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TWO COMPLEXES WHICH ARE HOMEOMORPHIC BUT COMBINATORIALLY DISTINCT

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Let L_q denote the 3-dimensional lens manifold of type $(7, q)$, suitably triangulated (see § 1), and let σ^n denote an n-simplex. A finite simplicial complex X_q is obtained from the product $L_q \times \sigma^n$ by adjoining a cone over the boundary $L_q \times \partial\sigma^n$. The dimension of X_q is $n + 3$.

THEOREM 1. *For $n + 3 \geqq 6$ the complex X_1 is homeomorphic to X_2.*

THEOREM 2. *No finite cell subdivision of the simplicial complex X_1 is isomorphic to a cell subdivision of X_2. In particular there is no piecewise linear homeomorphism from X_1 to X_2.*

The proof of Theorem 1 will be based on a recent result of B. Mazur. For the special case $n = 3$ (which is somewhat more difficult) the proof will make use of theorems of A. Haefliger and J. Stallings.

The proof of Theorem 2 will be based on the concept of "torsion" as defined by Reidemeister, Franz, and de Rham.

These two theorems show that the Hauptvermutung[2] for simplicial complexes of dimension $\geqq 6$ is false. On the other hand Papakyriakopoulos [10] has proved the Hauptvermutung for complexes of dimension $\leqq 2$.

The Hauptvermutung for manifolds remains open. However Moise [8] has proved the Hauptvermutung for manifolds of dimension $\leqq 3$; and Smale [13] has proved it for triangulations of the sphere S^n, $n \neq 4, 5, 7$, which look locally like the usual triangulation. A weak form of the Hauptvermutung for cells and spheres has been proved by Gluck [4].

As bi-products of the argument, two other curious phenomena appear. The symbols

$$S^{n-1} \subset D^n \subset R^n$$

will always denote the unit sphere bounding the unit disk in euclidean n-space.

THEOREM 3. *The manifold-with-boundary $L_1 \times D^5$ is not diffeomorphic to $L_2 \times D^5$. However the interiors of these two manifolds are diffeomorphic.*

[1] The author wishes to thank the Sloan Foundation for its support.

[2] See, for example, Alexandroff and Hopf [1, p. 152]. I do not know who originated the term "Hauptvermutung". The problem was clearly formulated by Tietze [18, pp. 13-14] in 1908. See also Steinitz [15, p. 23].

Two closed manifolds M_1 and M_2 will be called *h-cobordant* (ignoring orientation) if their disjoint sum $M_1 + M_2$ bounds a compact differentiable manifold W such that both M_1 and M_2 are deformation retracts of W. (The term "J-equivalent" has previously been used for this concept. Compare Thom [17], Smale [13].)

THEOREM 4. *The manifold $L_1 \times S^4$ is h-cobordant to $L_2 \times S^4$; but these two manifolds are not diffeomorphic.*

1. Mazur's theorem and lens manifolds

Let M_1 and M_2 be two closed differentiable manifolds of dimension k which are parallelizable[3] and have the same homotopy type.

THEOREM OF MAZUR [6]. *If $n > k$ then $M_1 \times R^n$ is diffeomorphic to $M_2 \times R^n$.*

An outline of the proof is given in § 2.

The *lens manifold* $L = L(p, q)$ can be constructed as follows. Let $p > q$ be relatively prime positive integers. Identify S^3 with the unit sphere in the complex plane, consisting of all (z_1, z_2) with $|z_1|^2 + |z_2|^2 = 1$. Let ω denote the complex number $\exp(2\pi i/p)$. Then the cyclic group Π of order p acts differentiably on S^3 without fixed points by the rule

$$T(z_1, z_2) = (\omega z_1, \omega^q z_2) ,$$

where T denotes a generator of Π. The quotient manifold S^3/Π is the required lens manifold.

This manifold L can be considered as a CW-complex with only four cells, namely the images $\bar{e}_m$ in L of:

(0) the point $e_0 = (1, 0)$,

(1) the set e_1 of $(e^{i\theta}, 0)$,

(2) the set e_2 of $(z_1, \sqrt{1 - |z_1|^2})$, and

(3) the set e_3 of $(z_1, e^{i\theta}\sqrt{1 - |z_1|^2})$;

where $0 < \theta < 2\pi/p$ and $|z_1| < 1$. (Compare de Rham [12].)

Alternatively L can be considered as a simplicial complex. Here is an example of a triangulation of L which is compatible both with the above cell subdivision and with the differentiable structure. Consider the convex polyhedron P spanned by the $2p$ points $(\omega^j, 0)$ and $(0, \omega^k)$ in the complex plane. The boundary ∂P is a simplicial complex which is homeomorphic

[3] Instead of parallelizability, it suffices to assume that the stable tangent bundles of M_1 and M_2 are compatible under some homotopy equivalence $M_1 \to M_2$.

to S^3 under radial projection from the origin. Taking two successive barycentric subdivisions of ∂P, and then collapsing under the action of Π, we obtain the required simplicial complex.

