Canadian Mathematical Society
Conference Proceedings
Volume 2, Part 2 (1982)

THE PROJECTIVE CLASS GROUP TRANSFER INDUCED BY AN S^1-BUNDLE

Hans J. Munkholm and Andrew A. Ranicki[1]

Introduction

This note gives an explicit algebraic description of the geometric transfer map induced in the (reduced) projective class groups by an S^1-bundle $S^1 \longrightarrow E \xrightarrow{\ p\ } B$

$$p^*_{K_0} : \tilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi])$$

with $\pi = \pi_1(E)$, $\rho = \pi_1(B)$. This is the transfer map (1.4) of the preceding paper, Munkholm and Pedersen [4], to which we refer for terminology and background material. In particular, $t \in \pi$ is the canonical generator of the cyclic group $\ker(p_*:\pi \twoheadrightarrow \rho)$ represented by the inclusion $S^1 \longrightarrow E$ of a fibre, $\phi : \mathbb{Z}[\pi] \twoheadrightarrow \mathbb{Z}[\pi]/(t-1) = \mathbb{Z}[\rho]$; $r \longmapsto \bar{r}$ is the projection of fundamental group rings induced by $p_*:\pi \twoheadrightarrow \rho$, and $\mathbb{Z}[\pi] \xrightarrow{\sim} \mathbb{Z}[\pi]$; $r \longmapsto r^t$ is a ring automorphism determined by the orientation class $w_1(p) \in H^1(B;\mathbb{Z}_2)$ such that $(t-1)r = r^t(t-1)$. In the orientable case $w_1(p) = 0$, $t \in \pi$ is central and $r^t = r$.

Our main results are:

Proposition 2.1 The projection of rings $\phi:\mathbb{Z}[\pi] \twoheadrightarrow \mathbb{Z}[\rho]$ gives rise to an algebraic transfer map in the projective class groups

$$\phi^!_0 : K_0(\mathbb{Z}[\rho]) \longrightarrow K_0(\mathbb{Z}[\pi]) \ ; \ [\mathrm{im}(\bar{X})] \longmapsto [\mathrm{im}(X^!)] - [\mathbb{Z}[\pi]^n] \ .$$

Here $\bar{X} \in M_n(\mathbb{Z}[\rho])$ is a projection (i.e. an $n \times n$ matrix $\bar{X}$ with entries in $\mathbb{Z}[\rho]$ such that $\bar{X}^2 = \bar{X}$) and $X^! \in M_{2n}(\mathbb{Z}[\pi])$ is the projection defined by

$$X^! = \begin{pmatrix} X & Y \\ t-1 & 1-X^t \end{pmatrix} \in M_{2n}(\mathbb{Z}[\pi])$$

for any $X,Y \in M_n(\mathbb{Z}[\pi])$ such that $\phi(X) = \bar{X}$, $X(1-X) = Y(t-1)$, $XY = YX^t$.

[]

1980 Mathematics Subject Classification. 57Q12, 18F25.

[1]Partially supported by NSF grants.

0731-1036/82/0000-0481/$07.00

Proposition 4.1 The algebraic and geometric transfer maps in the reduced projective class groups coincide, that is if B,E are finitely dominated CW complexes

$$\widetilde{\phi}_0^! = p^*_{K_0} : \widetilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi]) ;$$
$$[B] \longmapsto \widetilde{\phi}_0^!([B]) = p^*_{K_0}([B]) = [E]$$

with [B],[E] the Wall finiteness obstructions.

[]

We should like to thank the Nassau Inn, Princeton for the hospitality of its back steps.

*

Contents

§1. Rings with pseudostructure

Let R be an associative ring with 1. We shall be using the following conventions regarding matrices and morphisms over R.

Given (left) R-modules M,N let $\mathrm{Hom}_R(M,N)$ denote the additive group of R-module morphisms

$$f : M \longrightarrow N \ ; \ x \longmapsto f(x) \ .$$

For $m,n \geqslant 1$ let $M_{m,n}(R)$ be the additive group of $m\times n$ matrices $X = (x_{ij})$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ with entries $x_{ij} \in R$, and use the isomorphism of abelian groups

$$M_{m,n}(R) \xrightarrow{\ \simeq\ } \mathrm{Hom}_R(R^m,R^n) \ ;$$
$$X = (x_{ij}) \longmapsto (f:(r_1,r_2,\dots,r_m) \longmapsto (\sum_{i=1}^{m} r_i x_{i1}, \sum_{i=1}^{m} r_i x_{i2},\dots,\sum_{i=1}^{m} r_i x_{in}))$$

to identify

$$M_{m,n}(R) = \mathrm{Hom}_R(R^m,R^n) \ .$$

If the R-module morphisms $f \in \mathrm{Hom}_R(R^m,R^n)$, $g \in \mathrm{Hom}_R(R^n,R^p)$ have matrices $X = (x_{ij}) \in M_{m,n}(R)$, $Y = (y_{jk}) \in M_{n,p}(R)$ the composite R-module morphism

$$gf : R^m \xrightarrow{\ f\ } R^n \xrightarrow{\ g\ } R^p \ ; \ r \longmapsto g(f(r))$$

has the product matrix

$$XY = (\sum_{j=1}^{n} x_{ij}y_{jk}) \in M_{m,p}(R) \ .$$

The $n \times n$ matrix ring $M_n(R) = M_{n,n}(R)$ is thus identified with the endomorphism ring $\mathrm{Hom}_R(R^n,R^n)$ of the f.g. free R-module R^n of rank n, as usual.

A <u>projection over R</u> is a matrix $X \in M_n(R)$ such that

$$X(1-X) = 0 \in M_n(R) \ ,$$

so that $\mathrm{im}(X) \subseteq R^n$ is a f.g. projective R-module with

$$\mathrm{im}(X) \oplus \mathrm{im}(1-X) = R^n$$

and $\mathrm{im}(1-X)$ is a f.g. projective inverse of $\mathrm{im}(X)$. Let

$$P_n(R) = \{X \in M_n(R) \mid X(1-X) = 0\} \subseteq M_n(R)$$

denote the subset of $M_n(R)$ consisting of projections. Every f.g. projective R-module P is isomorphic to $\mathrm{im}(X)$ for some $X \in P_n(R)$.

A pseudostructure $\phi = (\alpha, t)$ on the ring R consists of an automorphism

$$\alpha : R \xrightarrow{\sim} R \; ; \; r \longmapsto r^t$$

and an element $t \in R$ such that

$$t^t = t \; , \; (t-1)r = r^t(t-1) \; .$$

Let ϕ also denote the projection onto the quotient of R by the two-sided principal ideal $(t-1) \triangleleft R$

$$\phi : R \longrightarrow\!\!\!\!\rightarrow \bar{R} = R/(t-1) \; ; \; r \longmapsto \bar{r} \; .$$

An S^1-bundle $S^1 \longrightarrow E \xrightarrow{p} B$ with $p_* = \phi : \pi_1(E) = \pi \longrightarrow\!\!\!\!\rightarrow \pi_1(B) = \rho$ determines a pseudostructure $\phi = (\alpha, t)$ on $R = \mathbb{Z}[\pi]$ with $\bar{R} = \mathbb{Z}[\rho]$ (cf. Munkholm and Pedersen [3],[4]).

Let then (R, ϕ) be a ring R with pseudostructure $\phi = (\alpha, t)$.

A pseudoprojection over (R, ϕ) is a pair of matrices over R

$$(X, Y) \in M_n(R) \times M_n(R)$$

such that

$$X(1-X) = Y(t-1) \; , \; XY = YX^t \in M_n(R) \; ,$$

where $X^t = \alpha(X) = (x_{ij}^t) \in M_n(R)$. The pseudoprojection (X,Y) gives rise to a projection over $\bar{R}$

$$\bar{X} \in P_n(\bar{R})$$

with $\bar{X} = \phi(X) = (\bar{x}_{ij}) \in M_n(\bar{R})$, and also to a projection over R

$$X^! = \begin{pmatrix} X & Y \\ t-1 & 1-X^t \end{pmatrix} \in P_{2n}(R) \; .$$

Let

$$P_n(R, \phi) = \{(X, Y) \in M_n(R) \times M_n(R) \,|\, X(1-X) = Y(t-1) \, , \, XY = YX^t\}$$

denote the subset of $M_n(R) \times M_n(R)$ consisting of the pseudoprojections over (R, ϕ).

Proposition 1.1 Every projection $\bar{X} \in P_n(\bar{R})$ over $\bar{R}$ lifts to a pseudoprojection $(X, Y) \in P_n(R, \phi)$ (non-uniquely), with $\phi(X) = \bar{X}$.

