Topology Vol. 26, No. 3, pp. 297–309, 1987
Printed in Great Britain.

0040–9383/87 $03.00 + .00

A SPANNING TREE EXPANSION OF THE JONES POLYNOMIAL

MORWEN B. THISTLETHWAITE

(*Received* 12 *February* 1986)

ABSTRACT

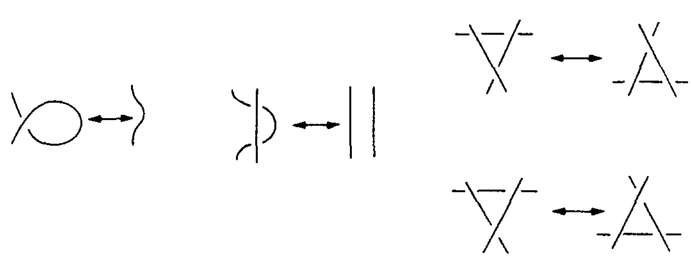
A NEW combinatorial formulation of the Jones polynomial of a link is used to establish some basic properties of this polynomial. A striking consequence of these properties is the result that a link admitting an alternating diagram with m crossings and with no "nugatory" crossing cannot be projected with fewer than m crossings.

§1. INTRODUCTION AND STATEMENT OF RESULTS

This article is concerned with classical links, that is to say closed 1-manifolds embedded piecewise-linearly in the oriented 3-sphere. The link itself may also be endowed with an orientation. Two oriented links L_1, L_2 are *isotopic* if there exists an autohomeomorphism of S^3 mapping L_1 to L_2, preserving the orientations of S^3 and the L_i. Much knot theory is devoted to the problem of finding efficient and effectively calculable isotopy invariants of links.

A *diagram* D of a link L is a regular projection of L in the plane, together with an overcrossing–undercrossing structure; an orientation of L is usually indicated by means of arrows suitably placed on the diagram. Diagrams D_1, D_2 will be considered to be *equivalent* if there is an autohomeomorphism of the extended plane $\mathbb{R}^2 \cup \{\infty\}$ mapping D_1 to D_2, preserving all orientations and, of course, the overcrossings and undercrossings. Where no confusion can arise, we shall not make the distinction between a diagram and its equivalence class.

It is a long-established fact (see, for instance [2]) that link diagrams D_1, D_2 represent isotopic links if and only if D_1 may be transformed to D_2 by means of a finite sequence of *Reidemeister moves*:



Type I Type II Type III

Fig. 1.

Now, if D is a diagram of an oriented link, the *writhe* or *twist* of D is the sum of the signs of the crossing-points of D, according to the convention explained in Fig. 2. It is clear that the type I Reidemeister move alters the writhe of a diagram, whereas the type II and type III moves do not. It is also true, but not entirely obvious, that if D_1, D_2 are diagrams of isotopic links with the same writhe, then D_1 may be transformed to D_2 using Reidemeister moves II, III only. Since the writhe of any diagram is readily computable, and can be altered at will without losing the isotopy class of the corresponding link by introducing *curls* (see Fig. 2), we have the far-reaching principle enunciated recently by L. H. Kauffman.

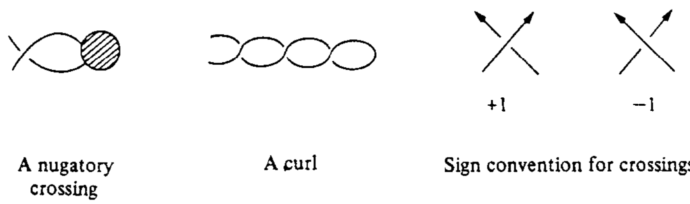


Fig. 2.

KAUFFMAN'S PRINCIPLE. *Any function defined on equivalence classes of link diagrams (oriented or unoriented) which is invariant under Reidemeister moves II, III yields an invariant of oriented link type.*

Kauffman calls such a function an invariant of *regular isotopy* of link diagrams in the plane. An example, central to this article, is Kauffman's "*bracket polyomial*" $\langle D\rangle$ of an unoriented link diagram D. He defines recursively, for each such diagram, a Laurent polynomial $\langle D\rangle$ in the ring $Z[A, A^{-1}]$ by the rules

(i) if D is a simple closed curve, $\langle D\rangle = 1$;
(ii) if D_1 is the disjoint union of D and a simple closed curve, then $\langle D_1\rangle = (-A^{-2} - A^2)\langle D\rangle$;
(iii) if D_1, D_2 are obtained from D by nullifying some particular crossing-point according to the pictures

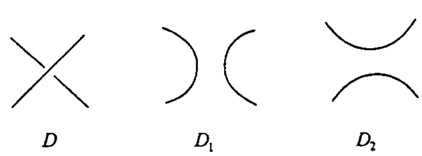


Fig. 3.

then $\langle D\rangle = A\langle D_1\rangle + A^{-1}\langle D_2\rangle$.

Kauffman proves with beautiful simplicity in [4] that this polynomial is indeed invariant under Reidemeister moves II and III; further, if the diagram D is endowed with an orientation, the Jones polynomial $V_L(t)$ of the corresponding oriented link L is given by the formula $V_L(A^{-4}) = (-A^3)^{-w}\langle D\rangle$, where w is the writhe of D. It transpires that $V_L(t)$ is independent of the orientation of L, apart from multiplication by powers of t. This so-called "reversing result" had been discovered previously by V. F. R. Jones; elementary "skein-theoretic" proofs are given in [5, 8]. Thus the bracket polynomial provides an excellent "neutral" way of looking at the Jones polynomial.

