_1} \to 0$.

Theorem 1.1(d) is reduced to the result for 2-groups by the following theorem.

Absolute Detection Theorem 2.2 : If π is a finite 2-group, then

$$L^p_i(\mathbb{Z}\pi) \to \bigoplus_{\substack{\text{special} \\ \text{subquotients}}} L^p_i(\mathbb{Z}G) \text{ is 1-1.}$$

The proof of (2.2) relies on Wall's reduction theorem which implies that $L^p_i(\hat{\mathbb{Z}}_2\pi) \cong L^p_i(\mathbb{Z}/2)$.

Section 3: Proof of the Relative Detection Theorem

π = finite 2-group

$\mathbb{Q}\pi = \times_\rho A_\rho$, where ρ varies over the $\mathbb{Q}$-irreducible representations of π and A_ρ = simple $\mathbb{Q}$-algebra.

Let $\mathfrak{n}_\rho = \text{image}\,(\mathbb{Z}\pi \to \mathbb{Q}\pi \to A_\rho)$, $\mathfrak{n} = \times_\rho \mathfrak{n}_\rho$. $\mathfrak{n}_\rho$ is a $\mathbb{Z}$-order of A_ρ.

Proposition 3.1

$$K_i(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \cong K_i(\mathfrak{n} \to \hat{\mathfrak{n}}_2) \cong \bigoplus_\rho K_i(\mathfrak{n}_\rho \to \hat{\mathfrak{n}}_{\rho(2)})$$

Proof: Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & & & & & K_{i-1}(\hat{\mathbb{Z}}_{odd}\pi) & & \\
 & & & & & & \uparrow\wr & & \\
\cdots \to & K_i(\mathbb{Z}\pi \to \hat{\mathbb{Z}}\pi) & \to & K_i(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) & \to & K_i(\hat{\mathbb{Z}}\pi \to \hat{\mathbb{Z}}_2\pi) & \to \cdots \\
 & \downarrow f_i & & \downarrow g_i & & \downarrow h_i & \\
\cdots \to & K_i(\mathfrak{n} \to \hat{\mathfrak{n}}) & \to & K_i(\mathfrak{n} \to \hat{\mathfrak{n}}_2) & \to & K_i(\hat{\mathfrak{n}} \to \hat{\mathfrak{n}}_2) & \to \cdots \\
 & \downarrow k_i & & & & \downarrow\wr & \\
 & K_i(\mathbb{Q}\pi \to \hat{\mathbb{Q}}\pi) & & & & K_{i-1}(\hat{\mathfrak{n}}_{odd}\pi) &
\end{array}$$

The Meyer-Vietoris sequences associated to the arithmetic squares

$$\begin{array}{ccc} \mathbb{Z}\pi & \to & \mathbb{Q}\pi \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}\pi & \to & \hat{\mathbb{Q}}\pi \end{array} \qquad \begin{array}{ccc} \mathfrak{n} & \to & \mathbb{Q}\pi \\ \downarrow & & \downarrow \\ \hat{\mathfrak{n}} & \to & \hat{\mathbb{Q}}\pi \end{array}$$

imply that f_i, $k_i \circ f_i$ and k_i are isomorphisms. Since π is a

2-group, $\hat{\mathbb{Z}}_{odd}\,\pi$ is a maximal $\hat{\mathbb{Z}}_{odd}$ - order (see [Re]), $\hat{\mathbb{Z}}_{odd}\,\pi = \hat{n}_{(odd)}$, and h_i is an isomorphism. Apply the 5-lemma.

Let

$$K_i^f(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) = \bigoplus_{\substack{\rho \\ \text{faithful}}} K_i(n_\rho \to \hat{n}_{\rho(2)})$$

$$K_i^u(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) = \bigoplus_{\substack{\rho \\ \text{unfaithful}}} K_i(n_\rho \to \hat{n}_{\rho(2)})$$

<u>Proposition 3.2</u> :

(a) $K_i^f(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \hookrightarrow K_i(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \to K_i(\pi/N \to \hat{\mathbb{Z}}_2\,\pi/N)$ is a trivial map, for any proper normal subgroup N.

