PROPER SURGERY GROUPS FOR NON-COMPACT MANIFOLDS OF FINITE DIMENSION

by Serge Maumary*

Introduction

This work first appeared in preprint form in 1972, with the goal of "computing" the formal open surgery obstruction groups (cf. Taylor [10]) in terms of the projective Wall groups introduced by Novikov [5]. The theory turned out to be quite complicated, both algebraically and geometrically. Despite its complexity the theory plays a role in at least two beautiful classical processes:

i) The transfer process, going from a surgery problem on a manifold M to one on a covering $\widetilde{M}$ of M. A typical case arises in the study of the L-groups of infinite groups. For a normal map $(f,b):M \longrightarrow X$ from a compact n-manifold M to a finite n-dimensional Poincaré complex X with $\pi_1(X) = \pi \times \mathbb{Z}$ the transfer map $t : L_n^h(\pi \times \mathbb{Z}) \longrightarrow L_{n-1}^p(\pi)$ sends the finite surgery obstruction $\sigma_*^h(f,b) \in L_n^h(\pi \times \mathbb{Z})$ in the finite Wall group of Shaneson [13] to the proper surgery obstruction $\sigma_*^p(\tilde{f},\tilde{b}) \in L_{n-1}^p(\pi)$ of the covering map $(\tilde{f},\tilde{b}):\widetilde{M} \longrightarrow \widetilde{X}$, with $\widetilde{X}$ the infinite cyclic covering of X such that $\pi_1(\widetilde{X}) = \pi$. Note that $\widetilde{M}$ is not compact and $\widetilde{X}$ is not finite, and that there is a dimension shift in the proper surgery obstruction.

ii) The deleting (or removing) process, going from a problem on a compact pair (M,K) to one on M-K with "conditions at ∞" or "boundary conditions". Typical cases arise in the study of knots and singularities, especially in dimension 4 (cf. the work of Cappell-Shaneson, Casson, Freedman etc.).

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These notes may serve as a general framework for particular cases.

On the algebraic side, the projective L-groups $L_*^p(\pi)$ appear in the analogue of the splitting theorem of Shaneson [13]

$$L_n^s(\pi \times \mathbb{Z}) = L_n^s(\pi) \oplus L_{n-1}^h(\pi)$$

obtained by Novikov [5] and Ranicki [14]

$$L_n^h(\pi \times \mathbb{Z}) = L_n^h(\pi) \oplus L_{n-1}^p(\pi) .$$

Work of Bak, Carlsson, Hambleton, Kolster, Milgram and Pardon (in various combinations) has shown that the computation of the projective L-groups $L_*^p(\pi)$ for finite groups π is easier than the computation of the finite L-groups $L_*^h(\pi)$ and of the original simple L-groups $L_*^s(\pi)$ of Wall [11], reducing to class group theory.

Pedersen and Ranicki [15] give a different geometric interpretation of the projective L-groups $L_*^p(\pi)$, in terms of normal maps $(f,b):M \longrightarrow X$ from compact n-dimensional manifolds M to finitely dominated n-dimensional Poincaré complexes X with $\pi_1(X) = \pi$.

A brief account of the main results of this paper may be found in Maumary [12].

Summary.

We consider non-compact connected manifolds M of finite dimension, which are countable union of compact subsets, and proper maps f of such manifolds (f^{-1} (compact) = compact). Given a proper normal map of open manifolds $f: M \to X$, we look for the obstruction to having a proper normal cobordism from f to some proper homotopy equivalence at ∞ $f': M' \to X$ (see [9] for definition). We shall need extensively mapping cylinder constructions, which change X into a properly homotopy equivalent CW-complex. So we have to study the proper homotopy invariant properties of the classical Poincaré duality in a non-compact manifold: this is taken care in Chapter I. Then we make f as connected as possible at ∞, by doing a sequence of ordinary surgeries $\to \infty$ and carving out a sequence of properly embedded q-spheres piped to ∞ as in Chapter II. Then, when $m = 2q+1$, we show (Th. III, 9) that for some sequence of cocompact submanifolds $M_n \to \infty$ the intersection pairing on the boundary, induces a non-singular quadratic form $\sigma_n \in L_{2q}(\pi_1 X_n)$ on a projective quotient of a submodule of $K_q(\partial M_r)^{\#}$ (coefficients $\pi_1 X_n$, $r > n$), and that the extension $\sigma_n^{\#}$ of σ_n to $\pi_1 X_{n-1}$ is canonically equivalent to σ_{n-1}. This is obtained by finding adequate cocompact subcomplexes $X_n \to \infty$ in X (up to mapping cylinder constructions) and an extensive use of Poincaré duality. The case $m = 2q+2$ can be divided in two cobordisms with common boundary U^{2q+1}, such that for some sequence of cocompact submanifolds $U_n \to \infty$ in U, the intersection form on $K_q(\partial U_r)^{\#}$ is canonically free hyperbolic and contains a distinguished projective Lagrangian

plane $\ell_n \in L_{2q+1}(\pi_1 X_n)$ (see notations and Th. IV. 4). Moreover, there is an essentially canonical equivalence between $\ell_n^{\#}$ and ℓ_{n-1}. More precisely, we get in this way an exact sequence $\varprojlim^1 L_m(\pi_1 X_n) \to L_m(\varepsilon X) \to \varprojlim L_{m-1}(\pi_1 X_n)$ where $L_m(\varepsilon X)$ is the proper surgery obstruction group at ∞ and $\varprojlim^1$ is as usual the cokernel of the map $1\text{-}s\colon \prod_{n\geq 1} L_m(\pi_1 X_n) \circlearrowleft$ given by $(1-s)(a_1,a_2,a_3,\ldots) = (a_1 - a_2^{\#},\ a_2 - a_3^{\#},\ldots)$. This can be globalized to the whole proper surgery group $L_m(X)$ (see e.g [10]) as an exact sequence

$$\longrightarrow \Pi_m \xrightarrow{1-s} L_m(\pi_1 X) \oplus \Pi_m \longrightarrow L_m(X) \longrightarrow \Pi_{m-1} \longrightarrow L_m(\pi_1 X) \oplus \Pi_m$$

where $\Pi_m = \prod_{n\geq 1} L_m(\pi_1 X_n)$ and $(1-s)(a_1,a_2,a_3\ldots) = (-a_1^{\#}, a_1 - a_2^{\#},\ a_2 - a_3^{\#},\ldots)$. Observe that although the map $1\text{-}s$ is in terms of $\pi_1 X_n$ for all n, nevertheless, Ker(1-s) and Coker(1-s) only depend on the equivalence class of the system $\pi_1 X_1 \leftarrow \pi_1 X_2 \leftarrow \pi_1 X_3 \leftarrow \ldots$. This exact sequence is the hermitian analog of a 5-terms exact sequence for K-theory obtained in [2] and [9]. Our method is geometric and uses a minimum of algebra (concentrated in Chapter V).

Let me thank W. Browder who encouraged me when I started this work at the I.A.S. (1969-71), Princeton. Let me thank also J. Wagoner for his helpful suggestions when I achieved this paper at U.C., Berkeley (1972). I also owe to R. Lee some useful conversations.

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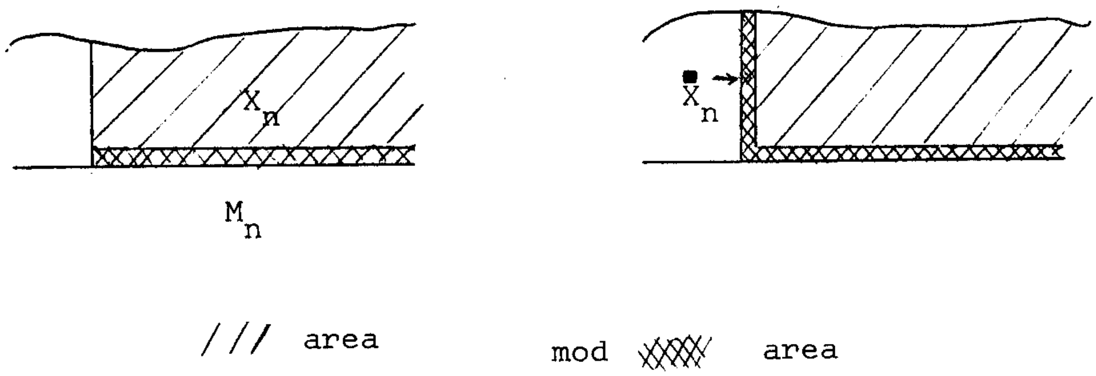
Notations and conventions.

1) For connected CW-complexes, all chain and cochain complexes, homology and cohomology modules are with universal coefficients. For non-connected CW-complexes, they are direct sum over the components. # means with some understood extended coefficients.

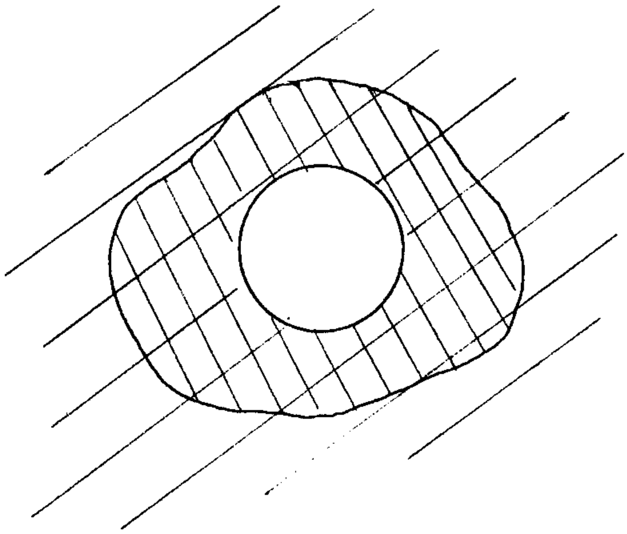
2) Our main geometrical situation will be the mapping cylinder of a map $f: M \to X$, with some understood subcomplexes X_n and $\overset{\blacksquare}{X}_n \subset X_n$. If $M_n \equiv X_n \cap M$, $\overset{\blacksquare}{M}_n \equiv \overset{\blacksquare}{X}_n \cap M$, we write $K_k(M_n)$ for $H_{k+1}(X_n, M_n)$, $K_k(M_m, \overset{\blacksquare}{M}_n)$ for $H_{k+1}(X_m, \overset{\blacksquare}{X}_n \cup M_m)$, $K_k(\overset{\blacksquare}{M}_n)$ for $H_{k+1}(\overset{\blacksquare}{X}_n, \overset{\blacksquare}{M}_n)$ and similarly for cohomology:



One should always remember what the X_n' are, as we shall have various X_n intersecting M along the same M_n.

3) For cocompact subcomplex (with relatively compact complement) a square $\blacksquare$ will mean a compact subcomplex containing

the frontier.



4) All L-groups are Wall-Novikov's groups (see [5]). Namely, given a group G, $L_{2q}(G)$ denotes the group of equivalence classes of quadratic finitely generated projective $\mathbb{Z}G$-modules (with the properties of the intersection pairing in a closed 2q-manifold). The nul element is represented by a quadratic module $\langle P \oplus X \rangle$ such that $\langle P,P \rangle = 0$, $\langle X,X \rangle = 0$ the induced composite isomorphism $P \cong X^* \cong P^{**}$, $X \cong P^* \cong X^{**}$ being $(-1)^q$ the evaluation map. Note that the dual is taken w.r.t. the involution $g \mapsto \omega(g)g^{-1}$ of ZG for some homomorphism $\omega : G \mapsto \pm 1$. This is also called a projective $(-1)^q$-hyperbolic module, and if P is free, a free (-1)-hyperbolic module. The opposite of a quadratic module $\langle Q \rangle$ is represented by Q with the opposite form $\langle x,y \rangle' = -\langle x,y \rangle$. Now, $L_{2q+1}(G)$ denotes the group of equivalence classes of projective Lagrangian planes ℓ in the standard free $(-1)^q$-hyperbolic module $\langle P \oplus X \rangle$ (ℓ is defined as a maximal direct summand of $P \oplus X$ such that $\langle \ell, \ell \rangle = 0$). The null element is represented by a Lagrangian plane ℓ which takes the trivial form $\ell_p \oplus \ell_x$ (ℓ_p, ℓ_x = direct summand of P, X respectively) after some Lagrangian transformation of $\langle P \oplus X \rangle$.

The latter is defined as follows: let $\langle t \oplus H\rangle$ be a hyperbolic module $(H \cong t^*,\ t \cong H^*)$, where t is projective, and $X \overset{\gamma}{\to} H$, $t \overset{\phi}{\to} H$ be linear maps, such that via $H \cong t^*$, ϕ becomes a $(-1)^{q+1}$-symmetric bilinear form on t (similar to the intersection pairing on a $2q+2$-manifold with boundary). A Lagrangian transformation of $\langle P \oplus X\rangle$ is the quadratic automorphism of $\langle P \oplus X\rangle + \langle t \oplus H\rangle$ defined by $(p,x,t,h) \to (p \pm \gamma^* t, x, t, h - x - \phi t)$ where $t \overset{\gamma^*}{\to} P$ is the dual of γ, and $p \in P$, $x \in X$, $t \in t$, $h \in H$. Note that the Lagrangian plane $\ell_0 \equiv P \oplus H = \{(p,0,0,h)\}$ is left fixed, while the image of the Lagrangian plane $X \oplus t = \{(0,x,t,0)\}$ is $\ell_1 \equiv \{(\pm\gamma^* t, x, t, -\gamma x - \phi t)\}$. These planes ℓ_0, ℓ_1 are considered as new "trivial" Lagrangian planes in $\langle P \oplus X\rangle \oplus \langle t \oplus H\rangle$ and the Lagrangian plane ℓ in $\langle P \oplus X\rangle$ represents 0 is $\ell \oplus t$ takes a trivial form w.r.t. ℓ_0 and ℓ_1. The opposite of a Lagrangian plane ℓ in $\langle P \oplus X\rangle$ is represented by ℓ^* in $\langle P \oplus X\rangle'$, where $\langle \ell \oplus \ell^*\rangle = \langle P \oplus X\rangle$ and $\langle p,x\rangle' = -\langle p,x\rangle$. When ℓ is free, then ℓ and ℓ^* in $\langle P \oplus X\rangle$ are equivalent, hence in this case (Wall groups) the inverse of ℓ in $\langle P \oplus X\rangle$ is also represented by ℓ in $\langle P \oplus X\rangle'$.

5) We shall often agree to reorder a sequence of integers $r_n \to \infty$ by replacing r_n by n.

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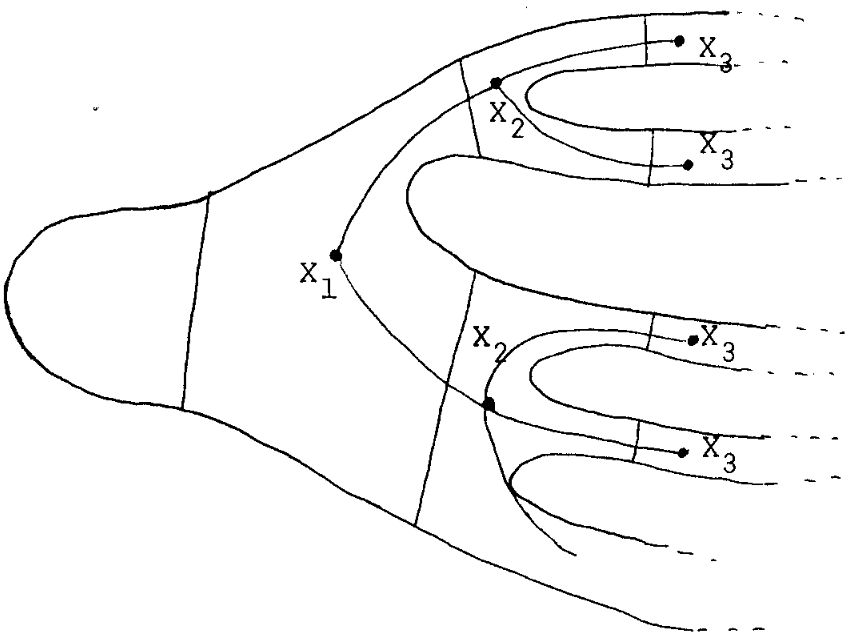
CHAPTER I. POINCARÉ DUALITY AT ∞

1. We work with the category of connected CW-complex X of finite dimension admitting a countable sequence $X_1 \supset X_2 \supset \ldots$ of subcomplexes, which is a fundamental system of ngbd of $\infty(\overline{X-X_n}$ is compact and $\bigcap_{n=1}^{\infty} X_n = \phi)$. By choosing a base point in each connected component of X_n, we let $\tilde{X}_n$ be the union of the universal covering of each pointed component. Then the $\pi_1 X_n$-chain complex $C(X_n)$ of cellular chains of $\tilde{X}_n$ is determined. Note that X_n has finitely many components.



Let $\overset{\blacksquare}{X}_n$ denote any finite subcomplex of X_n containing the frontier of X_n in X. We have a relative chain complex $C(X_n, \overset{\blacksquare}{X}_n)$ by taking $\tilde{X}_n$ mod the induced covering of $\overset{\blacksquare}{X}_n$. Similarly we have relative chain complexes $C(X_n, \overset{\blacksquare}{X}_n \cup X_r)$ for $r \geqslant n$, and we define $C_c^*(X_n,\overset{\blacksquare}{X}_n)$ by $\varinjlim_{r\to\infty} C^*(X_n,\overset{\blacksquare}{X}_n \cup X_r)$, where the dual is taken w.r.t. the anti-automorphism $g \mapsto \omega(g)g^{-1}$ of $\mathbb{Z}\pi_1 X_n$, for some fixed homomorphism $\pi_1 X \to \pm 1$. By joining the base points in X_{n+1} to the base points in X_n (forming a tree growing in each non-compact component of X_1), we get by excision canonical inverse systems of chain complexes $\{C(X_n)\}$ and $\{C_c^*(X_n,\overset{\blacksquare}{X}_n)\}$ well-defined up to an obvious notion of conjugate equivalence (see Chapter V). Given an element $[X] \in \varprojlim_r H_m(X,X_r;\mathbb{Z})$ (coefficients extended by $\pi_1 X \overset{w}{\to} \mathbb{Z}$), we find by excision $[X_{n,r}] \in H_m(X_n,\overset{\blacksquare}{X}_n \cup X_r;\ \mathbb{Z})$. The cap products by these latter homology classes induce a morphism of inverse systems $\{H_c^k(X_n,\overset{\blacksquare}{X}_n)\} \to$ $\to \{H_{m-k}(X_n)\}$ (see Chap. V and [1]). We shall say that $[X]$ is a <u>m-fundamental class</u> for X at infinity if $\cap\,[X]$ is an equivalence of inverse systems (see Chapter V). Observe that by taking a subsequence of (X_n) one can assume to have commutative diagrams

$$\begin{array}{ccc}
H_c^k(X_n,\overset{\blacksquare}{X}_n) & \xrightarrow{\ \cap\ } & H_{m-k}(X_n) \\
\uparrow & \overset{\psi_X}{\nwarrow} & \uparrow \\
H_c^k(X_{n+1},\overset{\blacksquare}{X}_{n+1}) & \xrightarrow{\ \cap\ } & H_{m-k}(X_{n+1}).
\end{array}$$

2. <u>Lemma</u>. Let $f\colon X \to X'$ be a proper homotopy equivalence. If $[X]$ is a m-fundamental class at ∞, then so is $[X'] = f_*[X]$.

For instance, if X has the proper homotopy type of a m-manifold, then X has a m-fundamental class at ∞. The proof of the lemma is clear.

3. If M, X are provided with m-fundamental classes at ∞ $[M]$, $[X]$, then we say that a proper map $f\colon M \to X$ is of degree 1 if $f_*[M] = [X]$. As f is proper we can find convergent sequences

of ngbd of ∞, M_n, X_n, such that $f(M_n) \subset X_n$ and choose $\dot{M}_n$, $\dot{X}_n$ such that $f(\dot{M}_n) \subset \dot{X}_n$. Then we have the modules $K_k(M_n)$ and $K_c^k(M_n, \dot{M}_n) \equiv H_c^{k+1}(X_n, \dot{X}_n \cup M_n)$ (see notations), which also form inverse systems, well-defined up to conjugate equivalence. When M is a manifold, we can choose the M_n to be cocompact submanifolds with boundary ∂M_n = closed bicollared submanifold. By enlarging $\dot{X}_n$, we can assume that $f(\partial M_n) \subset \dot{X}_n$. Now we identify X with the mapping cylinder of $M \overset{f}{\to} X$, so $X_n \cap M = M_n$ and $\dot{X}_n \cap M = \partial M_n$.

4. <u>Lemma</u>. Let M be an open manifold and $f: M \to X$ be a proper map of degree 1. Assume that $\dot{X}_n \cap M = \partial M_n$. Then the composition $K_{m-k}(M_n) \overset{\partial}{\to} H_{m-k}(M_n) \cong H_c^k(M_n, \partial M_n) \overset{\delta}{\to} K_c^k(M_n, \dot{M}_n)$ is a <u>canonical</u> equivalence of inverse systems, say $\psi: \{K_{m-k}(M_n)\} \to \{K_c^k(M_n, \partial M_n)\}$.

<u>Proof</u>. Choose an equivalence $\psi_X: H_{m-k}(X_{n+1}) \to H_c^k(X_n, \dot{X}_n)$ inverse to $\cap[X]$, and let α_n^* be the composition of morphisms

$$H_c^*(M_{n+1}, \dot{M}_{n+1}) \xrightarrow[\approx]{\cap[M]} H_*(M_{n+1}) \xrightarrow{f_{*n+1}} H_*(X_{n+1}) \xrightarrow{\psi_X} H_c^*(X_n, \dot{X}_n).$$

Then the square

$$\begin{array}{ccc} H_c^*(M_{n+1}, \partial M_{n+1}) & \xrightarrow{\alpha_n^*} & H_c^*(X_n, \dot{X}_n) \\ \approx \downarrow \cap[M] & & \\ H_*(M_{n+1}) & & \downarrow \cap[X] \\ \downarrow & & \\ H_*(M_n) & \xrightarrow{f_{*,n}} & H_*(X_n) \end{array}$$

is commutative, hence provides an equivalence $\mathrm{Ker}\ \alpha_n^* \xrightarrow{\cap} \mathrm{Ker}\ f_{*,n}$. Moreover, the composition $\alpha^* \circ f^*$ is just the canonical map ι.

Hence the map $\beta = \iota - f^* \circ \alpha^* : H_c^*(M_{n+2}, \partial M_{n+2}) \to H_c^*(M_{n+1}, \partial M_{n+1})$ induces a morphism Coker $f_{n+2}^* \xrightarrow{\beta}$ Ker α_n^*, which turns out to be inverse to the morphism Ker $\alpha_n^* \to H_c^*(M_{n+1}, \partial M_{n+1}) \to$ Coker f_{n+1}^*, hence is an equivalence. The composition $\beta \circ \psi_X$ (we skip some obvious map) reduces to $\iota \circ \psi_X$, because $f^* \circ \alpha^* \circ \cap^{-1} \circ \partial = f^* \circ \psi_X \circ \underbrace{f_* \circ \partial}_{0}$.

Hence $\cap \circ \beta \circ \psi : K_*(M_n) \to \text{Ker } f_{*,n}$ is the canonical map. But the latter turns out to be an equivalence, by introducing the composition of morphisms

$$\alpha_{*,n} : H_*(X_{n+1}) \xrightarrow{\psi_X} H_c^*(X_n, \dot{X}_n) \xrightarrow{f_n^*} H_c^*(M_n, \dot{M}_n) \xrightarrow[\approx]{\cap [M]} H_*(M_n)$$

which satisfies $f_* \circ \alpha_* =$ canonical map (use Chapter V). Similarly Coker $f_n^* \to K_c^*(M_n, \partial M_n)$ is an equivalence.

<u>Addendum</u>: ψ has an inverse equivalence $K_c^*(M_n, \partial M_n) \to K_{*-k}(M_n)$

<u>Proof</u>. By using α_* and α^*, check that the maps in the kernel systems of $K_*(M_n) \to \text{Ker } f_{*,n}$ and the cokernel system of Coker $f_n^* \to K_c^*(M_n, \partial M_n)$ vanish.

5. The above Poincaré duality has its dual counterpart. Namely, for a proper map $f: M \to X$ of degree 1, we have also the module $K_c^k(M_n) \equiv H_c^{k+1}(X_n, M_n)$ (see notations). If now # means with coefficients $\pi_1 X_n$, n fixed, then for $r \geq n$ $\{K_c^k(M_r)^\#\}_n$ and $\{K_k(M_r, \dot{M}_r)^\#\}_n$ are canonical direct systems (the latter by excision). Then the following holds.

6. <u>Lemma</u>. (Dual to lemma 4) With the above setting, if $\dot{M}_n \equiv \dot{X}_n \cap M = \partial M_n$, then the composition

$K_{m-k}(M_r,\partial M_r)^{\#} \overset{\partial}{\to} H_{m-k}(M_r,\partial M_r)^{\#} \cong H_c^k(M_r)^{\#} \overset{\delta}{\to} K_c^k(M_r)^{\#}$ is a <u>canonical</u> equivalence of direct systems, say $\bar{\psi}\colon \{K_{m-k}(M_r,\partial M_r)^{\#}\}_n \to \{K_c^k(M_r)^{\#}\}_n$.