In this article, an alternative formula for the bracket polynomial is given as a sum of monomials, indexed by the set of spanning trees of the graph associated with a black-and-white colouring of the regions of the link diagram. This formula is based on W. T. Tutte's concepts of internal and external activities of edges with respect to a spanning tree, and provides a convenient framework for proving certain properties of $V_L(t)$. The main results are

Theorems 1 and 2 below. Theorem 1(i) has been proved independently by L. H. Kauffman [4], and Theorems 1(i) and 2 have been proved independently by K. Murasugi [9]; their proofs are substantially different from the ones presented here.

Let the *breadth* of a non-zero Laurent polynomial f in an indeterminate t be the difference between the highest and lowest powers of t occurring in f. Clearly, this concept is meaningful for "polynomials" with terms involving fractional powers of t.

A link diagram will be called *irreducible* if it does not contain any "removable" or "nugatory" crossings, as illustrated in Fig. 2. Removing such crossings in the obvious way from an alternating diagram will eventually change it either to a diagram with no crossings or to an irreducible alternating diagram.

THEOREM 1. *If a link L admits a connected, irreducible, alternating diagram of m crossings, then*;

(i) *the breadth of $V_L(t)$ is precisely m*;
(ii) *$V_L(t)$ is an alternating polynomial*;
(iii) *the coefficients of the terms of $V_L(t)$ of maximal and minimal degree are both ± 1*;
(iv) *if L is prime in the sense of Schubert, and is not a (2, k) torus link, then $V_L(t)$ is of form $t^r \sum_{i=0}^{m} a_i t^i$, with each coefficient a_i non-zero.*

Let us say that a diagram is *prime* if it is connected, and there does not exist a simple closed curve in the plane meeting it transversely in just two points which lie in different arcs of the diagram. Clearly, any prime diagram of more than one crossing is irreducible; also, if D is any diagram of minimal crossing-number of a link which is prime (in the sense of Schubert), then D is prime.

THEOREM 2.

(i) *If a link L admits a connected diagram of m crossings, then the breadth of the Jones polynomial of L is $\leqslant m$.*
(ii) *If, further, the diagram is prime and non-alternating, then this inequality is strict. Therefore, if L is an m-crossing, prime non-alternating link, then the breadth of $V_L(t)$ is $<m$.*

The condition of primality in the statement of Theorem 2(ii) is necessary; this is evidenced by any connected sum of two alternating knots.

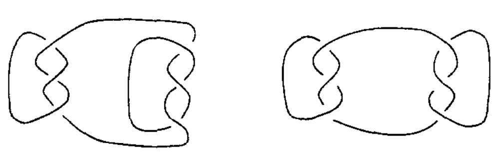


Fig. 4. Two 6-crossing diagrams of the "reef" or "square" knot, one alternating and the other non-alternating.

A link L in S^3 is *split* if it can be separated by a 2-sphere in $S^3 - L$. If L is separated in this way into links L_1, L_2, then $V_L(t) = (-t^{-1/2} - t^{1/2}) V_{L_1}(t) V_{L_2}(t)$; see, for instance [6]. This formula is used in Corollaries 1, 2 below, as is a striking theorem of W. Menasco [7], which allows us to avoid the qualification that links be non-split.

COROLLARY 1. *If a link L admits an alternating, irreducible diagram of m crossings, then L cannot be projected with fewer than m crossings.*

Proof. Let the alternating, irreducible diagram of L have r components. Then, from Theorem 1(i) and the formula given immediately above, the breadth of $V_L(t)$ is $m+r-1$. Now suppose that L admits a diagram with n crossings and s components. From Theorem 2(i) and the above formula, $m+r-1 \leqslant n+s-1$. From Theorem 1 of Menasco's paper [7], $s \leqslant r$. Therefore $m+r-1 \leqslant n+s-1 \leqslant n+r-1$, from which it follows that $m \leqslant n$.

COROLLARY 2. *If L_1, L_2 are alternating links with respective crossing-numbers m_1, m_2, then the crossing-number of any (Schubert) connected sum $L_1 \# L_2$ is m_1+m_2.*

Proof. From Corollary 1, L_1, L_2 admit alternating, irreducible diagrams with m_1, m_2 crossings respectively. Then $L_1 \# L_2$ has, by obvious construction, an alternating, irreducible diagram with m_1+m_2 crossings. The result now follows from a further use of Corollary 1.

Murasugi observes that Theorems 1 and 2, together with the well-known fact that the breadth of the Jones polynomial of an amphicheiral knot is even, yield

COROLLARY 3. *An alternating, amphicheiral knot has even crossing-number.*

The last corollary is of a more practical nature.

COROLLARY 4. *Amongst the* 12965 *unoriented prime knot types of up to* 13 *crossings, precisely* 6236 *are non-alternating.*

This follows from Theorem 1(i) and the author's own tabulations (which have not yet had the benefit of independent verification).

These theorems provide an interesting analogy with the Alexander polynomial: whereas the Alexander polynomial helps to determine the genus of a non-split alternating link (see [3]), the Jones polynomial helps to determine its crossing-number.

I would like to express my gratitude to John Conway, who helped to streamline the new formulation of the bracket polynomial, and to Norman Biggs, whose book on algebraic graph theory [1] introduced me to the world of the Tutte polynomial. I am also indebted to Joan Birman, who suggested in a letter that Theorem 1(iv) might be true.

§2. GRAPH-THEORETICAL BACKGROUND

Before re-defining the bracket polynomial, it is necessary to build some graph-theoretical machinery. Once this machinery has been set up, proofs of Theorems 1 and 2 will come quite naturally.

Figure 5 indicates how a planar graph G, with a valuation of the edges of G in the set $\{1, -1\}$, can be obtained from any connected diagram of an unoriented link L in S^3, by placing a vertex inside each region coloured black, and associating an edge with each crossing-point of the link diagram; each crossing-point (hence, each edge of G) is given a value ± 1 according to the convention illustrated. Of course, this sign convention is different from the one used in diagrams of *oriented* links, to calculate the writhe of the diagram.