(b) $K_i^u(\mathbb{Z}\pi \to \mathbb{Z}_2\pi) \hookrightarrow K_i(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \to \bigoplus_{N \trianglelefteq \pi} K_i(\mathbb{Z}\pi/N \to \hat{\mathbb{Z}}_2\pi/N)$ is 1-1.

<u>Proposition 3.3</u> : Assume π is a 2-group which is <u>not</u> special. Then π contains an index 2 subgroup π_o such that

(a) For any $\mathbb{Q}$-irreducible faithful representative ρ of π, $\rho \mid \pi_o = \rho_1 + \rho_2$ where ρ_1 and ρ_2 are nonisomorphic Q-irreducible representations.

(b) $\rho_1^\pi = \rho_2^\pi = \rho$, and

(c) $K_i^f(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \hookrightarrow K_i(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) \to K_i(\mathbb{Z}\pi_o \to \hat{\mathbb{Z}}_2\pi_o)$ is 1-1.

Theorem 2.1 (a) then follows from (3.2) and (3.3) by induction on the order of π. The proof of 2.1 (b) is similar.

In particular,

$$L_i^{C\ell_0(\pi)\to 0}(\mathbb{Z}\pi \to \mathbb{Z}_2\pi) \cong \bigoplus_\rho L_i^{\tilde{K}_0(n_\rho)\to I_\rho}(n_\rho \to \hat{n}_{\rho(2)}).$$

$$\text{where } I_\rho = \text{image } (\tilde{K}_0(n_\rho) \to \tilde{K}_0(\hat{n}_{\rho(2)}).$$

and

$$L_i^{C\ell_1(\pi)\to 0}(\mathbb{Z}\pi \to \hat{\mathbb{Z}}_2\pi) = \bigoplus_\rho L_i^{SK_1\to SK_1}(n_\rho \to \hat{n}_{\rho(2)}).$$

Section 4 : Special 2-Groups

Theorem 4.1

(a) If π is a special 2-group and $j>0$, then $\tilde{\mathcal{I}}_j^{\overline{C\ell}_1}$ and $\mathcal{I}_j^s$ are trivial.

(b) If π is cyclic or dihedral, then $\varkappa_j^s = 0$ for $j>1$.

If π is quaternionic, then $\varkappa_j^s = 0$ for $j \neq 0, 1, 3$ and $\varkappa_3^p = 0$.

If π is semi-dihedral, then $\varkappa_j^p = 0$ for $j>1$.

Remark : Cappell and Shaneson [C-S] have shown that $\varkappa_3^h \neq 0$ when π = quaternion group.

The following result of Oliver [O] is used to improve $\overline{Cl}_1$-results to s-results.

Theorem 4.2: If π is a special 2-group, then $C\ell_1(\pi)$ is trivial.

For the proof of 4.1 (b) we need to analyze what happens to $\varkappa_j$ under products.

For any pair of groups π_1 and π_2, we get a pairing of spectra

$$\mu : \underline{L}_o(Z\pi_1) \wedge \underline{L}^o(Z\pi_2) \to \underline{L}_o(\mathbb{Z}(\pi_1 \times \pi_2)) \text{ (see [R])}$$

such that the following diagram commutes

$$\begin{array}{ccc}
(\underline{L}_o(\mathbb{Z})\wedge B\pi_1^+)\wedge(\underline{L}^o(\mathbb{Z})\wedge B\pi_2^+) \cong \underline{L}_o(\mathbb{Z})\wedge \underline{L}^o(\mathbb{Z}) \wedge B(\pi_1\times\pi_2)^+ & \xrightarrow{\mu\wedge \mathrm{id}} & \underline{L}_o(\mathbb{Z})\wedge (B\pi_1\times\pi_2)^+ \\
4.3 \quad \Big\downarrow A_*\wedge A^* & & \Big\downarrow A_* \\
\underline{L}_o(\mathbb{Z}\pi_1)\wedge \underline{L}^o(\mathbb{Z}\pi_2) & \xrightarrow{\quad\mu\quad} & \underline{L}_o(\mathbb{Z}(\pi_1\times\pi_2))
\end{array}$$

If we introduce coefficients by doing surgery on $\mathbb{Z}/2$-manifolds, then we get an analogous diagram.