These complexes were discovered by Tietze [18, p. 110] in 1908. Tietze computed the fundamental group

$$\pi_1(L) \approx \Pi$$

and the homology of L. In particular he showed that the integer p is a topological invariant of $L = L(p, q)$.

In 1935 Reidemeister [11] classified the lens manifolds combinatorially. He showed that $L(p, q)$ is combinatorially equivalent to $L(p, q')$ if and only if either

$$q' \equiv \pm q \text{ or } \pm qq' \equiv 1 \qquad (\text{mod } p) .$$

(According to Moise [8] or Brody [2] two lens manifolds are homeomorphic if and only if they are combinatorially equivalent. This fact will not be needed in the present paper.)

In 1941 J. H. C. Whitehead [20] classified the lens manifolds up to homotopy type. (For a more recent version see Olum [9].) Whitehead showed that $L(p, q)$ has the homotopy type of $L(p, q')$ if and only if $\pm qq'$ is a quadratic residue modulo p. As an example, for $p = 7$, we obtain two distinct combinatorial manifolds $L(7, 1)$ and $L(7, 2)$; but only one homotopy type; since $1 \cdot 2 \equiv 3^2$ is a quadratic residue modulo 7.

All lens manifolds are parallelizable. This follows from the theorem of Stiefel [16] and Whitney that all orientable 3-manifolds are parallelizable. (For p odd the proof is quite easy since the obstructions to parallelizability lie in groups $H^m(L; \pi_{m-1}(SO_3))$ which are zero.)

Hence we can apply Mazur's theorem and conclude that:

LEMMA 1. *If $\pm qq'$ is a quadratic residue modulo p, and if $n > 3$, then $L(p, q) \times R^n$ is diffeomorphic to $L(p, q') \times R^n$.*

PROOF OF THEOREM 1 FOR $n > 3$. Recall the definition:

$$X_q = L_q \times \sigma^n \cup \text{Cone}\,(L_q \times \partial\sigma^n) ,$$

where $L_q = L(7, q)$. Let x_0 denote the vertex of the cone. The complement $X_q - x_0$ is homeomorphic to the product $L_q \times R^n$. In fact a specific homeomorphism $f: X_q - x_0 \to L_q \times R^n$ can be given as follows. Let $h: \sigma^n \to D^n$ be a homeomorphism, and define

$$f(y, z) = (y, h(z))$$
$$f(t(y, z') + (1 - t)x_0) = (y, h(z')/t)$$

for $y \in L_q$, $z \in \sigma^n$, $z' \in \partial\sigma^n$, and $0 < t \leqq 1$.

Therefore X_q is homeomorphic to the single point compactification of $L_q \times R^n$. Using Lemma 1, this implies that X_1 is homeomorphic to X_2; which completes the proof of Theorem 1 for $n > 3$.

2. *h*-cobordism

First let me outline a proof of Mazur's theorem. Given a homotopy equivalence $f: M_1 \to M_2$, choose a differentiable imbedding.

$$f' : M_1 \to \text{Interior}\,(M_2 \times D^n)$$

which approximates the function $x \to (f(x), 0)$. This is certainly possible if n is greater than the dimension k of M_q. Since both M_1 and M_2 are parallelizable, it follows that the normal bundle of $f'(M_1)$ is trivial providing that $n > k$. (See for example Milnor [7, Lemma 5].) Thus a tubular neighborhood of $f'(M_1)$ in Interior $(M_2 \times D^n)$ is diffeomorphic to $M_1 \times D^n$.

This gives an imbedding $i : M_1 \times D^n \to M_2 \times D^n$. Similarly, using a homotopy inverse to f, one obtains an imbedding $j : M_2 \times D^n \to M_1 \times D^n$ The main step in the proof is now the following.

LEMMA 1. *If $n > k > 1$ then any imbedding*

$$h : M_1 \times D^n \to \text{Interior}\, M_1 \times D^n$$

which is homotopic to the identity can be extended to a diffeomorphism of the pair $(M_1 \times 2D^n, M_1 \times D^n)$ onto the pair $(M_1 \times D^n, h(M_1 \times D^n))$. In particular this applies to the imbedding $h = ji$.

Here $2D^n$ denotes the disk of radius 2. The key step in the proof is to show that h restricted to $M_1 \times 0$ is differentiably isotopic to the standard inclusion map $M_1 \times 0 \to M_1 \times D^n$. For $n > k + 1$, this follows from a well known theorem of Whitney [23]. For the case $n = k + 1 > 2$, it follows from a recent theorem of A. Haefliger [5].