<u>Proof</u>. First we show that $[X]$ induces by cap products an equivalence of direct systems $\{K_c^k(X_r)^{\#}\}_n \to \{K_{m-k}(X_r,\overset{\blacksquare}{X}_r)^{\#}\}_n$. The dual of the cochain complex $C_c^*(X_r,\overset{\blacksquare}{X}_r)^{\#}$ is the chain complex $\bar{C}(X_r,\overset{\blacksquare}{X}_r) \equiv \varprojlim_s C(X_r,\overset{\blacksquare}{X}_r \cup X_s)^{\#}$ of locally finite $\pi_1 X_n$-cellular chains. Now $[X]$ comes from $\bar{C}_m(X;\mathbb{Z})$, because so does $[M]$. Then we get two morphisms $\{C^*(X_r)^{\#}\}_n \to \{\bar{C}(X_r,\overset{\blacksquare}{X}_r)^{\#}\}_n$, either by taking induced chain cap products $\bar{\cap}$, or by dualizing the induced former chain cap products $\{C_c^*(X_r,\overset{\blacksquare}{X}_r)^{\#}\}_n \overset{\cap}{\to} \{C(X_r)^{\#}\}_n$. On homology level, they are the same up to sign, hence $\{H^*(X_r)^{\#}\}_n \overset{\bar{\cap}}{\to} \{\bar{H}_*(X_r,\overset{\blacksquare}{X}_r)^{\#}\}_n$ is an equivalence of direct systems (See V, 12). In particular, $\varinjlim_r H^*(X_r)^{\#} \cong \varinjlim_r \bar{H}_*(X_r,\overset{\blacksquare}{X}_r)^{\#}$. The first member is the end cohomology $H_e^*(X_n)$, determined by the chain complex $\varinjlim_r C^*(X_r)^{\#}$, and the second member is say the end homology $H_*^e(X_n)$, determined by the chain complex $C^e(X_n) \equiv \varinjlim_r \bar{C}(X_r,\overset{\blacksquare}{X}_r)^{\#}$ which is nothing but the quotient $\bar{C}(X_n,\overset{\blacksquare}{X}_n) \big/ C(X_n,\overset{\blacksquare}{X}_n)$ (take $\varinjlim_r$ $\varprojlim_s$ of $0 \to C(\overline{X_n - X_r},\overset{\blacksquare}{X}_n) \to$
$\to C(X_n,\overset{\blacksquare}{X}_n \cup X_s) \to C(X_n,\overline{X_n - X_r} \cup X_s) \to 0$ where $\overline{X_n - X_r}$ is the subcomplex $(X_n - X_r) \cup \overset{\blacksquare}{X}_r)$. Then we have an exact $\pm$commutative ladder (see [1])

$$\begin{array}{ccccc} \xrightarrow{\delta} H^k_c(X_r)^\# & \longrightarrow & H^k(X_r)^\# & \longrightarrow & H^k_e(X_n) \xrightarrow{\delta} \\ \Big\downarrow \cap[X] & & \Big\downarrow \cap[X] & & \cong \Big\downarrow \cap[X] \\ \xrightarrow{\partial} H_{m-k}(X_r,\overset{\blacksquare}{X}_r) & \longrightarrow & \bar{H}_{m-k}(X_r,\overset{\blacksquare}{X}_r)^\# & \longrightarrow & H^e_{m-k}(X_n) \xrightarrow{\partial} . \end{array}$$

We have seen that the middle rung is an equivalence of direct systems, hence so is the left rung by V. 8. Now, we can dualize the proof of lemma 4 to get the assertion.

7. By taking a subsequence of (X_n), we can assume to have <u>simultaneous</u> equivalences $\psi_X: H_*(X_{n+1}) \to H^*_c(X_n,\overset{\blacksquare}{X}_n)$ and $\bar{\psi}_X: H_*(X_r,\overset{\blacksquare}{X}_r)^\# \to H^*_c(X_{r+1})^\#$. Then $\psi: K_*(M_n) \to K^*_c(M_n,\partial M_n)$ and $\bar{\psi}: K_*(M_r,\partial M_r)^\# \to K^*_c(M_r)^\#$ have inverses $K^*_c(M_n,\partial M_n) \to K_*(M_{n-4})$ and $K^*_c(M_r)^\# \to K_*(M_{r+4},\partial M_{r+4})^\#$. Hence, by taking again a subsequence of (X_n), we can assume that ψ and $\bar{\psi}$ have inverses $K^*_c(M_n,\partial M_n) \to K_*(M_{n-1})$ and $K^*_c(M_r)^\# \to K_*(M_{r+1},\partial M_{r+1})^\#$. Another important observation is that the square

$$\begin{array}{ccc} K^*_c(M_r,\partial M_r)^\# & \longrightarrow & K^*_c(M_r)^\# \\ \Big\uparrow \psi & & \Big\uparrow \bar{\psi} \\ K_*(M_r)^\# & \longrightarrow & K_*(M_r,\partial M_r)^\# \end{array}$$

is commutative. Now, let $\bar{C}(X_r,M_r)^\#$ be the chain complex dual to $C^*_c(X_r,M_r)^\#$, and $\bar{K}_k(M_r)^\#$ its k+1-homology. We have a canonical map $\overline{K}_k(M_r)^{\#} \to (K_c^k(M_r)^{\#})^*$, hence, by composition with the dual of $\overline{\psi}$, a map $\overline{\psi}^*\colon \overline{K}_k(M_r)^{\#} \longrightarrow K^{m-k}(M_r, \partial M_r)^{\#}$, which is a morphism of inverse systems. Similarly, the dual of ψ provides a morphism of direct systems $\psi^*\colon \overline{K}_k(M_r, \partial M_r)^{\#} \to K^{m-k}(M_r)^{\#}$. By taking the direct limit of the latter for $r \to \infty$, we get $K_k^e(M_r, \partial M_r)^{\#} \to K_e^{m-k}(M_r)$.

The exact ladders

$$\begin{array}{ccccccc}
\longrightarrow & K^{m-k}(M_r)^{\#} & \longrightarrow & K_e^{m-k}(M_r)^{\#} & \longrightarrow & K_c^{m-k+1}(M_r)^{\#} & \longrightarrow \\
 & \Big\uparrow \psi^* & & \Big\uparrow \lim\limits_{\to} \psi^* & & \Big\uparrow \overline{\psi} & \\
\longrightarrow & \overline{K}_k(M_r, \partial M_r)^{\#} & \longrightarrow & K_k^e(M_r, \partial M_r)^{\#} & \longrightarrow & K_{k-1}(M_r, \partial M_r)^{\#} & \longrightarrow
\end{array}$$

and

$$\begin{array}{ccccccc}
\longrightarrow & K^{m-k}(M_r, \partial M_r)^{\#} & \longrightarrow & K_e^{m-k}(M_r, \partial M_r)^{\#} & \longrightarrow & K_c^{m-k+1}(M_r, \partial M_r)^{\#} & \longrightarrow \\
 & \Big\uparrow \overline{\psi}^* & & \Big\uparrow \lim\limits_{\to} \psi^* & & \Big\uparrow \psi & \\
\longrightarrow & \overline{K}_k(M_r)^{\#} & \longrightarrow & K_k^e(M_r) & \longrightarrow & K_{k+1}(M_r) & \longrightarrow
\end{array}$$

where $K_k^e(M_r)^{\#} \equiv K_k^e(M_r, \partial M_r)^{\#}$ and $K_e^{m-k}(M_r)^{\#} \equiv K_e^{m-k}(M_r, \partial M_r)^{\#}$ by definition of H_e^* and H_*^e, are $\pm$ commutative. In general one knows nothing about ψ^* and $\overline{\psi}^*$.

CHAPTER II. PROPER SURGERY AT ∞

The data is a proper normal map $f: M \to X$ of degree 1, where M is a smooth open (oriented) m-manifold and X a complex with fundamental class $[X] = f_*[M]$ at ∞. Of course, "normal" means as in [1] that for some stable vector bundle ζ over X, f is covered by a map $\nu \to \zeta$, where ν is the stable normal bundle of M in euclidian space. A cobordism of such a data is the obvious thing (see III, 9 and IV, 3), and we look for the obstruction for f to be cobordant to a proper map $f': M' \to X$ such that

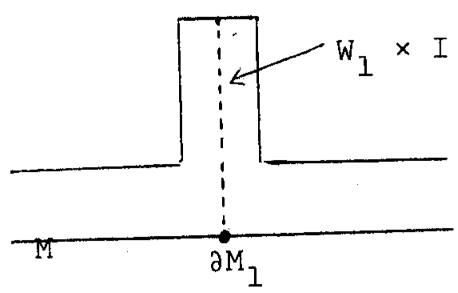
i) f' induces a bijection of ends spaces

ii) the morphism $f'_*: \{\pi_1 M'_n\} \to \{\pi_1 X_n\}$ of inverse systems of groups is an equivalence

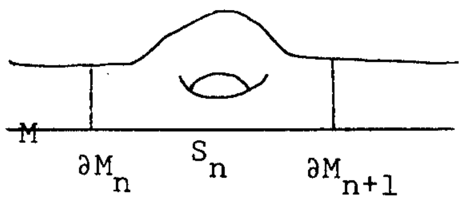
iii) all inverse systems $\{K_k(M'_n)\}$ are equivalent to 0.

Geometrically, this means that f' is a proper homotopy equivalence at ∞ (see [9]).

Recall first (see [11]) that, if $f: M \to X$ maps a bicollared closed submanifold $\dot{M}_1$ of M into a finite subcomplex $\overset{\blacksquare}{X}_1$ of X, then the restriction $\dot{M}_1 \xrightarrow{f} \overset{\blacksquare}{X}_1$ is normal, and every surgery on it extends to a surgery of f:

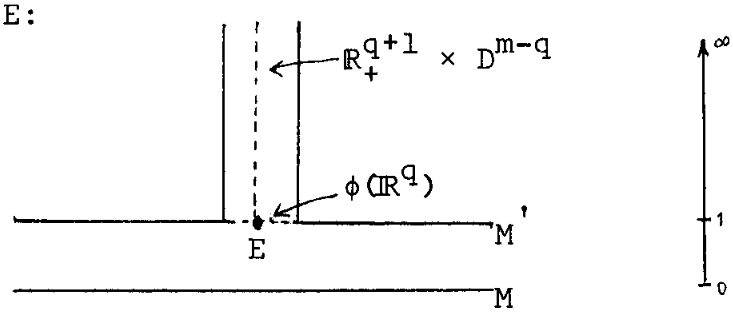


By doing this on a divergent sequence $\overset{\bullet}{M}_n \xrightarrow{f} \overset{\blacksquare}{X}_n$ we get obviously a cobordism of $f\colon M \to X$. If $\overset{\bullet}{M}_n \xrightarrow{f} \overset{\blacksquare}{X}_n$ and $\overset{\bullet}{M}_{n+1} \xrightarrow{f} \overset{\blacksquare}{X}_{n+1}$ bound a restriction $\overline{M_n - M_{n+1}} \xrightarrow{f} \overline{X_n - X_{n+1}}$ then every surgery on the latter rel. $\partial M_n \cup \partial M_{n+1}$ extends also to a surgery of f.



By doing this for each n, we get also a cobordism of f.

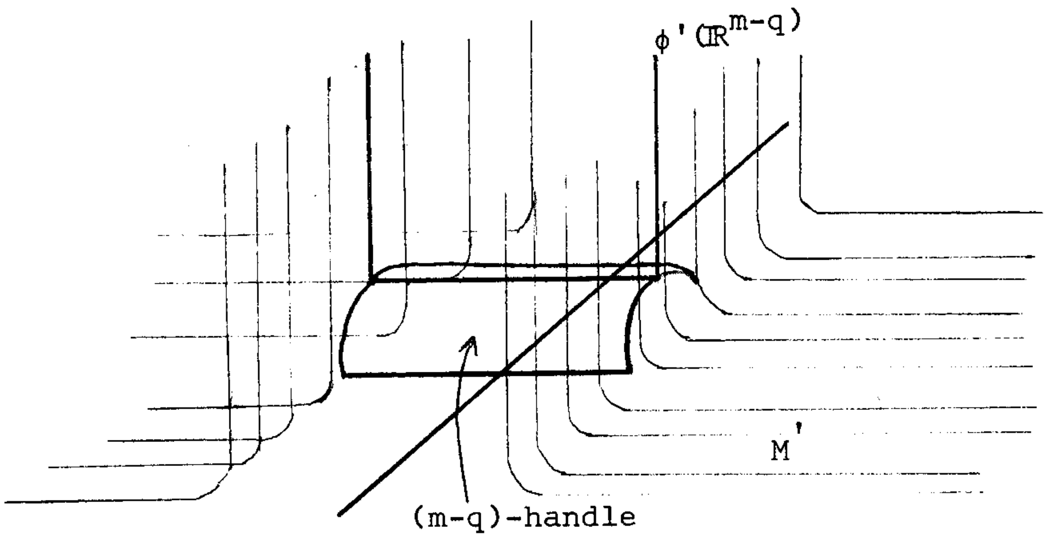
We consider still another particular kind of surgery . Suppose we have a proper embedding $\phi\colon \mathbb{R}^q \to M$ and a proper extension $\psi\colon \mathbb{R}^{q+1}_+ \to X$ of $f\circ\phi$, where $\mathbb{R}^{q+1}_+ \equiv \mathbb{R}^q \times [1,\infty)$. Then, if E is a tubular ngbd of $\phi(\mathbb{R}^q)$ in M, we have a trivialization $E \cong \mathbb{R}^q \times D^{m-q}$ (by contracting $\mathbb{R}^q$ into 0). Similarly, we have a trivialization of $\phi^*\nu$ which extends to a trivialization of $\psi^*\zeta$. Hence we can make a cobordism on f by gluing $\mathbb{R}^{q+1}_+ \times D^{m-q}$ to $M \times I$ along E:



and mapping the resulting (m+1)-manifold W to $X \times I$ by

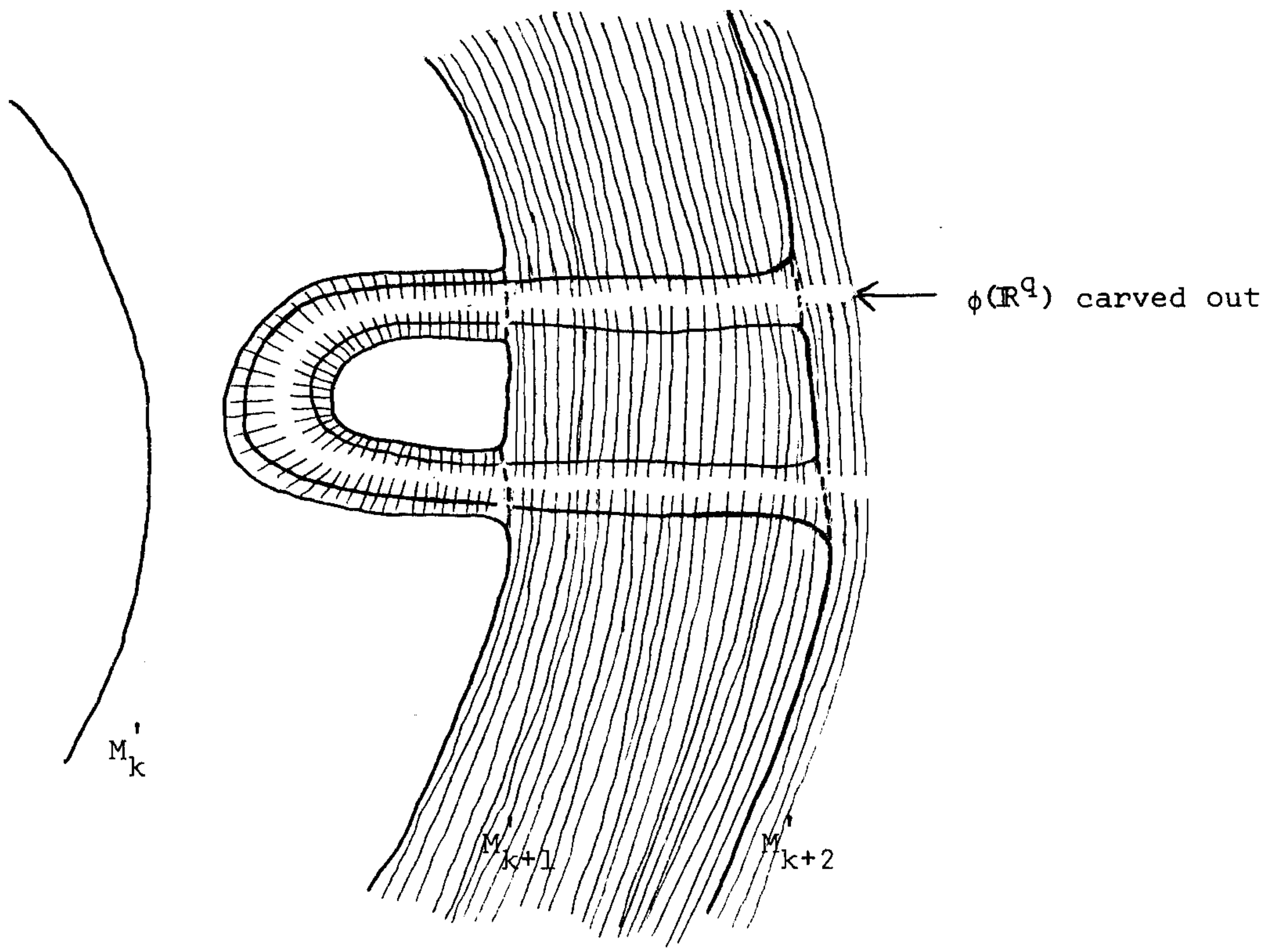
$$F = \begin{cases} f \times \text{id.} & \text{on } M \times I \\ \mathbb{R}^{q+1}_+ \times D^{m-q} \xrightarrow{\text{proj.}} \mathbb{R}^{q+1}_+ \xrightarrow{\psi} X & \text{on } \mathbb{R}^{q+1}_+ \times D^{m-q} \end{cases}$$

W is a cobordism from M to $M' \cong M - \phi(\mathbb{R}^q)$, and both inclusions $M \to W \leftarrow M' \cup D^{m-q}$ are homotopy equivalences (D^{m-q} = a fiber of E). Observe that W is constructed from M' by attaching first a $(m-q)$-handle along a framed sphere transverse to $\phi(\mathbb{R}^q)$ and then carving out $\phi'(\mathbb{R}^{m-q})$ in the result, i.e., attaching $(\phi'(\mathbb{R}^{m-q}) \times D^q) \times \mathbb{R}_+$



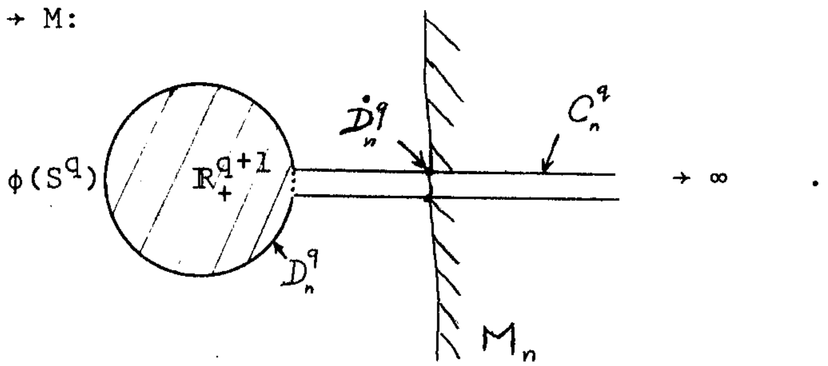
If $M_1 \supset M_2 \supset \ldots$ is a fundamental system of ngbd of ∞ in M, then $W_n \equiv (M_n \times I) \cup \left(\overline{\mathbb{R}_+^{q+1} - nD_+^{q+1}} \times D^{m-q}\right)$ is such a system in W, where nD_+^{q+1} is the half disc of radius n in $\mathbb{R}_+^{q+1}$. If $E \subset M_k$ but $E \not\subset M_{k+1}$, it is more convenient to replace above nD_+^{q+1} by $(n-k)D_+^{q+1}$. Then

$$M_n' \equiv W_n \cap M' = \begin{cases} M_n - \phi(\mathbb{R}^q) & \text{for } n \leqslant k \\ (M_n \cup q\text{-handle}) - \phi(\mathbb{R}^q) & \text{for } n > k. \end{cases}$$



This implies that for $q \geq 2$ and $m - q \geq 2$ the ends spaces of M and M' are the same. Moreover, for $q \geq 3$ and $m-q \geq 3$, $\pi_1 M_n \cong \pi_1 M_n'$.

We shall only use this kind of surgery in the case where (ϕ,ψ) comes from an embedding $S^q \xrightarrow{\phi'} M$ and an extension $D^{q+1} \xrightarrow{\psi'} X$ of $f\circ\phi'$, by piping $\phi(S^q)$ to ∞ along a proper embedding $[0,\infty) \to M$:



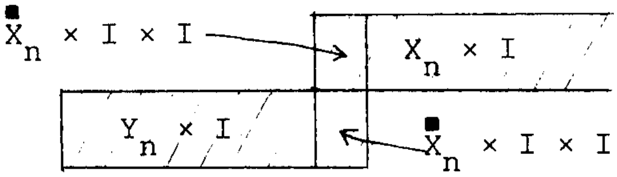
In this case, $\pi_1 M_n = \pi_1 M'_n$ already for $q \geqslant 2$, $m-q \geqslant 3$:

$$W_n = [0,1]\times M_n \bigcup_{1\times C^q_n \times D^{n-q}} [1,\infty)\times C^q_n \times D^{n-q} \bigcup_{[n,\infty)\times \dot{D}^q_n \times D^{n-q}} [n,\infty)\times D^q_n \times D^{n-q}$$

$$\dot{W}_n = [0,1]\times \dot{M}_n \bigcup_{1\times \dot{D}^q_n \times D^{n-q}} [1,n]\times \dot{D}^q_n \times D^{n-q} \bigcup_{n\times D^q_n \times D^{n-q}} n\times D^q_n \times D^{n-q} .$$

CHAPTER III. THE OPEN ODD DIMENSIONAL CASE

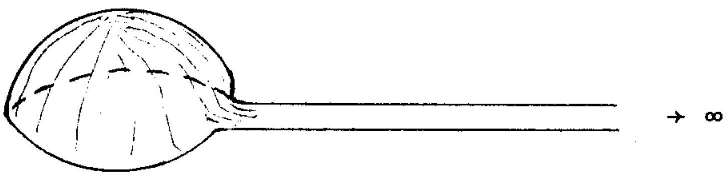
1. Let M be an open manifold of dimension $2q+1 \geqslant 7$, and $f: M \to X$ be a proper normal map of degree 1. We assume that X is connected, and choose a sequence of cocompact subcomplexes $X_n \to \infty$ in X, such that X_n has <u>only non-compact components</u>. Moreover, we can choose finite subcomplexes $\overset{\blacksquare}{X}_n$ of X_n containing the frontier such that $\overset{\blacksquare}{X}_n \cap$ (component of X_n) is connected. If $Y_n \equiv (X-X_n) \cup \overset{\blacksquare}{X}_n$, then by replacing X by $(X_n \times I) \cup (Y_n \times I)$ as follows:



we can assume that each $\overset{\blacksquare}{X}_n$ is bicollared in X. Putting f transverse on each $\overset{\blacksquare}{X}_n$, we get submanifolds $M_n = f^{-1}(X_n)$ with boundary $\partial M_n = f^{-1}(\overset{\blacksquare}{X}_n)$, such that $M_n \to \infty$. After surgering each map $\partial M_n \overset{f}{\to} \overset{\blacksquare}{X}_n$ and $\overline{M_n - M_{n+1}} \to \overline{X_n - X_{n+1}}$ we can assume that they are q-connected. In particular, f is bijective on ends spaces, by interpreting an end of X as a function $\{X_n\} \overset{\varepsilon}{\to} \{\pi_0 X_n\}$ such that $\varepsilon(X_{n+1}) \subset \varepsilon(X_n)$ and similarly for M. By van Kampen and Mayer-Vietoris, each map $M_n \overset{f}{\to} X_n$ is q-connected. Now, in the homotopy exact sequence

$$\pi_{q+1}(\overline{X_k - X_{k+1}}, \overline{M_k - M_{k+1}}) \to \pi_{q+1}(\overline{X_k - X_{k+1}}, \overset{\blacksquare}{X}_{k+1} \cup \overline{M_k - M_{k+1}}) \overset{\partial}{\to} \pi_q(\overset{\blacksquare}{X}_{k+1} \cup \overline{M_k - M_{k+1}}, \overline{M_k - M_{k+1}})$$

the last term vanishes by Hurewicz isomorphism and excision. The middle term is finitely generated because by the Hurewicz isomorphism it is the lowest homology of a finite complex. Hence each generator is represented by a map $(D^{q+1}, S^q) \to \left(\overline{X_k - X_{k+1}}, \overline{M_k - M_{k+1}}\right)$, and moreover, S^q can be embedded by general position. We can pipe the image of S^q to ∞ by a proper embedded pipe line