Proof: Every matrix $\bar{X} \in M_n(\bar{R})$ lifts to some $X \in M_n(R)$, with any two such lifts X_1, X_2 differing by

$$X_1 - X_2 = W(t-1) \in M_n(R)$$

for some $W \in M_n(R)$. Thus if $X \in M_n(R)$ is a lift of a projection $\bar{X} \in P_n(\bar{R})$ there exists $W \in M_n(R)$ such that

$$X(1-X) = W(t-1) \in M_n(R) \; .$$

Define the matrix

$$Z = \begin{pmatrix} X & W \\ t-1 & 1-X^t \end{pmatrix} \in M_{2n}(R) \ .$$

Now

$$Z(1-Z) = \begin{pmatrix} 0 & WX^t - XW \\ 0 & 0 \end{pmatrix} \in M_{2n}(R) \ ,$$

so that $(Z(1-Z))^2 = 0$ and

$$Z^2 + (1-Z)^2 = 1 - 2Z(1-Z) \in M_{2n}(R)$$

is invertible, with inverse

$$(Z^2 + (1-Z)^2)^{-1} = 1 + 2Z(1-Z) \in GL_{2n}(R) \ ,$$

so that there is defined a projection

$$X^! = (Z^2 + (1-Z)^2)^{-1} Z^2 \in P_{2n}(R) \ .$$

(The principal ideal $(Z(1-Z))$ of the matrix ring $M_{2n}(R)$ is nilpotent, and $X^! \in P_{2n}(R) \subset M_{2n}(R)$ is an idempotent (= projection) lifting the idempotent $[Z] \in M_{2n}(R)/(Z(1-Z))$ - cf. Bass [0,III.2.10], Swan [9,5.17]). Substituting the relation $Z^4 = 2Z^3 - Z^2$ we have

$$\begin{aligned} X^! &= (1 + 2Z(1-Z))Z^2 \\ &= (1 + 2Z)Z^2 - 2(2Z^3 - Z^2) \\ &= 3Z^2 - 2Z^3 \in P_{2n}(R) \ , \end{aligned}$$

with

$$\begin{aligned} X^! - Z &= (2Z-1)Z(1-Z) \\ &= \begin{pmatrix} 2X-1 & 2W \\ 2t-2 & 1-2X^t \end{pmatrix} \begin{pmatrix} 0 & WX^t - XW \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (2X-1)(WX^t - XW) \\ 0 & 0 \end{pmatrix} \in M_{2n}(R) \ . \end{aligned}$$

Defining

$$Y = W + (2X-1)(WX^t - XW) \in M_n(R) \ ,$$

we have

$$X^! = \begin{pmatrix} X & Y \\ t-1 & 1-X^t \end{pmatrix} \in P_{2n}(R)$$

with $\phi(X) = \overline{X}$, $X(1-X) = Y(t-1)$, $XY = YX^t$. The projection $\overline{X} \in P_n(\overline{R})$ has been lifted to a pseudoprojection $(X,Y) \in P_n(R,\phi)$.

[]

Given an $\bar{R}$-module $\bar{M}$ let $\phi^!\bar{M}$ be the R-module with the same additive group as $\bar{M}$ and

$$R \times \phi^!\bar{M} \longrightarrow \phi^!\bar{M} \ ; \ (r,\bar{x}) \longmapsto \bar{r}\,\bar{x} \ .$$

An $\bar{R}$-module morphism $\bar{f} \in \mathrm{Hom}_{\bar{R}}(\bar{M},\bar{N})$ also defines an R-module morphism

$$\phi^! f \ : \ \phi^!\bar{M} \longrightarrow \phi^!\bar{N} \ ; \ \bar{x} \longmapsto \bar{f}(\bar{x}) \ .$$

Given a pseudoprojection $(X,Y) \in P_n(R,\phi)$ define the f.g. projective $\bar{R}$-module $\bar{P} = \mathrm{im}(\bar{X})$, and define the associated <u>pseudoresolution</u> of the restricted R-module $\phi^!\bar{P}$ to be the 1-dimensional f.g. projective R-module chain complex $C^!$ with

$$d_{C^!} = \begin{bmatrix} 1-X \\ 1-t \end{bmatrix} : C_1^! = \mathrm{coker}(X^! = \begin{pmatrix} X & Y \\ t-1 & 1-X^t \end{pmatrix} : R^n \oplus R^n \longrightarrow R^n \oplus R^n)$$
$$\longrightarrow C_0^! = R^n \ .$$

The homology R-modules of $C^!$ are given by

$$H_0(C^!) = \mathrm{coker}(\begin{bmatrix} 1-X \\ 1-t \end{bmatrix} : R^n \oplus R^n \longrightarrow R^n) = \phi^!\bar{P} \ ,$$

$$H_1(C^!) = \ker((t-1 \ \ 1-X^t) : R^n \longrightarrow R^n \oplus R^n) \ ,$$

and in many respects $C^!$ is like a f.g. projective R-module resolution of $\phi^!\bar{P}$. However, $C^!$ is a genuine resolution of $\phi^!\bar{P}$ (with $H_1(C^!) = 0$) if and only if $t-1 \in R$ is a non-zero-divisor. By Proposition 1.1 there exists a pseudoresolution $C^!$ of $\phi^!\bar{P}$ for any f.g. projective $\bar{R}$-module $\bar{P}$. As for uniqueness, we have:

<u>Proposition 1.2</u> Given pseudoprojections $(X,Y) \in P_n(R,\phi)$, $(X',Y') \in P_{n'}(R,\phi)$ and a morphism of f.g. projective $\bar{R}$-modules

$$\bar{f} : \bar{P} = \mathrm{im}(\bar{X}) \longrightarrow \bar{P}' = \mathrm{im}(\bar{X}')$$

there is defined an R-module chain map of the associated pseudoresolutions

$$f^! : C^! \longrightarrow C'^!$$

uniquely up to chain homotopy, such that

$$(f^!)_* = \phi^!\bar{f} : H_0(C^!) = \phi^!\bar{P} \longrightarrow H_0(C'^!) = \phi^!\bar{P}' \ .$$

The construction of $f^!$ is functorial up to chain homotopy, with

$$1^! = 1 \ , \ (f'f)^! = f'^! f^!$$

up to chain homotopy. In particular, if $\bar{f} \in \mathrm{Hom}_{\bar{R}}(\bar{P},\bar{P}')$ is an isomorphism then $f^! : C^! \longrightarrow C'^!$ is a chain equivalence.

Proof: Let $\overline{F} \in M_{n,n'}(\overline{R})$ be the matrix of the composite $\overline{R}$-module morphism

$$\overline{F} : \overline{R}^n \xrightarrow{\text{projection}} \mathrm{im}(\overline{X}) = \overline{P} \xrightarrow{\overline{f}} \overline{P}' = \mathrm{im}(\overline{X}') \rightarrowtail^{\text{inclusion}} \overline{R}^{n'} .$$

Choose a lift $F \in M_{n,n'}(R)$ of $\overline{F}$ and define

$$F^! = \begin{pmatrix} XFX' & XFY' - YF^tX'^t \\ 0 & X^tF^tX'^t \end{pmatrix} \in M_{2n,2n'}(R)$$

such that

$$X^!F^! = F^!X'^! \in M_{2n,2n'}(R) .$$

The R-module chain map $f^! : C^! \longrightarrow C'^!$ is defined by

$$\begin{array}{ccccc} C^! : & C_1^! = \mathrm{coker}(X^!) & \xrightarrow{\begin{bmatrix} 1-X \\ 1-t \end{bmatrix}} & C_0^! = R^n \\ f^! \Big\downarrow & \Big\downarrow [F^!] & & \Big\downarrow XFX' \\ C'^! : & C_1'^! = \mathrm{coker}(X'^!) & \xrightarrow{\begin{bmatrix} 1-X' \\ 1-t \end{bmatrix}} & C_0'^! = R^{n'} \end{array} .$$

If $F_1, F_2 \in M_{n,n'}(R)$ are two different lifts of $\overline{F}$ there exists $G \in M_{n,n'}(R)$ such that

$$F_1 - F_2 = G(t-1) \in M_{n,n'}(R) ,$$

and the R-module morphism

$$g^! = [0 \quad XGX'^t] : C_0^! = R^n \longrightarrow C_1'^! = \mathrm{coker}(X'^!)$$

defines a chain homotopy

$$g^! : f_1^! \simeq f_2^! : C^! \longrightarrow C'^!$$

between the corresponding R-module chain maps $f_1^!, f_2^! : C^! \longrightarrow C'^!$.