Now consider the infinite direct sequence

$$M_1 \times D^n \xrightarrow{\ i\ } M_2 \times D^n \xrightarrow{\ j\ } M_1 \times D^n \xrightarrow{\ i\ } \cdots .$$

The "limit" or "union" of this sequence is non-compact manifold V. Using the lemma it is seen that V is diffeomorphic to the union $M_1 \times R^n$ of

$$M_1 \times D^n \subset M_1 \times 2D^n \subset M_1 \times 4D^n \subset \cdots .$$

But a similar proof shows that V is diffeomorphic to $M_2 \times R^n$. Hence $M_1 \times R^n$ is diffeomorphic to $M_2 \times R^n$. For details the reader is referred to Mazur's paper.

Now consider the region

$$W = M_2 \times D^n - \text{Interior}\, i(M_1 \times D^n) .$$

This is a compact differentiable manifold bounded by $M_2 \times S^{n-1}$ and $i(M_1 \times S^{n-1})$.

LEMMA 2. *If $n \geqq 3$ then both $M_2 \times S^{n-1}$ and $i(M_1 \times S^{n-1})$ are deformation retracts of W.*

PROOF. It will be convenient to denote the boundaries of W by W_2 and W_1 respectively. By a dimensional argument, any map of a 2-dimensional complex into $M_2 \times D^n$ can be deformed off $f'(M_1)$, and hence can be pushed into W.

This implies that

$$\pi_1(W) \xrightarrow{\approx} \pi_1(M_2 \times D^n)$$

and hence that

$$\pi_1(W_q) \xrightarrow{\approx} \pi_1(W) \qquad \text{for } q = 1, 2.$$

Given any system $\mathcal{S}$ of local coefficients on $M_2 \times D^n$ we have

$$H_*(W, W_1; \mathcal{S}) \xrightarrow{\approx} H_*(M_2 \times D^n, i(M_1 \times D^n); \mathcal{S})$$

by excision. But i is a homotopy equivalence, hence these groups are zero. Using Whitehead [21, Theorem 3] it follows that W_1 is a deformation retract of W.

The group $H_p(W, W_2; \mathcal{S})$ is isomorphic by Poincaré duality to $H^{n+k-p}(W, W_1; \mathcal{S})$, and therefore is zero. This implies that W_2 is a deformation retract of W, which completes the proof of Lemma 2.

Thus: *if M_1 and M_2 are closed parallelizable k-manifolds with the same homotopy type, and if $n > k > 1$, then $M_1 \times S^{n-1}$ is h-cobordant to $M_2 \times S^{n-1}$.*

In particular this shows that $L_1 \times S^4$ is h-cobordant to $L_2 \times S^4$; which proves half of Theorem 4.

Next we will see that most of the above arguments still work for the case $n = k = 3$. According to Haefliger [5], any homotopy equivalence

$$L_1 \rightarrow \text{Interior } (L_2 \times D^3)$$

is homotopic to an imbedding f'. The normal bundle of $f'(L_1)$ will be trivial, since the obstructions to triviality lie in groups

$$H^m(L_1; \pi_{m-1}(SO_3))$$

which are zero. Hence, according to Lemma 2, both $L_2 \times S^2$ and $i(L_1 \times S^2)$ are deformation retracts of the region

$$W = L_2 \times D^3 - \text{Interior } i(L_1 \times D^3) .$$

Thus $L_1 \times S^2$ is h-cobordant to $L_2 \times S^2$.

According to Stallings [14, Theorem 7.4] the space W, with the boundary $L_2 \times S^2$ removed, is homeomorphic to $i(L_1 \times S^2) \times [0, \infty)$. Filling in the region $i(L_1 \times D^3)$ it follows that $(L_2 \times D^3) - (L_2 \times S^2)$ is homeomorphic to

$$(L_1 \times D^3) \cup (L_1 \times S^2 \times [0, \infty))$$

where the two sets are matched along the boundary $L_1 \times S^2$. *Therefore $L_2 \times R^3$ is homeomorphic to $L_1 \times R^3$.*

It follows that X_2 is homeomorphic to X_1 for $n \geqq 3$. This completes the proof of Theorem 1.

3. Torsion

This section will describe the torsion invariant of Reidemeister [11], Franz [3] and de Rham [12]. The presentation will be close to that of de Rham.

Let Π be a discrete group which acts freely on a CW-complex K, and let

$$h : \Pi \to P$$

be a multiplicative homomorphism from Π to a commutative ring P. If

(1) the quotient complex K/Π has only finitely many cells, and

(2) the equivariant homology groups $H_i(P \otimes_\Pi C_*(K; Z))$ are all zero;

then the torsion $\Delta_h(K)$, will be defined. The torsion is a unit of P which is well defined up to multiplication by elements of the form $\pm h(\pi)$. We will use the notation

$$\Delta = \Delta_h(K) \in U/\pm h(\Pi) ,$$

where $U \subset P$ denotes the group of units. This element Δ is invariant under equivariant subdivision of K.