By using the techniques of [T-W], one can analyse

$$\mu : \underline{L}_o(\mathbb{Z};\mathbb{Z}/2)\wedge \underline{L}^o(\mathbb{Z};\mathbb{Z}/2) \to \underline{L}_o(\mathbb{Z};\mathbb{Z}/2)$$

localized at 2. This yields the following commutative diagram.

$$\begin{array}{ccc}
H_i(\pi_1;\mathbb{Z}/2)\times H_j(\pi_2;\mathbb{Z}/2) & \to & H_{i+j}(\pi_1\times\pi_2;\mathbb{Z}/2) \\
\Big\downarrow \bar{\kappa}_i\times \vartheta^j(\mathbb{Z}/2) & & \Big\downarrow \bar{\kappa}_{i+j} \\
L_{i+2}(\mathbb{Z}\pi_1;\mathbb{Z}/2)_{(2)}\times L^j(\mathbb{Z}\pi_2;\mathbb{Z}/2)_{(2)} & \to & L_{i+j+2}(\pi_1\times\pi_2;\mathbb{Z}/2)_{(2)}\,.
\end{array} \qquad (4.4)$$

where $\vartheta^j(\mathbb{Z}/2)$ is induced by

$$K(\mathbb{Z}/2;0)\wedge B\pi_2^+ \hookrightarrow L^o(\mathbb{Z};\mathbb{Z}/2)_{(2)}\wedge B\pi_2^+ \xrightarrow{A^*} \underline{L}^o(\mathbb{Z}\pi_2;\mathbb{Z}/2)_{(2)}\,.$$

$\bar{\kappa}_i = \kappa_i$ reduced mod 2.

<u>Action by the Center</u>

Suppose C is the center of a group π, then multiplication $\alpha : C\times\pi\to\pi$ is a homomorphism which induces a map $B\alpha : B(C\times\pi)\to B\pi$ plus a commutative diagram

4.5
$$\begin{array}{ccc}
\underline{L}_o(\mathbb{Z};\mathbb{Z}/2)\wedge B(C\times\pi)^+ & \xrightarrow{\ id\ \wedge B\alpha^+\ } & \underline{L}_o(\mathbb{Z};\mathbb{Z}/2)\wedge B\pi^+ \\
\downarrow A_* & & \downarrow A_* \\
\underline{L}_o(C\times\pi;\mathbb{Z}/2) & \xrightarrow{\ \alpha\ } & \underline{L}_o(\mathbb{Z}\pi;\mathbb{Z}/2)
\end{array}$$

If we combine 4.4 and 4.5, then we get the following commutative diagram

4.6
$$\begin{array}{ccc}
H_i(C;\mathbb{Z}/2)\times H_j(\pi;\mathbb{Z}/2) & \xrightarrow{\ \alpha_*\ } & H_{i+j}(\pi;\mathbb{Z}/2) \\
\downarrow \overline{K}\times\mathcal{J}^j(\mathbb{Z}/2) & & \downarrow \overline{K}_{i+j} \\
L_{i+2}(\mathbb{Z}C;\mathbb{Z}/2)_{(2)}\times L^j(\mathbb{Z}\pi;\mathbb{Z}/2)_{(2)} & \xrightarrow{\ \alpha_*\ } & L_{i+j+2}(\mathbb{Z}\pi;\mathbb{Z}/2)_{(2)}
\end{array}$$

The proof of 4.1 also involves the following result.

<u>Theorem 4.7</u> : If π is a special 2-group, then there is an exact sequence

$$L_{i+1}^{\tilde{K}_o\to I_\phi}(\mathfrak{n}_\phi\to\hat{\mathfrak{n}}_{\phi(2)}) \to L_i^P(\mathbb{Z}\pi) \to \bigoplus_{\substack{\text{proper}\\ \text{special}\\ \text{subquotients}}} L_i^P(\mathbb{Z}G)$$

where ϕ is the unique faithful, $\mathbb{Q}$-irreducible representation of π, and $I_\phi = \text{Image}: (\tilde{K}_o(\mathfrak{n}_\phi) \to \tilde{K}_o(\hat{\mathfrak{n}}_{\phi(2)})$. Also, there is an exact sequence,

$$L_i^{C\ell_1}(\mathfrak{n}_\phi\to\hat{\mathfrak{n}}_{\phi(2)}) \to L_i^{\overline{C\ell}_1}(\mathbb{Z}\pi\to\hat{\mathbb{Z}}_2\pi) \to \bigoplus_{\substack{\text{proper}\\ \text{special}\\ \text{subquotients}}} L_i^{\overline{C\ell}_1}(\mathbb{Z}G\to\hat{\mathbb{Z}}_2G)$$