By interchanging black regions with white regions, one obtains the planar dual G' of G, the values of whose edges are the negatives of those of their respective dual counterparts in the original graph. The diagram is alternating if and only if all crossing-points (edges of G) have the same value.

Some of the graph-theoretical terminology necessary for an understanding of the Tutte polynomial may not be familiar to readers, so a rapid survey now follows. A *graph* is a finite combinatorial structure G consisting of a set of vertices $V(G)$, a set of edges $E(G)$, and an

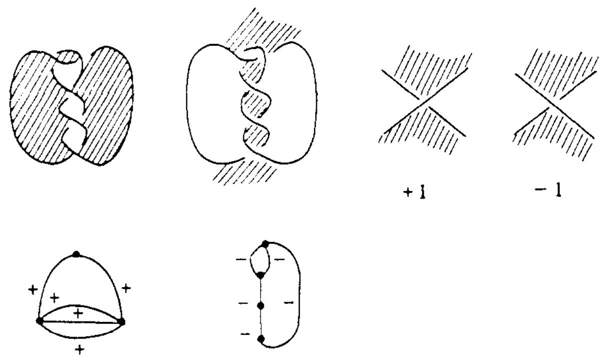


Fig. 5.

incidence function which assigns to each edge an unordered pair of vertices. The vertices of this unordered pair are the *ends* of this edge. If X is a subset of $E(G)$, the *subgraph generated by* X is the subgraph consisting of the edges of X, together with their incident vertices. A *spanning subgraph of* G is a subgraph of G containing all the vertices of G. A *path* in G from a vertex v_0 to a vertex v_r is an alternating sequence $v_0, e_1, v_1, \ldots, e_r, v_r$ of vertices and edges of G, all different, such that each edge e_i is incident to v_{i-1} and v_i. A *cycle* is a sequence $v_0, e_1, v_1, \ldots, e_r$ of pairwise distinct vertices and edges such that e_i is incident to v_{i-1} and $v_i (1 \leqslant i \leqslant r-1)$, and e_r is incident to v_{r-1} and v_0. A cycle consisting of one vertex and one edge is a *loop*. A graph containing no cycles is *acyclic*. Of course, any cycle is determined by its edges. A set X of edges is called a *cut* (or *cocycle*) if there exists a partition $V = V_1 \cup V_2$ of the vertices of G such that X is the set of edges of G with one end in V_1 and the other in V_2.

A graph G is *connected* if, given any distinct vertices v_0, v_1 of G, there is a path in G from v_0 to v_1. A *component* of G is a maximal connected subgraph of G. An *isthmus* of G is an edge the removal of which increases the number of components of G. A loopless graph is *non-separable* if it is connected, and cannot be disconnected by the removal of a single vertex together with its incident edges. The relevance of this concept lies in the fact that non-separable planar graphs correspond to prime link diagrams. A *block* of a loopless graph G is a maximal non-separable subgraph of G. A *tree* is a connected, acyclic graph. A *spanning tree* of G is a spanning subgraph of G which is also a tree.

If G is connected, with n vertices, then an acyclic subgraph H of G is a spanning tree of G if and only if H has $n-1$ edges. If T is a spanning tree of G and e is an edge of G not in T, then $T \cup e$ contains a single cycle, containing e and denoted cyc(T, e). If, on the other hand, e is an edge of T, then $T-e$ has two components; the resulting partition of the vertices of G into two subsets corresponds to a cut, containing e and denoted $\operatorname{cut}(T, e)$. It is easily checked from these definitions that $e \in \operatorname{cyc}(T, f)$ if and only if $f \in \operatorname{cut}(T, e)$.

Next, we state a technical proposition, which is used in the proofs of Theorems 1(iv) and 2(ii). This proposition is probably well known to graph theorists, but I have not found it in the literature. A proof is given in the Appendix.

Suppose we are given a loopless graph G together with a spanning tree T. If H is any subgraph of G containing at least one edge, let $\alpha(H)$ be the union of H with the subgraph of G generated by $\bigcup_{e \in H-T} \{\operatorname{cyc}(T, e)\}$, and let $\beta(H)$ be the union of H with the subgraph of G generated by $\bigcup_{e \in H \cap T} \{\operatorname{cut}(T, e)\}$.

PROPOSITION 1. *Let G, T be as above, and let H be a subgraph of G which contains at least* *one edge, and which is contained in some block B of G. Then the union of the (increasing) sequence of subgraphs H, $\beta\alpha(H)$, $\beta\alpha\beta\alpha(H)$, . . . is B.*

§3. THE TUTTE POLYNOMIAL AND SOME OF ITS PROPERTIES

Let G be a connected graph, with edges $e_1, e_2, \ldots, e_m$. The order in which the m edges of G appear in this list has been chosen arbitrarily, but will remain fixed for the moment. The edge e_i is deemed to *precede* the edge e_j if and only if $i<j$. The crucial concepts of *internal* and *external activity* of an edge with respect to a spanning tree of G will now be defined. An edge e_i in a spanning tree T is *internally active* with respect to T if e_i precedes all other edges in $\text{cut}(T, e_i)$, and an edge e_j not in T is *externally active* with respect to T if e_j precedes all other edges in $\text{cyc}(T, e_j)$. If G is planar, there is a dual relationship between internal and external activity, which is summed up as follows. Let T' be the spanning tree of the dual graph G' generated by the duals of those edges of G not in T, and let e_i' denote the dual edge of e_i. Then $\text{cut}(T, e_i)$ is the dual subgraph of $\text{cyc}(T', e_i')$, and so e_i is internally active with respect to T if and only if e_i' is externally active with respect to T'.