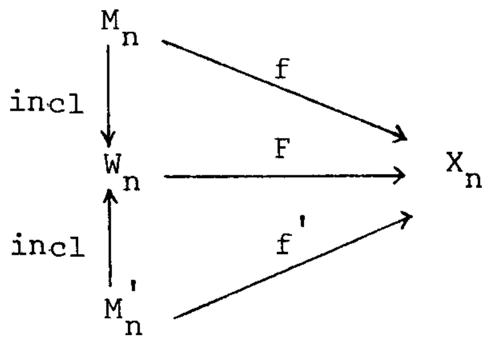
to get a proper map $(\mathbb{R}^{q+1}_+, \mathbb{R}^q) \to (X_k, M_k)$ which is an embedding on $\mathbb{R}^q$. Let us do surgery on this map, as in Chapter III. In the diagramm

$$\begin{array}{ccc} \partial M_n & \xrightarrow{\;f\;} & \\ \cap & & \\ \overset{\bullet}{W}_n & \xrightarrow{\;F\;} & \overset{\blacksquare}{X}_n \\ \cup & & \\ \partial M_n' & \xrightarrow{\;f'\;} & \end{array} \qquad (\overset{\bullet}{W}_n = \text{frontier})$$

we have

$$\begin{cases} \overset{\bullet}{W}_n = \partial M_n \times I & \text{if } n < k \\ (\partial M_n \vee S^q) \sim \overset{\bullet}{W}_n \sim (\partial M_n' \cup D^{q+1}) & \text{if } n \geqslant k \end{cases}$$

(we can assume that $\phi(\mathbb{R}^q)$ meets ∂M_n along the sphere of radius $n-k$, when $n > k$). Hence the maps $\mathring{W}_n \xrightarrow{F} \mathring{X}_n$ and $\partial M_n' \xrightarrow{f'} \mathring{X}_n$ are also q-connected, for any n. In the diagram



W_n has the homotopy type of M_n and $M_n' \cup D^{q+1}$ for $n < k$, and of $M_n \vee S^q$ and $M_n' \cup D^{q+1}$ for $n \geqslant k$. Hence F_n and f_n' are also q-connected. Now, if we write $K_q(M_k', M_{k+1}') \equiv H_{q+1}(X_k, X_{k+1} \cup M_k')$ in the mapping cylinder of f', we have

$$K_q(M_n', M_{n+1}') = \begin{cases} 0 & \text{for } n = k \\ K_q(M_n, M_{n+1}) & \text{for } n \neq k, \end{cases}$$

as easily verified. By induction on k, we can assume that $K_q(M_n, M_{n+1}) = 0$ for each n. An immediate consequence is that $K_c^q(M_n) = 0$, hence the direct system $\{K_{q+1}(M_r, \partial M_r)^\#\}$ is equivalent to $\{0\}$, by duality. Another consequence is that $\bar{K}_q(M_n) \equiv \bar{H}_{q+1}(X_n, M_n)$ vanishes: because $K_k(M_n) = 0$ for $k < q$, each n, one can eliminate by Whitehead's trick (see [6]) all cells of dimension $\leqslant q$ in $X_n - M_n$ and this by a proper (simple) homotopy equivalence of X rel. M. Moreoever, for each (n,r), separately, one can also eliminate the $q+1$-cells in $X_n - (M_n \cup X_r)$, because $K_q(M_n, M_r) = 0$. Hence each chain complex

$C(X_n,M_n \cup X_r)$ has the chain homotopy type of one chain complex $C(n,r)$ which vanishes in dimensions $\leqslant q+1$. Moreover, we can get commutative squares

$$\begin{array}{ccc} C(X_n,M_n \cup X_r) & \xrightarrow{\text{h.e.}} & C(n,r) \\ \uparrow & & \uparrow \\ C(X_n,M_n \cup X_{r+1}) & \xrightarrow{\text{h.e.}} & C(n,r+1) \end{array}$$

as follows: having eliminated in $X_n - (M_n \cup X_r)$, getting X_n', we choose the elimination in $X_n - (M_n \cup X_{r+1})$ by first eliminating in $X_r - (M_r \cup X_{r+1})$ getting X_r', and then extending this formal deformation to X_n', getting X_n''. This provides the required commutative diagram

$$\begin{array}{ccccc} C(X_n,M_n \cup X_r) & \longrightarrow & C(X_n',M_n \cup X_r) & \equiv & C(n,r) \\ & & \downarrow \approx & & \\ \uparrow \text{can.} & & C(X_n'',M_n \cup X_r') & & \\ & & \uparrow \text{can.} & & \\ C(X_n,M_n \cup X_{r+1}) & \longrightarrow & C(X_n'',M_n \cup X_{r+1}) & \equiv & C(n,r+1). \end{array}$$

Now, the chain mapping cone of each homotopy equivalence $C(X_n,M_n \cup X_r) \to C(n,r)$ is free acyclic, and for n fixed, r variable, they form an induced inverse system. Because each cone splits completely, their inverse limit is an acyclic chain complex, which is nothing but the chain mapping cone of $\varprojlim_r C(X_n,M_n \cup X_r) \to \varprojlim C(n,r)$, hence the latter map is a homology isomorphism. This proves that the $(q+1)$-dimensional homology of $\lim\limits_{\overleftarrow{r}} C(X_n, M_n \cup X_r)$ vanishes, i.e. $\overline{K}_q(M_n) = 0$. By duality, this implies that the inverse system $\{K^{q+1}(M_n, \partial M_n)\}$ is equivalent to $\{0\}$.

2. <u>Proposition</u>: The inverse system $\{K_q(M_n)\}$ and the direct system $\{K_q(M_r, \partial M_r)^{\#}\}_n$ (in the latter # means with $\pi_1 M_n$-coefficients, for n fixed) are equivalent to systems of projective countably generated modules.

<u>Proof</u>. By using the duality equivalence, one has to prove the same assertion for $\{K_c^{q+1}(M_n, \partial M_n)\}$ and $\{K_c^{q+1}(M_r)^{\#}\}$. As above, we can assume that $X_n - M_n$ contains no cells of dimension $\leq q$, for each n. Moreover, for each (n,r) separately, one can eliminate the $(q+1)$-cells in $X_n - (\overset{\blacksquare}{X}_n \cup M_n \cup X_r)$, because $K_q(M_n, \partial M_n \cup M_r) \equiv H_{q+1}(X_n, \overset{\blacksquare}{X}_n \cup M_n \cup X_r) = 0$ in virtue of the homology exact sequence $\underbrace{K_q(M_n, M_r)}_{0} \to K_q(M_n, \partial M_n \cup M_r) \xrightarrow{\partial} \underbrace{K_{q-1}(\partial M_n)^{\#}}_{0}$.

Hence each chain complex $C(X_n, \overset{\blacksquare}{X}_n \cup M_n \cup X_r)$ has the chain homotopy type of one chain complex $C(n,r)$ say, which vanishes in dimension $\leq q+1$. Moreover, we can get commutative diagram

$$\begin{array}{ccc} C(X_n, \overset{\blacksquare}{X}_n \cup M_n \cup X_r) & \xrightarrow{\text{h.e.}} & C(n,r) \\ \uparrow & & \uparrow \\ C(X_n, \overset{\blacksquare}{X}_n \cup M_n \cup X_{r+1}) & \xrightarrow{\text{h.e.}} & C(n,r+1) \end{array}$$

as follows: having eliminated in $X_n - (\overset{\blacksquare}{X}_n \cup M_n \cup X_r)$ getting X_n', choose the elimination in $X_n - (\overset{\blacksquare}{X}_n \cup M_n \cup X_{r+1})$ by first eliminating in $X_r - (M_r \cup X_{r+1})$ getting X_r', and then extending this formal deformation to X_n', getting X_n''. This provides the required commutative diagram

$$\begin{array}{ccc}
C(X_n,\overset{\blacksquare}{X}_n \cup M_n \cup X_r) & \longrightarrow & C(X_n',\overset{\blacksquare}{X}_n \cup M_n \cup X_r) \equiv C(n,r) \\
 & & \downarrow \approx \\
\uparrow \text{can.} & & C(X_n'',\overset{\blacksquare}{X}_n \cup M_n \cup X_r') \\
 & & \uparrow \text{can.} \\
C(X_n,\overset{\blacksquare}{X}_n \cup M_n \cup X_{r+1}) & \longrightarrow & C(X_n,\overset{\blacksquare}{X}_n \cup M_n \cup X_{r+1}) \equiv C(n,r+1)
\end{array}$$

If $C^*(n) \equiv \varprojlim_r C^*(n,r)$, we then have a chain map $C_c^*(X_n,\overset{\blacksquare}{X}_n \cup M_n) \to C^*(n)$ which is a homology isomorphism. But the above maps $C(n,r+1) \to C(n,r)$ are such that $C^*(n)$ is free of countable rank (up to isomorphism of $C(n,r)$, they are cellular embeddings), and $C_c^*(X_n,\overset{\blacksquare}{X}_n \cup M_n)$ is also free. So actually the map $C_c^*(X_n,\overset{\blacksquare}{X}_n \cup M_n) \to C^*(n)$ is a chain homotopy equivalence. Using homotopy inverse maps, we get an inverse system $C^*(n+1) \to C^*(n)$ whose associated homology systems are isomorphic to $\{K_c^*(M_n,\partial M_n)\}$ (although the diagram

$$\begin{array}{ccc}
C_c^*(X_n,\overset{\blacksquare}{X}_n \cup M_n) & \longrightarrow & C^*(n) \\
\uparrow & & \uparrow \\
C_c^*(X_{n+1},\overset{\blacksquare}{X}_{n+1} \cup M_{n+1}) & \longrightarrow & C^*(n+1)
\end{array}$$

is only chain homotopy commutative). Hence Prop. V, 9 applies to $\{C^*(n)\}$, proving the assertion for $\{K_q(M_n)\}$. For the other system, the proof is similar.

3. We are now at the point where we cannot do further surgeries, but we can still work on the subcomplexes $(X_n, \overset{\blacksquare}{X}_n)$ to improve the canonical square

$$\begin{array}{ccc} K_c^{q+1}(M_r, \partial M_r)^{\#} & \longrightarrow & K_c^{q+1}(M_r)^{\#} \\ \uparrow \psi & & \uparrow \overline{\psi} \\ K_q(M_r)^{\#} & \longrightarrow & K_q(M_r, \partial M_r)^{\#} \end{array}$$

that we have so far.

4. Lemma. Ker ψ and Ker $\overline{\psi}$ are finitely generated.

Proof. In the proof of Proposition 2 above, we have shown that $\{K_c^{q+1}(M_r, \partial M_r)^{\#}\}_n$ is the top homology system associated to some system of free chain complexes $\{C(r)\}$ (this is not so for $C_c^{*}(X_r, \overset{\blacksquare}{X}_r \cup M_r)^{\#}$ as $C_c^{q+1}(X_r, \overset{\blacksquare}{X}_r \cup M_r) \neq 0$). Then V. 10 applies to $\{C(r)\}$, giving an equivalence $\{K_c^{q+1}(M_r, \partial M_r)^{\#}\} \to \{P_r\}$ which is injective for each r, where each P_r is projective, as well as the image P_r' of $P_{r+2} \to P_r$. Hence the composition $K_q(M_r)^{\#} \overset{\psi}{\to} K_c^{q+1}(M_r, \partial M_r)^{\#} \to P_r$ has kernel equal to Ker ψ. Moreover, its image is P_r', because ψ and the injection into P_r are both equivalences, hence we have the commutative diagram

$$\begin{array}{ccc} K^q(M_r)^{\#} & \xrightarrow{\ \psi\ } & P_r \\ \text{surj.} \uparrow & \nwarrow \text{surj.} & \uparrow \\ K_q(M_{r+2})^{\#} & \xrightarrow{\ \psi\ } & P_{r+2} \end{array}$$

Then the exact sequence $0 \to \operatorname{Ker} \psi_r \to K_q(M_r)^{\#} \to P_r' \to 0$ splits. In particular, $\ker \psi_r$ is a retract of $K_q(M_r)^{\#}$. But the commutative triangle

$$\begin{array}{ccc} K_q(M_r)^{\#} & \xrightarrow{\quad \iota \quad} & K_q(M_{r-1})^{\#} \\ & \searrow^{\psi} \qquad \nearrow & \\ & K_c^{q+1}(M_r, \partial M_r)^{\#} & \end{array}$$

shows that $\ker \psi_r \subset \ker \iota = \partial K_{q+1}(M_{r-1}, M_r)^{\#}$. As $K_{q+1}(M_{r-1}, M_r)^{\#}$ is a finitely generated module, so is $\operatorname{Ker} \iota$. But $\ker \psi_r$ becomes a retract of $\operatorname{Ker} \iota$, hence is also finitely generated. The same argument applies to $\ker \overline{\psi}$.

<u>Remarks 1^0</u>. If above we knew that ψ was <u>already injective</u>, then $K_q(M_r)^{\#}$ is isomorphic to P_r' hence is projective. Moreover, by V. 10, one can assume that P_r' is a direct summand of P_r, hence ψ splits. Similarly for $\overline{\psi}$.

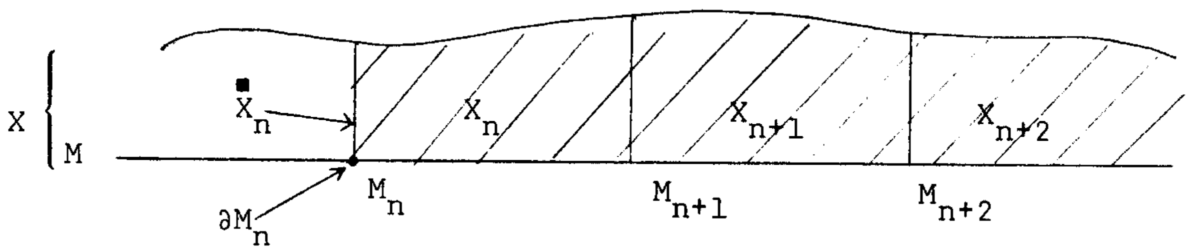
<u>2^0</u>. We have shown that for each n, there is some $r_n > n$ such that $K_q(M_{r_n})^{\#} \xrightarrow{\psi} K_c^{q+1}(M_{r_n}, \partial M_{r_n})^{\#}$ and $K_q(M_{r_n}, \partial M_{r_n})^{\#} \xrightarrow{\overline{\psi}} K_c^{q+1}(M_{r_n})^{\#}$ ($\pi_1 X_n$-coefficients) have finitely generated kernels. Up to taking a subsequence, one can assume that $r_n = n + 1$.

5. <u>Main Lemma</u>. By enlarging X_{n+1} inside X_n, and $\overset{\blacksquare}{X}_{n+1}$ inside $\overline{X_n - X_{n+2}}$, one can get commutative squares

$$\begin{array}{ccc} K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n) & \longrightarrow & K_c^{q+1}(M_n) \\ \uparrow \psi & & \uparrow \overline{\psi} \\ K_q(M_n) & \longrightarrow & K_q(M_n, \overset{\blacksquare}{M}_n) \end{array} \qquad \overset{\blacksquare}{M}_n \equiv \overset{\blacksquare}{X}_n \cap M = \overline{M_n - M_r}$$

where $K_q(M_n)$ and $K_q(M_n, \overset{\blacksquare}{M}_n)$ are projective (countably generated) and ψ is bijective.

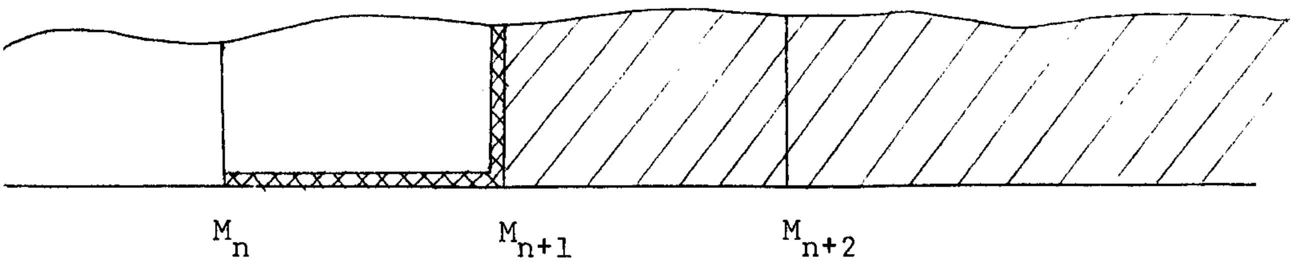
<u>Proof</u>. Our starting situation is as in §3



$X_n \cap M = M_n$, $\overset{\blacksquare}{X}_n \cap M = \partial M_n$, a square

$$\begin{array}{ccc} K_c^{q+1}(M_n, \partial M_n) & \longrightarrow & K_c^{q+1}(M_n) \\ \uparrow \psi & & \uparrow \overline{\psi} \\ K_q(M_n) & \longrightarrow & K_q(M_n, \partial M_n) \end{array}$$

and inverses $K_c^{q+1}(M_n, \partial M_n) \to K_q(M_{n-1})$, for ψ, and $K_c^{q+1}(M_n) \to K_q(M_{n+1}, \partial M_{n+1})^{\#}$ for $\overline{\psi}$. Choose new X_n', $\overset{\blacksquare}{X}{}_n'$ as follows

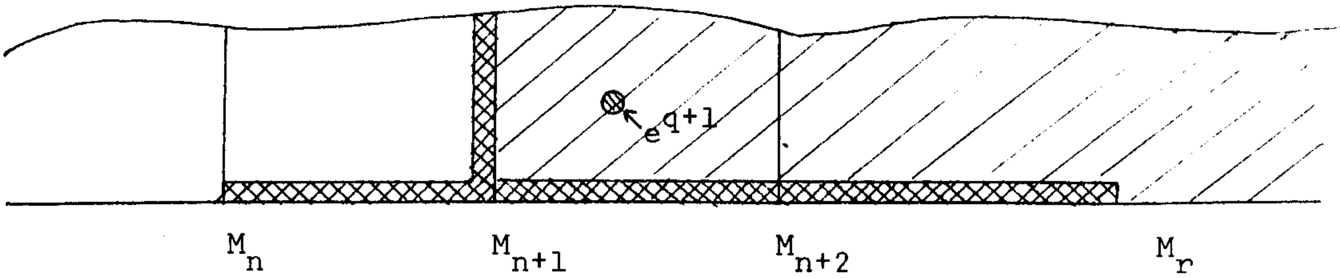


$$X_n' \equiv X_{n+1} \cup M_n \qquad \overset{\blacksquare}{X}{}_n' \equiv \overset{\blacksquare}{X}_{n+1} \cup \overline{M_n - M_{n+1}}$$

Then we get a new square

$$\begin{array}{ccc} K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n)' & \longrightarrow & K_c^{q+1}(M_n)' \\ \uparrow \psi' & & \uparrow \overline{\psi}' \\ K_q(M_n)' & \longrightarrow & K_q(M_n, \overset{\blacksquare}{M}_n)' \end{array} \qquad \overset{\blacksquare}{M}_n = \overset{\blacksquare}{X}{}_n' \cap M = \overline{M_n - M_{n+1}}$$

by taking the old square for $n+1$, with $\pi_1 X_n$-coefficients. By §4, $\ker \overline{\psi}'$ is finitely generated. Each generator can be represented by a map $(D^{q+1}, S^q) \xrightarrow{\alpha} (X_n', \overset{\blacksquare}{X}{}_n' \cup M_n)$, by Hurewicz. But the inverse of $\overline{\psi}$ shows that α represents 0 in $K_q(M_{n+1}, \overset{\blacksquare}{M}_{n+1})'$, i.e. α can be deformed into $\overline{X_{n+1} - X_{n+2}} \cup M_n$. By mapping cylinder constructions, one can assume that α is the characteristic map of a cell e^{q+1} in $\overline{X_{n+1} - X_{n+2}}$, attached to $\overset{\blacksquare}{X}{}_n' \cup \overline{M_n - M_r}$ for some r large enough (good for a finite set of generators of Ker $\overline{\psi}'$). Choose new X_n'', $\overset{\blacksquare}{X}{}_n''$ as follows



$$X_n'' \equiv X_n' \qquad \overset{\blacksquare}{X}{}_n'' = \overset{\blacksquare}{X}{}_n' \cup \overline{M_n - M_r} \cup e^{q+1}.$$

By passing to the quotient, $\overline{\psi}'$ induces now injections $\overline{\psi}'': K_q(M_n,\overset{\blacksquare}{M}_n)'' \to K_c^{q+1}(M_n)''$. This is still an equivalence with inverse $K_c^{q+1}(M_n)'' \to K_q(M_{n+1},\overset{\blacksquare}{M}_{n+1})''^{\#}$. Consider the diagram of exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & K^q(M_n,\overset{\blacksquare}{M}_n)'' & \longrightarrow & K_e^q(M_n,\overset{\blacksquare}{M}_n)'' & \longrightarrow & K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)'' & \\
& \uparrow {\scriptstyle \overline{\psi}''^*} & & \uparrow {\scriptstyle \varinjlim \psi^*} & & & \\
& \overline{K}_{q+1}(M_n)'' & \longrightarrow & K^e_{q+1}(M_n)'' & \longrightarrow & K_q(M_n)'' & \longrightarrow 0
\end{array}$$

where $K_e^q(M_n,\overset{\blacksquare}{M}_n)'' \cong K_e^q(M_n)$, $K^e_{q+1}(M_n)'' \cong K^e_{q+1}(M_n,\overset{\blacksquare}{M}_n)$ by definition of K_*^e and K_e^* (see I.7). <u>Claim</u>: ψ^* and $\overline{\psi}''$ are equivalences. In fact, by using the proof of V.6, the canonical map $\overline{K}_{q+1}(M_n,\partial M_n) \to (K_c^{q+1}(M_n,\partial M_n))^*$ is an equivalence, and by V. 9, the dual of ψ is an equivalence hence so is the composed map ψ^*. This implies that $\varinjlim \psi^*$ is an isomorphism. Similarly, $\overline{\psi}''^*$

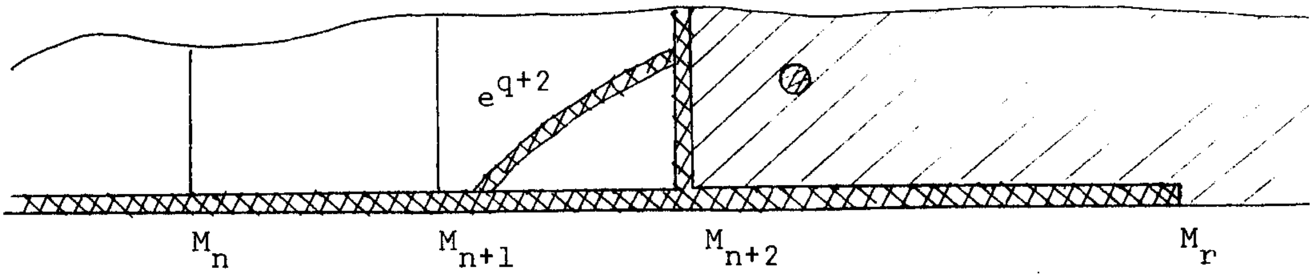
is an equivalence. We get an induced equivalence $\psi'' : K_q(M_n)'' \to K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n)''$ (apply V.5) and a commutative square

$$\begin{array}{ccc} K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)'' & \longrightarrow & K_c^{q+1}(M_n)'' \\ \Big\uparrow{\scriptstyle \psi''} & & \Big\uparrow{\scriptstyle \bar{\psi}''} \quad \text{(injective)} \\ K_q(M_n)'' & \longrightarrow & K_q(M_n,\overset{\blacksquare}{M}_n)'' \end{array} .$$

Observe that $K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)'' \xrightarrow{\text{restr.}} K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)' \to K_q(M_{n-1})' \equiv K_q(M_{n-1})''$ is an inverse for ψ'', because the square

$$\begin{array}{ccc} K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)'' & \longrightarrow & K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)' \\ \Big\uparrow{\scriptstyle \psi''} & & \Big\uparrow{\scriptstyle \psi'} \\ K_q(M_n)'' & = & K_q(M_n)' \end{array}$$

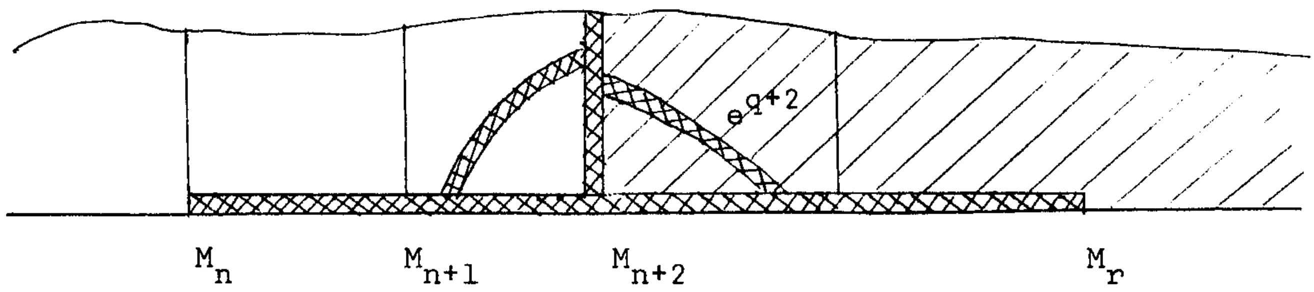
is commutative. Now, by §4, ker ψ'' is finitely generated. As $\bar{\psi}''$ is injective, it is certainly contained in the image of $K_q(\overset{\blacksquare}{M}_n)''^{\#}$, but also in the image of $K_{q+1}(M_{n-1}, M_n)$ in virtue of the inverse $K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)'' \to K_q(M_{n-1})''$ for ψ''. By Hurewicz and mapping cylinder construction each generator of ker ψ'' can be represented by a cell e^{q+1} in $\overset{\blacksquare}{X}_n''$ attached to M, which is the boundary mod M of a cell e^{q+2} in X_{n-1}''. Choose new $X_n''', \overset{\blacksquare}{X}_n'''$ as follows