Proof of (4.1) when $\pi = \mathbb{Z}/2$

Facts

1. $\operatorname{tor} L_j^{C\ell_1 \to 0}(\mathbb{Z}\,\mathbb{Z}/2 \to \hat{\mathbb{Z}}_2\,\mathbb{Z}/2) = 0$ unless $j \equiv 1(4)$.

2. $H_{2k}(\mathbb{Z}/2, \mathbb{Z}_{(2)}) = 0$ for $k \neq 0$.

3. $L_j^s(\mathbb{Z}\,\mathbb{Z}/2) \xrightarrow{P_* \oplus \iota^*} L_j^s(\mathbb{Z}e) \oplus L_j^s(\mathbb{Z}e)$ is 1-1 for $j \not\equiv 3(4)$. $(P : \mathbb{Z}/2 \to e,\ \iota : e \to \mathbb{Z}/2)$.

4. The Pontryagin product $\alpha_* : H_{2i}(\mathbb{Z}/2\,;\,\mathbb{Z}/2) \times H_1(\mathbb{Z}/2\,;\,\mathbb{Z}/2) \to H_{2i+1}(\mathbb{Z}/2\,;\,\mathbb{Z}/2)$ is onto. $2\,L_3^s(\mathbb{Z}\,\mathbb{Z}/2) = 0$.

Facts 1 and 2 imply that $\tilde{\mathfrak{I}}_j^{\overline{C\ell}_1} = 0$ for $j > 0$. Fact 3 plus naturality of κ_j^s, imply $\kappa_j^s = 0$ for $j > 0$ and $j \not\equiv 1(4)$. Fact 4 plus commutativity of (4.6) imply $\kappa_j^s = 0$ for $j \equiv 1(4)$ and $j > 1$.

Proof of (4.1) when $\pi = D_n$, the dihedral group:

Lemma 4.8: If $A = \mathbb{Z}/2$ or $\mathbb{Z}_{(2)}$, then

$$\bigoplus_{\substack{\text{elementary}\\ \text{abelian}\\ \text{2-groups}\\ E \subset D_n}} H_i(E\,;\,A) \to H_i(D_n\,;\,A) \text{ is onto.}$$

Proof: (See Quillen [Q], 4.6)

In Section 5, we show that for $\pi = E$, $\kappa_j^s = 0$ for $j > 1$, and $\mathfrak{I}_j^{C\ell_1} = 0$ for $j > 0$.

Proof of (4.1) when $\pi = \mathbb{Z}/2^i$, $(i > 1)$ or SD_n:

Lemma 4.9 : If $\pi = \mathbb{Z}/2^i$ $(i > 1)$ or SD_n, then
$\text{tor } L_i(\eta_\phi \to \hat{\eta}_{\phi(2)}) = (0)$.

Apply (4.7). $L_*^s(\mathbb{Z}\,\mathbb{Z}/2^i) \to L_*^P(\mathbb{Z}\,\mathbb{Z}/2^i)$ is 1-1. (See [B]).

Proof of (4.1) for $\pi = Q_n$, generalized quaternionic:

Facts

1. $\text{tor } L_{j+1}^{C\ell_1 \to 0}(\mathbb{Z}Q_n \to \hat{\mathbb{Z}}_2 Q_n) \xrightarrow{P_*} L_{j+1}^{C\ell_1 \to 0}(\mathbb{Z}D_{n-1} \to \hat{\mathbb{Z}}D_{n-1})$ is 1-1 for $j \not\equiv 1$ or $2(4)$. $P_* : Q_n \to Q_n/C = D_{n-1}$ (see (4.7)).

2. $H_{4\ell+2}(Q_n, \mathbb{Z}_{(2)}) = 0$.