The *internal* (respectively *external*) *activity* of a spanning tree T is the number of edges of G which are internally (respectively externally) active with respect to T. It is a remarkable theorem of W. T. Tutte (see [10, 11]) that, given natural numbers r, s, the number of spanning trees with internal activity r and external activity s is independent of the choice of ordering of the edges of G. The *Tutte polynomial* $\chi_G(x, y)$ of the graph G is the polynomial $\sum_{T \subset G} x^r y^s$, where the sum is taken over all spanning trees T of G, and r, s are respectively the internal and external activities of T. From the discussion of the previous paragraph, if G, G' are planar duals, then $\chi_G(x, y) = \chi_{G'}(y, x)$. By examining Tutte's proof in [10] of the invariance of $\chi_G(x, y)$ with respect to different edge-orderings, it will be seen that a related polynomial Γ_G in one variable, also invariant, can be defined for a connected graph G with *signed* edges. It turns out that Γ_G is simply the Kauffman bracket polynomial of the link diagram associated with G. We shall defer the definition of Γ_G until the next section, as there is still work to do on unsigned graphs.

Tutte's original proof of the invariance of $\chi_G(x, y)$ relies on an examination of the effect of interchanging the labels of edges which are adjacent in the ordering, say e_i and e_{i+1}. Thus he considers the effect of defining $e_i' = e_{i+1}$, $e_{i+1}' = e_i$, and $e_j' = e_j$ for $j \neq i, i+1$. He observes that, for any spanning tree T of G, the activity (or non-activity) of any edge e_j $(j \neq i, i+1)$ is unaltered by this interchange of labels, and shows that a change in the activity of e_i or e_{i+1} is only possible if (i) one of these edges (say e_i) is in T and the other is not in T, (ii) $e_i \in \text{cyc}(T, e_{i+1})$ [equivalently $e_{i+1} \in \text{cut}(T, e_i)$], and (iii) each edge $e_j (j \neq i, i+1)$ has the same activity with respect to T as it does with respect to the spanning tree $\sigma(T)$ obtained from T by substituting

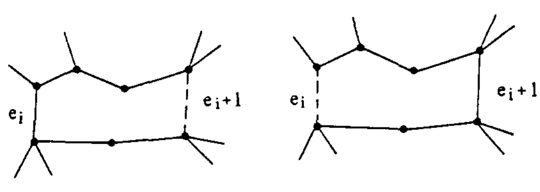


Fig. 6.

e_{i+1} for e_i. Under these restrictive circumstances, certain changes in activity are possible, as set out in Table 1 below.

Table 1

		Old ordering of edges		New ordering of edges	
		e_i	e_{i+1}	e'_i	e'_{i+1}
Case 1	T	L	d	d	D
	$\sigma(T)$	d	D	L	d
Case 2	T	D	d	ℓ	D
	$\sigma(T)$	ℓ	D	D	d
Case 3	T	L	d	ℓ	D
	$\sigma(T)$	ℓ	D	L	d

L denotes "internally active", i.e. "live"; D denotes "internally inactive", i.e. "dead"; ℓ denotes "externally active"; and d denotes "externally inactive".

It is evident that in each of these three cases $\chi_G(x, y)$ is unaltered. All that happens is that the activities of certain pairs of trees are interchanged. The excellent notation expressing the various states of activity of edges was devised by John Conway, and is also capable of distinguishing between positive and negative edges, when the need arises, by means of a bar placed above the symbol in the case of a negative edge.

The polynomial $\chi_G(x, y)$, out of which the bracket polynomial naturally springs, will now be examined a little further.

First, it is clear that an isthmus of G is in every spanning tree of G, and is always internally active (it is the only member of its cut). Similarly, a loop of G is always externally active. Let us now introduce some notation which is standard in graph theory: G'_j is the graph obtained from G by deleting the edge e_j, and G''_j is the graph obtained by contracting e_j (it is assumed here that e_j is not a loop). Any ordering of the edges of G induces, in a natural way, orderings of the edges of G'_j, and the edges of G''_j. From these remarks, if e_j is an isthmus, $\chi_G = x\,.\,\chi_{G''}$, and if e_j is a loop, $\chi_G = y\,.\,\chi_{G'}$.

The Tutte polynomial satisfies a very simple recurrence relation: if e_j is any edge of G which is not an isthmus or a loop, then $\chi_G = \chi_{G'_j} + \chi_{G''_j}$. To verify this, let the edges of G be ordered so that e_j is the highest-ranking edge which is not an isthmus or a loop. Then, a spanning tree not containing e_j becomes, on deletion of e_j, a spanning tree of G'_j with the same internal and external activities. A spanning tree containing e_j becomes, on contraction of e_j, a spanning tree of G''_j with the same internal and external activities. Moreover, any spanning tree of G'_j or of G''_j arises in this fashion.

This reduction process generates a "binary tree", which is set forth for the case of the triangle graph, corresponding to the trefoil knot, in Fig. 7. The Tutte polynomials of the "terminal" graphs in the tree of Fig. 4 are indicated. Using the recurrence relation, the Tutte polynomial of the triangle graph is $\chi_G(x, y) = x^2 + x + y$. Note, incidentally, that $\chi_G(-t, -t^{-1}) = t^2 - t - t^{-1}$, which is $-t^{-2}$ times the Jones polynomial of the right-handed trefoil! The reason for this will be clear, presently.

The next proposition will be used in the proof of Theorem 1, parts (i) and (iii).