$X_n^{'''} = X_{n+1}^{'} \cup M_n \cup e^{q+2}$, $\overset{\blacksquare}{X}{}_n^{'''} = \overset{\blacksquare}{X}{}_{n+1}^{''} \cup \overline{M_n - M_r} \cup e^{q+2}$. By passing to quotient, $\psi^{''}$ induces an injective equivalence $\psi^{'''}: K_q(M_n)^{'''} \to K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n)^{'''}$. Then again the duality argument used in the previous step provides an equivalence $\overline{\psi}^{'''}: K_q(M_n, \overset{\blacksquare}{M}_n)^{'''} \to K_c^{q+1}(M_n)^{'''}$. <u>Claim</u>: $K_q(M_n)^{'''}$ and $K_q(M_n, \overset{\blacksquare}{M}_n)^{'''}$ are projective. In fact, as $\overline{\psi}^{''}$ and $\psi^{'''}$ are injective, $K_q(M_n)^{'''}$ and $K_q(M_n, \overset{\blacksquare}{M}_n)^{''}$ are projective, by §4. But $K_q(M_n, \overset{\blacksquare}{M}_n)^{'''} \cong K_q(M_{n+1}, \overset{\blacksquare}{M}_{n+1})^{''\#}$ by excision. Then the exact sequence

$$K_q(\overset{\blacksquare}{M}_n)^{'''\#} \to K_q(M_n)^{'''} \to K_q(M_n, \overset{\blacksquare}{M}_n)^{'''} \to 0$$

implies that the image of the first map is projective, hence its kernel is a retract of $K_q(\overset{\blacksquare}{M}_n)^{'''\#}$, in particular, finitely generated. By Hurewicz and mapping cylinder constructions, each generator of this kernel can be represented by a cell e^{q+1} in $\overset{\blacksquare}{X}{}_n^{'''}$ attached to M_n, which is the boundary mod M of a cell e^{q+2} in $X_n^{'''}$ (meeting $X_{n+1}^{'''}$ only along M). Choose new $X_n^{'v}$, $\overset{\blacksquare}{X}{}_n^{'v}$ as follows



$X_n^{\prime v} \equiv X_n^{\prime\prime\prime}$, $\overset{\blacksquare}{X}{}_n^{\prime v} \equiv \overset{\blacksquare}{X}{}_n^{\prime\prime\prime} \cup e^{q+2}$. We get the same K-groups as before, except that we have an injective restriction $K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n)^{\prime v} \to K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n)^{\prime\prime\prime}$. But $\psi^{\prime\prime\prime}$ factors through this injection, because so does $K_q(M_s)^{\prime\prime\prime} \xrightarrow{\#\ \text{surj.}} K_c^{q+1}(M_s, \overset{\blacksquare}{M}_s)^{\prime\prime\prime} \xrightarrow{\text{restr}} K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n)^{\prime v}$ for large s. So we get a final square

$$\begin{array}{ccc} K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n)^{\prime v} & \longrightarrow & K_c^{q+1}(M_n)^{\prime v} \\ \uparrow{\scriptstyle \psi^{\prime v}} & & \uparrow{\scriptstyle \overline{\psi}^{\prime v}} \\ K_q(M_n)^{\prime v} & \longrightarrow & K_q(M_n, \overset{\blacksquare}{M}_n)^{\prime v} \end{array}$$

where $\psi^{\prime v}$ is injective. <u>Claim</u>: $\psi^{\prime v}$ is also surjective. As ψ is an equivalence, it suffices to show that the maps $K_c^{q+1}(M_{n+1}, \overset{\blacksquare}{M}_{n+1})^{\prime v} \longrightarrow K_c^{q+1}(M_n, \overset{\blacksquare}{M}_n)^{\prime v}$ are surjective. In the exact sequence

$$K_{q+1}(M_n)^{\prime v} \longrightarrow K_{q+1}(M_n, \overset{\blacksquare}{M}_n)^{\prime v} \xrightarrow{\partial} K_q(\overset{\blacksquare}{M}_n)^{\prime v} \xrightarrow{\#\ \text{inj}} K_q(M_n)^{\prime v} \longrightarrow K_q(M_n, \overset{\blacksquare}{M}_n)^{\prime v}$$

the first map vanishes, because the inverse system $\{K_{q+1}(M_n)^{\prime v}\}$ is equivalent to 0, and we have commutative squares

$$\begin{array}{ccc} K_{q+1}(M_k)^{\prime\vee} & \longrightarrow & K_{q+1}(M_k,\overset{\blacksquare}{M}_k)^{\prime\vee} \\ \uparrow{\scriptstyle 0} & & \downarrow \\ K_{q+1}(M_n)^{\prime\vee} & \longrightarrow & K_{q+1}(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee}. \end{array} \qquad k << n$$

This implies $K_{q+1}(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee} = 0$, hence also $K^{q+1}(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee} = 0$ because $K_q(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee}$ is projective. Then we have the exact sequence

$$0 \longrightarrow K^q(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee} \longrightarrow K_e^q(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee} \xrightarrow{\delta} K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee} \longrightarrow 0 \ .$$

This implies that $K_c^{q+1}(M_{n+1},\overset{\blacksquare}{M}_{n+1})^{\prime\vee\#} \longrightarrow K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee}$ is surjective, because $K_e^q(M_{n+1},\overset{\blacksquare}{M}_{n+1})^{\prime\vee\#} \cong K_e^q(M_n,\overset{\blacksquare}{M}_n)^{\prime\vee}$ by definition of K_e^q.

6. Lemma. If in the squares

$$\begin{array}{ccc} K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n) & \longrightarrow & K_c^{q+1}(M_n) \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \overline{\psi}} \\ K_q(M_n) & \longrightarrow & K_q(M_n,\overset{\blacksquare}{M}_n) \end{array}$$

ψ is bijective, then $\overline{\psi}$ is injective.

Proof. By considering the diagram

$$\begin{array}{ccccccc}
K^q(M_n) & \longrightarrow & K^q_e(M_n) & \longrightarrow & K^{q+1}_c(M_n) & & \\
\uparrow \psi^* & & \uparrow \lim\limits_{\rightarrow} \psi^* & & \uparrow \bar{\psi} & & \\
\bar{K}_{q+1}(M_n,\dot{M}_n) & \longrightarrow & K^e_{q+1}(M_n,\dot{M}_n) & \longrightarrow & K_q(M_n,\dot{M}_n) & \longrightarrow & 0
\end{array}$$

it suffices to show that ψ^* is a surjective equivalence, i.e. that the canonical map $\bar{K}_{q+1}(M_n,\dot{M}_n) \to (K_c^{q+1}(M_n,\dot{M}_n))$ is onto. This is an equivalence by V.9 and V.12, and it is surjective, because (with notations as in V.7) $K_c^{q+1}(M_n,\dot{M}_n) \cong P_n'$ is a retract of P_n, which is a retract of $E_\ell(n)$, hence $K_c^{q+1}(M_n,\dot{M}_n)$ is a retract of $C_\ell(n) \subset E_\ell(n)$. In particular, all linear forms on $K_c^{q+1}(M_n,\dot{M}_n)$ extend. The middle map $\lim\limits_{\rightarrow} \psi^*$ is an isomorphism (because $\bar{\psi}'$ is an equivalence of direct systems), hence $\bar{\psi}$ is injective.

7. <u>Proposition</u>. Let us come back to the initial situation of lemma 5, obtained after preliminary surgery: $\dot{X}_n \cap M = \partial M_n$. For each n, and sufficiently large $r > n$ there is a certain non-trivial submodule $A \subset K_q(\partial M_r)^\#$ ($\pi_1 X_n$-coefficients) such that the restriction to A of the intersection pairing $\dot{\psi}$: $K_q(\partial M_r)^\# \overset{\partial}{\to} H_q(\partial M_r)^\# \cong H^q(\partial M_r)^\# \overset{\delta}{\to} K^q(\partial M_r)^\#$ induces a non-singular quadratic form on a projective finitely generated quotient of A.

<u>Proof</u>. By the two preceeding lemma, we can assume that in the square

$$\begin{array}{ccc}
K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n) & \longrightarrow & K_c^{q+1}(M_n) \\
\uparrow \psi & & \uparrow \bar{\psi} \\
K_q(M_n) & \longrightarrow & K_q(M_n,\overset{\blacksquare}{M}_n)
\end{array}$$

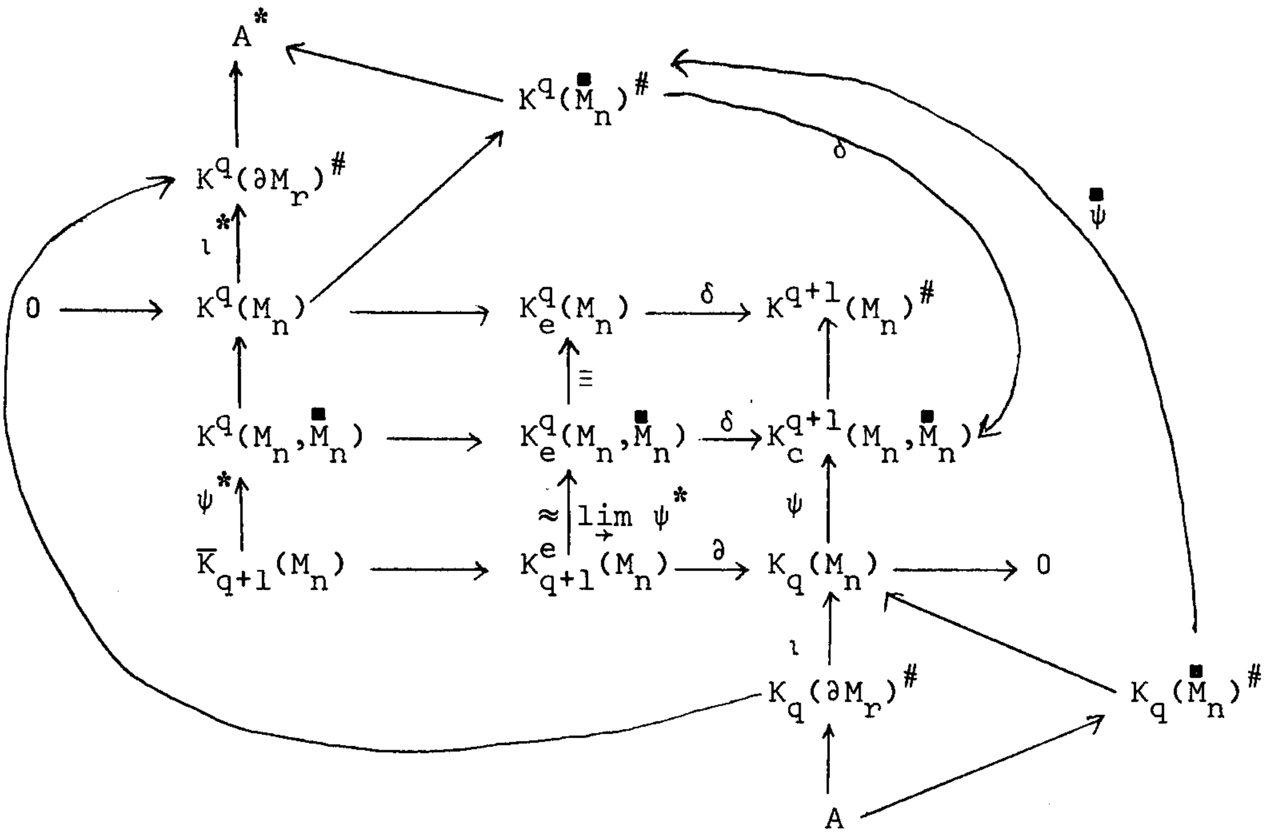
ψ is an isomorphism, $\bar{\psi}$ injective, $K_q(M_n)$ and $K_q(M_n,\overset{\blacksquare}{M}_n)$ are projective, and $K_{q+1}(M_n,\overset{\blacksquare}{M}_n) = 0$. The horizontal maps are part of the exact sequences of $(M_n,\overset{\blacksquare}{M}_n)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^q(\overset{\blacksquare}{M}_n)^{\#} & \xrightarrow{\delta} & K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n) & \longrightarrow & K_c^{q+1}(M_n) & & \\
 & & \uparrow \overset{\blacksquare}{\psi} & & \uparrow \psi & & \uparrow \bar{\psi} & & \\
0 & \longrightarrow & K_q(\overset{\blacksquare}{M}_n)^{\#} & \longrightarrow & K_q(M_n) & \longrightarrow & K_q(M_n,\overset{\blacksquare}{M}_n) & \longrightarrow & 0
\end{array}$$

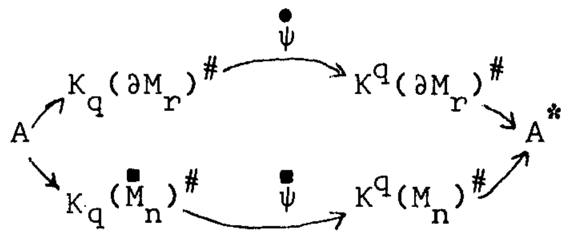
hence we get an induced isomorphism $\overset{\blacksquare}{\psi}$ of $K_q(\overset{\blacksquare}{M}_n)$ with its dual, i.e. a non-singular bilinear form on $K_q(\overset{\blacksquare}{M}_n)^{\#}$. The lower exact sequence shows that $K_q(\overset{\blacksquare}{M}_n)^{\#}$ is projective (finitely generated). One should remember that all the above K-groups refer to the last choice $\tilde{X}_n$, $\overset{\blacksquare}{\tilde{X}}_n$ in the proof of 5. But we are interested in the initial choice $X_n,\overset{\blacksquare}{X}_n$. Choose r so large that $\overset{\blacksquare}{X}_r$ meet $\overset{\blacksquare}{\tilde{X}}_n$ only along M, or not at all. Then by excision we have a canonical map $H_{q+1}(\tilde{X}_n,\overset{\blacksquare}{\tilde{X}}_n \cup M_n) \to H_{q+1}(X_r,\overset{\blacksquare}{X}_r \cup M_r)^{\#}$. In the exact sequence

$$\underbrace{H_{q+1}(\overset{\blacksquare}{X}_r,\partial M_r)^{\#}}_{K_q(\partial M_r)^{\#}} \xrightarrow{\iota} \underbrace{H_{q+1}(\tilde{X}_n,M_n)}_{K_q(M_n)} \longrightarrow H_{q+1}(\tilde{X}_n,\overset{\blacksquare}{X}_r \cup M_n) \longrightarrow 0$$

the last term is $H_{q+1}(X_r,\overset{\blacksquare}{X}_r \cup M_r)^{\#}$ by Mayer-Vietoris, because $H_{q+1}(\tilde{X}_n, X_r \cup M_n) = 0$. Hence the second map factors through $K_q(M_n,\overset{\blacksquare}{M}_n)$ hence $K_q(\overset{\blacksquare}{M}_n)^{\#}$ is contained in the image of $K_q(\partial M_r)^{\#}$. Let A be the preimage $\iota^{-1}(K_q(\overset{\blacksquare}{M}_n))$ in $K_q(\partial M_r)^{\#}$, and consider the diagram

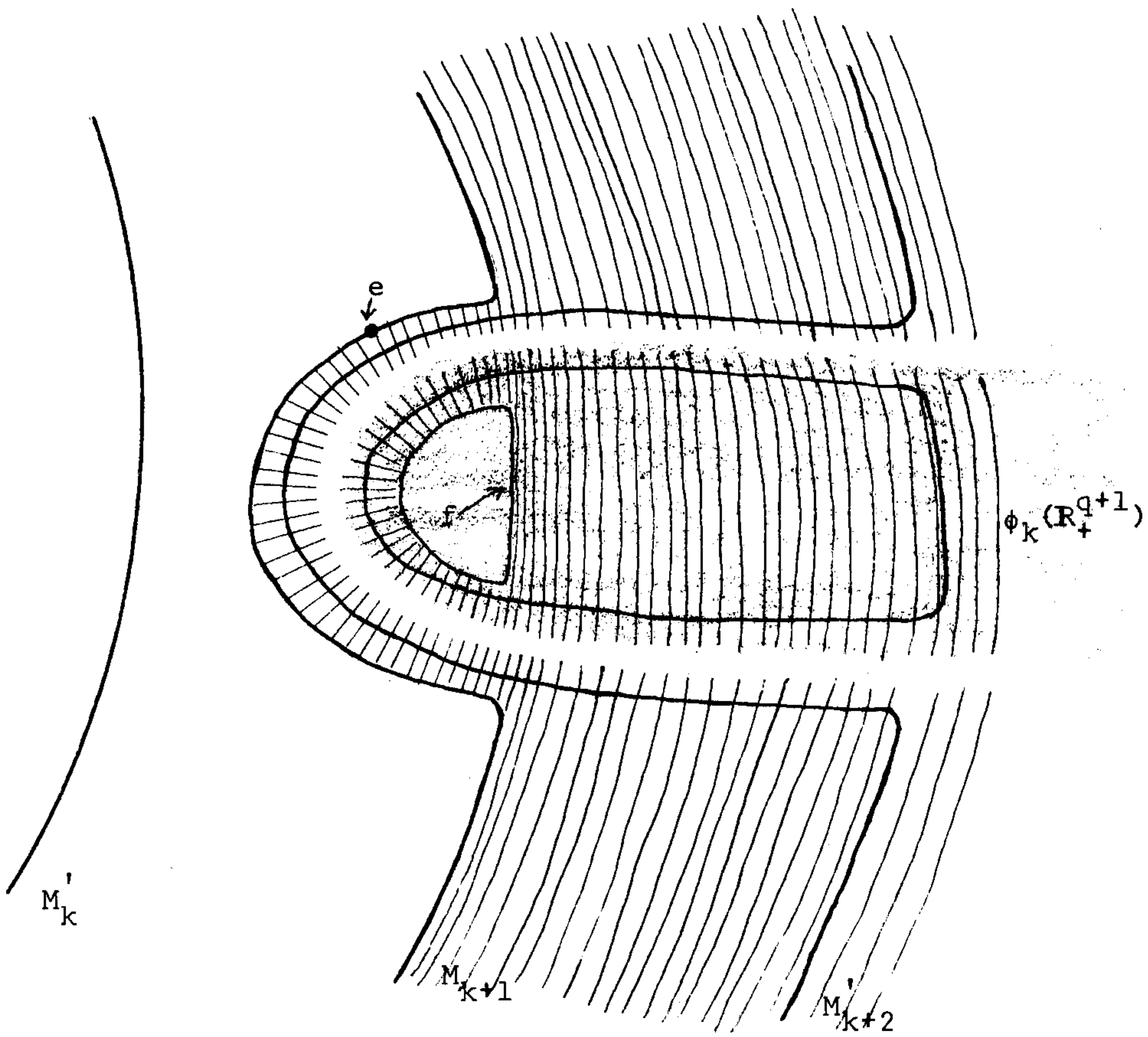


The fact that the two maps



are equal is a result of diagram chasing $A \longrightarrow K_q(M_n) \dashrightarrow K^q(M_n) \longrightarrow A^*$.

8. Trivial surgery. Let us come back again to the situation obtained after preliminary surgery. Choose a proper embedding $\phi_k : \mathbb{R}_+^{q+1} \longrightarrow M_k$ and let us do surgery on $(f \circ \phi_k, \phi_k | \mathbb{R}^q)$ as in Chapter II. From the picture



we see that $K_q(M_n') = \begin{cases} K_q(M_n) \oplus [e] & \text{for } n \leq k \\ K_q(M_n) \oplus [e] \oplus [f] & \text{for } n > k \end{cases}$

where [e], [f] denote free modules of rank 1 generated by e,f. Moreover, the map $K_q(M_{n+1}') \to K_q(M_n')$ sends e to e, for all n, and f to f for $n > k$, and f to 0 for $n = k$. Similarly,

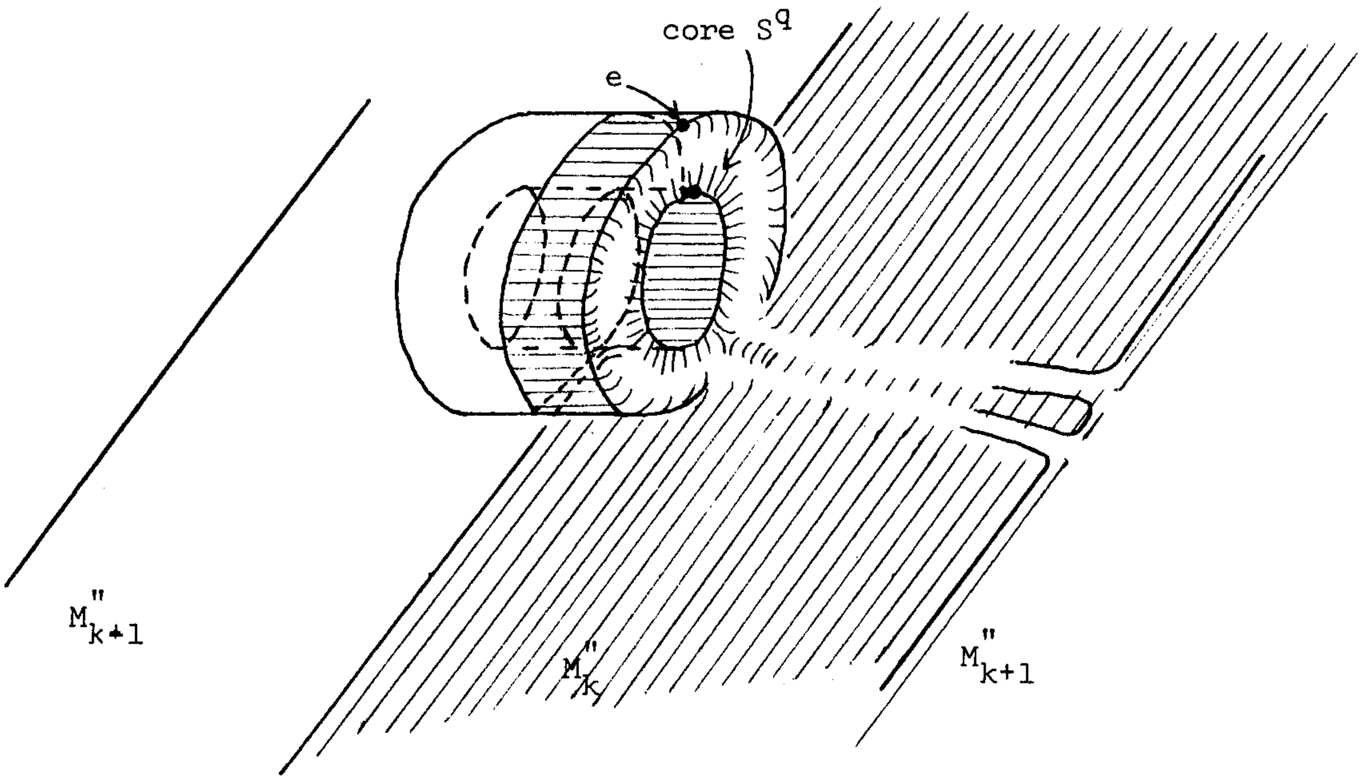
we see that $K_q(M_n', \partial M_n') = \begin{cases} K_q(M_n, \partial M_n) \oplus [e] & \text{for } n \leqslant k \\ K_q(M_n, \partial M_n) & \text{for } n > k. \end{cases}$

Hence, if we do this operation for $k \to \infty$, we get $K_q(M_n') = K_q(M_n) \oplus E \oplus F_n$, $K_q(M_n', \partial M_n') = K_q(M_n, \partial M_n) \oplus E_n$, where E is a free module of countable rank, E_n the free module generated by all but a finite number say s_n of basis elements of E, F_n a free module of finite rank s_n. The map $K_q(M_{n+1}') \to K_q(M_n')$ sends E to E identically F_{n+1} onto F_n with a basis element mapped to itself or to 0. The map $K_q(M_n', \partial M_n') \to K_q(M_{n+1}', \partial M_{n+1}')$ is onto, a basis element being mapped to itself or to 0. Now, each $e \times (I, \partial I)$ introduces a new basis element in $K_c^{q+1}(M_n, \partial M_n)$ for $n > k$, and each $f \times (I, \partial I)$ also, for all n, hence $K_c^{q+1}(M_n', \partial M_n') = K_c^{q+1}(M_n, \partial M_n) \oplus (E/E_n')^* \oplus F_c^*$, where $F_c^* \equiv \varinjlim_s F_s^*$ is a free module of countable rank. Similarly we have $K_c^{q+1}(M_n') = K_c^{q+1}(M_n) \oplus (F_c^*/F_n^*)$. The canonical map

$\psi': K_q(M_n') \to K_c^{q+1}(M_n', \partial M_n')$ induces an isomorphism

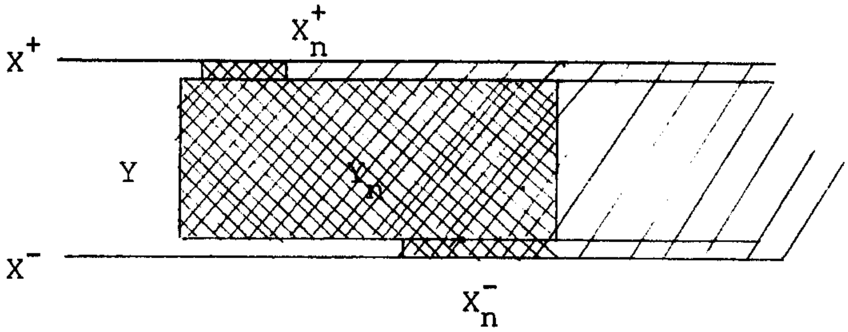
$$E \oplus F_n \longrightarrow (E/E_n')^* \oplus F_c^*.$$

Hence, on the kernel of $K_q(M_n') \to K_q(M_n', \partial M_n')$ we have added the free hyperbolic module $(E/E_n') \oplus F_n$. The reciprocal trivial surgery consists in the following: do surgery on a trivial (q-1)-sphere in ∂M_k, getting M' by extending it to M, then carve out the core S^q (piped to ∞) of the q-handle in M_k', getting M'':

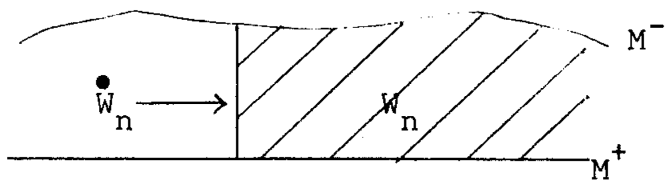


If e is the transverse q-sphere to the core S^q and f the q-sphere parallel to this core; we have the same situation as above, with e and f exchanged (note that e bounds a transverse $q+1$-disc in M''_{k-1}).