In practice K is taken to be the universal covering space of a finite cell complex, and $\Pi = \pi_1(K)$ the group of covering transformations. In particular, letting $K = \tilde{L}(p, q)$ be the universal covering space of a lens manifold, and letting P be the field of complex numbers, the $\Delta_h L(p, q)$ were used by Reidemeister to give the complete combinatorial classification of the lens manifolds.

The proof of Theorem 2 will be based on a more general concept of torsion in which the CW-complex K is replaced by a CW-pair (K, L). The group Π must act cellularly on K and freely on $K - L$. The resulting torsion

$$\Delta_h(K, L) \in U/\pm h(\Pi)$$

is still a combinatorial invariant. That is:

THEOREM A. *If the* CW-*pair* (K', L') *is a* Π-*equivariant subdivision of* (K, L), *and if* $\Delta_h(K, L)$ *is defined, then*

$$\Delta_h(K', L') = \Delta_h(K, L) .$$

The proof will be given in § 4.

In this generality, the torsion is definitely not a topological invariant: it depends on the cell structure of (K, L). However in the classical case, with L vacuous, it is not known whether or not $\Delta_h(K)$ really depends on the cell structure of K. (If K/Π is a compact differentiable manifold then $\Delta_h(K)$ can also be defined. See [12], [19].)

For the definition of torsion, it will be convenient to assume that P is a principal ideal domain. The more general case is considered in the appendix.

DEFINITION. Let F be a free P-module of finite rank q. A *volume* v in F will mean a generator for the q^{th} exterior power $\Lambda_P^q(F)$. If $q>0$, then any volume can be written in the form $b_1 \wedge \cdots \wedge b_q$ where $b_1, \cdots, b_q$ form a basis for F. If $q = 0$ then a volume is defined to be a unit of P.

Now let $0 \to F' \to F \to F'' \to 0$ be a short exact sequence of free, finitely generated modules. Let $v' = b'_1 \wedge \cdots \wedge b'_p$ and $v'' = b''_1 \wedge \cdots \wedge b''_r$ be volumes in F' and F'' respectively. Then each basis element b''_i can be lifted to an element b_i of F. Thus we obtain a well defined volume

$$v = b_1 \wedge \cdots \wedge b_r \wedge b'_1 \wedge \cdots \wedge b'_p$$

in F. It is clear that any two of the volumes v', v, and v'' determine the the third uniquely. In particular we will write

$$v'' = v/v'$$

to indicate the dependence of v'' on v and v'. If F' or F'' is zero then this notation, suitably interpreted, still makes sense. For example if

$$0 \longrightarrow F' \xrightarrow{\approx} F \longrightarrow 0 \longrightarrow 0$$

then v' and v can be considered as generators of the same module. Their ratio v/v' is a unit of P.

Now consider an exact sequence

$$0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$

of free P-modules, and suppose that a volume v_i is given in each C_i. Since P is assumed to be a principal ideal domain, it follows that each submodule $\partial C_i \subset C_{i-1}$ is free. Using the exact sequence

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \partial C_{n-1} \longrightarrow 0 ,$$

the volumes v_n and v_{n-1} give rise to a volume v_{n-1}/v_n in ∂C_{n-1}. Now using the sequence

$$0 \longrightarrow \partial C_{n-1} \longrightarrow C_{n-2} \longrightarrow \partial C_{n-2} \longrightarrow 0 ,$$

the volumes v_{n-1}/v_n and v_{n-2} give rise to a volume

$$v_{n-2}/(v_{n-1}/v_n)$$

in ∂C_{n-2}. Continuing by induction we obtain a volume

$$v_1/(v_2/\cdots/(v_{n-2}/(v_{n-1}/v_n))\cdots)$$

in $\partial C_1 = C_0$. Comparing this with the given volume v_0 in C_0 the ratio

$$v_0/(v_1/(v_2/\cdots/(v_{n-2}/(v_{n-1}/v_n))\cdots))$$

is a well defined unit of P. The unit obtained in this way will be denoted briefly by

$$[v_0 v_1^{-1} v_2 v_3^{-1} \cdots v_n^{\pm 1}] \in U \subset P .$$

Now consider a CW-complex K on which the group Π operates.

Hypothesis 1. Π permutes the cells of K freely. The quotient complex K/Π has only finitely many cells.