PROPOSITION 2. (i) *If G has n vertices and m edges, then χ_G is of degree $n-1$ in x, and degree $m-n+1$ in y.* (ii) *If, in addition, G has no isthmuses or loops, then χ_G has just one term of maximal degree in x, namely x^{n-1}, and just one term of maximal degree in y, namely y^{m-n+1}.*

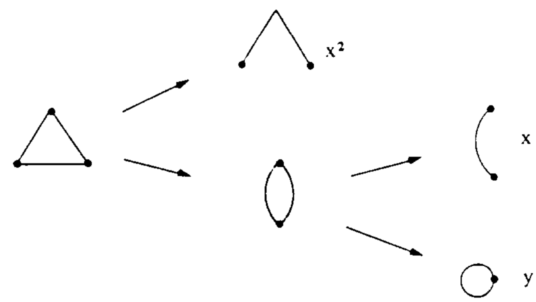


Fig. 7.

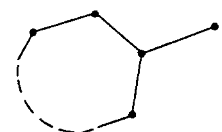
Proof. (i) Let T be any spanning tree of G. Since T contains $n-1$ edges, its internal activity cannot exceed $n-1$, and its external activity cannot exceed $m-n+1$. If the edges of G are ordered so that the edges of T are $e_1, e_2, \ldots, e_{n-1}$, then all edges of T are internally active, resulting in a term of degree $n-1$ in x; if, instead, the edges of G are ordered so that the edges not in T are $e_1, e_2, \ldots, e_{m-n+1}$, then all these edges are externally active, resulting in a term of degree $m-n+1$ in y.

(ii) Suppose now that G contains no isthmuses or loops. If the edges of a spanning tree T are $e_1, e_2, \ldots, e_{n-1}$, then no edge e_r outside T is externally active, as $\mathrm{cyc}(T, e_r)$ contains at least one edge of T. Therefore T yields a term x^{n-1} in χ_G. Let T' be a spanning tree not equal to T, and let e_r be the first edge of T which is not in T'. Then $\mathrm{cyc}(T', e_r)$ contains at least one edge of $T'-T$: otherwise T itself would contain a cycle. Let this edge of $T'-T$ be e_s. Then, since $r<s$, e_s is not internally active with respect to T', so the internal activity of T' is less than that of T. This confirms the claim that x^{n-1} is the only term of degree $n-1$ in x. The corresponding claim concerning y is dealt with similarly.

The final proposition of this section is the midway stage between Proposition 1 and Theorem 1(iv). It ensures that, for suitable graphs G, $\chi_G(-t, -t^{-1})$ is of form $t^r \sum_{i=0}^{m} a_i t^i$, with each coefficient a_i non-zero.

Proposition 3. *Let G be a loopless, non-separable graph with n vertices and m edges.*

(i) *For each $1 \leqslant i \leqslant n-1$, the coefficient in $\chi_G(x, y)$ of x^i is strictly positive, and for each $1 \leqslant j \leqslant m-n+1$ the coefficient of y^j is strictly positive.*
(ii) *Suppose further that G contains a subgraph K consisting of a cycle with an isthmus of K attached, as in Fig. 8. Then the coefficient in $\chi_G(x, y)$ of xy is strictly positive.*



K

Fig. 8.

Proof. (i) Take any spanning tree T of G, and label any set of i edges of T $e_1, \ldots, e_i$ $(1 \leqslant i \leqslant n-1)$. Then each of these i edges will be internally active with respect to T. Let H_1 be the subgraph of G consisting of these edges together with their incident vertices. Let

$H_2 = \beta(H_1)$, $H_3 = \alpha(H_2)$. $H_4 = \beta(H_3)$, and so on, where α, β are as in Proposition 1. The union of the H_j is G by Proposition 1, so we can label the edges of $G - H_1$ so that they are all inactive with respect to T: we simply arrange that each edge in $H_j - H_{j-1}$ precedes each edge in $H_{j+1} - H_j$ $(j > 1)$. Therefore, with this labelling scheme, T contributes x^i to $\chi_G(x, y)$. The corresponding result concerning y^j is dealt with similarly: one starts by labelling any j edges of $G - T$ $e_1, \ldots, e_j$. (ii) Let e_1 be the isthmus of K, and let e_2 be any edge of the cycle of K. Let $K' = K - e_2$. Then K' is acyclic, so there exists a spanning tree T of G containing K'. e_1 is internally active with respect to T, and e_2 is externally active. Taking H_1 to be the subgraph generated by e_1 and e_2, and proceeding as in (i), we get a labelling of edges such that T has internal activity and external activity both equal to 1.

§4. THE POLYNOMIAL Γ_G

Let G be a connected graph, with signed edges ordered somehow. Given a spanning tree T_i of G, each edge e_j of G has one of eight possible states, depending on whether (i) it is active or inactive, (ii) it is in T_i or not in T_i, (iii) its sign is $+1$ or -1. These eight states will be denoted by the shorthand symbols L, D, ℓ, d, $\bar{\text{L}}$, $\bar{\text{D}}$, $\bar{\ell}$, $\bar{\text{d}}$, as explained immediately below Table 1.

The definition of the polynomial Γ_G now follows. For each spanning tree T_i and each edge e_j of G, a monomial μ_{ij} in $Z[A, A^{-1}]$ is defined according to the table below.

Table 2

State of e_j	L	D	ℓ	d	$\bar{\text{L}}$	$\bar{\text{D}}$	$\bar{\ell}$	$\bar{\text{d}}$
μ_{ij}	$-A^{-3}$	A	$-A^3$	A^{-1}	$-A^3$	A^{-1}	$-A^{-3}$	A

Then
$$\Gamma_G = \sum_{\substack{\text{all spanning trees} \\ T_i \subset G}} \left(\prod_{e_j \in G} \mu_{ij} \right).$$

The product $w(T_i) = \prod_{e_j \in G} \mu_{ij}$ will be referred to as the *weight* of T_i, and the exponent of A in $w(T_i)$ the *exponent* of T_i. The *state* of T_i is the number of edges of G of each of the above eight kinds, and will be denoted by an appropriate word in the shorthand symbols. Here is a simple example.