9. <u>Cobordism invariance</u>. Suppose we have a proper normal cobordism $F: W^{2q+2} \to Y$ between $f^+: M^+ \to X^+$ and $f^-: M^- \to X^-$ (Y has a $2q+2$-fundamental class mod $X^+ \cup X^-$ at ∞ and the inclusion $X^{\pm} \subset Y$ are simple homotopy equivalences). Choose $(X_n^{\pm}, \dot{X}_n^{\pm})$ arbitrarily in $X^{\pm}$. By using a collar along $X^{\pm}$, we can find ngbd of ∞ Y_n in Y, and finite subcomplexes $\dot{Y}_n$ containing the frontier, such that $Y_n \cap X^{\pm} = X_n^{\pm}$, $\dot{Y}_n \cap X^{\pm} = \dot{X}_n^{\pm}$.



Now, by a standard construction (see beginning of §1) we can assume that $\dot{X}_n^{\pm}$ is bicollared in $X^{\pm}$ and $\dot{Y}_n$ is bicollared in Y. We can put then $f^{\pm}$ and F transverse on these subcomplexes. Then $F^{-1}(Y_n)$ is a submanifold W_n (ngbd of ∞) with boundary $\partial W_n = M_n^+ \cup \dot{W}_n \cup M_n^-$

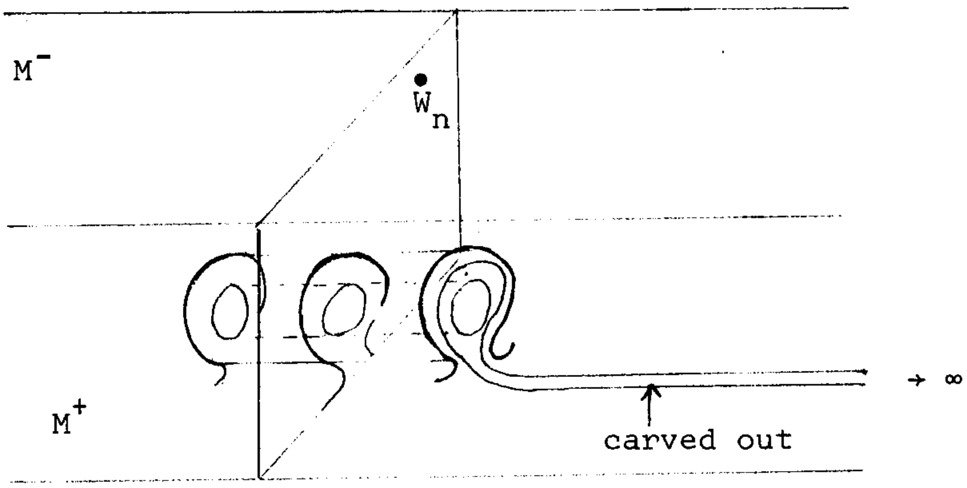


where the frontier $\dot{W}_n$ is a compact bicollared submanifold with boundary $\partial\dot{W}_n = \partial M_n^+ \cup \partial M_n^-$, and $M_n^{\pm} = f^{\pm-1}(X_n^{\pm})$. The relativization of Chapter I is clear and we get canonical squares

$$\begin{array}{ccc} K_c^{2q+2-k}(W_n,\partial W_n) & \to & K_c^{2q+2-k}(W_n,M_n^+ \cup M_n^-) \\ \uparrow \psi & & \uparrow \psi_{\shortmid} \\ K_k(W_n) & \longrightarrow & K_k(W_n,\dot{W}_n) \end{array} \qquad \begin{array}{ccc} K_c^{2q+2-k}(W_n,\dot{W}_n) & \to & K_c^{2q+2-k}(W_n) \\ \uparrow \psi_{=} & & \uparrow \psi_{\sqsubset} \\ K_k(W_n,M_n^+ \cup M_n^-) & \to & K_k(W_n,\partial W_n) \end{array}$$

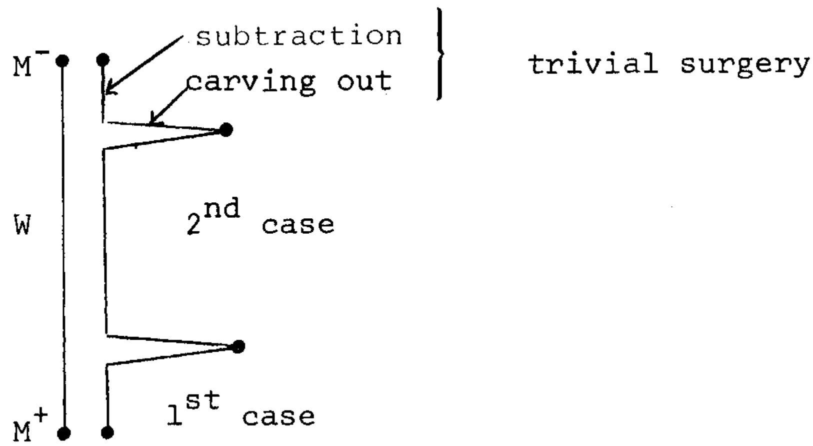
where ψ, $\psi_{=}$ are equivalences of inverse system and $\psi_{\shortmid}$, $\psi_{\sqsubset}$ equivalences of direct systems (all with inverse shifting the indice by ± 1).

We can assume that the preliminary surgery on $f^{\pm}: M^{\pm} \to X^{\pm}$ (see §1), are already done. Then by doing surgery on $\overset{\bullet}{W}_n \xrightarrow{F} \overset{\blacksquare}{Y}_n$ $\overline{W_n - W_{n+1}} \xrightarrow{F} \overline{Y_n - Y_{n+1}}$ rel $M^+ \cup M^-$, one can assume that $\overset{\bullet}{W}_n \xrightarrow{F} \overset{\blacksquare}{Y}_n$ is q-connected and $\overline{W_n - W_{n+1}} \xrightarrow{F} \overline{Y_n - Y_{n+1}}$ q+1-connected. Now, by handles substraction in $\overset{\bullet}{W}_n$ (see [11]) extended to W, and carving out construction (see §7) one divides the cobordism invariance problem in two cases:

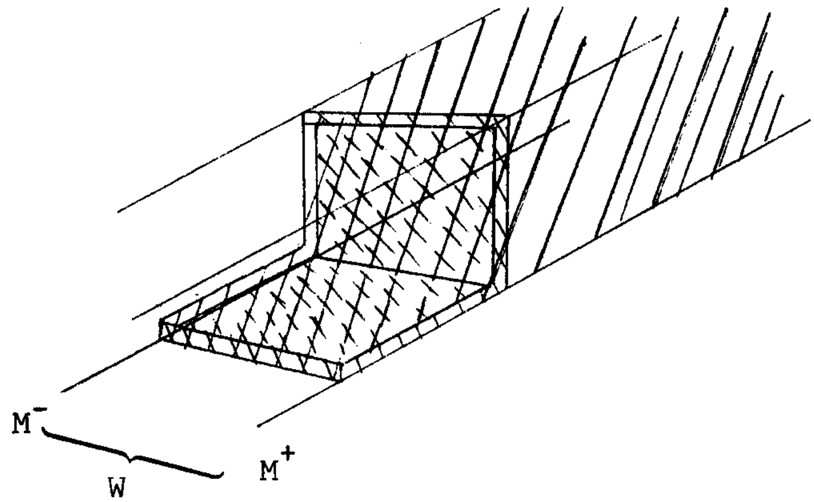


1^{st} case: invariance by trivial surgery and $X^+ \equiv X^-$

2^{nd} case: invariance by cobordism satisfying the additional condition $K_q(\overset{\bullet}{W}_n, \partial M_n^+ \cup \partial M_n^-) = 0$. Schematically:



Let us concentrate on the 2^{nd} case. Claim$_F$: the construction in the proof of lemma 5 extend to $W \to Y$. We have to follow the whole proof of lemma 5, and we use the same notations, with an additional $\pm$. The first operation $X_n^{\pm'} = X_{n+1}^{\pm} \cup M_n^{\pm}$, $\overset{\blacksquare}{X}{}_n^{\pm'} = \overset{\blacksquare}{X}{}_{n+1}^{\pm} \cup \overline{M_n - M_{n+1}}^{\pm}$ is induced by $Y_n' = Y_{n+1} \cup W_n$, $\overset{\blacksquare}{Y}{}_n' = \overset{\blacksquare}{Y}_{n+1} \cup \overline{W_n - W_{n+1}}$

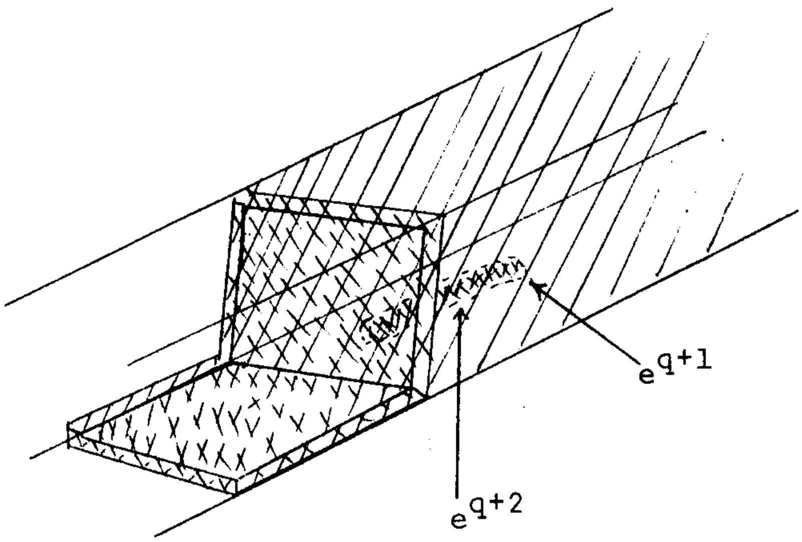


We get the K'-squares by taking the above K-square for $n+1$ with $\pi_1 X_n$-coefficients.

$$\begin{array}{ccc} K^{q+1}(W_n, \partial W_n)' & \longrightarrow & K^{q+1}(W_n, M_n^+ \cup M_n^-)' \\ \uparrow \psi' & & \uparrow \psi'_{\blacksquare} \\ K_{q+1}(W_n)' & \longrightarrow & K_{q+1}(W_n, \overset{\blacksquare}{W}_n)' \end{array} \qquad \begin{array}{ccc} K^{q+1}(W_n, \overset{\blacksquare}{W}_n) & \longrightarrow & K_c^{q+1}(W_n) \\ \uparrow \psi_{=} & & \uparrow \psi_{\blacksquare} \\ K_{q+1}(W_n, M_n^+ \cup M_n^-) & \longrightarrow & K_{q+1}(W_n, \partial W_n) \end{array}$$

where $\partial W_n = (X_n^{'+} \cup \overset{\blacksquare}{Y}{}_n' \cup X_n^{'-}) \cap W = M_n^+ \cup \overline{W_n - W_{n+1}} \cup M_n^-$

The second operation is $X_n^{\pm''} \equiv X_n^{\pm'}$, $\overset{\blacksquare}{X}{}_n^{\pm''} \equiv \overset{\blacksquare}{X}{}_n^{\pm'} \cup \overline{M_n - M_r^{\pm}} \cup e^{q+1}$, where e^{q+1} describes generators of ker $(K_q(M_n^{\pm}, \overset{\blacksquare}{M}{}_n^{\pm})' \to K_c^{q+1}(M_n^{\pm})')$ contained in $\overline{X_{n+1} - X}{}_{n+2}^{\pm} \cup \overline{M_n - M_r}^{\pm}$. By connectivity of f, e^{q+1} bounds a cell e^{q+2} in $\overline{X_{n+1} - X_{n+2}} \cup \overline{W_n - W_r}$ mod $\overset{\blacksquare}{Y}_{n+1} \cup \overline{W_n - W_r}$ (up to mapping cylinder constructions). Then take $Y_n'' \equiv Y_n'$, $\overset{\blacksquare}{Y}{}_n'' \equiv \overset{\blacksquare}{Y}{}_n' \cup \overline{W_n - W_r} \cup e^{q+2} \cup e^{q+1}$



As $\overset{\blacksquare}{Y}{}_n''$ collapses onto $\overset{\blacksquare}{Y}{}_n' \cup \overline{W_n - W_r}$, we have $K_{q+1}(W_n, \overset{\blacksquare}{W}_n)'' \cong K_{q+1}(W_n, \overset{\blacksquare}{W}_n)'$. So the map $\psi_{\blacksquare}' : K_{q+1}(W_n, \overset{\blacksquare}{W}_n)' \to K_c^{q+1}(W_n, M_n^{+} \cup M_n^{-})'$ becomes a map $\psi_{\blacksquare}'' : K_{q+1}(W_n, \overset{\blacksquare}{W}_n)'' \to K_c^{q+1}(W_n, M_n^{+} \cup M_n^{-})''$. We get $\psi_{=}''$ by duality:

$$\begin{array}{ccccc}
0 \to K_c^{q+1}(W_n, \overset{\blacksquare}{W}_n)'' & \longrightarrow & K^{q+1}(W_n, \overset{\blacksquare}{W}_n)'' & \longrightarrow & K_e^{q+1}(W_n, \overset{\blacksquare}{W}_n)'' \\
\uparrow \psi_{=}'' & & \uparrow \psi_{\blacksquare}''^{*} & & \uparrow \lim_{\to} \psi^{*} \\
K_{q+1}(W_n, M_n^{+} \cup M_n^{-})'' & \longrightarrow & \bar{K}_{q+1}(W_n, M_n^{+} \cup M_n^{-})'' & \longrightarrow & K_{q+1}^{e}(W_n, M_n^{+} \cup M_n^{-})''.
\end{array}$$

Then $\psi''_{=}$ induces a ψ'':

$$
\begin{array}{ccccc}
0 \to K_c^{q+1}(W_n,\partial W_n)'' & \longrightarrow & K_c^{q+1}(W_n,\overset{\blacksquare}{W}_n)'' & \longrightarrow \underset{\pm}{\oplus} & K_c^{q+1}(M_n,\overset{\blacksquare}{M}_n)'' \\
\uparrow \psi'' & & \uparrow \psi''_{=} & & \uparrow \psi''^{\pm} \\
K_{q+1}(W_n)'' & \longrightarrow & K_{q+1}(W_n,M_n^{+} \cup M_n^{-})'' & \longrightarrow \underset{\pm}{\oplus} & K_q(M_n)''
\end{array}
$$

and then we get $\psi''_{\sqsubset}$ by duality again. This provides the convenient K''-squares. Before extending the third operation, we need some preparation inside Y, rel $X^{\pm}$. Observe that the direct system $\{K_{q+1}(W_s,\overset{\blacksquare}{W}_s)''^{\#}\}_n$ is composed of surjections. Then we can apply the argument of §4 to see that $\ker \psi''_{\blacksquare}$ is finitely generated. As in the operation " of lemma 5, we can add cells e^{q+2} to $\overset{\blacksquare}{Y}''_n$ to get $\psi''_{\blacksquare}$ (split) injective, without altering anything on $X^{+} \cup X^{-}$. When $\psi''_{\blacksquare}$ is (split) injective, so is $\psi''_{\sqsubset}$ in virtue of the diagram

$$
\begin{array}{ccccc}
0 \to K_c^{q+1}(W_n,M_n^{+} \cup M_n^{-}) & \longrightarrow & K_c^{q+1}(W_n)'' & \longrightarrow \underset{\pm}{\oplus} & K_c^{q+1}(M_n^{\pm})'' \\
\uparrow \psi''_{\blacksquare} & & \uparrow \psi''_{\sqsubset} & & \uparrow \psi''^{\pm} \\
K_{q+1}(W_n,\overset{\blacksquare}{W}_n)'' & \longrightarrow & K_{q+1}(W_n,\partial W_n)'' & \longrightarrow \underset{\pm}{\oplus} & K_q(M_n^{\pm},\overset{\blacksquare}{M}{}_n^{\pm})''
\end{array}
\qquad \text{(injective)}
$$

Then the duality diagram

$$\begin{array}{ccccc}
0 \to K_c^{q+1}(W_n,\partial W_n)'' & \longrightarrow & K^{q+1}(W_n,\partial W_n)'' & \longrightarrow & K_e^{q+1}(W_n,\partial W_n)'' \\
\Big\uparrow \psi & & \text{surj.} \Big\uparrow (\psi_{\llcorner})^* & & \approx \Big\uparrow \lim\limits_{\to} (\psi''_{=}) \\
K_{q+1}(W_n)'' & \longrightarrow & \bar{K}_{q+1}(W_n)'' & \longrightarrow & K^e_{q+1}(W_n)''
\end{array}$$

shows that ψ is surjective. Now, we are ready to extend the third operation $X_n^{\pm'''} \equiv X_{n+1}^{\pm'} \cup M_n^{\pm} \cup e_{\pm}^{q+2}$, $\overset{\blacksquare}{X}{}_n^{\pm'''} \equiv \overset{\blacksquare}{X}{}_{n+1}^{\pm''} \cup \overline{M_n - M_r} \cup e_{\pm}^{q+2}$ where $e_{\pm}^{q+2}$ is a null homotopy of a generator e^{q+1} of ker $\psi''^{\pm}: K_q(M_{n+1}^{\pm})'' \to K_c^{q+1}(M_{n+1}^{\pm}, \overset{\blacksquare}{M}{}_{n+1}^{\pm})''$. Actually e^{q+1} lies in $K_q(\overset{\blacksquare}{M}{}_{n+1}^{\pm})''^{\#}$ and we look at its image $\underline{e}^{q+1}$ in $K_q(\overset{\blacksquare}{W}_{n+1})''^{\#}$. The exact sequence $K_{q+1}(W_{n+1},\overset{\blacksquare}{W}_{n+1})'' \to K_q(\overset{\blacksquare}{W}_{n+1})''^{\#} \to 0$, shows that $\underline{e}^{q+1}$ comes from some $e^{q+2} \in K_{q+1}(W_{n+1},\overset{\blacksquare}{W}_{n+1})''$. Under the composition $K_{q+1}(W_{n+1},\overset{\blacksquare}{W}_{n+1})'' \xrightarrow{\psi''_{\shortmid}} K^{q+1}(W_{n+1},M_{n+1}^{+} \cup M_{n+1}^{-})'' \to$

$\to K_c^{q+1}(\partial W_{n+1},M_{n+1}^{+} \cup M_{n+1}^{-})''^{\#} \cong K^{q+1}(\overset{\blacksquare}{W}_{n+1},\overset{\blacksquare}{M}{}_{n+1}^{+} \cup \overset{\blacksquare}{M}{}_{n+1}^{-})''^{\#}$, e^{q+2} is mapped to 0 (so does $\underline{e}^{q+1}$), hence $\psi''_{\shortmid}(e^{q+2})$ lies in $K_c^{q+1}(W_{n+1},\partial W_{n+1})''$ (see top exact sequence below). But we have by the above preparation the diagram

$$\begin{array}{ccccc}
0 \to K_c^{q+1}(W_{n+1},\partial W_{n+1})'' & \to & K_c^{q+1}(W_{n+1},M_{n+1}^{+} \cup M_{n+1}^{-})'' & \to & K^{q+1}(\overset{\blacksquare}{W}_{n+1},\overset{\blacksquare}{M}{}_{n+1}^{+} \cup \overset{\blacksquare}{M}{}_{n+1}^{-})'' \\
\Big\uparrow \psi''(\text{surjective}) & & \Big\uparrow \psi''_{\shortmid}(\text{injective}) & & \Big\uparrow \\
K_{q+1}(W_{n+1})'' & \longrightarrow & K_{q+1}(W_{n+1},\overset{\blacksquare}{W}_{n+1})'' & \longrightarrow & K_q(\overset{\blacksquare}{W}_{n+1})''^{\#}
\end{array}$$

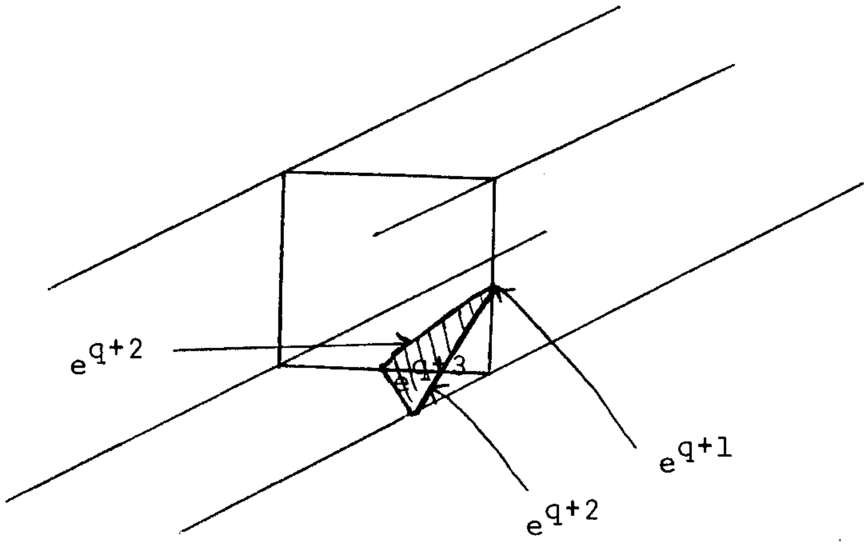
from which one deduces that e^{q+2} comes from $K_{q+1}(W_{n+1})''$, i.e. $\underline{e}^{q+1} = 0$. As a result, $e^{q+1} = \partial e^{q+2}$ with $e^{q+2} \in K_{q+1}(\overset{\blacksquare}{W}_{n+1}, \overset{\blacksquare}{M}{}^{+}_{n+1} \cup \overset{\blacksquare}{M}{}^{-}_{n+1})''^{\#}$, because of the exact sequence

$$K_{q+1}(\overset{\blacksquare}{W}_{n+1}, \overset{\blacksquare}{M}{}^{+}_{n+1} \cup \overset{\blacksquare}{M}{}^{-}_{n+1})''^{\#} \xrightarrow{\partial} \bigoplus_{\pm} K_q(\overset{\blacksquare}{M}{}^{\pm}_{n+1})''^{\#} \longrightarrow K_q(\overset{\blacksquare}{W}_{n+1})''^{\#} \to 0 .$$

Now, $K_{q+1}(\overset{\blacksquare}{W}_{n+1}, \overset{\blacksquare}{M}{}^{+}_{n+1} \cup \overset{\blacksquare}{M}{}^{-}_{n+1})''^{\#} \xrightarrow{\psi_=} K^q(\overset{\blacksquare}{W}_{n+1})''^{\#}$ maps e^{q+2} to 0 in virtue of the following diagram

$$\begin{array}{ccccc}
K^q(\overset{\blacksquare}{W}_{n+1})''^{\#} & \xrightarrow{\text{inj.}} & \bigoplus_{\pm} K^q(\overset{\blacksquare}{M}{}^{\pm}_{n+1})''^{\#} & \xrightarrow{\text{inj.}} \bigoplus_{\pm} & K_c^{q+1}(M^{\pm}_{n+1}, \overset{\blacksquare}{M}{}^{\pm}_{n+1}) \\
\uparrow \psi_= & & \uparrow & & \uparrow \\
K_{q+1}(\overset{\blacksquare}{W}_{n+1}, \overset{\blacksquare}{M}{}^{+}_{n+1} \cup \overset{\blacksquare}{M}_{n+1})''^{\#} & \longrightarrow & \bigoplus_{\pm} K_q(\overset{\blacksquare}{M}{}^{\pm}_{n+1})''^{\#} & \longrightarrow \bigoplus_{\pm} & K_q(M_{n+1}).
\end{array}$$

In particular, the image of e^{q+2} in $K_{q+1}(W_{n+1}, M^{+}_{n+1} \cup M^{-}_{n+1})''$ is mapped to 0 by $\psi_=$, and by using an inverse $K_c^{q+1}(W_{n+1}, \overset{\blacksquare}{W}_{n+1})''^{\#} \to K_{q+1}(W_n, M^{+}_n \cup M^{-}_n)''^{\#}$ we see that e^{q+2} vanishes in $K_{q+1}(W_n, M^{+}_n \cup M^{-}_n)''$. This means that the cell e^{q+2} in $\overset{\blacksquare}{Y}{}''_{n+1}$ can be deformed over a cell e^{q+3} in Y''_n into a cell $e^{q+2}_{\pm}$ in $X''^{\pm}_n$, that we can assume to coincide with the initial ones.