Thus the integral chain groups $C_i(K; Z)$ can be considered as free modules of finite rank over the integral group ring $Z\Pi$. In fact each i-cell of K/Π gives rise to a basis element of $C_i(K; Z)$ which is well defined up to sign, and up to multiplication, by elements of Π.

Using the homomorphism $h : \Pi \longrightarrow U \subset P$ we can form the chain complex

$$C_* = P \otimes_\Pi C_*(K; Z)$$

where the subscript Π indicates that

$$\rho h(\pi) \otimes c - \rho \otimes \pi_\#(c)$$

is set equal to zero for each $\rho \in P$, $\pi \in \Pi$, and $c \in C_i(K; Z)$. Thus C_i is a free P-module of finite rank with one basis element for each i-cell of K/Π. Taking the exterior product of these basis elements, we obtain a volume v_i in C_i which is well defined up to multiplication by elements of the form $\pm h(\pi)$.

Hypothesis 2. The homology groups $H_i(P \otimes_\Pi C_*(K; Z))$ are all zero, so that the sequence

$$0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \longrightarrow 0$$

is exact.

Then the torsion $\Delta_h(K)$ can be defined as the residue class of

$$[v_0 v_1^{-1} v_2 v_3^{-1} \cdots v_n^{\pm 1}] \in U$$

modulo the multiplicative subgroup $\pm h(\Pi)$.

The definition of torsion for a CW-pair (K, L) is similar. In this case one assumes that Π is a group of automorphisms of the pair; that Π operates freely on the cells of $K - L$; that $(K - L)/\Pi$ has only finitely many cells; and that

$$H_i(P \otimes_\Pi C_*(K, L; Z)) = 0 \qquad \text{for all } i.$$

(The group Π is definitely allowed to have fixed points in L.) The torsion

$$\Delta_h(K, L) \in U/\pm h(\Pi)$$

is defined just as above; using the chain complex $C_* = P \otimes_\Pi C_*(K, L; Z)$.

As an example let K be the 3-sphere considered as the universal covering space $\tilde{L}$ of $L(p, q)$ and let Π be the cyclic group of covering transformations. As described in § 1, $\tilde{L}$ has a Π-equivariant cell structure with $4p$ cells; so that)$L(p, q) = \tilde{L}/\Pi$ has only 4 cells. Thus $C_*(\tilde{L}; Z)$ is a free $Z\Pi$-module with 4 generators: e_0, e_1, e_2 and e_3. The boundary relations are easily seen to be as follows:

$$\partial e_1 = (T - 1)e_0$$
$$\partial e_2 = (1 + T + T^2 + \cdots + T^{p-1})e_1$$
$$\partial e_3 = (T^r - 1)e_2 \,,$$

where r is determined by the congruence $qr \equiv 1 \pmod p$.

A homomorphism h from Π to the complex numbers P takes the generator T into some p^{th} root of unity τ. If $\tau \neq 1$ then

$$1 + \tau + \tau^2 + \cdots + \tau^{p-1} = 0 \,;$$

so that the boundary relations in

$$C_* = P \otimes_\Pi C_*(\tilde{L}; Z)$$

become

$$\partial e_1 = (\tau - 1)e_0 \,, \qquad \partial e_2 = 0 \,, \qquad \partial e_3 = (\tau^r - 1)e_2 \,.$$

Clearly the chain complex C_* is acyclic. The torsion

$$\Delta_h(\tilde{L}) = [e_0 e_1^{-1} e_2 e_3^{-1}] \in U/\pm h(\Pi)$$

is defined; and is equal to $(\tau - 1)^{-1}(\tau^r - 1)^{-1}$. This complex number is well defined up to multiplication by numbers of form $\pm\tau^k$. Taking the absolute valute of $\Delta_h(\tilde{L})$ we obtain a well defined real number $|\Delta|$.

Applying this construction to $L(7, 1)$ we obtain $|\Delta| = 1.33$ or 0.41 or 0.26 (to two decimal places) depending on the choice of h. On the other hand for $L(7, 2)$ we obtain $|\Delta| = 0.74$ or 0.59 or 0.33. Thus the torsion invariant distinguishes $L(7, 1)$ from $L(7, 2)$. Together with Theorem A, it follows that no CW-subdivision of $L(7, 1)$ is isomorphic to a CW-subdivision of $L(7, 2)$.

Next consider the complexes X_1 and X_2 defined in the beginning of this paper. Each X_q is a manifold except at one exceptional point x_0. Removing this point we obtain a space $X_q - x_0$ which is homeomorphic to $L(7, q) \times R^n$. The fundamental group Π of $X_q - x_0$ is cyclic of order 7.

Let K_q denote the single point compactification of the universal covering space of $X_q - x_0$. Thus the fundamental group Π of $X_q - x_0$ operates on K_q with a single fixed point. The quotient space K_q/Π is equal to X_q. Any cell structure on the pair (X_q, x_0) gives rise to Π-equivariant cell structure on K_q.