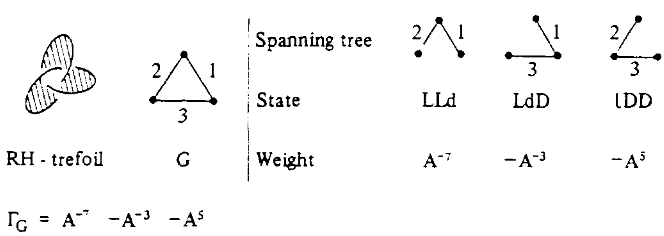


Fig. 9.

It is of interest to see directly why Γ_G is independent of the choice of ordering of the edges of G. To demonstrate this invariance, it is sufficient to consider the three cases of Table 1, when e_i and e_{i+1} have opposite signs. In cases 1 and 2, the weights of T and $\sigma(T)$ sum to zero in both the old and the new orderings, whereas in case 3 the respective weights of T and $\sigma(T)$

are unaltered by the change of ordering. It transpires that Γ_G is invariant under the change of ordering, even though the collection of tree-weights might not be invariant.

Henceforth, we shall assume that G is planar. If G' is the dual graph of G, with all edge-signs reversed, it is clear from Table 2, and from the property $\chi_G(x, y) = \chi_{G'}(y, x)$, that $\Gamma_G = \Gamma_{G'}$. Therefore Γ_G is independent of the choice of black-and-white colouring of the corresponding link diagram.

Of course, it is often more convenient to speak of the polynomial of a connected link diagram, rather than the polynomial of its associated connected graph. One then realizes that it is necessary to define Γ_D for a *disconnected* diagram D. The requirement that Γ_D be invariant under type II Reidemeister moves which alter the number of components of a diagram dictates to us a formula for Γ_D in terms of the polynomials of its components. Specifically, if D has components $D_1, \ldots, D_r$, then $\Gamma_D = (-A^{-2} - A^2)^{r-1}\ \Gamma_{D_1} \ldots \Gamma_{D_r}$. Taking on board this extended definition of Γ_D, it is a simple matter to check that Γ_D is invariant under Reidemeister moves II and III. However, we shall not pursue this, as there is a quick proof that Γ_D really is equal to Kauffman's bracket polynomial, without even having to check invariance under edge ordering. As the above formula for the polynomial of a disconnected diagram agrees with that of Kauffman, it is sufficient to check that the polynomials agree for connected diagrams.

First, observe that if G is a "terminal graph" in a deletion-contraction "binary tree", consisting of p positive isthmuses, q negative isthmuses, r positive loops and s negative loops, then, from Table 2, $\Gamma_G = (-A^3)^{-p+q+r-s}$. The diagram corresponding to this graph is a diagram of the unknot, with writhe $-p+q+r-s$, so by Theorem 2.5 of [4] Γ_G is equal to the bracket polynomial in this case. Now let G be any connected planar graph, and let e_j be the highest-ranking edge of G which is not an isthmus or a loop (the case where there is no such e_j has just been dealt with). Then, since this edge e_j is always inactive, from Table 2 we have $\Gamma_G = A^{-\varepsilon} \cdot \Gamma_{G'_j} + A^{\varepsilon} \cdot \Gamma_{G''_j}$, where $\varepsilon = \pm 1$ is the sign of e_j. Since this agrees with the recursion formula in the definition of the bracket polynomial, the verification that these polynomials are equal is complete.

It is interesting to note that Γ_G has been defined for an *arbitrary* graph with signed edges, not necessarily planar. I do not know whether this polynomial has any application in the case that G is non-planar. The proofs which follow do not use planarity.

§5. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1

Suppose we are given a link L admitting a connected, irreducible, alternating diagram of m crossings, with associated graph G. Without loss of generality, we can assume that the m edges of G all have positive sign. Further, since the diagram is irreducible, G has no isthmuses or loops. Use will be made, without further reference, of the formula given in the Introduction which connects $V_L(t)$ with the (bracket) polynomial Γ_G.

Part (*ii*). The state of any spanning tree of G is of form $\mathrm{L}^p\mathrm{D}^q\ell^r\mathrm{d}^s$, where $p+q=n-1$ and $r+s=m-n+1$; from Table 2, the weight of this spanning tree is $(-1)^{p+r}A^{-3p+q+3r-s}$. Putting $u=p-r$ and $k=2(n-1)-m$, it is easily checked that this weight is $(-1)^u A^{k-4u}$. Since k is constant for the given graph, the sign of the weight of a spanning tree of G is determined by its exponent, and the weights of two trees have the same sign if and only if their exponents differ by a multiple of 8. This confirms part (ii) of Theorem 1, and tells us also that the Jones polynomial of the alternating link L is, up to multiplication by a power of t, equal to $\pm\chi_G(-t, -t^{-1})$. This last fact is of vital importance in the proof of part (iv).

Part (iv). Recall that the graph, G say, of an irreducible, prime diagram (with a choice of black-and-white colouring) is non-separable. From the preceding remarks, and Proposition 3, $V_L(t)$ can only fail to satisfy the required condition if G fails to contain a subgraph of the type described in the statement of Proposition 3(ii), and illustrated in Fig. 8. But in this event either G or its planar dual would consist of a single cycle, and such a graph corresponds to a diagram of a $(2, k)$ torus link.