Take $Y_n^{'''} \equiv Y_{n+1}^{''} \cup W_n \cup e_{\pm}^{q+2} \cup e^{q+3}$, $\overset{\blacksquare}{Y}_n^{'''} \equiv \overset{\blacksquare}{Y}_{n+1}^{''} \cup \overline{W_n - W_r} \cup e_{\pm}^{q+2} \cup e^{q+3}$. As $Y_n^{'''}$ collapses on $Y_{n+1}^{''} \cup W_n$, we have $K_{q+1}(W_n)^{'''} \cong K_{q+1}(W_{n+1})^{''}$ and by excision $K_{q+1}(W_n, \overset{\blacksquare}{W}_n)^{'''} = K_{q+1}(W_{n+1}, \overset{\blacksquare}{W}_{n+1})^{''\#}$, $K_{q+1}(W_n, \partial W_n)^{'''} = K_{q+1}(W_{n+1}, \partial W_{n+1})^{''\#}$. Then $\psi^{''}, \psi_{\sqsubset}^{''}$ become $\psi^{'''}$, $\psi_{\sqsubset}^{'''}$, and $\psi_{=}^{''}$ passes to the quotient by e^{q+2}, to give $\psi_{=}^{'''}$. We get $\psi_{|}^{'''}$ by duality. To extend the last operation $'^{\vee}$, we just add the same cells e^{q+2} to $\overset{\blacksquare}{Y}_n^{'''}$ as to form $\overset{\blacksquare}{X}^{\pm '\vee}$: this doesn't change the K_* and K^*-modules of W_n, $(W_n, \partial W_n)$, $(W_n, M_n^+ \cup M_n^-)$ and we get $\psi_{|}^{'\vee}$ from the diagram

$$\begin{array}{ccccc}
0 \to K_c^{q+1}(W_n, M_n^+ \cup M_n^-)^{'\vee} & \longrightarrow & K_c^{q+1}(W_n)^{'\vee} & \longrightarrow & \bigoplus_{\pm} K_c^{q+1}(M_n^{\pm})^{'\vee} \\
\uparrow \vdots & & \uparrow \psi_{\sqsubset}^{'\vee} & & \uparrow \psi^{\pm '\vee} \\
K_{q+1}(W_n, \overset{\blacksquare}{W}_n)^{'\vee} & \longrightarrow & K_{q+1}(W_n, \partial W_n)^{'\vee} & \longrightarrow & \bigoplus_{\pm} K_q(M_n^{\pm}, \overset{\blacksquare}{M}_n^{\pm})^{'\vee}.
\end{array}$$

Now that we have proved that the operations of lemma 5 extend, we can assume that the subcomplexes Y_n, $\overset{\blacksquare}{Y}_n$ of Y intersect $X^{\pm}$ along $X_n^{\pm}, \overset{\blacksquare}{X}{}_n^{\pm}$, which satisfy the conditions of lemma 5, and moreover, that we have the squares

$$\begin{array}{ccc} K_c^{q+1}(W_n,\partial W_n) & \longrightarrow & K_c^{q+1}(W_n,M_n^+ \cup M_n^-) \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi_{|}} \\ K_{q+1}(W_n) & \longrightarrow & K_{q+1}(W_n,\overset{\blacksquare}{W}_n) \end{array} \qquad \begin{array}{ccc} K_c^{q+1}(W_n,\overset{\blacksquare}{W}_n) & \longrightarrow & K_c^{q+1}(W_n) \\ \uparrow{\scriptstyle \psi_{=}} & & \uparrow{\scriptstyle \psi_{\sqsubset}} \\ K_{q+1}(W_n,M_n^+ \cup M_n^-) & \longrightarrow & K_{q+1}(W_n,\partial W_n) \end{array}$$

with inverse for $\psi, \psi_{|}, \psi_{=}, \psi$ shifting the indice by ± 1.

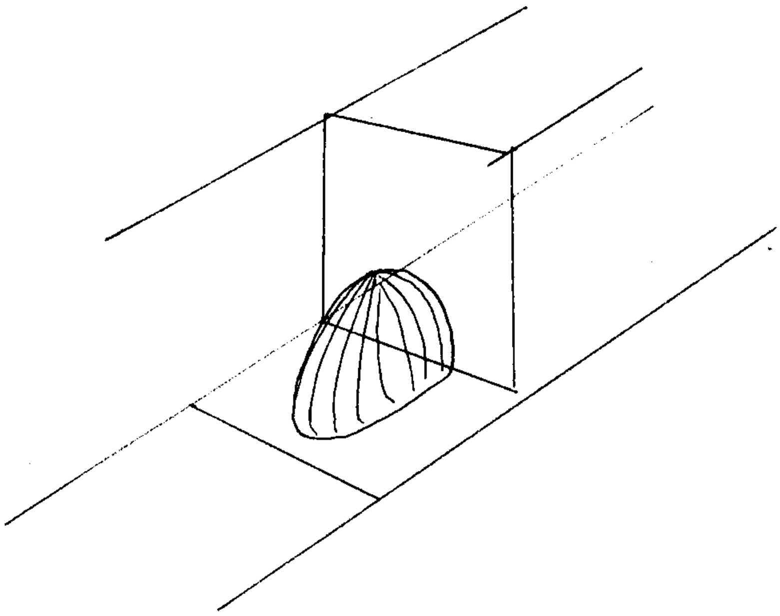
<u>Claim</u>: By changing the Y_n, $\overset{\blacksquare}{Y}_n$ rel. $X^+ \cup X^-$ one can assume that $\psi, \psi_{=}$ are isomorphisms and $\psi_{|}$, $\psi_{\sqsubset}$, injective. We proceed as in lemma 5, but skip quite a bit through it. The direct system $\{K_{q+1}(W_s,\overset{\blacksquare}{W}_s)\}_n$ is composed of surjections, hence by the argument of §4, ker $\psi_{|}$ is finitely generated and we can kill it by enlarging $\overset{\blacksquare}{Y}_n$ (keeping the squares as above). Once $\psi_{|}$ is injective (split by §4), so is $\psi_{\sqsubset}$ by a previous argument, and moreover, by duality, ψ and $\psi_{=}$ are (split) surjective. In particular, ker $\psi_{=}$ is a retract. But it is contained in the image of $K_{q+1}(\overset{\blacksquare}{W}_n,\overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^{\#}$, in virtue of the diagram

$$\begin{array}{ccccccc} & & K_c^{q+1}(W_n,\overset{\blacksquare}{W}_n) & \longrightarrow & K_c^{q+1}(W_n) & & \\ & & \uparrow{\scriptstyle \psi_{=}} & & |{\scriptstyle \psi_{\sqsubset}} & & \text{(injective)} \\ K_{q+1}(\overset{\blacksquare}{W}_n,\overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^{\#} & \longrightarrow & K_{q+1}(W_n,M_n^+ \cup M_n^-) & \longrightarrow & K_{q+1}(W_n,\partial W_n) & \longrightarrow & 0 \end{array}$$

hence is finitely generated. Actually, $\ker \psi_=$ comes from $K_{q+1}(\overset{\bullet}{W}_n)^\#$ in virtue of the diagram

$$\begin{array}{ccccc}
 & & K^q(\overset{\bullet}{W}_n)^\# & \longrightarrow & \bigoplus_{\pm} K^q(\overset{\bullet}{M}{}_n^{\pm})^\# \\
 & & \uparrow & & \uparrow \approx \\
K_{q+1}(\overset{\bullet}{W}_n)^\# & \longrightarrow & K_{q+1}(\overset{\bullet}{W}_n, \overset{\bullet}{M}{}_n^{+} \cup \overset{\bullet}{M}{}_n^{-})^\# & \longrightarrow & \bigoplus_{\pm} K_q(\overset{\bullet}{M}{}_n^{\pm})^\# .
\end{array}$$

Hence, as in the operation '" of lemma 5, we can add cells e^{q+3} to both Y_n and $\overset{\bullet}{Y}_n$ to get $\psi_=$ bijective



The same argument also applies to $\ker \psi$: it is a retract, and contained in the image of $\bigoplus_{\pm} K_{q+1}(M_n^{\pm})$ in virtue of the diagram

$$\begin{array}{ccccc} & & K_c^{q+1}(W_n,\partial W_n) & \longrightarrow & K_c^{q+1}(W_n,\overset{\blacksquare}{W}_n) \\ & & \Big\uparrow \psi & & \approx \Big\uparrow \psi_= \\ \bigoplus_{\pm} K_{q+1}(M_n^{\pm}) & \longrightarrow & K_{q+1}(W_n) & \longrightarrow & K_{q+1}(W_n, M_n^+ \cup M_n^-) \end{array} .$$

But, because $K_{q+1}(M_n^{\pm},\overset{\blacksquare}{M}{}_n^{\pm}) = 0$, $K_{q+1}(M_n^{\pm})$ is a quotient of $K_{q+1}(\overset{\blacksquare}{M}{}_n^{\pm})^{\#}$, and the latter is finitely generated because $K_q(\overset{\blacksquare}{M}{}_n^{\pm})^{\#}$ is projective (if in a finite chain complex the lowest homology H_k is projective then H_{k+1} is finitely generated because the $k+1$-cycles are direct summand). Now that we have shown how to prove the second claim, we can assume to have the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_c^{q+1}(W_n,\partial M_n) & \rightarrow & K_c^{q+1}(W_n, M_n^+ \cup M_n^-) & \rightarrow & K^{q+1}(\overset{\blacksquare}{W}_n,\overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^{\#} & & \\ \Big\uparrow & & \psi \Big\uparrow \text{(isom.)} & & \psi_{\blacksquare} \Big\uparrow \text{(inj.)} & & \Big\uparrow & & \\ K_{q+1}(\overset{\blacksquare}{W}_n)^{\#} & \longrightarrow & K_{q+1}(W_n) & \longrightarrow & K_{q+1}(W_n,\overset{\blacksquare}{W}_n) & \rightarrow & K_q(\overset{\blacksquare}{W}_n)^{\#} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccc} 0 \rightarrow K^q(\overset{\blacksquare}{W}_n)^{\#} & \longrightarrow & K_c^{q+1}(W_n,\overset{\blacksquare}{W}_n) & \longrightarrow & K_c^{q+1}(W_n) & \longrightarrow & K_q(\overset{\blacksquare}{W}_n)^{\#} \\ \Big\uparrow & & \psi_= \Big\uparrow \text{(isom.)} & & \psi_{\blacksquare} \Big\uparrow \text{(inj.)} & & \Big\uparrow \\ K_{q+1}(\overset{\blacksquare}{W}_n,\overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^{\#} & \rightarrow & K_{q+1}(W_n, M_n^+ \cup M_n^-) & \rightarrow & K_{q+1}(W_n,\partial W_n) & \longrightarrow & 0 \end{array}$$

Observe that the exact sequence

$$K_{q+1}(\overset{\blacksquare}{W}_n)^{\#} \to K_{q+1}(\overset{\blacksquare}{W}_n,\overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^{\#} \to \bigoplus_{\pm} K_q(\overset{\blacksquare}{M}{}_n^{\pm})^{\#} \to K_q(\overset{\blacksquare}{W}_n)^{\#} \to 0$$

remains exact is one replaces $K_{q+1}(\overset{\blacksquare}{W}_n)^{\#}$ by its image 0 in $K_{q+1}(W_n)$, and $K_{q+1}(\overset{\blacksquare}{W}_n,\overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^{\#}$ by its image E in

$K_{q+1}(W_n, M_n^+ \cup M_n^-)$, in virtue of the diagram

$$\begin{array}{ccccc}
K_{q+2}(W_n,\dot{W}_n) & \longrightarrow & K_{q+2}(W_n,\partial W_n) & \longrightarrow & \bigoplus_{\pm} \underbrace{K_{q+1}(M_n^{\pm},\dot{M}_n^{\pm})}_{0} \\
 & & \downarrow & & \\
K_{q+1}(\dot{W}_n)^{\#} & \longrightarrow & K_{q+1}(\dot{W}_n,\dot{M}_n^+ \cup \dot{M}_n^-)^{\#} & & \\
 & & \downarrow & & \\
 & & K_{q+1}(W_n,M_n^+ \cup M_n^-) \ . & &
\end{array}$$

But the former diagrams provides isomorphisms $E \cong K^q(\dot{W}_n)^{\#}$ and $K_q(\dot{W}_n) \cong E^*$. Hence, putting $K_q(\dot{W}_n)^{\#} \equiv F$, the above exact sequence reduces to $0 \to E \to \bigoplus_{\pm} K_q(\dot{M}_n^{\pm})^{\#} \to F \to 0$ where the quadratic form on the middle module induces isomorphism $E \cong F^*$, $F \cong E^*$. <u>Claim</u>: this sequence splits. To construct a section $F \to \bigoplus_{\pm} K_q(\dot{M}_n^{\pm})^{\#}$, consider the diagram of exact sequences

$$\begin{array}{cccccccc}
 & & & & & 0 & & \\
 & & & & & \downarrow & & \\
 & K_{q+1}(\dot{W}_n)^{\#} & \longrightarrow & K_{q+1}(\dot{W}_n,\dot{M}_n^+ \cup \dot{M}_n^-)^{\#} & \longrightarrow & \bigoplus_{\pm} K_q(\dot{M}_n^{\pm})^{\#} & & \\
 & {\scriptstyle 0}\downarrow & & \downarrow & & \downarrow & & \\
 & K_{q+1}(W_n) & \longrightarrow & K_{q+1}(W_n,M_n^+ \cup M_n^-) & \longrightarrow & \bigoplus_{\pm} K_q(M_n^{\pm}) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \uparrow & & \downarrow & & \\
0 \to & K_{q+1}(W_n,\dot{W}_n) & \longrightarrow & K_{q+1}(W_n,\partial W_n) & \longrightarrow & \bigoplus_{\pm} K_q(M_n^{\pm},\dot{M}_n^{\pm}) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & K_q(\dot{W}_n)^{\#} & \longrightarrow & 0 & & 0 & & \\
 & \downarrow & & & & & & \\
 & 0 & & & & & &
\end{array}$$

It contains a commutative triangle

$$\begin{array}{ccc} K_{q+1}(W_n) & \longrightarrow & K_{q+1}(W_n, M_n^+ \cup M_n^-) \\ & \searrow^{\text{(injective)}} & \\ & & K_{q+1}(W_n, \partial W_n). \end{array}$$

In particular, $K_{q+1}(W_n)$ is a submodule of $K_{q+1}(W_n, M_n^+ \cup M_n^-)$ which meets the image E of $K_{q+1}(\overset{\blacksquare}{W}_n, \overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^\#$ only at 0. But, as $K_{q+1}(W_n, \partial W_n)$ and $K_q(M_n^\pm)$ are projective, E and $K_q(W_n)$ are direct summands of $K_{q+1}(W_n, M_n^+ \cup M_n^-)$. Hence, the preimage of $K_{q+1}(W_n)$ by the map

$K_{q+1}(W_n, M_n^+ \cup M_n^-) \to K_{q+1}(W_n, \partial W_n)$ is $E \oplus K_q(W_n)$. We construct a map $K_q(\overset{\blacksquare}{W}_n)^\# \to \bigoplus_\pm K_q(\overset{\blacksquare}{M}{}_n^\pm)^\#$ by representing $x \in K_q(\overset{\blacksquare}{W}_n)^\#$ into $K_{q+1}(W_n, \overset{\blacksquare}{W}_n)$ taking the image x' into $K_{q+1}(W_n, \partial W_n)$, then a section $K_{q+1}(W_n, \partial W_n) \xrightarrow{s} K_{q+1}(W_n, M_n^+ \cup M_n^-)$, and taking the image of $x'' = s(x')$ in $\bigoplus_\pm K_q(M_n^\pm)$, which actually lies in $\bigoplus_\pm K_q(\overset{\blacksquare}{M}{}_n^\pm)^\#$. This does not depend on the way of representing x, because if x' comes from $K_{q+1}(W_n)$, then on one side $s(x') \in E \oplus K_q(W_n)$, but on the other side the E-component of $s(x')$ is 0, hence $s(x')$ comes from $K_q(W_n)$, i.e., vanishes in $\bigoplus_\pm K_q(M_n^\pm)$. This achieves the proof that, in the second case of cobordism, the quadratic module $\langle\!\langle K_q(M_n^+)\rangle\!\rangle \oplus \langle\!\langle K_q(M_n^-)\rangle\!\rangle$ is isomorphic to a projective hyperbolic module $\langle E \oplus F\rangle$, i.e., $K_q(\overset{\blacksquare}{M}{}_n^+)^\#$ and $K_q(\overset{\blacksquare}{M}{}_n^-)^\#$ are equivalent. As for the first case of cobordism, i.e., a trivial surgery from $f^+ \colon M^+ \to X$ to $f^- \colon M^- \to X$, observe that the <u>same</u> operations on X can be used to satisfy lemma 5 for both $f^{\pm}$, and then one readily sees the equivalence.

10. Theorem. Let M be an open manifold of dim $2q+1 \geqslant 7$ and $f: M \to X$ a proper normal map of degree 1. Then, to the cobordism class $[f]$ of f are associated canonically a sequence $(\mathcal{J}_n) \in \varprojlim L_{2q}(\pi_1 X_n)$ and, if all $\mathcal{J}_n$ vanish, an element $(\ell_n) \in \varprojlim^1 L_{2q+1}(\pi_1 X_n)$, such that $[f]$ contains a proper homotopy equivalence at ∞ iff all $\mathcal{J}_n = 0$ and $(\ell_n) = 0$. By definition, if $\{A_n\}$ is an inverse system of abelian groups, $\varprojlim^1 A_n$ is the cokernel of the map $\prod_{n\geqslant 1} A_n \xrightarrow{1-S} \prod_{n\geqslant 1} A_n$ sending $(a_1, a_2, a_3, \ldots.)$ to $(a_1 - a_2^{\#}, a_2 - a_3^{\#}, a_3 - a_4^{\#}, \ldots.)$, where $a_n^{\#}$ is the image of a_n in A_{n-1}. A subsequence gives the same result, e.g. $\varprojlim^1 A_{2n+1} \cong \varprojlim^1 A_n$ by sending $(a_1, a_2, a_3, \ldots.)$ to $(a_1 + a_2^{\#}, a_3 + a_4^{\#} \ldots.)$ in the range product. Note that the choice of base points and paths has no influence on the inverse system $\{L_*(\pi_1 X_n)\}$ because an inner automorphism of a group G induces $\pm$ identity on $L_*(G)$, according to whether $\omega: \pi_1 X_n \to \pm 1$ is trivial or not.

Proof. Define $\mathcal{J}_n$ by the quadratic module $K_q(\dot{M}_n)^{\#}$ obtained in Proposition 7. A canonical equivalence between $\mathcal{J}_{n+1}^{\#}$ and $\mathcal{J}_n$ is given by the exact sequence

$$K_{q+1}(\overline{M_n - M_{n+1}}, \dot{M}_n \cup \dot{M}_{n+1})^{\#} \to K_q(\dot{M}_n)^{\#} \oplus K_q(\dot{M}_{n+1}) \to K_q(\overline{M_n - M_{n+1}})^{\#} \to 0.$$

Actually, $K_q(\overline{M_n - M_{n+1}})^{\#}$ is projective in virtue of the exact sequence

$$\underbrace{K_{q+1}(M_{n+1},\overset{\blacksquare}{M}_{n+1})^{\#}}_{0} \longrightarrow K_q(\overline{M_n - M_{n+1}})^{\#} \longrightarrow K_q(M_n) \longrightarrow K_q(M_{n+1},\overset{\blacksquare}{M}_{n+1})^{\#} \longrightarrow 0.$$

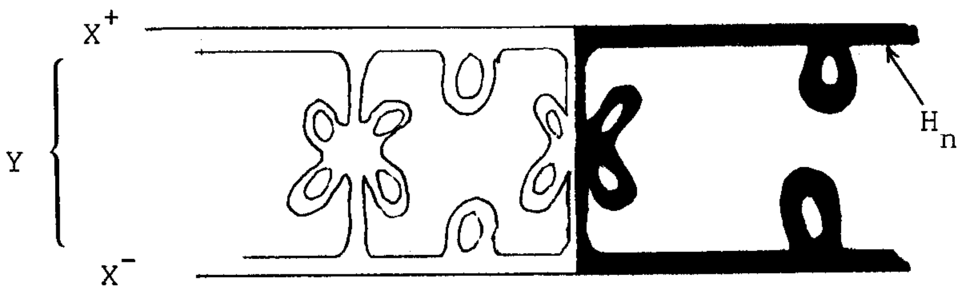
A reciprocal duality between $F \equiv K_q(\overline{M_n - M_{n+1}})^{\#}$ and the image E of $K_{q+1}(\overline{M_n - M_{n+1}},\overset{\blacksquare}{M}_n \cup \overset{\blacksquare}{M}_{n+1})^{\#}$ comes from the diagram

$$\begin{array}{ccccccc}
0 \to K^q(\overline{M_n - M_{n+1}})^{\#} & \longrightarrow & K^q(\overset{\blacksquare}{M}_n)^{\#} & \oplus & K^q(\overset{\blacksquare}{M}_{n+1})^{\#} & \longrightarrow & K^{q+1}(\overline{M_n - M_{n+1}},\overset{\blacksquare}{M}_n \cup \overset{\blacksquare}{M}_{n+1}) \\
\uparrow & & \uparrow \approx & & \uparrow \approx & & \uparrow \\
K_{q+1}(\overline{M_n - M_{n+1}},\overset{\blacksquare}{M}_n \cup \overset{\blacksquare}{M}_{n+1})^{\#} & \to & K_q(\overset{\blacksquare}{M}_n)^{\#} & \oplus & K_q(\overset{\blacksquare}{M}_{n+1})^{\#} & \longrightarrow & K_q(\overline{M_n - M_{n+1}})^{\#} \to 0
\end{array}$$

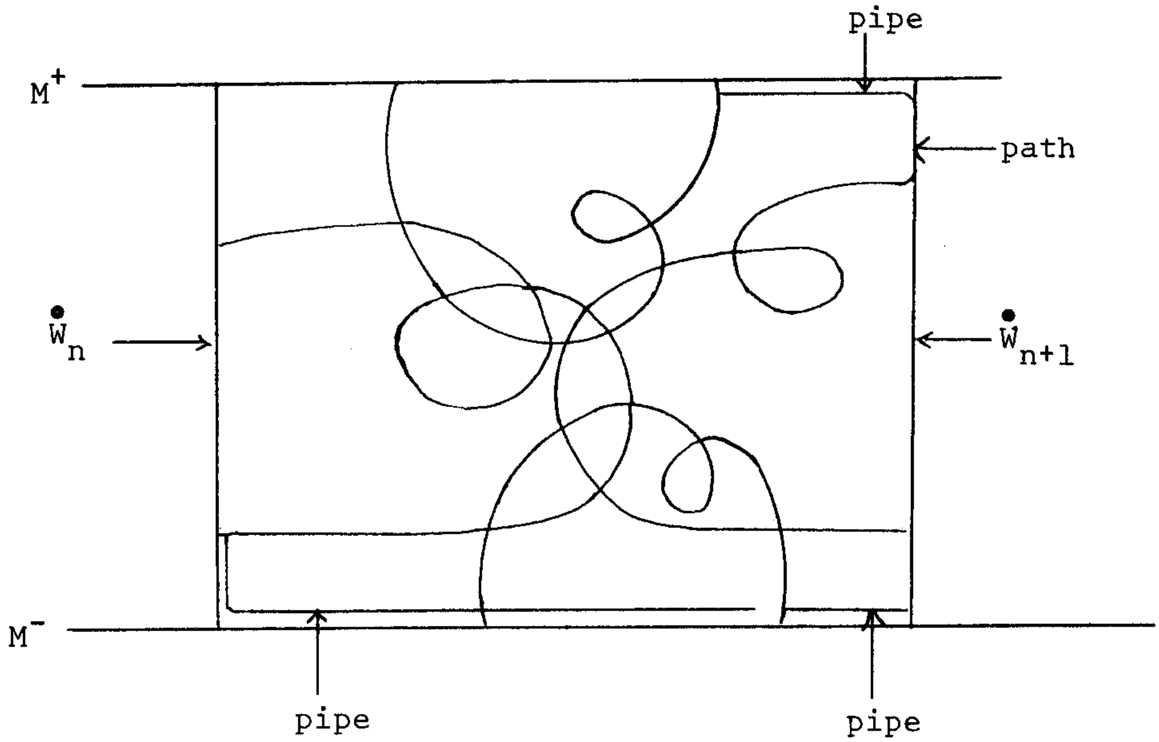
and its dual. This exhibits $K_q(\overset{\blacksquare}{M}_n)^{\#} \oplus K_q(\overset{\blacksquare}{M}_{n+1})^{\#}$ as the hyperbolic module $E \oplus F$, i.e. $\mathcal{q}^{\#}_{n+1} = \mathcal{q}_n$. By §8, the element $(\mathcal{q}_n) \in \varprojlim L_{2q}(\pi_1 X_n)$ is independent of all choices and invariant by cobordism. If all $\mathcal{q}_n = 0$, then by choosing a trivialization $E_n \oplus F_n$ for $K_q(\overset{\blacksquare}{M}_n)^{\#}$, the plane $K_q(\overline{M_n - M_{n+1}})^{\#}$ becomes a Lagrangian plane L_n in the standard module $(E_n \oplus E_{n+1}) \oplus (F_n \oplus F_{n+1})$, i.e. $L_n \in L_{2q+1}(\pi_1 X_n)$. Another choice of trivializationsmodify the sequence (L_n) by a sequence in the image of $1-S$. The same is true if one alters f by a cobordism, and we sketch the proof as follows. Let $f^{\pm}: M^{\pm} \to X^{\pm}$ be cobordant by $F: W \to Y$, in the final setting of §8. Then we have quadratic modules $K_q(\overset{\blacksquare}{M}{}^{\pm}_n)^{\#}$, trivial by assumption, and Lagrangian planes $K_q(\overset{\blacksquare}{W}_n)^{\#}$ in $\langle K_q(\overset{\blacksquare}{M}{}^{+}_n)^{\#}\rangle \oplus \langle K_q(\overset{\blacksquare}{M}{}^{-}_n)^{\#}\rangle'$, and $K_q(\overline{M_n - M_{n+1}}{}^{\pm})^{\#}$ in $\langle K_q(\overset{\blacksquare}{M}{}^{\pm}_n)^{\#}\rangle \oplus \langle K_q(\overset{\blacksquare}{M}{}^{\pm}_{n+1})^{\#}\rangle'$. By choosing a trivialization $\langle E^{\pm}_n \oplus F^{\pm}_n\rangle$ of $\langle K_q(\overset{\blacksquare}{M}{}^{\pm}_n)^{\#}\rangle$, the planes

$K_q(\dot{W}_n)^\#$ and $K_q(\overline{M_n - M^\pm_{n+1}})^\#$ become elements $\omega_n, \ell_n \in L_{2q+1}(\pi_1 X_n)$. We have to show that $\ell_n^+ - \ell_n^- = \omega^\#_{n+1} - \omega_n$, i.e., that the Lagrangian plane $\ell_n \equiv K_q(\dot{W}_n)^\# \oplus K_q(\overline{M_n^+ - M_{n+1}^+})^\# \oplus K_q(\overline{M_n^- - M_{n+1}^-})^{\#*} \oplus K_q(W_{n+1})^{\#*}$ in $\langle H \rangle \oplus \langle H \rangle'$ is equivalent to 0 where

$\langle H \rangle \equiv \langle K_q(\dot{M}_n^+)^\# \rangle \oplus \langle K_q(\dot{M}_n^-)^\# \rangle \oplus \langle K_q(\dot{M}_{n+1}^+)^\# \rangle \oplus \langle K_q(\dot{M}_{n+1}^-)^\# \rangle$ is equivalent to a trivial one. We consider this problem as the bounded case of Chapter 1. For this, we need to choose the very initial $X^\pm$, $X_n^\pm$, $\dot{X}_n^\pm$, Y, Y_n, $\dot{Y}_n$ as follows. By infinite simple homotopy type theory, $X^\pm$ is simply homotopy equivalent to a CW-complex of the form $X^{0\pm} \cup_{\partial H^\pm} H^\pm$, where $H^\pm$ is a locally finite 2q+1-handlebody of 0 and 1-handles, which is a thickened tree (see [10]). Moreover, Y is simply homotopy equivalent rel $X^+ \cup X^-$ to a CW-complex of the form $Y^0 \cup_{\partial H} H$, where H is a locally finite 2q+2-handlebody of 0 and 1-handles, such that $H \cap X^\pm = H^\pm$, and $\partial H = H^+ \cup H^- \cup \partial^r H$, $Y^0 \cap X^\pm = X^{0\pm}$.