If $(X,Y) = (X',Y') \in P_n(R,\phi)$ and $\overline{f} = 1 : \overline{P} = \mathrm{im}(\overline{X}) \longrightarrow \overline{P} = \mathrm{im}(\overline{X})$ then $F = X \in M_n(R)$ is a lift of the composite $\overline{R}$-module morphism

$$\overline{F} = \overline{X} : \overline{R}^n \xrightarrow{\text{projection}} \overline{P} \rightarrowtail^{\text{inclusion}} \overline{R}^n ,$$

so that

$$F^! = \begin{pmatrix} X^3 & 0 \\ 0 & (X^t)^3 \end{pmatrix} \in M_{2n}(R)$$

and the R-module morphism

$$h = [1+X+X^2 \quad 0] : C_0^! = R^n \longrightarrow C_1^! = \mathrm{coker}(X^!)$$

defines a chain homotopy

$$h : f^! \simeq 1 : C^! \longrightarrow C^! \ .$$

Given pseudoprojections $(X,Y) \in P_n(R,\phi)$, $(X',Y') \in P_{n'}(R,\phi)$, $(X'',Y'') \in P_{n''}(R,\phi)$ and $\bar{R}$-module morphisms

$$\bar{f} : \bar{P} = \text{im}(\bar{X}) \longrightarrow \bar{P}' = \text{im}(\bar{X}') \ , \ \bar{f}' : \bar{P}' = \text{im}(\bar{X}') \longrightarrow \bar{P}'' = \text{im}(\bar{X}'')$$

let

$$\bar{f}'' = \bar{f}'\bar{f} : \bar{P} \xrightarrow{\bar{f}} \bar{P}' \xrightarrow{\bar{f}'} \bar{P}''$$

be the composite $\bar{R}$-module morphism. If $F \in M_{n,n'}(R)$ and $F' \in M_{n',n''}(R)$ are lifts of the composite $\bar{R}$-module morphisms

$$\bar{F} : \bar{R}^n \twoheadrightarrow \bar{P} \xrightarrow{\bar{f}} \bar{P}' \rightarrowtail \bar{R}^{n'}$$

$$\bar{F}' : \bar{R}^{n'} \twoheadrightarrow \bar{P}' \xrightarrow{\bar{f}'} \bar{P}'' \rightarrowtail \bar{R}^{n''}$$

then the product

$$F'' = FX'^2F' \in M_{n,n''}(R)$$

is a lift of the composite $\bar{R}$-module morphism

$$\bar{F}'' : \bar{R}^n \twoheadrightarrow \bar{P} \xrightarrow{\bar{f}''} \bar{P}'' \rightarrowtail \bar{R}^{n''}$$

such that

$$F''^! = F^!F'^! \in M_{2n,2n''}(R) \ ,$$

and so

$$f''^! = f'^!f^! : C^! \xrightarrow{f^!} C'^! \xrightarrow{f'^!} C''^! \ .$$

[]

§2. The projective class transfer

Proposition 2.1 Given a ring R with pseudostructure $\phi = (\alpha,t)$ there is defined an algebraic transfer map in the projective class groups

$$\phi_0^! : K_0(\bar{R}) \longrightarrow K_0(R) \ ; \ [\bar{P}] \longmapsto [\text{im}(X^!)] - [R^n] \ ,$$

sending a f.g. projective $\bar{R}$-module $\bar{P} = \text{im}(\bar{X})$ $(\bar{X} \in P_n(\bar{R}))$ to the projective class $[C^!] = [\text{im}(X^!)] - [R^n] \in K_0(R)$ $((X,Y) \in P_n(R,\phi))$ of any pseudoresolution $C^!$ of $\phi^!\bar{P}$. If $\bar{P}$ is a (stably) f.g. free $\bar{R}$-module then $\phi_0^!([\bar{P}]) = 0 \in K_0(R)$, so that there is also defined an algebraic transfer map in the reduced projective class groups

$$\tilde{\phi}_0^! : \tilde{K}_0(\bar{R}) \longrightarrow \tilde{K}_0(R) \ ; \ [\bar{P}] \longmapsto [\text{im}(X^!)] \ .$$

Proof: Given a f.g. projective $\overline{R}$-module $\overline{P}$ use Proposition 1.1 to lift a projection $\overline{X} \in P_n(\overline{R})$ such that $\overline{P} = \mathrm{im}(\overline{X})$ to a pseudoprojection $(X,Y) \in P_n(R,\phi)$, and let $C^! : \mathrm{im}(X^!) \longrightarrow R^n$ be the corresponding pseudoresolution of $\phi^!\overline{P}$. Up to R-module isomorphism

$$\mathrm{im}(X^!)\oplus\mathrm{coker}(X^!) = \mathrm{im}(X^!)\oplus\mathrm{im}(1-X^!) = R^{2n} ,$$

so that

$$[C^!] = [R^n] - [\mathrm{coker}(X^!)]$$
$$= [\mathrm{im}(X^!)] - [R^n] = \phi_0^!([\overline{P}]) \in K_0(R) .$$

An element of $K_0(\overline{R})$ is the formal difference $[\overline{P}] - [\overline{P}']$, for some f.g. projective $\overline{R}$-modules $\overline{P} = \mathrm{im}(\overline{X})$, $\overline{P}' = \mathrm{im}(\overline{X}')$. Now $[\overline{P}] - [\overline{P}'] = 0 \in K_0(\overline{R})$ if and only if there exists an $\overline{R}$-module isomorphism $\overline{f} : \overline{P}\oplus\overline{Q} \xrightarrow{\simeq} \overline{P}'\oplus\overline{Q}$ for some f.g. projective $\overline{R}$-module $\overline{Q}$, in which case Proposition 1.2 gives a chain equivalence $f^! : C^! \xrightarrow{\simeq} C'^!$ of the corresponding pseudoresolutions of $\phi^!\overline{P},\phi^!\overline{P}'$. As the projective class of a chain complex is a chain homotopy invariant it follows that

$$\phi_0^!([\overline{P}] - [\overline{P}']) = [C^!] - [C'^!] = 0 \in K_0(R) ,$$

and so $\phi_0^!:K_0(\overline{R}) \longrightarrow K_0(R)$ is well-defined.

For $\overline{P} = \overline{R}^n$ take $\overline{X} = 1 \in P_n(\overline{R})$, $(X,Y) = (1,0) \in P_n(R,\phi)$, so that the projection

$$X^! = \begin{pmatrix} 1 & 0 \\ t-1 & 0 \end{pmatrix} : R^n\oplus R^n \longrightarrow R^n\oplus R^n$$

has $\mathrm{im}(X^!) \cong R^n$ and so

$$\phi_0^!([\overline{R}^n]) = [R^n] - [R^n] = 0 \in K_0(R) .$$

Thus $\widetilde{\phi}_0^!:\widetilde{K}_0(\overline{R}) \longrightarrow \widetilde{K}_0(R)$ is also well-defined.

[]

The original algebraic description in terms of matrices of the Whitehead group S^1-bundle transfer map

$$p^*_{Wh} = \widetilde{\phi}_1^! : Wh(\rho) \longrightarrow Wh(\pi)$$

due to Munkholm and Pedersen [3] was reformulated by Ranicki [6,§7.8] in terms of the theory of pseudo chain complexes. We shall now recall this theory, and show how it applies to the projective class group S^1-bundle transfer.

Given an R-module M let M^t denote the R-module with the same additive group and

$$R \times M^t \longrightarrow M^t \; ; \; (r,x) \longmapsto r^{-t}x \; ,$$

where $\alpha^{-1}: R \xrightarrow{\sim} R; r \longmapsto r^{-t}$ is the inverse of the ring automorphism $\alpha: R \xrightarrow{\sim} R; r \longmapsto r^t$ in the pseudostructure $\phi = (\alpha, t)$. An R-module morphism $f \in \mathrm{Hom}_R(M,N)$ also defines an R-module morphism

$$f^t : M^t \longrightarrow N^t \; ; \; x \longmapsto f(x) \; ,$$

such that

$$f(t-1) = (t-1)f^t : M^t \longrightarrow N$$

with $t-1 \in \mathrm{Hom}_R(M^t, M)$ defined by

$$t-1 : M^t \longrightarrow M \; ; \; x \longmapsto tx - x \; .$$

For $M = R^n$ use the R-module isomorphism

$$M^t \xrightarrow{\sim} R^n \; ; \; (r_1, r_2, \ldots, r_n) \longmapsto (r_1^t, r_2^t, \ldots, r_n^t)$$

to identify $M^t = R^n$, so that $t-1 \in \mathrm{Hom}_R(M^t, M)$ has matrix $t-1 \in M_n(R)$. If $f \in \mathrm{Hom}_R(R^m, R^n)$ has matrix $X = (x_{ij}) \in M_{m,n}(R)$ then $f^t \in \mathrm{Hom}_R((R^m)^t, (R^n)^t) = \mathrm{Hom}_R(R^m, R^n)$ has matrix $X^t = (x_{ij}^t) \in M_{m,n}(R)$.