Parts (i), (iii). From Table 2 and Proposition 2, the unique spanning tree which contributes x^{n-1} to $\chi_G(x, y)$ is also the unique spanning tree of lowest exponent with respect to Γ_G, and the unique spanning tree which contributes y^{m-n+1} to $\chi_G(x, y)$ is also the unique spanning tree of highest exponent. These exponents are $-3(n-1)-(m-n+1) = -m-2n+2$ and $3(m-n+1)+(n-1) = 3m-2n+2$ respectively. The difference between these exponents is $4m$, so the breadth of $V_L(t)$ is m.

Proof of Theorem 2

Part (i). As before, we shall assume that the graph G associated with the diagram of L has no isthmuses or loops, as these correspond to removable crossings. We are interested in the difference of the exponents of two spanning trees T_1, T_2 of G, which shall remain fixed. Considering the definition of the polynomial Γ_G, it is seen that each edge e_j of G contributes a certain integer to this difference of exponents. The absolute value σ_j of this integer is given by Table 3. This table does not show the possible signs of the edge e_j, as σ_j is independent of this sign. To prove part (i), it is sufficient to show that $\sum_{e_j \in G} \sigma_j \leqslant 4m$. Let s_k be the number of integers j for which $\sigma_j = k$. Then $\Sigma\sigma_j \leqslant 4m$ if and only if $2s_6 \leqslant 2s_2 + 4s_0$. It will be shown that $s_6 \leqslant s_2$. In the notation of Table 3, let there be r_1 edges of type ${}^{\ell}_{\mathrm{L}}$ and r_2 edges of type ${}^{\mathrm{L}}_{\ell}$. Then $s_6 = r_1 + r_2$. Now, let $C_i = \bigcup_{e_j \in E_i} \mathrm{cyc}(T_i, e_j)$ $(i = 1, 2)$, where E_1, E_2 are the sets of edges of types ${}^{\ell}_{\mathrm{L}}$, ${}^{\mathrm{L}}_{\ell}$ respectively. C_1 contains r_1 independent cycles; since T_2 is acyclic, C_1 contains at least r_1 edges of T_1 not in T_2. Each of these edges is D in T_1, so from Table 3 their "scores" are 2 each. Similarly, we get r_2 different edges of score 2 from C_2. Therefore $s_2 \geqslant r_1 + r_2 = s_6$, so (i) is proved.

Table 3

State of e_j with respect to T_1	L	ℓ	L	D	ℓ	d	L	d	ℓ	D	D	d	x
State of e_j with respect to T_2	ℓ	L	D	L	d	ℓ	d	L	D	ℓ	d	D	x
σ_j	6	6	4	4	4	4	2	2	2	2	2	2	0

Part (ii). Here, it is convenient to prove the following, of which part (ii) is a special case: if the breadth of $V_L(t)$ is m, and L admits an m-crossing diagram [necessarily irreducible by part (i)], then this diagram is a connected sum of alternating diagrams. Translating into graph-theoretical language, let G be a graph with m edges, and with no isthmuses or loops; we shall show that if the breadth of Γ_G is $4m$, then within each block of G all edges have the same sign. Suppose, therefore, that the difference between the exponents of spanning trees T_1, T_2 of G is $4m$. We shall make some observations concerning the rigid constraints placed on activities of edges of G by this condition. In the notation employed in the proof of part (i), we have $s_6 = s_2$ and $s_0 = 0$. Now, let X be a component of some C_i. From the condition $s_6 = s_2$, together with the proof of part (i), the number of edges in $(T_2 - T_1) \cap X$ equals the number of edges in $(T_1 - T_2) \cap X$. Therefore, from the mode of construction of the C_i, $T_1 \cap X$ and $T_2 \cap X$ are both spanning trees of X. It follows also that each edge of C_1 is of type ${}^{\ell}_{\mathrm{L}}$, ${}^{\mathrm{D}}_{\mathrm{L}}$ or ${}^{\mathrm{D}}_{\mathrm{d}}$, and each edge of C_2 is of type ${}^{\mathrm{L}}_{\ell}$, ${}^{\mathrm{L}}_{\mathrm{D}}$ or ${}^{\mathrm{d}}_{\mathrm{D}}$ (hence C_1, C_2 have no edge in common). The edge e_1 belongs to some C_i, so we may suppose, without loss of generality, that C_1 is non-empty and contains a positive edge. To maintain the difference $4m$ between the exponents of T_1 and T_2, (i) each edge of type ${}^{\ell}_{\mathrm{L}}$, ${}^{\mathrm{D}}_{\mathrm{L}}$, ${}^{\mathrm{D}}_{\mathrm{d}}$ or ${}^{\ell}_{\mathrm{d}}$ must be positive, and each edge of type ${}^{\mathrm{L}}_{\ell}$, ${}^{\mathrm{L}}_{\mathrm{D}}$, ${}^{\mathrm{d}}_{\mathrm{D}}$ or ${}^{\mathrm{d}}_{\ell}$ must be negative; (ii) each edge outside $C_1 \cup C_2$ must have "score" equal to 4; hence there are no edges in G of type ${}^{\mathrm{L}}_{\mathrm{d}}$, ${}^{\mathrm{d}}_{\mathrm{L}}$, ${}^{\ell}_{\mathrm{D}}$, ${}^{\mathrm{D}}_{\ell}$, or indeed ${}^{x}_{x}$; also, $T_1-(C_1 \cup C_2)=T_2-(C_1 \cup C_2)$.