As ngbd of ∞ in H, we take subhandlebodies H_n with relative frontier a disjoint union of $D^{2q} \times I$ (see figure). Choose ngbd of ∞ Y^0_n in Y^0, and finite subcomplexes $\overset{\blacksquare}{Y}{}^0_n$ containing the frontier. The $X^{0\pm}_n \equiv Y^0_n \cap X^\pm$ are ngbd of ∞ in $X^{0\pm}$, and let $\overset{\blacksquare}{X}{}^{0\pm}_n \equiv \overset{\blacksquare}{Y}{}^0_n \cap X^\pm$. By the construction in §1, we can assume that $\overset{\blacksquare}{Y}{}^0_n$ is bicollared in Y^0 (and $\overset{\blacksquare}{X}{}^{\pm 0}_n$ bicollared in $X^\pm$). Now, $X^{\pm 0}_n \cup (H_n \cap X^\pm)$ is a ngbd of ∞ $X^\pm_n$ in $X^\pm$, and we choose $\overset{\blacksquare}{X}{}^\pm_n \equiv \overset{\blacksquare}{X}{}^{0\pm}_n$. Similarly, $Y_n \equiv Y^0_n \cup H_n$ is a ngbd of ∞ in Y such that $Y_n \cap X^\pm = X^\pm_n$, and by using a collar along ∂H, we can assume that $\overset{\blacksquare}{Y}_n \equiv \overset{\blacksquare}{Y}{}^0_n \cup \overset{\blacksquare}{H}_n$ is bicollared in Y. Then we do all the necessary preliminary surgery (as in §1) first on $M^\pm \xrightarrow{f^\pm} X^\pm$, then on $W \xrightarrow{F} Y$ rel. $M^+ \cup M^-$. Then one meets the modules $K_q(\overline{M^\pm_n - M^\pm_{n+1}})$ and $K_q(\overset{\bullet}{W}_n)$. Represent each generator by an embedded q-sphere, and extend them into immersed q+1-discs in $\overline{W_n - W_{n+1}}$ (see IV. 1). Then pipe the left discs and upper and lower discs to ∞ as in the figure



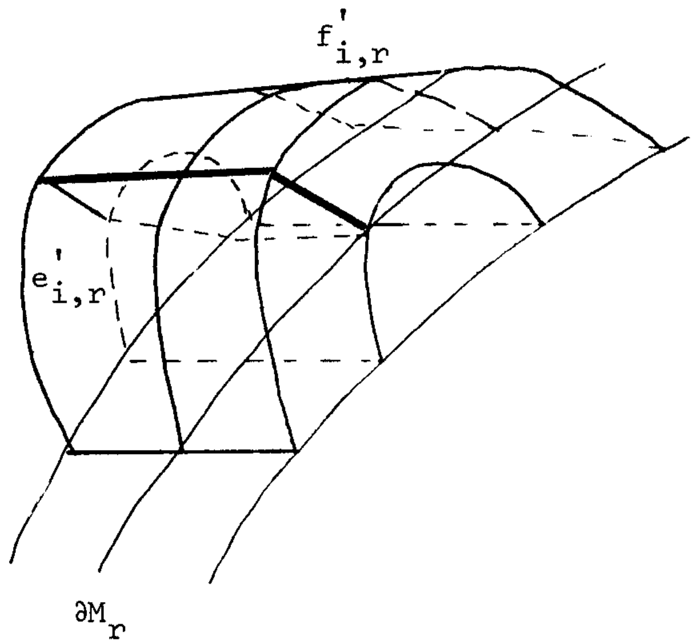
We also connect up by path all the q-spheres so obtained in a connected component of $\dot{W}_{n+1}$. Then take a regular ngbd V of this connected union of images of immersions. Let $V_n = V \cap W_n$, $U_n^\pm = V_n \cap M^\pm$, $U_n = V_n \cap \dot{W}_n$, $\partial^r V_n = \overline{\partial \bar{V} \cap W_n - U_n^+ \cup U_n^-}$, $\partial^r U^\pm = \partial U_n^\pm \cap M_n^\pm$, $M^{0\pm} = \overline{M^\pm - U^\pm}$, $W^0 = \overline{W-V}$. Then, as in IV. 1, $\overline{V_n - V_{n+1}}$ is a handlebody on $U_{n\pm} \cup \overline{U_n^\pm - U_{n+1}^\pm} \cup U_{n+1}$ composed of 1 and q+1-handles. This allows (by standard geometrical arguments like in [7]) to arrange F and $f^\pm$ so that they induce maps $M^{0\pm} \to X^{0\pm}$, $U^\pm \to H^\pm$, $\partial U^\pm \to \partial H^\pm$, $W^0 \to Y^0$, $V \to H$, $\partial^r V \to \partial^r H$ (now, H may be smaller). Apply §5 to $M^{0\pm} \to X^{0\pm}$ rel $\partial M^{0\pm} \equiv \partial U^\pm$ and §9 to $W^0 \to Y^0$ rel ∂W^0. Then, by IV. 2, we get a projective Lagrangian plane $K_q(\overset{\blacksquare}{W}{}^0_n)$ in $K_q(\partial U_n)$. By the argument of [11, lemma 7.2], the Lagrangian plane $K_q(\overset{\blacksquare}{W}{}^0_n) \oplus K_q(\overset{\blacksquare}{W}{}^0_{n+1})^*$ in $\langle K_q(\partial U_n)\rangle \oplus \langle K_q(\partial U_{n+1})\rangle'$ is equivalent to L_n in $\langle H\rangle \oplus \langle H\rangle'$. But the former is equivalent to 0 by IV. 4. Hence $\ell_n \in \varprojlim^1 L_{2q+1}(\pi_1 X_n)$ associated to [f] in a well-defined way. If all $\sigma_n = 0$ and $(\ell_n) = 0$, then by [11] (realizing Lagrangian transformation) one can arrange so that actually the plane $K_q(\overline{M_n - M_{n+1}})^{\#}$ is actually "trivial",

for all n. This means that the map: $K_q(\overset{\blacksquare}{M}_n)^{\#} \oplus (\overset{\blacksquare}{M}_{n+1})^{\#} \longrightarrow K_q(\overline{M_n - M_{n+1}})^{\#}$ (the first summand mapping by inj., the second by surj.)

is nothing but the canonical projection $(E_n \oplus F_n) \oplus (E_{n+1} \oplus F_{n+1}) \to$ $\to {}^{\ell}E_n \oplus F_{n+1} \oplus {}^{\ell}E_{n+1} \oplus F_n$ (see notations). Then, the map

$K_q(\overset{\blacksquare}{M}_r)^{\#} \to K_q(M_n)$ injects E_r onto a direct summand and projects F_r onto F_n, for $r \to \infty$. But these image generated $K_q(M_n)$ hence $K_q(M_n) \cong E \oplus F_n$, as in §8. Once we know that, we can do surgery as follows: 1°) make E free by trivial surgery, 2°) each basis element $e_i \in E$ is in E_r for $r \to \infty$, but moreover, we saw at the end of §6, that $K_q(\overset{\blacksquare}{M}_n)$ is in the image of $K_q(\partial M_r)^{\#}$ for large r, hence e can be represented by a sequence of maps $(D^{q+1}, S^q) \xrightarrow{\alpha_r} (\partial X_r, \partial M_r)$ for $r \to \infty$. Now, we know that the intersection form between elements of E vanishes with $\pi_1 M_n$-coefficients. Hence it already vanishes with some $\pi_1(\overline{M_n - M_s}) = \pi_1 \overset{\blacksquare}{M}_n$-coefficients, because the group functor L_* commutes with direct limits.

By modifying ∂M_r inside $\overset{\blacksquare}{M}_n$ with 1 and 2-handles we can assume that $\pi_1 \partial M_r \cong \pi_1 \overset{\blacksquare}{M}_n$ (see [8]). Now, the interestion between the e's vanish with $\pi_1 \partial M_r$ coefficients, so we can do a sequence of surgeries on α_r.



To each e_i is substituted in $K_q(M_n)$ a free module generated by the $e_{i,r}$ and a corresponding free module over $f'_{i,r}$ appears as $K_{q+1}(M_n)$. So the new K-systems look like

$$\begin{array}{ccccc} K_q(M'_n) & = & F_n & \oplus & E'_n \\ \uparrow & & \uparrow \text{surj.} & & \uparrow \text{inj.} \\ K_q(M'_{n+1})^{\#} & = & F_{n+1} & \oplus & E_{n+1} \end{array} \qquad \begin{array}{ccc} K_{q+1}(M'_n) & = & F'_n \\ \uparrow & & \uparrow \text{inj.} \\ K_{q+1}(M'_{n+1})^{\#} & = & F'_{n+1} \end{array}$$

where the injection of free modules are of the form $A \xrightarrow{1^{st} \text{comp}} A \oplus B$, with B free of finite rank. Hence $K_q(M'_n)$ is free, and the cokernel of $K_q(M'_{n+1})^{\#} \longrightarrow K_q(M'_n)$ is free of finite rank.

In other words, one can write $K_q(M_n') = \bigoplus_{k \geq n} A_n$ where each A_n is free of finite rank. Observe that $K_q(M_n') \cong K_q(M_n', \dot{M}_n')$ now.

3°) represent each basis element of A_n by an element in $K_q(M_n')$, and do surgery on it. This gives a cobordism $W \xrightarrow{F} X$ from $M' \xrightarrow{f'} X$ to $M'' \xrightarrow{f'} X$, such that $K_{q+1}(W_n, M_n') \xrightarrow{\partial} K_q(M_n')$ is an isomorphism for each n, and $K_k(W_n, M_n') = 0$ for $k \neq q+1$.

The exact sequence

$$0 \to K_{q+1}(M_n') \to K_{q+1}(W_n) \to K_{q+1}(W_n, M_n') \underset{\approx}{\overset{\partial}{\to}} K_q(M_n') \to K_q(W_n) \to 0$$

shows that $K_q(W_n) = 0$, and $K_{q+1}(W_n) \cong K_{q+1}(M_n')$. On the other side, we get the exact sequence

$$0 \to K_{q+1}(M_n'') \to K_{q+1}(W_n) \to K_{q+1}(W_n, M_n'') \to K_q(M_n'') \to 0.$$

<u>Claim</u>: The middle map is an equivalence of inverse systems. We can take $\dot{W}_n = \dot{M}_n' \times I$, hence $K_{q+1}(W_n, M_n'') \cong K_{q+1}(W_n, \dot{W}_n \cup M_n'')$. As $K_{q+1}(W_n) \cong K_{q+1}(M_n') \cong K_{q+1}(M_n', \dot{M}_n')$, we have the commutative square

$$\begin{array}{ccc} K_{q+1}(W_n) & \longrightarrow & K_{q+1}(W_n, M_n'') \\ \text{equiv.} \Big\downarrow \psi & & \psi \Big\downarrow \text{equiv.} \\ K_c^q(M_n') & \xrightarrow{\delta} & K_c^{q+1}(W_n, M_n') \end{array} .$$

By construction, $K_{q+1}(W_r,M_r')^{\#} \xrightarrow{\partial} K_q(M_r')^{\#}$ is an isomorphism, $\forall r$, so is its dual, hence we get by direct limit over r an isomorphism $K_e^q(M_n') \underset{\approx}{\xrightarrow{\delta}} K_e^{q+1}(W_n,M_n')$. Then the diagram

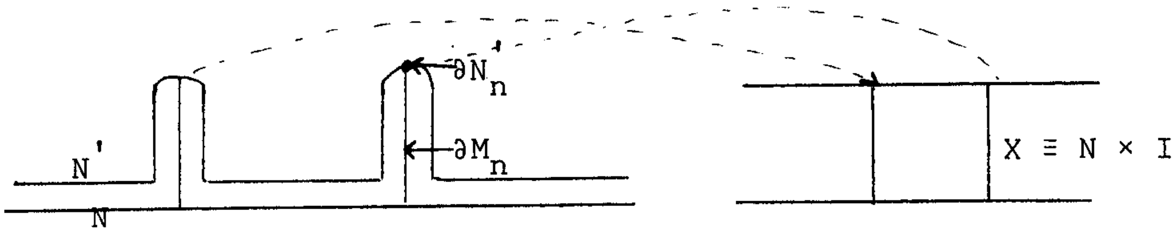
$$\begin{array}{ccccccccc}
0 \to K_e^q(M_n') & \longrightarrow & K_c^q(M_n') & \longrightarrow & K^q(M_n') & \longrightarrow & K_e^{q+1}(M_n') \\
\downarrow \approx & & \downarrow \delta & & \downarrow \approx & & \downarrow \\
0 \to K_e^{q+1}(W_n,M_n') & \longrightarrow & K_c^{q+1}(W_n,M_n') & \longrightarrow & K^{q+1}(W_n,M_n') & \longrightarrow & 0
\end{array}$$

where $K_e^{q+1}(M_n') = \varinjlim_{r} F_r'^{*} = 0$ shows that the middle map δ is an isomorphism, and this implies the assertion. Hence the inverse systems $\{K_q(M_n'')\}$ and $\{K_{q+1}(M_n'')\}$ are equivalent to 0, which implies that $M'' \xrightarrow{f''} X$ is a proper homotopy equivalence at ∞. This achieves the proof of the theorem. A more refined formulation of Theorem 9 is given by the following result.

11. Corollary. Let $L_{2q+1}(\varepsilon X)$ be the obstruction group for our problem, i.e., to each surgery data $(M,^{2q+1},\partial M)^f \to (X,\partial X)$ rel. boundary $(f|\partial M$ is already a proper homotopy equivalence at $\infty)$ is associated $\sigma(f) \in L_{2q+1}(\varepsilon X)$ which vanishes iff f is cobordant rel ∂M to a proper homotopy equivalence at ∞, and each element of $L_{2q+1}(\varepsilon X)$ is equal to $\sigma(f')$ for some surgery data rel. boundary $(M',\partial M') \xrightarrow{f'} (X',\partial X')$, where $\{\pi_1 X_n'\}$ is conjugate equivalent to εX in a specific way. Then we have an exact sequence

$$0 \to \varprojlim{}^1 L_{2q+1}(\pi_1 X_n) \to L_{2q+1}(\varepsilon X) \xrightarrow{\mathcal{q}} \varprojlim L_{2q}(\pi_1 X_n) \to L_{2q}(\pi_1 X).$$

Sketch of proof: For the first map, take an open 2q-manifold N such that $\{\pi_1 N_n\}$ is conjugate equivalent to $\{\pi_1 X_n\}$. Then, following [11], do surgery on $N \xrightarrow{id} N$ to kill enough trivial (q-1)-spheres in each $\overline{N_n - N_{n+1}}$ (r_n say, if the Lagrangian plane ℓ_n is in the free hyperbolic module of rank r_n). Let $N' \xrightarrow{f'} N$ be the result of this surgery. Then $K_q(\overline{N'_n - N'_{n+1}})$ is free of finite rank, and $K_q(N'_n) \cong \bigoplus_{r \geq 0} K_q(N'_{n+r} - N'_{n+r+1})^{\#}$ is free of countable rank. By definition, ℓ'_n is a Lagrangian plane in $K_q(\overline{N'_n - N'_{n+1}})$, so we can do surgery on $N' \xrightarrow{f'} N$ killing a finite set of generators of ℓ_n. The result $N'' \xrightarrow{f''} N$ of this surgery is a proper homotopy equivalence (see end of IV.4). If $M^{2q+1} \xrightarrow{f} N \times I$ is the cobordism so obtained between $N \xrightarrow{id.} N$ an $N'' \xrightarrow{f''} N$, $M \xrightarrow{f} N \times I$ provides the surgery data $(M, \partial M) \to (X, \partial X)$ we are looking for in $L_{2q+1}(\varepsilon X)$. For the second map: if $(M', \partial M') \to (X', \partial X')$ is a surgery data, we take the sequence of quadratic forms $\mathcal{q}_n \in \varprojlim L_{2q}(\pi_1 X'_n) \cong \varprojlim L_{2q}(\pi_1 X_n)$. Observe that in this case with boundary, where the map on the boundary is already a proper homotopy equivalence, everything looks like if M' were open. The composition of the two first maps is 0 by construction. The composition of the two last maps is 0, by the argument proving that $\mathcal{q}^{\#}_{n+1} = \mathcal{q}_n$ (case $\dot{M}_n = \phi$, see proof of Theorem 10). For the exactness at $\varprojlim L_{2q}(\pi_1 X_n)$, note that any element of this limit can be represented by a free (singular) quadratic module. Take N as above, and by [11] again, do surgery on each identity map $\partial N_n \to \partial N_n$ to some map (which would be a homotopy equivalence iff the quadratic form on the free module were nonsingular) so that the cobordism map $\partial M_n \longrightarrow \partial N_n \times I$ has obstruction σ_n



The condition $\sigma^{\#}_{n+1} = \sigma_n$ and $\sigma_1 = 0$ in $L_{2q}(\pi_1 X)$ allows to do surgery by strips on the other side N', rel ∂N_n to get a proper homotopy equivalence $N'' \to N$. This construction provides a cobordism $M \to X$ between $N \xrightarrow{id.} N$ and some proper homotopy equivalence $N'' \to N$

This shows that (σ_n) comes from a surgery data $(M,\partial M) \to (X,\partial X)$. For the exactness at $L_{2q+1}(\varepsilon X)$ Theorem 10 gives an injective retraction of the map $\varprojlim^1 L_{2q+1}(\pi_1 X_n)$ Ker σ, hence the latter is an isomorphism.

12. Globalization. If $L_{2q+1}(X)$ denotes the formal obstruction group for surgering maps to proper homotopy equivalences then we have an exact sequence

$$\varprojlim_n L_{2q+1}(\pi_1 X_n) \longrightarrow L_{2q+1}(\pi_1 X) \longrightarrow L_{2q+1}(X) \longrightarrow L_{2q+1}(\varepsilon X) \longrightarrow 0$$

where $L_{2q+1}(\pi_1 X) \longrightarrow L_{2q+1}(X)$ is the usual realization map, and the composition of the first two maps is 0 by the "alternated sequence" trick. Moreover, if one takes care of $X_0 = X$ in constructing the sequence of Corollary 11, then one gets the sequence

$$\Pi_{2q+1} \xrightarrow{1-S} L_{2q+1}(\pi_1 X) \oplus \Pi_{2q+1} \longrightarrow L_{2q+1}(X) \longrightarrow \Pi_{2q} \xrightarrow{1-S} L_{2q}(\pi_1 X) \oplus \Pi_{2q}$$

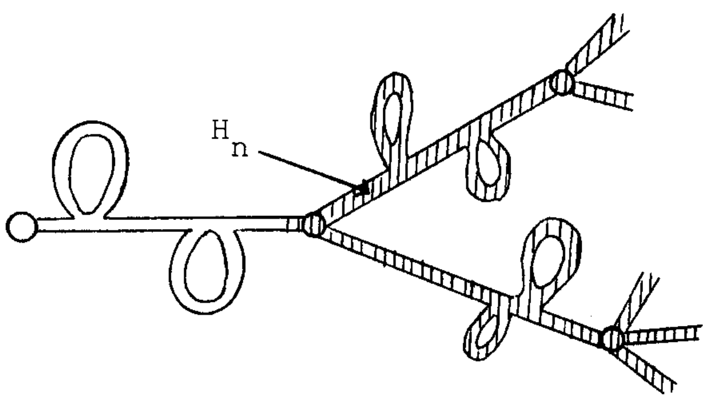
where $\Pi_* = \prod_{n \geqslant 1} L_*(\pi_1 X_n)$ and

$$(1-S)(a_1, a_2, a_3, \ldots) = (-a_1^{\#}, a_1 - a_2^{\#}, a_2 - a_3^{\#}, \ldots) .$$

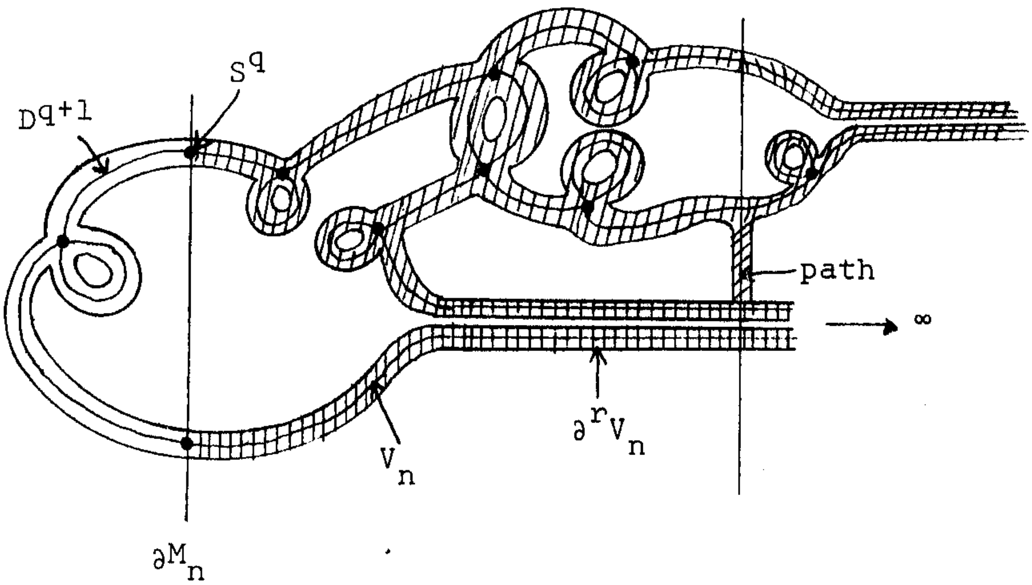
This sequence is exact by virtue of the previous exact sequence and 11.