3. $\bigoplus_{\substack{\text{cyclic} \\ \text{subgroups} \\ H}} H_{4\ell+1}(H, \mathbb{Z}_{(2)}) \to H_{4\ell+1}(Q_n, \mathbb{Z}_{(2)})$ is onto.

4. The Pontryagin product $H_{4i}(C\,;\mathbb{Z}/2) \times H_\epsilon(Q_n, \mathbb{Z}/2) \to H_{4i+\epsilon}(Q_n; \mathbb{Z}/2)$ is onto for $\epsilon \le 3$. $2 \text{ tor } L_i^s(\mathbb{Z}\,Q_n) = 0$ for $i \not\equiv 1(4)$.

5. $\text{tor } L_o^{\overline{C\ell_1}}(\mathbb{Z}\,Q_n) = 0$

6. $L_1^P(\mathbb{Z}\,Q_n) \xrightarrow{P_* \oplus \iota} L_1^P(\mathbb{Z}\,D_{n-1}) \oplus L_1^P(\mathbb{Z}\,Q_{n+1})$ is 1 - 1

   where $P_* : Q_n \to Q_n/C = D_{n-1}$, and $\iota : Q_n \to Q_{n+1}$ is the inclusion map.

7. $H_{4k+3}(Q_n\,;\,\mathbb{Z}/2) \xrightarrow{\iota_*} H_{4k+3}(Q_{n+1}\,;\,\mathbb{Z}/2)$ is trivial for all k.

8. $H_{4k+3}(Q_n\,;\mathbb{Z}) \to H_{4k+3}(Q_n\,;\mathbb{Z}/2)$ is onto for all k.

Facts 1, 2, and 3 plus naturality imply $\tilde{\mathfrak{I}}_j = 0$ for $j > 0$.

Fact 4 plus the commutativity of (4.6) imply $\varkappa^s_j = 0$ for $j > 3$ and $j \neq 3$ (4). Facts 4 and 8 plus the commutativity of

$$\begin{array}{ccc} H_{4k}(\mathbb{Z}/2\,;\,\mathbb{Z}/2) \times H_3(Q_n) & \xrightarrow{\alpha_*} & H_{4k+3}(Q_n\,;\,\mathbb{Z}/2) \\ \Big\downarrow \varkappa_{4k} + \vartheta^3 & & \Big\downarrow \varkappa_{4k+3} \\ L_2(\mathbb{Z}\,\mathbb{Z}/2)_{(2)} \times L^3(\mathbb{Z}\,Q_n)_{(2)} & \xrightarrow{\alpha_*} & L_1(\mathbb{Z}\,Q_n)_{(2)} \end{array}$$

imply $\varkappa^s_{4k+3} = 0$ for $k > 0$.

$$\left(\begin{array}{l} \vartheta^* \text{ is induced by} \\ K(\mathbb{Z}\,;\,0) \wedge BQ_n^+ \hookrightarrow \underline{L}^\circ(\mathbb{Z})_{(2)} \wedge BQ_n^+ \xrightarrow{A^*} \underline{L}^\circ(\mathbb{Z}\,Q_n)_{(2)} \end{array} \right)$$

Fact 5 implies $\varkappa^s_2 = 0$.

Facts 6 and 7 imply $\varkappa^p_3$ is trivial.

Section 5 : Proof of Theorem 1.2

As always it suffices to assume that π is a 2-group. We first do the case of an elementary abelian 2-group, $E \cong \mathbb{Z}/2 \oplus \ldots \oplus \mathbb{Z}/2$.

Lemma 5.1 : $\mathcal{J}_j^s : H_j(BE ; \mathbb{Z}_{(2)}) \to L_j^s(\mathbb{Z}E)_{(2)}$ is trivial for $j > 0$.

Proof: $SK_1(\mathbb{Z}E) = 0$, so 1.1 (b) proves the result.

The fact that $SK_1(\mathbb{Z}E) = 0$ shows that $L_*^s(\mathbb{Z}E) \to L_*'(\mathbb{Z}E)$ is an isomorphism so by Wall's calculations [W2] the torsion in $L_*^s(\mathbb{Z}E)$ has exponent 2.