The proof of part (ii) will be complete once we have shown that the set of edges of types ${}^{\ell}_{\mathrm{L}}$, ${}^{\mathrm{D}}_{\mathrm{L}}$, ${}^{\mathrm{D}}_{\mathrm{d}}$, ${}^{\ell}_{\mathrm{d}}$, i.e. the set of all positive edges of G, is fixed by the operations α, β of Proposition 1, these operations being taken with respect to the spanning tree T_2. First, consider an edge e of type ${}^{\ell}_{\mathrm{L}}$ or ${}^{\mathrm{D}}_{\mathrm{L}}$. Then any edge of cut$(T_2, e)$ is d with respect to T_2, so it must be positive. Next, let e be of type ${}^{\mathrm{D}}_{\mathrm{d}}$. Then e is in some component X of C_1; since, as explained above, $T_2 \cap X$ is a spanning tree of X, cyc(T_2, e) lies in C_1 and consists of positive edges. Finally, suppose that e is of type ${}^{\ell}_{\mathrm{d}}$. Then e is outside $C_1 \cup C_2$, and we need to exclude the possibility that cyc(T_2, e) might contain a negative edge of T_2, i.e. an edge of type ${}^{\mathrm{L}}_{\mathrm{D}}$ or ${}^{\mathrm{d}}_{\mathrm{D}}$. Recall that, for each component X of either C_i, $T_1 \cap X$ and $T_2 \cap X$ are both spanning trees of X; also, $T_1-(C_1 \cup C_2)=T_2-(C_1 \cup C_2)$. It follows that, for each such X and for each e outside $C_1 \cup C_2$, cyc(T_1, e) has an edge in X if and only if cyc(T_2, e) has an edge in X; moreover, cyc (T_1, e) agrees with cyc(T_2, e) outside $C_1 \cup C_2$. Now if, for our edge e of type ${}^{\ell}_{\mathrm{d}}$, cyc(T_2, e) contains an edge of type ${}^{\mathrm{L}}_{\mathrm{D}}$ outside C_2, then this edge is also in cyc(T_1, e), contradicting the fact that e is ℓ with respect to T_1. Also, if cyc(T_2, e) contains an edge (of type ${}^{\mathrm{L}}_{\mathrm{D}}$ or ${}^{\mathrm{d}}_{\mathrm{D}}$) inside C_2, then cyc(T_1, e) also contains an edge of C_2, which is automatically L in T_1, similarly contradicting the given state of e. We have now examined all possibilities, so part (ii) is proved.

REFERENCES

1. N. Biggs: *Algebraic Graph Theory*. Cambridge University Press, Cambridge (1974).
2. G. Burde and H. Zieschang: *Knots*. De Gruyter, Berlin (1985).
3. R. H. Crowell: Genus of alternating link types. *Ann. Math.* **69** (1959), 258–275.
4. L. H. Kauffman: State models for knot polynomials. Preprint.
5. W. B. R. Lickorish and K. C. Millett: The reversing result for the Jones polynomial. Preprint.
6. W. B. R. Lickorish and K. C. Millett: A polynomial invariant of knots and links. *Topology*, to appear.
7. W. Menasco: Closed incompressible surfaces in alternating knot and link complements. *Topology* **23** (1984), 37–44.
8. H. R. Morton: The Jones polynomial for unoriented links. *Q. J. Math. Oxford* **37** (1986), 55–60.
9. K. Murasugi: Jones polynomials and classical conjectures in knot theory. Preprint.
10. W. T. Tutte: A contribution to the theory of chromatic polynomials. *Can. J. Math.* **6** (1954), 80–91.
11. W. T. Tutte: Graph theory. *Encyclopedia of Mathematics* (1985).

APPENDIX: PROOF OF PROPOSITION 1

Firstly, if the subgraph H is in the block B, then both $\alpha(H)$ and $\beta(H)$ are in B, as neither a cycle, nor a cut of form cut(T, e), can contain edges from more than one block.

Now suppose that H is strictly contained in B and that H contains at least one edge. It is sufficient to show that $\alpha(H) \neq H$ or $\beta(H) \neq H$. Let K be any component of H. Since B is non-separable, there exist distinct vertices v_1, v_2 of K, together with a path in B from v_1 to v_2 consisting entirely of edges in $B-K$. Let ω_1 be the unique path in the spanning tree T from v_1 to v_2. Every edge of ω_1 is in B.

Suppose ω_1 contains an edge not in K. Then, by altering v_1, v_2 if necessary, we can assume that ω_1 contains no edge of K. Let η be a path in K from v_1 to v_2. As T cannot contain a cycle, η must contain at least one edge not in T; let us suppose that η has been chosen with a minimal number of such edges. In the journey along η from v_1 to v_2, let e be the first edge of η not in T. Let ω_3 be the part of η joining v_1 to the beginning, v_3 say, of e, and let ω_4 be the unique path in T from the other end of e to v_2. If ω_4 does not contain v_1, there is a cycle C in B consisting of e, ω_3, part or all of ω_1, and part or all of ω_4. Each edge of C apart from e is in T, and C contains an edge e' not in H, namely the edge of ω_1 incident to v_1. Since $e' \in \alpha(H)$, it follows that $\alpha(H) \neq H$ in this case. If, on the other hand, ω_4 contains v_1, then, by the minimal property of η, the part of ω_4 between the edge e and v_1 cannot lie entirely in K; hence this part of ω_4 contains an edge e' say of $B-H$. There is now a cycle consisting of e, part or all of ω_3 and a part of ω_4 containing e'. As before, we conclude that $e' \in \alpha(H)$, so $\alpha(H) \neq H$.

Suppose now that ω_1 lies entirely in K. By similar reasoning, using the fact that there exists a path in $B-K$ from v_1 to v_2, there is a cycle C consisting entirely of edges of T except for some edge e not in K, and containing an edge e' in K. If e is not in H, then $\beta(H) \neq H$ as $e \in \beta(H)$. If e is in a component of H different from K, then C contains an edge e'' not in H; then $e'' \in \alpha(H)$, so $\alpha(H) \neq H$.

Department of Mathematical Sciences and Computing
Polytechnic of the South Bank
London SE1 0AA, U.K.