The simplest cell structure on X_q has five cells: namely the four cells $\bar{e}_i \times R^n$ of $L(7, q) \times R^n \approx X_q - x_0$; together with the vertex x_0. The corresponding cell structure on K_q has 28 cells of the form $T^r e_i \times R^n$; together with one vertex which will be denoted by k_0

Consider the chain complex $C_*(K_q, k_0; Z)$. This complex is free over the group ring $Z\Pi$ with 4 preferred generators $e_i \times R^n$. It is isomorphic to the chain complex $C_*(\tilde{L}(7, q); Z)$ except for a shift in dimension. Hence the torsion $\Delta_h(K_p, k_0)$ is defined and is equal to $\Delta_h(\tilde{L}(7, q))^{\pm 1}$. (The exponent is $+1$ or -1 according as n is even or odd.) Therefore the torsion invariant distinguishes $(K_1, k_0; \Pi)$ from $(K_2, k_0; \Pi)$. It follows that no CW-subdivison of the CW-complex X_1 is isomorphic to a CW-subdivision of X_2. Since the simplicial structure on X_q defined in § 1 is a subdivision of the above cell structure, this completes the proof of Theorem 2; except for the verification that torsion is invariant under subdivision (Theorem A).

4. Invariance under subdivision

The proof of Theorem A will be based on three lemmas.

First consider a commutative diagram of short exact sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{11} & \longrightarrow & F_{12} & \longrightarrow & F_{13} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{21} & \longrightarrow & F_{22} & \longrightarrow & F_{23} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{31} & \longrightarrow & F_{32} & \longrightarrow & F_{33} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The F_{ij} are to be free P-modules of finite rank.

LEMMA 3. *Given volumes v_{ij} in F_{ij} for $i, j \leqq 2$ the identity*

$$(v_{22}/v_{12})/(v_{21}/v_{11}) = \pm\ (v_{22}/v_{21})/(v_{12}/v_{11})$$

is satisfied.

PROOF. Choose a basis $\{b_1, \cdots, b_p, \cdots, b_q, \cdots, b_r, \cdots, b_s\}$ for F_{22} so that $\{b_1, \cdots, b_p\}$ forms a basis for F_{11}, so that $\{b_1, \cdots, b_q\}$ forms a basis for F_{12} and so that $\{b_1, \cdots, b_p, b_{q+1}, \cdots, b_r\}$ forms a basis for F_{21} (using the same symbol for corresponding elements in different groups). Set

$$v_{11} = u_{11}b_1 \wedge \cdots \wedge b_p \qquad v_{12} = u_{12}b_1 \wedge \cdots \wedge b_q$$
$$v_{21} = u_{21}b_1 \wedge \cdots \wedge b_p \wedge b_{q+1} \wedge \cdots \wedge b_r \qquad v_{22} = u_{22}b_1 \wedge \cdots \wedge b_s\ ,$$

where the u_{ij} are units. Then it is easily verified that both $(v_{22}/v_{12})/(v_{21}/v_{11})$ and $(v_{22}/v_{21})/(v_{12}/v_{11})$ are equal to $\pm(u_{22}u_{12}^{-1}u_{21}^{-1}u_{11})b_{r+1} \wedge \cdots \wedge b_s$. This proves Lemma 3.

LEMMA 4. *Suppose that* Π *operates cellularly on a* CW-*triple* (K, L, M). *Then*

$$\Delta_h(K, M) = \Delta_h(K, L)\Delta_h(L, M)\ .$$

To be more precise: if two of these three invariants are defined, then the third is also defined and equality holds.

PROOF. If two of the three invariants are defined, then certainly Π permutes the cells of $K - M$ freely; and $(K - M)/\Pi$ has only finitely many cells. Let

$$C'_* = P \otimes_\Pi C_*(L, M; Z)$$
$$C_* = P \otimes_\Pi C_*(K, M; Z)$$
$$C''_* = P \otimes_\Pi C_*(K, L; Z)\ .$$

Then there is an exact sequence

$$0 \rightarrow C'_* \rightarrow C_* \rightarrow C''_* \rightarrow 0$$

of chain mappings. Since two of these three chain complexes are acyclic, it follows that the third is also. Let v'_i, v_i, v''_i denote the preferred volumes in C'_i, C_i, C''_i which are determined by the preferred bases. Each of these is well defined up to multiplication by elements of the form $\pm h(\pi)$. Furthermore it is clear that

$$v_i/v'_i = \pm\, h(\pi)v''_i$$

for some π. Applying Lemma 3 to each of the diagrams

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \partial C'_{i+1} & \longrightarrow & \partial C_{i+1} & \longrightarrow & \partial C''_{i+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C'_i & \longrightarrow & C_i & \longrightarrow & C''_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \partial C'_i & \longrightarrow & \partial C_i & \longrightarrow & \partial C''_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array},$$

it follows by induction on i that

$$(v_i/(v_{i-1}/\cdots/v_n\cdots))/(v'_i/(v'_{i-1}/\cdots/v'_n\cdots)) = \pm\, h(\pi_i)(v''_i/(v''_{i-1}/\cdots/v''_n\cdots))$$

for some π_i. This completes the proof of Lemma 4.