A <u>pseudo chain complex over (R,ϕ)</u> $\mathcal{C} = (C,d,e)$ consists of a collection of R-modules $\{C_r \,|\, r \geq 0\}$ and two collections of R-module morphisms $\{d \in \mathrm{Hom}_R(C_r, C_{r-1}) \,|\, r \geq 1\}$, $\{e \in \mathrm{Hom}_R(C_r, C_{r-2}^t) \,|\, r \geq 2\}$ such that

$$d^2 = (t-1)e : C_r \longrightarrow C_{r-2} \; , \; d^t e = ed : C_r \longrightarrow C_{r-3}^t \; .$$

Note that $\mathcal{C}$ determines an $\bar{R}$-module chain complex $\bar{C}$ with

$$d_{\bar{C}} = 1 \otimes d : \bar{C}_r = \bar{R} \otimes_R C_r \longrightarrow \bar{C}_{r-1} = \bar{R} \otimes_R C_{r-1} \; ; \; a \otimes x \longmapsto a \otimes d(x) \; ,$$

and an R-module chain complex $C^!$ with

$$d_{C^!} = \begin{pmatrix} d & (-)^r e \\ (-)^r (t-1) & d^t \end{pmatrix}$$

$$: C_r^! = C_r \oplus C_{r-1}^t \longrightarrow C_{r-1}^! = C_{r-1} \oplus C_{r-2}^t \; ;$$

$$(x,y) \longmapsto (d(x) + (-)^r (t-1)(y) \; , \; (-)^r e(x) + d^t(y)) \; .$$

Proposition 7.8.8 of Ranicki [6] associates to an S^1-bundle of CW complexes $S^1 \longrightarrow E \xrightarrow{p} B$ with $p_* = \phi : \pi_1(E) = \pi \twoheadrightarrow \pi_1(B) = \rho$ a pseudo chain complex $\mathcal{C}(p) = (C,d,e)$ over $(\mathbb{Z}[\pi], \phi)$ with C_r $(r \geq 0)$ the f.g. free $\mathbb{Z}[\pi]$-module of rank the number of r-cells in B, such that the cellular chain complexes of the universal covers $\tilde{B}, \tilde{E}$ of B,E are given by

$$C(\widetilde{B}) = \bar{C}\ ,\ C(\widetilde{E}) = C^! \ .$$

If B is finitely dominated then so is E, and the Wall finiteness obstructions are given by the reduced projective classes

$$[B] = [C(\widetilde{B})] = [\bar{C}] \in \widetilde{K}_0(\mathbb{Z}[\rho])\ ,$$
$$[E] = [C(\widetilde{E})] = [C^!] \in \widetilde{K}_0(\mathbb{Z}[\pi])\ .$$

The <u>geometric transfer map</u> $p^*_{K_0} : \widetilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi])$ is defined by

$$p^*_{K_0}([B]) = [E] \in \widetilde{K}_0(\mathbb{Z}[\pi])\ ,$$

so that it will follow from the identification $p^*_{K_0} = \widetilde{\phi}^!_0$ in §4 below that

$$[C^!] = [C(\widetilde{E})] = [E]$$
$$= p^*_{K_0}([B]) = \widetilde{\phi}^!_0([\bar{C}]) \in \widetilde{K}_0(\mathbb{Z}[\pi])\ .$$

In Ranicki [8] it will be shown algebraically that for any finitely dominated pseudo chain complex $\mathscr{C} = (C,d,e)$ over a ring with pseudostructure (R,ϕ) the algebraic transfer map $\phi^!_0 : K_0(\bar{R}) \longrightarrow K_0(R)$ sends the projective class $[\bar{C}] \in K_0(\bar{R})$ to

$$\phi^!_0([\bar{C}]) = [C^!] \in K_0(R)$$

(which will give an alternative proof of $p^*_{K_0} = \widetilde{\phi}^!_0$ on setting $R = \mathbb{Z}[\pi]\ , \mathscr{C} = \mathscr{C}(p)$). At any rate, for any pseudoprojection $(X,Y) \in P_n(R,\phi)$ there is defined a finitely dominated pseudo chain complex $\mathscr{C} = (C,d,e)$ over (R,ϕ) with

$$d = \begin{cases} 1-X : C_{2i+1} = R^n \longrightarrow C_{2i} = R^n \\ X : C_{2i+2} = R^n \longrightarrow C_{2i+1} = R^n \end{cases} \quad (i \geqslant 0)$$
$$e = Y : C_j = R^n \longrightarrow C^t_{j-2} = R^n \quad (j \geqslant 2)$$

for which

$$[\bar{C}] = [\mathrm{im}(\bar{X})] \in K_0(\bar{R})\ ,$$
$$[C^!] = [\mathrm{im}(X^!)] - [R^n] = \phi^!_0([\bar{C}]) \in K_0(R)\ .$$

Note that $C^!$ is an infinite f.g. free R-module chain complex which is chain equivalent to the f.g. projective pseudoresolution $C^!$ of $\phi^!(\mathrm{im}(\bar{X}))$ associated to $(X,Y) \in P_n(R,\phi)$ in §1 above.

In the case when $t-1 \in R$ is a non-zero-divisor (which for a group ring $R = \mathbb{Z}[\pi]$ is equivalent to $t \in \pi$ being of infinite order) $\phi^!\bar{R}$ is an R-module of homological dimension 1, with a f.g. free R-module resolution

$$0 \longrightarrow R \xrightarrow{\ t-1\ } R \xrightarrow{\ \phi\ } \phi^!\bar{R} \longrightarrow 0\ .$$

If $\bar{P}$ is a f.g. projective $\bar{R}$-module then $\phi^!\bar{P}$ is therefore an R-module of homological dimension 1, with a f.g. projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \phi^!\bar{P} \longrightarrow 0 \ .$$

The <u>classical transfer map</u> in the projective class groups is defined by

$$\phi^! : K_0(\bar{R}) \longrightarrow K_0(R) \ ; \ [\bar{P}] \longmapsto [P_0] - [P_1] \ ,$$

and this definition extends by the Bass-Quillen resolution theorem to transfer maps in the higher K-groups

$$\phi^! : K_m(\bar{R}) \longrightarrow K_m(R) \quad (m \geqslant 1) \ .$$

(More generally, the classical methods give transfer maps $\phi^!:K_*(\bar{R}) \longrightarrow K_*(R)$ for any morphism of rings $\phi:R \longrightarrow \bar{R}$ such that $\phi^!\bar{R}$ is an R-module of finite homological dimension).

<u>Proposition 2.2</u> If (R,ϕ) is a ring with pseudostructure such that $t-1 \in R$ is a non-zero-divisor the projective class group transfer map $\phi^!_0$ defined above agrees with the classical transfer map

$$\phi^!_0 = \phi^! : K_0(\bar{R}) \longrightarrow K_0(R) \ .$$

<u>Proof</u>: In this case the pseudoresolution $C^!$ of $\phi^!(\mathrm{im}(\bar{X}))$ associated to a pseudoprojection $(X,Y) \in P_n(R,\phi)$ in §1 above is a 1-dimensional f.g. projective R-module resolution of $\phi^!(\mathrm{im}(\bar{X}))$

$$0 \longrightarrow \mathrm{coker}(X^!) \longrightarrow R^n \xrightarrow{\bar{X}\phi} \phi^!(\mathrm{im}(\bar{X})) \longrightarrow 0 \ ,$$

so that

$$\phi^!_0([\mathrm{im}(\bar{X})]) = [C^!] = \phi^!([\mathrm{im}(\bar{X})]) \in K_0(R) \ .$$

[]

For a group ring $R = \mathbb{Z}[\pi]$ the identification $\tilde{\phi}^!_0 = \tilde{\phi}^! : \tilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi])$ given by Proposition 2.2 may also be obtained by combining the identifications $\tilde{\phi}^!_0 = p^*_{K_0}$ of §4 and $p^*_{K_0} = \tilde{\phi}^!$ of Munkholm and Pedersen [2].

In Proposition 3.2 below the algebraic S^1-bundle transfer map $\phi^!_1 : K_1(\bar{R}) \longrightarrow K_1(R)$ of Munkholm and Pedersen [3] in the case when $t-1 \in R$ is a non-zero-divisor will be similarly identified with the classical transfer map $\phi^! : K_1(\bar{R}) \longrightarrow K_1(R)$. It would be interesting to know if the definitions of $\phi^!_0$ and $\phi^!_1$ extend to algebraic transfer maps in the higher K-groups

$$\phi^!_m : K_m(\bar{R}) \longrightarrow K_m(R) \quad (m \geqslant 2)$$

in the case when $t-1 \in R$ is a zero divisor, so that $\phi^!\overline{R}$ is an R-module of infinite homological dimension and the classical methods fail.

§3. The Whitehead torsion transfer

The Whitehead torsion transfer map of Munkholm and Pedersen [3] was defined for any ring with pseudostructure (R,ϕ) to be

$$\phi_1^! : K_1(\overline{R}) \longrightarrow K_1(R) \ ; \ \tau(\overline{X}) \longmapsto \tau\begin{pmatrix} X & -Z \\ t-1 & Y^t \end{pmatrix}$$

with $X \in M_n(R)$ a lift of $\overline{X} \in GL_n(\overline{R})$ and $Y,Z \in M_n(R)$ such that

$$XY = 1 - Z(t-1) \in M_n(R) \ .$$

In Ranicki [6,§7.8] $\phi_1^!(\tau(\overline{X})) \in K_1(R)$ was interpreted as the torsion $\tau(C^!)$ of the based acyclic R-module chain complex

$$C^! : R^n \xrightarrow{(1-t \ \ X^t)} R^n \oplus R^n \xrightarrow{\begin{pmatrix} X \\ t-1 \end{pmatrix}} R^n$$

associated to the pseudo chain complex $\mathcal{C} = (C,d,e)$ with

$$d = X : C_1 = R^n \longrightarrow C_0 = R^n \ , \ C_r = 0 \ (r \geqslant 2) \ , \ e = 0 \ ,$$

for which

$$\tau(\overline{C}) = \tau(\overline{X}:\overline{R}^n \xrightarrow{\sim} \overline{R}^n) \in K_1(\overline{R}) \ .$$

(The identification $\phi_1^!(\tau(\overline{X})) = \tau(C^!) \in K_1(R)$ is immediate from the observation that $\begin{pmatrix} -Z \\ Y^t \end{pmatrix} : R^n \oplus R^n \twoheadrightarrow R^n$ is a splitting map for $(1-t \ \ X^t) : R^n \rightarrowtail R^n \oplus R^n$). It will be shown in Ranicki [8] that for any finite pseudo chain complex $\mathcal{C} = (C,d,e)$ over (R,ϕ) with each C_r $(r \geqslant 0)$ a based f.g. free R-module with $\overline{C}$ (and hence $C^!$) acyclic

$$\phi_1^!(\tau(\overline{C})) = \tau(C^!) \in K_1(R) \ .$$

We shall now interpret $\phi_1^!$ in terms of the pseudoresolution construction $(X,Y) \longmapsto C^!$ of §1.