Lemma 5.2 : $\kappa_j^s : H_j(BE ; \mathbb{Z}/2) \to L_{j+2}^s(\mathbb{Z}E)_{(2)}$ is trivial for $j > 1$.

Proof.

$$\begin{array}{ccc} H_i(B\mathbb{Z}/2 ; \mathbb{Z}/2) \otimes H_j(BE ; \mathbb{Z}/2) & \to & H_{i+j}(B(E \times \mathbb{Z}/2) ; \mathbb{Z}/2) \\ \downarrow \kappa_i \times \mathcal{J}^j(\mathbb{Z}/2) & & \downarrow \kappa_{i+j} \\ L_{i+2}^s(\mathbb{Z}\mathbb{Z}/2 ; \mathbb{Z}/2) \otimes L^j(\mathbb{Z}E ; \mathbb{Z}/2) & \to & L_{i+j+2}^s(\mathbb{Z}[E \times \mathbb{Z}/2] ; \mathbb{Z}/2) \end{array}$$

commutes. Since the result is true for $\mathbb{Z}/2$ we can begin an induction.

Since $L_0^s(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2])$ is torsion-free [W2], κ_2^s must be trivial for $\mathbb{Z}/2 \times \mathbb{Z}/2$. It is not hard to finish.

We need a generalization of a trick in Stein [S].

Lemma 5.3: Let π_1 and π_2 be finite groups and suppose the torsion in $L_*^X(\mathbb{Z}(\pi_1 \times \pi_2))$ is annihilated by $\mathbb{Z}/2^r$. Assume further that $H_*(B\pi_1 ; \mathbb{Z}/2^r)$ is a free $\mathbb{Z}/2^r$ module.

Then, if ϑ_j^X is trivial for $j > 0$ for π_1 and for π_2, then ϑ_j^X is trivial for $j > 0$ and $\pi_1 \times \pi_2$.

Proof: By the universal coefficients theorem

$$\begin{array}{c}\displaystyle\bigoplus_{s \geq r}{}_s \bigoplus_i H_i(B\pi_1 ; \mathbb{Z}/2^s) \otimes H_{n+1-i}(B\pi_2 ; \mathbb{Z}/2^s) \to \bigoplus_s H_{n+1}(B\pi_1 \times \pi_2) ; \mathbb{Z}/2^s) \\ \Big\downarrow \beta \\ H_n(B(\pi_1 \times \pi_2) ; \mathbb{Z}_{(2)})\end{array}$$

is onto the torsion in H_n.

The lemma follows from the commutativity of

(see next page)

$$\begin{array}{ccc}
H_i(B\pi_1\,;\,\mathbb{Z}/2^s)\otimes H_{n+1-i}(B\pi_2\,;\,\mathbb{Z}/2^s) & \longrightarrow & H_n(B(\pi_1\times\pi_2)\,;\,\mathbb{Z}_{(2)}) \\
\downarrow{\scriptstyle \beta\otimes 1\pm 1\otimes\beta} & & \downarrow{\scriptstyle \mathcal{J}^x_n} \\
H_{i-1}(B\pi_1\,;\,\mathbb{Z}_{(2)})\otimes H_{n+1-i}(B\pi_2\,;\,\mathbb{Z}/2^s)\oplus H_i(B\pi_1\,;\,\mathbb{Z}/2^s)\otimes H_{n-i}(B\pi_2\,;\,\mathbb{Z}_{(2)}) & & L^x_n(\mathbb{Z}[\pi_1\times\pi])_{(2)} \\
\downarrow{\scriptstyle \mathcal{J}^x_{i-1}\otimes\mathcal{J}^{n+1-i}\oplus\mathcal{J}^i\otimes\mathcal{J}^x_{n-i}} & & \downarrow \\
L^x_{i-1}(\mathbb{Z}\pi_1)_{(2)}\otimes L^{n+1-i}(\mathbb{Z}\pi_2\,;\,\mathbb{Z}/2^s)_{(2)}\oplus L^i(\mathbb{Z}\pi_1\,;\,\mathbb{Z}/2)_{(2)}\otimes L^x_{n-i}(\mathbb{Z}\pi_2)_{(2)} & \longrightarrow & L^x_n(\mathbb{Z}[\pi_1\times\pi_2]\,;\,\mathbb{Z}/2^s)_{(2)}\,.
\end{array}$$

Lemma 5.4 : If A is an abelian 2-group, then

$$\vartheta_j^s : H_j(BA\,;\mathbb{Z}_{(2)}) \to L_j^s(\mathbb{Z}A) \text{ is trivial for } j>0.$$

Proof: The lemma follows from the Stein trick (lemma 5.3) and induction on the rank of A once we observe:

(i) the result is true if A is elementary abelian (5.1)

(ii) by Wall [W2], $L_*^s(\mathbb{Z}A)$ has torsion of exponent at most 4.