LEMMA 5. *If* Π *permutes the components of* $K - L$ *freely, and if* $H_*(K, L; Z) = 0$, *then* $\Delta_h(K, L) = 1$.

PROOF. Let K_0 denote the union of L with one component of $K - L$. Then the injection

$$P \otimes C_*(K_0, L; Z) \to P \otimes_\Pi (K, L; Z)$$

is an isomorphism. Thus the torsion

$$\Delta_h(K, L) \in U/\pm h(\Pi)$$

is the image in $U/\pm h(\Pi)$ of the torsion invariant

$$\Delta_1(K_0, L) \in U/\pm 1 ,$$

where the subscript 1 denotes the homomorphism from the trivial group to U. But this is in turn the image of a corresponding invariant with the ring P replaced by the ring Z of integers. Since the only units in Z are ± 1, it follows that $\Delta_h(K, L)$ is trivial.

PROOF OF THEOREM A (following Whitehead [22]). Choose a sequence

$$L = K_0 \subset K_1 \subset \cdots \subset K_r = K$$

of subcomplexes of K so that each $K_{i+1} - K_i$ consists a single cell, together with its translates under Π. Let I denote the unit interval considered as CW-complex, with Π acting trivially.

Given a subdivision K' of K let (A, B) denote the CW-pair formed from $(K \times I, L \times I)$ by subdividing $K \times 1$ only. Let A_i denote the subcomplex of A formed from

$$(K \times 0) \cup (K_i \times I)$$

by subdividing $K_i \times 1$.

The inclusion $C_*(K \times 0, L \times 0) \to C_*(A_0, B)$ is an excision isomorphism, and hence

$$\Delta_h(A_0, B) = \Delta_h(K, L) .$$

Each pair (A_{i+1}, A_i) clearly satisfies the conditions of Lemma 5. Hence by Lemma 4

$$\Delta_h(A_0, B) = \Delta_h(A_1, B) = \cdots = \Delta_h(A_r, B) ,$$

where $A_r = A$. Thus $\Delta_h(A, B)$ is equal to $\Delta_h(K, L)$.

Now let $\bar{A}_i$ denote the subcomplex of A formed from $(K \times 1) \cup (K_i \times I)$ by subdividing $K \times 1$. Then by a similar argument

$$C_*(K' \times 1, L' \times 1) \xrightarrow{\approx} C_*(\bar{A}_0, B)$$

hence $\Delta_h(K', L') = \Delta_h(\bar{A}_0, B)$, and

$$\Delta_h(\bar{A}_0, B) = \Delta_h(\bar{A}_1, B) = \cdots = \Delta_h(\bar{A}_r, B)$$

where $\bar{A}_r = A$. Therefore

$$\Delta_h(K', L') = \Delta_h(A, B) = \Delta_h(K, L) ,$$

which completes the proof of Theorem A.

In conclusion, here is a theorem concerning the torsion of a product.

Let A be a finite CW-complex with Euler characteristic $\chi(A)$. Assume that Π acts trivially on A.

THEOREM B. *If $\Delta_h(K)$ is defined then $\Delta_h(K \times A)$ is defined and is equal to $\Delta_h(K)^{\chi(A)}$.*

PROOF. Choose subcomplexes $A_0 \subset A_1 \subset \cdots \subset A_r = A$ so that A_0 is vacuous and each $A_{i+1} - A_i$ consists of a single cell. The chain complex

$$C_*(K \times A_{i+1}, K \times A_i; Z)$$

is isomorphic to $C_*(K; Z)$ except for a shift in dimension; hence

$$\Delta_h(K \times A_{i+1}, K \times A_i) = \Delta_h(K)^{\pm 1}$$

where the exponent is exactly the difference $\chi(A_{i+1}) - \chi(A_i)$. Now by Lemma 4,

$$\Delta_h(K \times A) = \Pi_{i=0}^{r-1}\Delta_h(K \times A_{i+1}, K \times A_i) = \Delta_h(K)^{\chi(A)} ;$$

which completes the proof.