Proposition 3.1 The Whitehead torsion transfer map

$$\phi_1^! : K_1(\overline{R}) \longrightarrow K_1(R)$$

sends the torsion $\tau(\overline{f}) \in K_1(\overline{R})$ of an automorphism $\overline{f} \in \mathrm{Hom}_{\overline{R}}(\overline{P},\overline{P})$ of a f.g. projective $\overline{R}$-module $\overline{P}$ to the torsion

$$\phi_1^!(\tau(\overline{f})) = \tau(f^!) \in K_1(R)$$

of the induced self chain equivalence $f^!:C^! \xrightarrow{\sim} C^!$, with $C^!$ the pseudoresolution of $\phi^!\bar{P}$ associated to any pseudoprojection $(X,Y)\in P_n(R,\phi)$ with $\bar{P}=\mathrm{im}(\bar{X})$.

Proof: Stabilizing $\bar{f}$ by $1\in \mathrm{Hom}_{\bar{R}}(\mathrm{im}(1-\bar{X}),\mathrm{im}(1-\bar{X}))$ it may be assumed that $\bar{P}=\bar{R}^n$ is a f.g. free $\bar{R}$-module, and $(X,Y)=(1,0)\in P_n(R,\phi)$, so that $C^!:R^n\xrightarrow{1-t}R^n$.

If $\bar{f}\in \mathrm{Aut}_{\bar{R}}(\bar{R}^n,\bar{R}^n)$ has matrix $\bar{X}\in GL_n(\bar{R})$ then

$$\begin{array}{ccccc} C^! & : & R^n & \xrightarrow{1-t} & R^n \\ f^! \downarrow \wr & & \downarrow X^t & & \downarrow X \\ C^! & : & R^n & \xrightarrow{1-t} & R^n \end{array}$$

for any lift $X\in M_n(R)$ of $\bar{X}$, so that

$$\tau(f^!) = \tau(C(f^!):R^n\xrightarrow{(1-t\ \ X^t)}R^n\oplus R^n\xrightarrow{\begin{pmatrix}X\\ t-1\end{pmatrix}}R^n)$$
$$=\phi_1^!(\tau(\bar{X})) = \phi_1^!(\tau(\bar{f}))\in K_1(R)\ .$$

[]

By analogy with Proposition 2.2:

Proposition 3.2 If $t-1\in R$ is a non-zero-divisor the Whitehead torsion transfer map $\phi_1^!$ agrees with the classical transfer map

$$\phi_1^! = \phi^! : K_1(\bar{R})\longrightarrow K_1(R)\ .$$

Proof: Given an automorphism $\bar{f}\in \mathrm{Aut}_{\bar{R}}(\bar{R}^n,\bar{R}^n)$ note that the self chain equivalence $f^!:C^!\xrightarrow{\simeq}C^!$ defined in the proof of Proposition 3.1 is a resolution of the automorphism $\phi^!\bar{f}\in \mathrm{Aut}_R(\phi^!\bar{R}^n,\phi^!\bar{R}^n)$, so that

$$\phi_1^!(\tau(\bar{f})) = \tau(f^!) = \phi^!(\tau(\bar{f}))\in K_1(R)\ .$$

[]

For a group ring $R=\mathbb{Z}[\pi]$ the identification $\tilde{\phi}_1^!=\tilde{\phi}^!:Wh(\rho)\longrightarrow Wh(\pi)$ given by Proposition 3.2 may also be obtained by combining the identifications $\tilde{\phi}_1^!=p_{Wh}^*$ of Munkholm and Pedersen [3] and $p_{Wh}^*=\tilde{\phi}^!$ of Munkholm [1].

In §4 we shall make use of the following relation between the projective class group transfer $\phi_0^!:K_0(\bar{R})\longrightarrow K_0(R)$ for a ring with pseudostructure (R,ϕ), the Whitehead torsion transfer $(\phi\times 1)_1^!:K_1(\bar{R}[z,z^{-1}])\longrightarrow K_1(R[z,z^{-1}])$ for the polynomial extension ring with pseudostructure $(R[z,z^{-1}],\phi\times 1)$ and the canonical Bass-Heller-Swan injections

$$h_R : K_0(R) \rightarrowtail K_1(R[z,z^{-1}]) \; ; \; [P] \longmapsto \tau(z : P[z,z^{-1}] \xrightarrow{\sim} P[z,z^{-1}])$$

and $h_{\bar{R}} : K_0(\bar{R}) \rightarrowtail K_1(\bar{R}[z,z^{-1}])$ defined similarly.

<u>Proposition 3.3</u> There is defined a commutative diagram

$$\begin{array}{ccc} K_0(\bar{R}) & \xrightarrow{\quad \phi_0^! \quad} & K_0(R) \\ {\scriptstyle h_{\bar{R}}} \big\downarrow & & \big\downarrow {\scriptstyle h_R} \\ K_1(\bar{R}[z,z^{-1}]) & \xrightarrow{(\phi \times 1)_1^!} & K_1(R[z,z^{-1}]) \end{array} \quad .$$

<u>Proof</u>: Given a f.g. projective $\bar{R}$-module $\bar{P}$ let $(X,Y) \in P_n(R,\phi)$ be a pseudoprojection such that $\bar{P} = \mathrm{im}(\bar{X})$, and let $C^!$ be the corresponding pseudoresolution of $\phi^!\bar{P}$. Now

$$\begin{aligned} (\phi\times 1)_1^! h_{\bar{R}}([\bar{P}]) &= (\phi\times 1)_1^!(\tau(z:\bar{P}[z,z^{-1}] \xrightarrow{\sim} \bar{P}[z,z^{-1}])) \\ &= \tau(z : C^![z,z^{-1}] \xrightarrow{\sim} C^![z,z^{-1}]) \quad \text{(by Proposition 3.1)} \\ &= h_R([C^!]) = h_R\phi_0^!([\bar{P}]) \in K_1(R[z,z^{-1}]) \;, \end{aligned}$$

so that $(\phi\times 1)_1^! h_{\bar{R}} = h_R \phi_0^!$.

[]

§4. <u>The algebraic and geometric transfer maps coincide</u>

Let $S^1 \longrightarrow E \xrightarrow{\;p\;} B$ be an S^1-bundle with $p_* = \phi : \pi_1(E) = \pi \longrightarrow\!\!\!\!\rightarrow \pi_1(B) = \rho$, and let $(R = \mathbb{Z}[\pi], \phi)$ be the corresponding ring with pseudostructure.

<u>Proposition 4.1</u> The algebraic and geometric transfer maps in the reduced projective class groups coincide, that is

$$\tilde{\phi}_0^! = p^*_{K_0} \; : \; \tilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi]) \; .$$

<u>Proof</u>: We offer two proofs, in fact.

i) Given a pseudoprojection $(X,Y) \in P_n(\mathbb{Z}[\pi],\phi)$ and a number $m \geqslant 2$ the proof of Theorem F of Wall [10] gives an S^1-bundle of CW pairs

$$S^1 \longrightarrow (E,F) \xrightarrow{\;(p,q)\;} (B,K)$$

with K finite and B finitely dominated, such that $\pi_1(B) = \pi_1(K) = \rho$ and such that the relative pseudo chain complex $\mathcal{E}(p,q) = (C,d,e)$ is given by

$$C_r = \begin{cases} \mathbb{Z}[\pi]^n & \text{if } r \geqslant 2m \\ 0 & \text{if } r \leqslant 2m-1 \end{cases}$$

$$d = \begin{cases} 1-X : C_{2i+1} \longrightarrow C_{2i} \\ X : C_{2i+2} \longrightarrow C_{2i+1} \end{cases} \quad (i \geqslant m)$$

$$e = Y : C_r \longrightarrow C^t_{r-2} \quad (r \geqslant 2m+2) .$$

The finiteness obstruction of B (= the reduced projective class of $C(\widetilde{B}) = \overline{C}$) is given by

$$[B] = [\overline{C}] = [\mathrm{im}(\overline{X})] \in \widetilde{K}_0(\mathbb{Z}[\rho]) ,$$