We now take up the results for κ_j. To fix notation let A be our abelian group. Let $i : E \to A$ be the inclusion of the subgroup of elements of order ≤ 2. Let $j : \mathbb{Z}^r \to A$ be a map of a free abelian group of rank = r = rank of A which is onto. Then we have

(i)

$$\begin{array}{c} \oplus\, H_i(BE\,;\mathbb{Z}/2) \otimes H_{n-i}(B\mathbb{Z}^r\,;\mathbb{Z}_{(2)}) \\ \downarrow \\ H_n(BA\,;\mathbb{Z}/2) \end{array}$$

is onto, where the map is defined using the H-space structure of BA

(ii)

$$\begin{array}{ccc} H_i(BE\,;\mathbb{Z}/2)\otimes H_{n-i}(B\mathbb{Z}^r\,;\mathbb{Z}_{(2)}) & \longrightarrow & H_n(BA\,;\mathbb{Z}/2) \\ \downarrow{\scriptstyle \kappa\otimes\vartheta} & & \downarrow{\scriptstyle \kappa} \\ L^x_{i+2}(\mathbb{Z}E)_{(2)}\otimes L^{n-i}(\mathbb{Z}[\mathbb{Z}^r])_{(2)} & \longrightarrow & L^x_{n+2}(\mathbb{Z}A)_{(2)} \end{array}$$

commutes.

An easy induction plus 5.2 shows that any $c \in H_j(BA ; \mathbb{Z}/2)$ such that $\varkappa_j^x(c) = 0$ must be equal to $j_*(\bar{c})$ for the unique element $\bar{c} \in H_j(B\mathbb{Z}^r ; \mathbb{Z}/2)$ such that $j_*(\bar{c}) = c$.

<u>Lemma 5.4</u>: The maps

$$\varkappa_j' : H_j(BA ; \mathbb{Z}/2) \to L'_{j+2}(\mathbb{Z}A)_{(2)}$$

are trivial for $j > 2$.

<u>Proof</u>: Bak [B] shows $L'_*(\mathbb{Z}A) \to L^h_*(\mathbb{Z}A)$ is monic so we prove the result for $\varkappa^h$.

The result just above the lemma implies that it is enough to show that the problem $(T^2 \to S^2) \times T^j$ is solvable for $j > 2$ over BA.

We can write our problem as $(T^2 \to S^2) \times T^{j-1}) \times S^1$ where $j-1>1$. Now 1.1(d) plus Ranicki's result [R1] that

$L^h_{j-1+2}(\mathbb{Z}[G]) \to L^h_{j+2}(\mathbb{Z}[G \times \mathbb{Z}])$ factors through $L^P_{j-1+2}(\mathbb{Z}[G])$ finishes the proof.

An entirely similar trick shows 1.2 (c). We now do the promised determination of $\varkappa_2$.

<u>Theorem 5.5</u>: The sequence

$$H_2(BE ; \mathbb{Z}/2) \xrightarrow{i_*} H_2(BA ; \mathbb{Z}/2) \xrightarrow{\varkappa_2^h} L_0^h(\mathbb{Z}[A])$$

is exact.

<u>Proof</u>: Naturality of $\varkappa_2^h$ plus 5.2 shows that we have a zero sequence. Naturality again reduces exactness for A to exactness for $\mathbb{Z}/2 \times \mathbb{Z}/4$.

For $\mathbb{Z}/2 \times \mathbb{Z}/4$ the cokernel of i_* is $\mathbb{Z}/2$. Morgan-Pardon showed that $\varkappa_2^h \neq 0$ by example.

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