COROLLARY 1. *For any n the differentiable manifold $L_1 \times D^n$ is not diffeomorphic with $L_2 \times D^n$.*

PROOF. The triangulation of L_q described in § 1 is a C^1-triangulation in the sense of Whitehead [19]. Choosing any C^1-triangulation of D^n, consider the resulting product triangulation of $L_q \times D^n$. According to Theorem B

$$\Delta_h(\widetilde{L}_q \times D^n) = \Delta_h(\widetilde{L}_q)^1$$

hence $L_1 \times D^n$ (in this triangulation) is not combinatorially equivalent to $L_2 \times D^n$. But, according to Whitehead, if two manifolds are diffeomorphic then any C^1-triangulation of one is combinatorially equivalent to any C^1-triangulation of the other. Therefore $L_1 \times D^n$ is not diffeomorphic to $L_2 \times D^n$. This proves Corollary 1, and (together with Lemma 1) completes the proof of Theorem 3.

COROLLARY 2. *For n even the manifold $L_1 \times S^n$ is not diffeomorphic to $L_2 \times S^n$*

(I do not know what happens for n odd.) The proof is similar except that

$$\Delta_h(\widetilde{L}_q \times S^n) = \Delta_h(\widetilde{L}_q)^2 ,$$

since the Euler characteristic of an even dimensional sphere is $+2$. The absolute value of the torsion distinguishes L_1 from L_2, hence its square will also distinguish L_1 from L_2. This completes the proof of Corollary 2, and hence of Theorem 4.

Appendix: Torsion and simple homotopy type

The definition of torsion in § 3 can be extended to the case where P is an arbitrary commutative ring with unit as follows. Call a P-module M *quasi-free of rank r* if the direct sum of M with a free module of rank n is free of rank $r + n$ for large n. It follows easily that $\Lambda^r M$ is free on one generator, so that volumes can be defined as before. Furthermore, using the exact sequences

$$0 \to \partial C_{i+1} \to C_i \to \partial C_i \to 0 ,$$

it follows by induction on i that each ∂C_i is quasi-free. The definition of torsion now proceeds as in § 3.

In his study of simple homotopy types, Whitehead has defined a sharper torsion invariant which makes sense even over a non-commutative ring. In this construction the group U of units is replaced by an abelian group $W(P)$ which is defined as follows.

Let G_n denote the group of all non-singular $n \times n$ matrices over P. Using the standard imbeddings

$$U = G_1 \subset G_2 \subset G_3 \subset \cdots ,$$

one can form the union G: the infinite general linear group of P. Let E denote the subgroup of G generated by all elementary matrices (i.e., all matrices which coincide with the identity matrix except for one off-diagonal element). Whitehead shows that E is exactly the commutator subgroup of G. Define the *Whitehead group* $W(P)$ to be the quotient G/E. Thus each non-singular matrix $A \in G_n$ determines an element of $W(P)$ which will be denoted by $w(A)$. (Note that $w(A)$ behaves very much like a determinant of A.)

EXAMPLES. If P is an euclidean domain then $W(P) = U$; however I do not know whether or not this is true for a principal ideal domain. In general, if P is a commutative ring, then $W(P)$ splits as the direct sum of U and a second group $W_0(P)$. If P is a skew-field, then $W(P)$ is the commutator quotient group of the multiplicative group U.

The definition of torsion using $W(P)$ in place of U can be carried out as soon as one has a suitable concept of "volume". Let M be a quasi-free left P-module of rank r and let F denote the free P-module generated by countably many elements $b_1, b_2, b_3, \cdots$. A *quasi-basis* for M will mean an ordered basis $(m_1, m_2, m_3, \cdots)$ for the free module $M \oplus F$, which satisfies the condition $m_{r+i} = b_i$ for large i. An *elementary transformation* of such a quasi-basis will mean the operation of adding a left multiple of m_i to m_j, $i \neq j$. Define a *volume* in M to be an equivalence class of quasi-bases, where two quasi-bases are equivalent if and only if one can be obtained from the other by a finite sequence of elementary transformations. For the special case $M = 0$, note that a volume in M can be considered as an element of the Whitehead group $W(P)$.

Proceeding just as in § 3 one can now define the torsion invariant

$$\Delta_h(K, L) \in W(P)/w(\pm h\Pi) .$$

The hypotheses are the same as those of § 3 except that the ring P need not be commutative.

As a case of particular interest suppose that Π operates freely on the simply connected complexes $K \supset L$, and suppose that $H_*(K, L; Z) = 0$.

Let $i: \Pi \to Z\Pi$ denote the inclusion homomorphism. Then the torsion

$$\Delta_i(K, L) \in W(Z\Pi)/w(\pm\Pi)$$

is defined. This invariant plays a fundamental role in Whitehead's theory. It vanishes if and only if the inclusion map

$$L/\Pi \to K/\Pi$$

is a simple homotopy equivalence.

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