and that of E by

$$[E] = [C^!] = [\mathrm{im}(X^!)] \in \widetilde{K}_0(\mathbb{Z}[\pi]) ,$$

so that

$$\begin{aligned} p^*_{K_0}([B]) &= [E] = [\mathrm{im}(X^!)] \\ &= \widetilde{\phi}^!_0([\mathrm{im}(\overline{X})]) = \widetilde{\phi}^!_0([B]) \in \widetilde{K}_0(\mathbb{Z}[\pi]) . \end{aligned}$$

ii) Consider the commutative diagram preceding Corollary 2.3 of Munkholm and Pedersen [4]

$$\begin{array}{ccc} \widetilde{K}_0(\mathbb{Z}[\pi]) & \overset{\overline{h}_\pi}{\twoheadleftarrow} & Wh(\pi\times\mathbb{Z}) \\ p^*_{K_0} \uparrow & & \uparrow (p\times 1)^*_{Wh} = (\widetilde{\phi\times 1})^!_1 \\ \widetilde{K}_0(\mathbb{Z}[\rho]) & \overset{h_\rho}{\rightarrowtail} & Wh(\rho\times\mathbb{Z}) \end{array}$$

in which $\overline{h}_\pi$ (resp. h_ρ) is the canonical Bass-Heller-Swan surjection (resp. injection). From Proposition 3.3 we have $(\widetilde{\phi\times 1})^!_1 h_\rho = h_\pi \widetilde{\phi}^!_0$, so that

$$\begin{aligned} p^*_{K_0} &= \overline{h}_\pi (\widetilde{\phi\times 1})^!_1 h_\rho \\ &= \overline{h}_\pi h_\pi \widetilde{\phi}^!_0 = \widetilde{\phi}^!_0 : \widetilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi]) . \end{aligned}$$

[]

§5. The relative transfer exact sequence

A ring morphism $\phi : R \longrightarrow S$ induces morphisms in the algebraic K-groups

$$\phi_! : K_0(R) \longrightarrow K_0(S) \; ; \; [P] \longmapsto [\phi_! P] \; , \; \phi_! P = S\otimes_R P$$

$$\phi_! : K_1(R) \longrightarrow K_1(S) \; ; \; \tau(X) \longmapsto \tau(\phi(X)) \; , \; X \in GL_n(R)$$

which are related by a change of rings exact sequence

$$K_1(R) \xrightarrow{\phi_!} K_1(S) \xrightarrow{j} K_1(\phi_!) \xrightarrow{\partial} K_0(R) \xrightarrow{\phi_!} K_0(S)$$

with $K_1(\phi_!)$ the relative K-group of stable isomorphism classes of pairs (P,f) consisting of a f.g. projective R-module P and an S-module isomorphism $f : \phi_! P \xrightarrow{\sim} S^n$, with $(R^n,1) = 0 \in K_1(\phi_!)$ and

$$j : K_1(S) \longrightarrow K_1(\phi_!) \ ; \ \tau(Z) \longmapsto (R^n,Z) \ , \ Z \in GL_n(S)$$

$$\partial : K_1(\phi_!) \longrightarrow K_0(R) \ ; \ (P,f) \longmapsto [P] - [R^n] \ .$$

We shall now obtain an analogous exact sequence for the transfer maps

$$K_1(\bar{R}) \xrightarrow{\phi_1^!} K_1(R) \xrightarrow{j} K_1(\phi^!) \xrightarrow{\partial} K_0(\bar{R}) \xrightarrow{\phi_0^!} K_0(R) \ ,$$

relating the projective class group transfer $\phi_0^!$ of §2 to the Whitehead torsion transfer $\phi_1^!$ of §3.

A <u>base</u> (S,T) for a pseudoprojection $(X,Y) \in P_n(R,\phi)$ is a pair of matrices

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \in M_{2n,m}(R) \quad , \quad T = (T_1 \ T_2) \in M_{m,2n}(R)$$

with $S_1, S_2 \in M_{n,m}(R)$, $T_1, T_2 \in M_{m,n}(R)$ such that

$$ST = X^! \in M_{2n}(R) \quad , \quad TS = 1 \in M_m(R) \ .$$

The factorization of R-module morphisms

$$X^! = \begin{pmatrix} X & Y \\ t-1 & 1-X^t \end{pmatrix} : R^n \oplus R^n \xrightarrow{S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}} R^m \xrightarrow{T = (T_1 \ T_2)} R^n \oplus R^n$$

shows that a base (S,T) of (X,Y) determines a base (in the usual sense) of the f.g. projective R-module $\text{im}(X^!) \subseteq R^n \oplus R^n$ consisting of m elements. Conversely, if $\text{im}(X^!)$ is a f.g. free R-module of rank m then a choice of base for $\text{im}(X^!)$ determines a factorization

$$X^! : R^n \oplus R^n \xrightarrow{S} R^m \xrightarrow{T} R^n \oplus R^n$$

with S onto and T one-one; it follows from the identity

$$\begin{aligned} S(TS-1)T &= ST(ST-1) \\ &= X^!(X^!-1) = 0 \in M_{2n}(R) \end{aligned}$$

that $TS = 1 \in M_m(R)$, and so (S,T) defines a base of (X,Y). There is thus a natural one-one correspondence between the bases (S,T) of the pseudoprojection (X,Y) and the bases of the f.g. projective R-module $\text{im}(X^!)$, if any such exist. In dealing with bases of pseudoprojections we shall assume that (R,ϕ) satisfies the following two conditions:

i) f.g. free $\bar{R}$-modules have a well-defined rank,

ii) $\bar{\alpha}^2 : \bar{R} \xrightarrow{\sim} \bar{R} ; \bar{r} \longmapsto (\bar{r}^t)^t$ is an inner automorphism of $\bar{R}$,

in which case $m = n$ for any pseudoprojection base (S,T): by i) $[\bar{R}] \in K_0(\bar{R})$ generates an infinite cyclic subgroup of $K_0(\bar{R})$, and by ii) $\bar{\alpha}_! : K_0(\bar{R}) \xrightarrow{\sim} K_0(\bar{R}) ; [\bar{P}] \longmapsto [\bar{P}^t]$ is an involution of $K_0(\bar{R})$ fixing $[\bar{R}]$, so that if $(S,T) \in M_{2n,m}(R) \times M_{m,2n}(R)$ is a base for the pseudoprojection $(X,Y) \in P_n(R,\phi)$ the f.g. projective $\bar{R}$-module $\bar{P} = \mathrm{im}(1-\bar{X})$ is such that up to $\bar{R}$-module isomorphism

$$\bar{R}^m = \phi_!(\mathrm{im}(X^!)) = \mathrm{im}(\bar{X}) \oplus \bar{P}^t \quad , \quad \bar{R}^n = \mathrm{im}(\bar{X}) \oplus \bar{P} \quad ,$$

and it is clear from the action of $\bar{\alpha}_!$ on the identity

$$[\bar{P}] - [\bar{P}^t] = [\bar{R}^n] - [\bar{R}^m] \in K_0(\bar{R})$$

that $m = n$. In particular, the conditions i) and ii) are satisfied by the group rings with pseudostructure $(R = \mathbb{Z}[\pi], \phi)$ arising in topology.

A <u>based pseudoprojection</u> (X,Y,S,T) is a pseudoprojection $(X,Y) \in P_n(R,\phi)$ together with a base $(S,T) \in M_{2n,n}(R) \times M_{n,2n}(R)$. Given such an object define the associated <u>based pseudoresolution</u> of the R-module $\phi^!(\mathrm{im}(\bar{X}))$ to be the 1-dimensional based f.g. free R-module chain complex

$$D^! : R^n \xrightarrow{S_2} R^n$$

which is chain equivalent to the projective pseudoresolution $C^!$ of $\phi^!(\mathrm{im}(\bar{X}))$ associated to (X,Y) in §1. Explicitly, a chain equivalence $C^! \xrightarrow{\sim} D^!$ is defined by

$$\begin{array}{cccc} C^! : & \mathrm{coker}(X^!) & \xrightarrow{\begin{bmatrix} 1-X \\ 1-t \end{bmatrix}} & R^n \\ \wr\downarrow & \downarrow \begin{bmatrix} Y \\ -X^t \end{bmatrix} & & \downarrow XS_1+YS_2 \\ D^! : & R^n & \xrightarrow{S_2} & R^n \end{array} .$$

(This is the composite $C^! \xrightarrow{\sim} B^! \xrightarrow{\sim} D^!$ of the chain equivalence

$$\begin{array}{cccc} C^! : & \mathrm{coker}(X^!) & \xrightarrow{\begin{bmatrix} 1-X \\ 1-t \end{bmatrix}} & R^n \\ \wr\downarrow & \downarrow \begin{bmatrix} Y \\ -X^t \end{bmatrix} & & \downarrow [X \;\; Y] \\ B^! : & R^n & \xrightarrow{[t-1 \;\; 1-X^t]} & \mathrm{im}(X^!) \end{array}$$

(defined for any pseudoprojection (X,Y)) and the chain isomorphism

$$\begin{array}{ccccc} B^! & : & R^n & \xrightarrow{[t-1 \;\; 1-X^t]} & \mathrm{im}(X^!) \\ \simeq\big\downarrow & & \simeq\big\downarrow 1 & & \simeq\big\downarrow \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \\ D^! & : & R^n & \xrightarrow{\quad S_2 \quad} & R^n \end{array} \qquad) \; .$$

A <u>morphism</u> of based pseudoprojections over (R,ϕ)

$$f \; : \; (X,Y,S,T) \longrightarrow (X',Y',S',T')$$

is just a morphism of the associated f.g. projective $\bar{R}$-modules

$$\bar{f} \; : \; \mathrm{im}(\bar{X}) \longrightarrow \mathrm{im}(\bar{X}') \; .$$

Replacing the projective pseudoresolutions $C^!, C'^!$ in the construction of Proposition 1.2 by the chain equivalent based pseudoresolutions $D^!, D'^!$ there is obtained an R-module chain map

$$f^! \; : \; D^! \longrightarrow D'^!$$

inducing the R-module morphism

$$(f^!)_* = \phi^! \bar{f} \; : \; H_0(D^!) = \phi^!(\mathrm{im}(\bar{X})) \longrightarrow H_0(D'^!) = \phi^!(\mathrm{im}(\bar{X}')) \; ,$$

uniquely up to chain homotopy. More precisely, $f^!$ is defined by

$$\begin{array}{ccccc} D^! & : & R^n & \xrightarrow{\quad S_2 \quad} & R^n \\ f^!\big\downarrow & & \big\downarrow XFX' & & \big\downarrow TF^!S \\ D'^! & : & R^{n'} & \xrightarrow{\quad S_2' \quad} & R^{n'} \end{array}$$

with $F \in M_{n,n'}(R)$ the matrix of any R-module morphism $F \in \mathrm{Hom}_R(R^n, R^{n'})$ lifting the composite $\bar{R}$-module morphism

$$\bar{F} \; : \; \bar{R}^n \xrightarrow{\text{projection}} \mathrm{im}(\bar{X}) \xrightarrow{\quad \bar{f} \quad} \mathrm{im}(\bar{X}') \xrightarrow{\text{injection}} \bar{R}^{n'}$$

and

$$F^! = \begin{pmatrix} XFX' & XFY' - YF^tX'^t \\ 0 & X^tF^tX'^t \end{pmatrix} \in M_{2n,2n'}(R)$$

as before.

An <u>isomorphism</u> of based pseudoprojections is a morphism

$$f \; : \; (X,Y,S,T) \xrightarrow{\;\sim\;} (X',Y',S',T')$$

which is defined by an $\bar{R}$-module isomorphism $\bar{f} \in \mathrm{Hom}_{\bar{R}}(\mathrm{im}(\bar{X}), \mathrm{im}(\bar{X}'))$, in which case $f^! : D^! \xrightarrow{\;\sim\;} D'^!$ is a chain equivalence of based

R-module chain complexes and the torsion of f is defined by

$$\tau(f) = \tau(f^!:D^! \xrightarrow{\simeq} D'^!) \in K_1(R) \ .$$

In general, the torsion is an invariant of f but not of $\bar{f}$. However, if f is an automorphism (i.e. $(X,Y,S,T) = (X',Y',S',T')$) then the torsion $\tau(\bar{f}:\mathrm{im}(\bar{X}) \xrightarrow{\simeq} \mathrm{im}(\bar{X})) \in K_1(\bar{R})$ is defined, and Proposition 3.1 shows that

$$\tau(f) = \tau(f^!) = \phi_1^!(\tau(\bar{f})) \in K_1(R) \ .$$

An isomorphism $f:(X,Y,S,T) \xrightarrow{\simeq} (X',Y',S',T')$ is simple if

$$\tau(f) = 0 \in K_1(R) \ .$$

Define the relative transfer group $K_1(\phi^!)$ to be the abelian group with one generator for each simple isomorphism class of based pseudoprojections (X,Y,S,T) over (R,ϕ), with relations

$$(X,Y,S,T) + (X',Y',S',T') = (X\oplus X',Y\oplus Y',S\oplus S',T\oplus T') \in K_1(\phi^!) \ .$$

Proposition 5.1 The relative transfer group $K_1(\phi^!)$ fits into an exact sequence

$$K_1(\bar{R}) \xrightarrow{\phi_1^!} K_1(R) \xrightarrow{j} K_1(\phi^!) \xrightarrow{\partial} K_0(\bar{R}) \xrightarrow{\phi_0^!} K_0(R)$$

with

$$j : K_1(R) \longrightarrow K_1(\phi^!) \ ;$$

$$\tau(Z) \longmapsto (0,0,\begin{pmatrix} 0 \\ Z \end{pmatrix},(Z^{-1}(t-1) \ \ Z^{-1})) \quad (Z \in GL_n(R))$$

$$\partial : K_1(\phi^!) \longrightarrow K_0(\bar{R}) \ ; \ (X,Y,S,T) \longmapsto [\mathrm{im}(\bar{X})]$$

Proof: If $\bar{P},\bar{Q}$ are f.g. projective $\bar{R}$-modules such that

$$[\bar{P}] - [\bar{Q}] \in \ker(\phi_0^!:K_0(\bar{R}) \longrightarrow K_0(R))$$

let $-\bar{Q}$ be a f.g. projective inverse for $\bar{Q}$, so that $\bar{Q}\oplus-\bar{Q} = \bar{R}^m$ is a f.g. free $\bar{R}$-module, and let $(X,Y) \in P_n(R,\phi)$ be a pseudoprojection such that $\bar{P}\oplus-\bar{Q} = \mathrm{im}(\bar{X})$. Now

$$\begin{aligned}[\mathrm{im}(X^!)] - [R^n] &= \phi_0^!([\mathrm{im}(\bar{X})]) \\ &= \phi_0^!([\bar{P}] - [\bar{Q}] + [\bar{R}^m]) = 0 \in K_0(R) \ ,\end{aligned}$$

so that $\mathrm{im}(X^!)$ is a stably f.g. free R-module. Stabilizing $\bar{P},\bar{Q}$ if necessary it may be assumed that $\mathrm{im}(X^!)$ is an unstably f.g. free R-module. Choosing a base (S,T) for (X,Y) there is obtained an element $(X,Y,S,T) - (1,0,\begin{pmatrix} 1 \\ t-1 \end{pmatrix},(1 \ \ 0)) \in K_1(\phi^!)$ $(1 \in GL_m(R))$ such that

$$\begin{aligned}[\bar{P}] - [\bar{Q}] &= [\bar{P}\oplus\text{-}\bar{Q}] - [\bar{R}^m] \\ &= [\mathrm{im}(\bar{X})] - [\bar{R}^m] \\ &= \partial((X,Y,S,T) - (1,0,\begin{pmatrix}1\\ t-1\end{pmatrix},(1\ 0))) \\ &\in \mathrm{im}(\partial : K_1(\phi^!) \longrightarrow K_0(\bar{R})) ,\end{aligned}$$

verifying exactness at $K_0(\bar{R})$.

If $(X,Y,S,T),(X',Y',S',T')$ are based pseudoprojections such that

$$(X',Y',S',T') - (X,Y,S,T) \in \ker(\partial : K_1(\phi^!) \longrightarrow K_0(\bar{R}))$$

there exists a (stable) isomorphism

$$f : (X,Y,S,T) \xrightarrow{\sim} (X',Y',S',T') .$$

The torsion $\tau(f) \in K_1(R)$ is such that

$$\begin{aligned}(X',Y',S',T') - (X,Y,S,T) &= j(\tau(f)) \\ &\in \mathrm{im}(j : K_1(R) \longrightarrow K_1(\phi^!)) ,\end{aligned}$$

verifying exactness at $K_1(\phi^!)$.

If $Z \in GL_n(R)$ is such that $\tau(Z) \in \ker(j : K_1(R) \longrightarrow K_1(\phi^!))$ there exists a based pseudoprojection (X,Y,S,T) with a simple isomorphism

$$f : (X,Y,S,T)\oplus j\tau(Z) \xrightarrow{\sim} (X,Y,S,T) .$$

The automorphism of based pseudoprojections

$$g : (X,Y,S,T) \xrightarrow{\sim} (X,Y,S,T)$$

defined by the automorphism $\bar{f} \in \mathrm{Hom}_{\bar{R}}(\mathrm{im}(\bar{X}),\mathrm{im}(\bar{X}))$ is such that

$$\begin{aligned}\tau(Z) = \tau(g^!) &= \phi_1^!(\tau(\bar{f})) \\ &\in \mathrm{im}(\phi_1^! : K_1(\bar{R}) \longrightarrow K_1(R)) ,\end{aligned}$$

verifying exactness at $K_1(R)$.

[]

For the group ring with pseudostructure $(R = \mathbb{Z}[\pi],\phi)$ associated to an S^1-bundle $S^1 \longrightarrow E \xrightarrow{p} B$ with $p_* = \phi : \pi_1(E) = \pi \longrightarrow \pi_1(B) = \rho$, $\bar{R} = \mathbb{Z}[\rho]$ there is also defined a reduced version of the exact sequence of Proposition 5.1

$$Wh(\rho) \xrightarrow{\tilde{\phi}_1^!} Wh(\pi) \xrightarrow{\tilde{j}} Wh(\phi^!) \xrightarrow{\tilde{\partial}} \tilde{K}_0(\mathbb{Z}[\rho]) \xrightarrow{\tilde{\phi}_0^!} \tilde{K}_0(\mathbb{Z}[\pi])$$

in the Whitehead and reduced projective class groups, with $Wh(\phi^!)$ defined by

$$Wh(\phi^!) = K_1(\phi^!)/j(\pm\pi)+(1,0,\begin{pmatrix}1\\ t-1\end{pmatrix},(1\ 0)) \ .$$

See Ranicki [7,§7] for the geometric interpretation of this sequence.

Appendix: Connection with L-theory

We note the following connection between the algebraic K-theory S^1-bundle transfer maps

$$\tilde{\phi}_0^! : \tilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi]) \ , \ \tilde{\phi}_1^! : Wh(\rho) \longrightarrow Wh(\pi)$$

and the algebraic L-theory S^1-bundle transfer maps of Munkholm and Pedersen [3],[4] and Ranicki [6],[8]

$$\phi_L^! : L_n^X(\rho) \longrightarrow L_{n+1}^{\tilde{\phi}_m^!(X)}(\pi) \quad (m = 0 \text{ or } 1)$$

which are defined for duality-invariant subgroups $X \subseteq \tilde{K}_0(\mathbb{Z}[\rho])$ $(m=0)$ and $X \subseteq Wh(\rho)$ $(m=1)$. (The geometric interpretation of $\phi_L^!$ for $m=1$ in terms of finite surgery obstruction theory extends to $m=0$ using the projective surgery obstruction theory of Pedersen and Ranicki [5]). The duality involutions on the algebraic K-groups are defined by

$$* : \tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi]) \ ; \ [im(X)] \longmapsto [im(X^*)]$$

$$* : Wh(\pi) \longrightarrow Wh(\pi) \ ; \ \tau(X) \longmapsto \tau(X^*)$$

$$* : Wh(\phi^!) \longrightarrow Wh(\phi^!) \ ;$$

$$(X,Y,\begin{pmatrix}S_1\\ S_2\end{pmatrix},(T_1\ T_2)) \longmapsto -(1-X^*,-t^{-1}Y^*,\begin{pmatrix}-t^{-1}T_2^*\\ T_1^*\end{pmatrix},(-tS_2^*\ S_1^*)) \ ,$$

using the group ring involution

$$* : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi] \ ; \ \sum_{g\in\pi} n_g g \longmapsto \sum_{g\in\pi} w(g)n_g g^{-1} \quad (w = \text{orientation})$$

and the corresponding matrix ring involutions

$$* : M_n(\mathbb{Z}[\pi]) \longrightarrow M_n(\mathbb{Z}[\pi]) \ ; \ X = (x_{ij}) \longmapsto X^* = (x_{ji}^*) \ .$$

The maps in the exact sequence of §5

$$Wh(\rho) \xrightarrow{\tilde{\phi}_1^!} Wh(\pi) \xrightarrow{\tilde{j}} Wh(\phi^!) \xrightarrow{\tilde{\partial}} \tilde{K}_0(\mathbb{Z}[\rho]) \xrightarrow{\tilde{\phi}_0^!} \tilde{K}_0(\mathbb{Z}[\pi])$$

are such that

$$\tilde{\phi}_m^! * = -*\tilde{\phi}_m^! \ (m = 0,1) \ , \ \tilde{j}* = *\tilde{j} \ , \ \tilde{\partial}* = *\tilde{\partial} \ .$$

The short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$0 \longrightarrow \mathrm{coker}(\tilde{\phi}_1^!) \xrightarrow{\tilde{j}} \mathrm{Wh}(\phi^!) \xrightarrow{\tilde{\partial}} \ker(\tilde{\phi}_0^!) \longrightarrow 0$$

gives rise to connecting maps in the Tate $\mathbb{Z}_2$-cohomology groups

$$\phi_H^! = \delta : \hat{H}^n(\mathbb{Z}_2;\ker(\tilde{\phi}_0^!)) \longrightarrow \hat{H}^{n+1}(\mathbb{Z}_2;\mathrm{coker}(\tilde{\phi}_1^!))$$

which appear in a transfer map of generalized Rothenberg exact sequences

$$\begin{array}{ccccccccc}
\dots \longrightarrow & L_n^h(\rho) & \longrightarrow & L_n^{\ker\tilde{\phi}_0^!}(\rho) & \longrightarrow & \hat{H}^n(\mathbb{Z}_2;\ker\tilde{\phi}_0^!) & \longrightarrow & L_{n-1}^h(\rho) & \longrightarrow \dots \\
 & \downarrow \phi_L^! & & \downarrow \phi_L^! & & \downarrow \phi_H^! & & \downarrow \phi_L^! & \\
\dots \longrightarrow & L_{n+1}^{\mathrm{im}\tilde{\phi}_1^!}(\pi) & \longrightarrow & L_{n+1}^h(\pi) & \longrightarrow & \hat{H}^{n+1}(\mathbb{Z}_2;\mathrm{coker}\tilde{\phi}_1^!) & \longrightarrow & L_n^{\mathrm{im}\tilde{\phi}_1^!}(\pi) & \longrightarrow \dots .
\end{array}$$

In particular, for the trivial S^1-bundle $E = B \times S^1$, $\pi = \rho\times\mathbb{Z}$, $t = z$, $\tilde{\phi}_m^! = 0$ $(m = 0,1)$ and the exact sequence

$$0 \longrightarrow \mathrm{Wh}(\rho\times\mathbb{Z}) \xrightarrow{\tilde{j}} \mathrm{Wh}(\phi^!) \xrightarrow{\tilde{\partial}} \tilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow 0$$

is split by the map

$$\tilde{\Sigma} : \tilde{K}_0(\mathbb{Z}[\rho]) \rightarrowtail \mathrm{Wh}(\phi^!) ;$$

$$[\mathrm{im}(X)] \longmapsto \left(X,0,\begin{pmatrix} -X \\ X-z \end{pmatrix},(z^{-1}(1-X)-1 \ \ -z^{-1}(1-X))\right) ,$$

which is related to the duality involutions $*$ by

$$\tilde{\Sigma}* - *\tilde{\Sigma} = \tilde{j}h'* : \tilde{K}_0(\mathbb{Z}[\rho]) \longrightarrow \mathrm{Wh}(\phi^!)$$

with

$$h' : \tilde{K}_0(\mathbb{Z}[\rho]) \rightarrowtail \mathrm{Wh}(\rho\times\mathbb{Z}) ; \ [\mathrm{im}(X)] \longmapsto \tau(-zX+1-X) .$$

The transfer map in this case consists of split injections

$$\begin{array}{ccccccccc}
\dots \longrightarrow & L_n^h(\rho) & \longrightarrow & L_n^p(\rho) & \longrightarrow & \hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\rho])) & \longrightarrow & L_{n-1}^h(\rho) & \longrightarrow \dots \\
 & \downarrow \phi_L^! & & \downarrow \phi_L^! & & \downarrow \phi_H^! = \hat{h}' & & \downarrow \phi_L^! & \\
\dots \longrightarrow & L_{n+1}^s(\rho\times\mathbb{Z}) & \longrightarrow & L_{n+1}^h(\rho\times\mathbb{Z}) & \longrightarrow & \hat{H}^{n+1}(\mathbb{Z}_2;\mathrm{Wh}(\rho\times\mathbb{Z})) & \longrightarrow & L_n^h(\rho\times\mathbb{Z}) & \longrightarrow \dots ,
\end{array}$$

although not the standard such injections - see Ranicki [7] for a further discussion.

BIBLIOGRAPHY

[0] H.Bass Algebraic K-theory, Benjamin (1968)

[1] H.J.Munkholm Transfer on algebraic K-theory and Whitehead torsion for PL fibrations, J. Pure and App. Alg. 20, 195 - 225 (1981)

[2] and E.K.Pedersen On the Wall finiteness obstruction for the total space of certain fibrations, Trans. A.M.S. 261, 529 - 545 (1980)

[3] Whitehead transfers for S^1-bundles, an algebraic description, Comm. Math. Helv. 56, 404 - 430 (1981)

[4] Transfers in algebraic K- and L-theory induced by S^1-bundles, these proceedings

[5] E.K.Pedersen and A.A.Ranicki Projective surgery theory, Topology 19, 234 - 254 (1980)

[6] A.A.Ranicki Exact sequences in the algebraic theory of surgery, Mathematical Notes 26, Princeton (1981)

[7] Algebraic and geometric splittings of the K- and L-groups of polynomial extensions, preprint

[8] Splitting theorems in the algebraic theory of surgery, in preparation

[9] R.G.Swan K-theory of finite groups and orders, Springer Lecture Notes 149 (1970)

[10] C.T.C.Wall Finiteness conditions for CW complexes, Ann. of Maths. 81, 56 - 69 (1965)

INSTITUTE FOR ADVANCED STUDY, PRINCETON (H.J.M. & A.A.R.)
ODENSE UNIVERSITET, DENMARK (H.J.M.)
PRINCETON UNIVERSITY (A.A.R.)

ABCDEFGHIJ–AMS–898765432