CHAPTER IV. THE OPEN EVEN DIMENSIONAL CASE

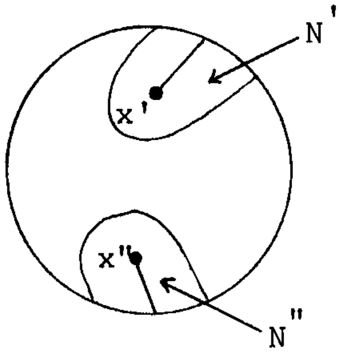
1. If in the data of Chapter III,.1 one lets $m = 2q+2 \geqslant 6$, then one can also do preliminary surgery to make $\partial M_n \xrightarrow{f} \dot{X}_n$ q-connected and $\overline{M_n - M_{n+1}} \xrightarrow{f} \overline{X_n - X_{n+1}}$ (q+1)-connected. Then f is bijective on ends. spaces, and each map $M_n \to X_n$ is (q+1)-connected: $K_k(M_n) = 0$ for $k \leqslant q$. This implies $K_k(M_n, \partial M_n) = 0$ for $k \leqslant q$ (because $K_k(\partial M_n) = 0$ for $k \leqslant q-1$), and $K_k(M_n, \partial M_n \cup M_r) = 0$ for $k \leqslant q$. Hence $K_c^k(M_n, \partial M_n) = 0$ for $k \in q$, and the duality equivalence shows that $\{K_{q+1}(M_n)\}$ is the only inverse system not equivalent to 0. Similarly, $\{K_{q+1}(M_r, \partial M_r)^{\#}\}_n$ is the only direct system not equivalent to 0. Now, the data $M \xrightarrow{f} X$ can be decomposed into two cobordisms with common boundary. By infinite simple homotopy type theory, X is simply homotopy equivalent to a CW-complex of the form $X^0 \underset{\partial H}{\cup} H$, where H is a locally finite m-handlebody of 0 and 1-handles (see [10]):



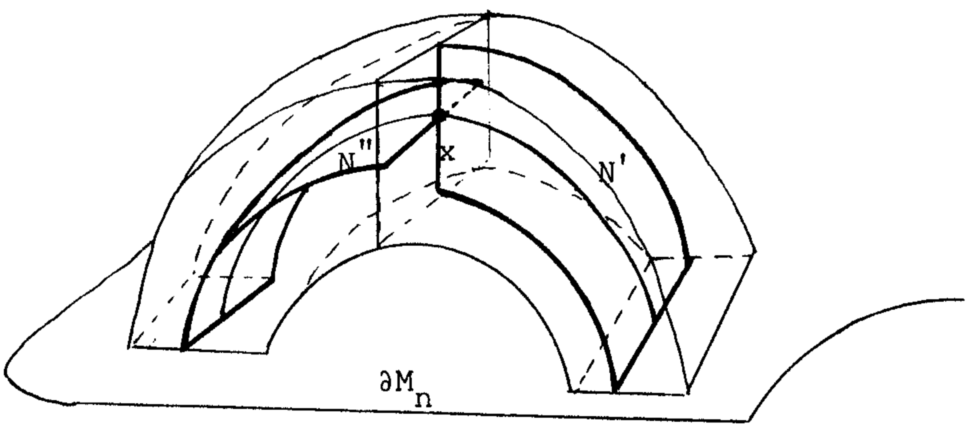
(∂H is collared in X^0). As ngbd of ∞ in H, we can take subhandlebodies H_n, with relative frontier $\overset{\bullet}{H}_n$ a disjoint union of $2q+1$-discs. Denote $X^0 \underset{\partial H}{\cup} H$ by X again, and choose ngbd of ∞ X_n^0 in X^0, with finite subcomplexes $\overset{\blacksquare}{X}{}_n^0$ containing the frontier. Then $X_n^0 \cup H_n$ is a ngbd of ∞ X_n in X, and $\mathring{X}_n \cup \overset{\bullet}{H}_n$ a finite subcomplex $\overset{\blacksquare}{X}_n$ containing the frontier of X_n. By using a collar, we can assume that X_n resp. $\overset{\blacksquare}{X}_n$, meets X^0 and H along X_n^0 and H_n resp. $\overset{\blacksquare}{X}{}_n^0$ and $\overset{\bullet}{H}_n$ and that $\overset{\blacksquare}{X}{}_n^0$ is bicollared in X^0, $\overset{\blacksquare}{X}_n$ bicollared in X. After preliminary surgery on $M \xrightarrow{f} X$ as above, we meet $K_q(\partial M_n)$. Represent each generator by an embedded q-sphere $S^q \subset \partial M_n$ (nulhomotopic in $\overset{\blacksquare}{X}_n$). By the argument of [11, lemma 8.1], these spheres bound immersed (right) $q+1$-discs in $\overline{M_{n-1}-M_n}$, that one can assume to generate $K_{q+1}(\overline{M_{n-1}-M_n}, \partial M_n)$. Similarly, they bound immersed (left) $q+1$-discs in $\overline{M_n-M_{n+1}}$, that one can assume to generate $K_{q+1}(\overline{M_n-M_{n+1}}, \partial M_n)$. The immersed left and right discs which coincide along their boundary S^q form an immersion $S^{q+1} \to \overline{M_{n-1}-M_{n+1}}$ that we pipe to ∞,



getting an immersion $\mathbb{R}^{q+1} \to M_{n-1}$. We also connect by paths all the S^q contained in a connected component of ∂M_n. Let V be a regular ngbd of this connected union of images of immersions, and let $M^0 \equiv \overline{M-V}$. The ngbd of ∞ in V are $V_n \equiv V \cap M_n$, with frontier $U_n \equiv V \cap \partial M_n$ (connected union of $S^q \times D^{q+1}$ and those in ∂V are $\partial^r V_n \equiv \partial V \cap M_n$. Observe that the regular ngbd of the left and right q+1-discs is a handlebody on ∂M_n, with only 1 and q+1-handles. One sees that by taking the preimage $x' \underset{\text{disj.}}{\cup} x''$ in D^{q+1} of a self intersection point x, joining each of them to S^q by a path and taking a regular ngbd N', N'' of



each path. In the image in M, a regular ngbd of N' matches with a regular ngbd of N'' around x, to form a 1-handle.



As $\overline{D^{q+1}-N' \cup N''}$ is an embedded disc, its regular ngbd forms a q+1-handle attached to $\partial M_n \cup$ 1-handle. As a result, by using standard geometrical arguments like in [7], one can arrange f to induce proper maps of degree 1 $V \xrightarrow{f} H$, $\partial V \xrightarrow{f} \partial H$, $M^0 \xrightarrow{f} X^0$ (H may be smaller now). We want to apply III. 9 to $M^0 \xrightarrow{f} X^0$, by considering it as a cobordism. But first of all, what is the connectivity of the maps $\partial V \xrightarrow{f} \partial H$, $V \xrightarrow{f} H$ and $M^0 \xrightarrow{f} X^0$? The map $U_n \xrightarrow{f} \overset{\bullet}{H}_n$ is obviously q-connected. The map $\overline{V_n - V_{n+1}} \longrightarrow \overline{H_n - H_{n+1}}$ is only q-connected, but it satisfies at least $K_q(V_n, V_{n+1}) = 0$. The map $\partial U_n \xrightarrow{f} \partial H_n$ is q-connected, because ∂U_n is a union of $S^q \times S^q$, and ∂H_n a union of S^{2q}. The map

$\overline{\partial^r V_n - \partial^r V_{n+1}} \xrightarrow{f} \overline{\partial^r H_n - \partial^r H_{n+1}}$ is q-connected ($\partial^r H_n = \partial H \cap X_n$),

because by general position, the connectivity in this range is the same as for $\overline{V_n - V_{n+1}} \xrightarrow{f} \overline{H_n - H_{n+1}}$. But moreover $K_q(\partial^r V_n, \partial^r V_{n+1}) = 0$ because, on one hand, a transverse q-sphere to $S^q \subset \partial M_n$ can be translated across the left disc and along the pipe to ∞, and one the other hand, the equatorial $S^q \subset \partial M_n$ itself is homotop over the left disc to a "slice" of the pipe, which can be translated to ∞. So the map $\partial V \xrightarrow{f} \partial H$, has the required connectivity (see III. 1). Now as for the connectivity of $M^0 \xrightarrow{f} X^0$, note that $\pi_1(\partial U_n) = \pi_1(U_n) = \{e\}$ and $\pi_1(\overline{\partial^r V_n - \partial^r V_{n+1}}) \cong \pi_1(\overline{V_n - V_{n+1}})$, hence by van Kampen, we have

$\pi_1(\overset{\bullet}{M}{}^0_n) = \pi_1(\partial M_n)$ and $\pi_1(\overline{M^0_n - M^0_{n+1}}) \cong \pi_1(\overline{M_n - M_{n+1}})$. Similarly,

$\pi_1(\overset{\bullet}{X}{}^0_n) \cong \pi_1(\overset{\bullet}{X}_n)$, and $\pi_1(\overline{X^0_n - X^0_{n+1}}) \cong \pi_1(\overline{X_n - X_{n+1}})$. Hence the maps

$\overset{\bullet}{M}{}^0_n \xrightarrow{f} \overset{\blacksquare}{X}{}^0_n$ and $\overline{M^0_n - M^0_{n+1}} \xrightarrow{f} \overline{X^0_n - X^0_{n+1}}$ are π_1-isomorphisms. The exact sequence

$$K_q(U_n) \xrightarrow{\text{surj.}} K_q(\partial M_n) \to K_q(\partial M_n, U_n) \to \underbrace{K_{q-1}U_n)}_{0} \to \cdots$$

shows, together with excision, that $K_k(\overset{\bullet}{M}{}^0_n, \partial U_n) = 0$ for $k \leq q$, hence also $K_k(\overset{\bullet}{M}{}^0_n) = 0$ for $k \leq q-1$. Next, the exact sequence of $(\overline{M_n - M_{n+1}}, \overline{M^0_n - M^0_{n+1}})$ shows similarly that $K_k(\overline{M^0_n - M^0_{n+1}}) = 0$ but only for $k \leq q-1$, while for $k = q$ we get

$$K_{q+1}(\overline{M_n - M_{n+1}}) \to K_{q+1}(\overline{V_n - V_{n+1}}, \overline{\partial^r V_n - \partial^r V_{n+1}}) \xrightarrow{\partial} K_q(\overline{M^0_n - M^0_{n+1}}) \longrightarrow 0$$

where the first map is non-trivial in general, for intersection reason. We can do surgery in the interior of $\overline{M^0_n - M^0_{n+1}}$ to kill $K_q(\overline{M^0_n - M^0_{n+1}})$ without altering anything on $\overset{\bullet}{M}{}^0_n$.

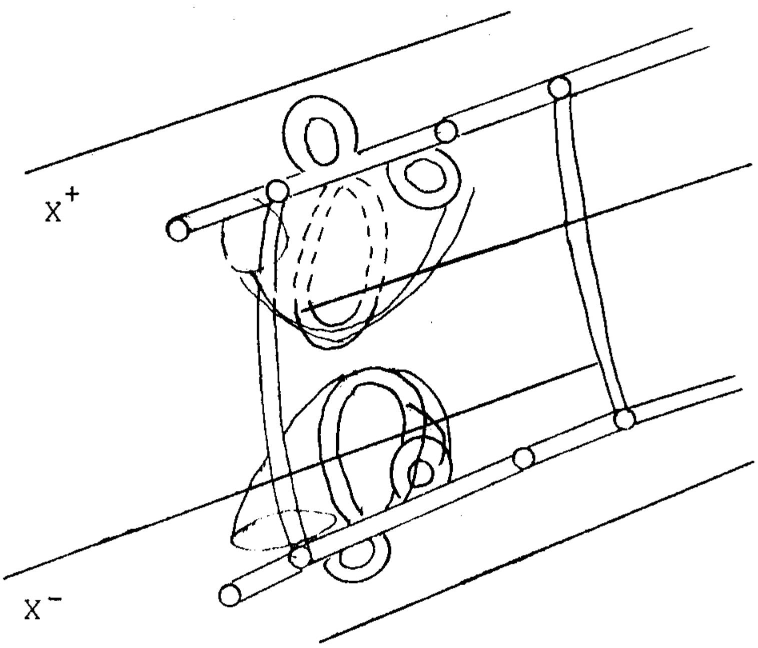
2. <u>Proposition</u>. If $M^0 \underset{U}{\cup} V \xrightarrow{f} X^0 \underset{\partial H}{\cup} H$ is a Mayer-Vietoris decomposition of $M \xrightarrow{f} X$ as above, and if $M^0 \xrightarrow{f} X$ is made q-connected, then for some convenient $\overset{\blacksquare}{X}{}^0_n$, $K_q(\overset{\bullet}{M}{}^0_n)^{\#}$ is a projective Lagrangian plane in $K_q(\partial U_n)^{\#}$.

<u>Proof</u>. First note that, as ∂V and ∂H are both manifolds, the canonical equivalences $\psi\colon K_q(\partial^r V_n) \to K_c^{q+1}(\partial^r V_n, \partial U_n)$ and $\bar{\psi}\colon K_q(\partial^r V_n, \partial U_n) \to K_c^{q+1}(\partial^r V_n)$ are actually isomorphisms. The induced quadratic module $K_q(\partial U_n)$ is clearly free hyperbolic, because of the exact sequence

$$0 \to K_{q+1}(U_n, \partial U_n) \to K_q(\partial U_n) \to K_q(U_n) \to 0$$

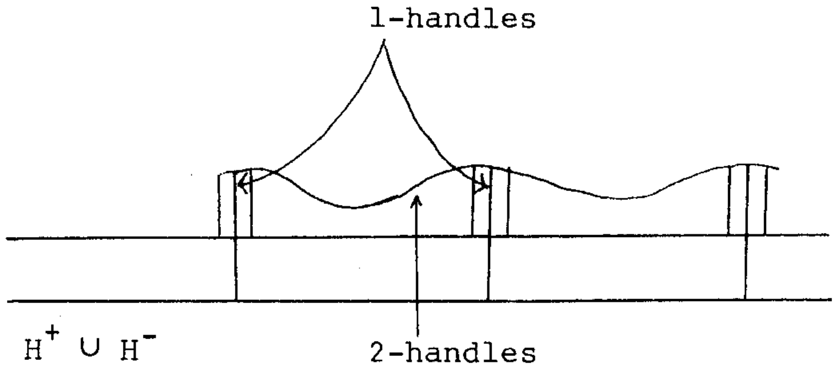
and the duality isomorphisms between the extreme terms. We apply III. 9 to modify X_n^0, $\overset{\blacksquare}{X}{}_n^0$ rel. ∂H, and get a split exact sequence $0 \to E \to K_q(\partial U_n)^{\#} \to K_q(\overset{\blacksquare}{M}{}_n^0) \to 0$ exhibiting $K_q(\overset{\blacksquare}{M}{}_n^0)^{\#}$ as a Lagrangian plane in a standard hyperbolic module (E is the image of $K_{q+1}(\overset{\blacksquare}{M}{}_n^0, \partial U_n)^{\#}$).

3. <u>Cobordism invariance</u>. Let $F: W^{2q+3} \to Y$ be a cobordism between $f^+: M^+ \to X^+$ and $f^-: M^- \to X^-$ (Y has a 2q+3-fundamental class mod $X^+ \cup X^-$ at ∞, and the inclusions $X^{\pm} \subset Y$ are simple homotopy equivalences). <u>Claim</u>: the Mayer-Vietoris decompositions of $f^{\pm}$ (see §1) extends to F. One can assume that $X^{\pm}$ is already decomposed into $X^{0\pm} \underset{\partial H^{\pm}}{\cup} H^{\pm}$. By infinite simple homotopy type theory, Y is simply homotopy equivalent to rel $X^+ \cup X^-$ to a CW-complex of the form $Y^0 \underset{\partial H}{\cup} H$, where H is a locally finite 2q+3-handlebody on $H^+ \cup H^-$ composed of 1 and 2-handles.

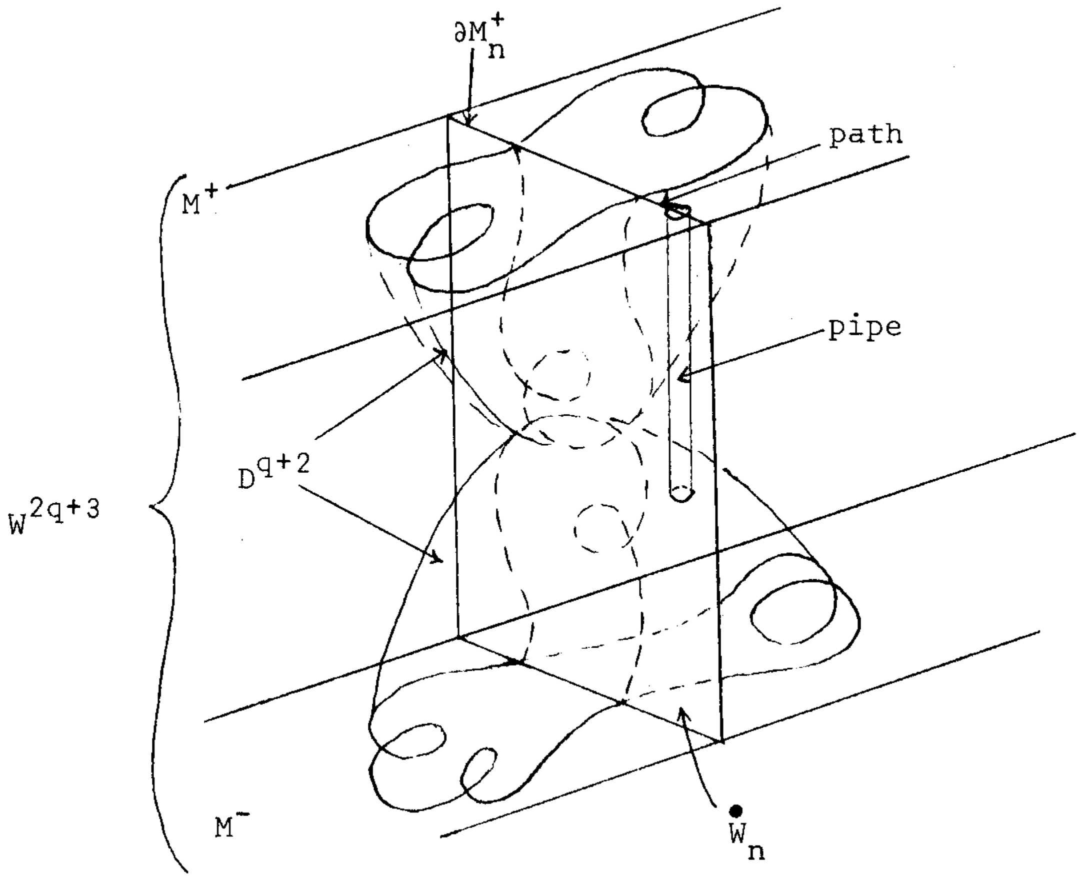


Moreoever, H is a cobordism on $H^+ \cup H^-$ resulting by surgery on (trivial) 0-spheres in $\dot{H}_n^+ \cup \dot{H}_n^-$ and then by surgery on 1-spheres in $\overline{H_n^+ - H_{n+1}^+} \cup \overline{H_n^- - H_{n+1}^-}$ (see Chapter II.). Let

$\partial^r H \equiv \overline{\partial H - H^+ \cup H^-}$.

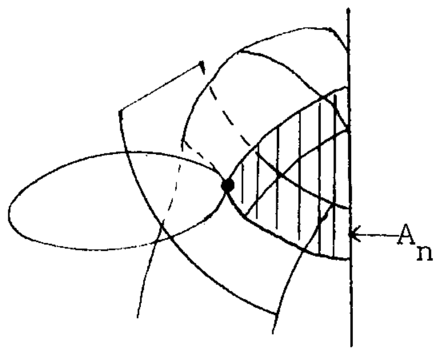


Denote by H_n the cobordism so obtained on $H^+ \cup H_n^-$, which is a ngbd of ∞, with frontier $\dot{H}_n$ = cobordism obtained on $\dot{H}_n^+ \cup \dot{H}_n^-$. Choose subcomplexes Y_n^0 in Y^0 (ngbd of ∞) and finite subcomplexes $\overset{\blacksquare}{Y}{}_n^0$ containing the frontier of Y_n^0 . By using a collar along $\partial^r H$, we can assume that the subcomplexes $Y_n \equiv Y_n^0 \cup H_n$, $\overset{\blacksquare}{Y}_n \equiv \overset{\blacksquare}{Y}{}_n^0 \cup \dot{H}_n$ meet H actually along H_n and $\dot{H}_n$. Moreoever, that $\overset{\blacksquare}{Y}{}_n^0$ is bicollared in Y^0 (hence also $\overset{\blacksquare}{Y}_n$ and Y). At this stage, we do all the necessary preliminary surgeries, first on $f^\pm : M^\pm \to X^\pm$, then on $F: W \to Y$ rel $M^+ \cup M^-$. In particular, one can kill $K_{q+1}(\overline{W_n - W_{n+1}}, \overline{M_n^\pm - M_{n+1}^\pm})$ by representing each generator by an embedded $q+1$-sphere inside $\overline{W_n - W_{n+1}}$ piped to $\overline{M_n^{0\pm} - M_{n+1}^{0\pm}}$ and subtracting them. This preserves $K_q(\overline{M_n^{0\pm} - M_{n+1}^{0\pm}}) = 0$, and $V^\pm$ is unaltered. Also, as in III. 1, we can kill $K_{q+1}(W_n, W_{n+1})$ rel $M_n^+ \cup M_n^-$. Then we decompose $f^\pm$ into $M^{0\pm} \cup_{U^\pm} V^\pm \longrightarrow X^{0\pm} \cup_{\partial H^\pm} H^\pm$. To extend this, consider the q-spheres $S^q \subset \partial M_n$. They bound immersed q+1-discs in $\dot{W}_n$, which bound, together with the left and right q+1-discs in $M^{\pm}$, immersed q+2-discs in $\overline{W_n - W_{n+1}}$, because $K_{q+1}(\overline{W_n - W_{n+1}}, \overline{M_n^{\pm} - M_{n+1}^{\pm}}) = 0$ and $K_{q+1}(W_n, W_{n+1}) = 0$.

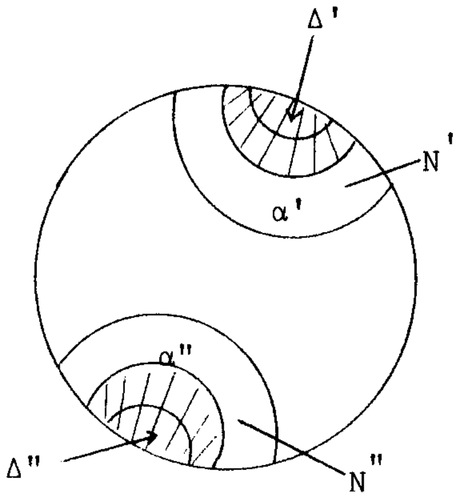


Moreoever, one can assume that the q+1-discs in $\dot{W}_n$ generate $K_{q+1}(\dot{W}_n, \partial M_n^+ \cup \partial M_n^-)$ (see lemma 8.1 of [11]. Next we pipe the lower q+2-discs to M^+ (see figure), connect the S^q's contained in $\partial M_n^{\pm}$ and take a regular ngbd V of this connected union of immersions $D^{q+2} \to W$. Let $V_n = V \cap W_n$, $U_n = V \cap \dot{W}_n$, $\partial^r V = \overline{\partial V - \partial V^+ \cup \partial V^-}$ and $\partial^r U_n = \partial^r V \cap \dot{W}_n = \overline{\partial U_n - \partial U_n^+ \cup \partial U_n^-}$.

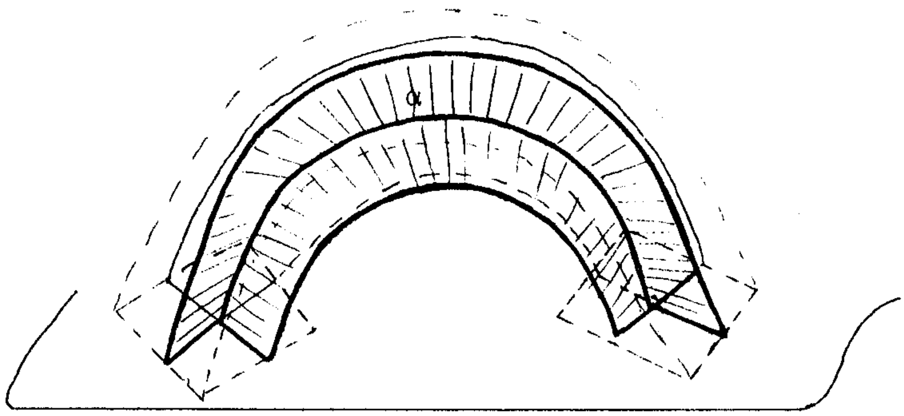
Now, $\overline{V_n - V_{n+1}}$ is a handlebody on $A_n \equiv U_n \underset{\pm}{\cup} \overline{V_n^{\pm} - V_{n+1}^{\pm}} \cup U_{n+1}$ formed by 1, 2 and $q+2$-handles as follows. The self intersections of D^{q+2} are arcs with both ends in A_n and circles. One can exchange the circles into arcs by joining a point of the circle to A_n by one path in each of the two branches crossing through the point, getting an arc with ends in A_n, and then isotoping everything along a 2-disc bounded by the arc mod A_n



The preimage of an arc α in D^{q+2} is the disjoint union of two arcs α', α'' in D^{q+2}, with both ends in ∂D^{q+2}.



Let N', N'' be regular ngbd of α', α'' respectively. Then in W, a regular ngbd of N' coincides with a regular ngbd of N'' to form a 1-handle with core α



Next, α', α'' bound 2-discs Δ', Δ'' mod ∂D^{q+2}, the image of Δ' being in one of the two branches through α and the image of Δ'', in the other. Let A', A'' be regular ngbd of Δ', Δ'' in D^{q+2}, embedded in W. Then a regular ngbd of A' and a regular ngbd of A'' in W matches along the 1-handle with core α, and form a 2-handle attached to $V^+ \cup \dot{W}_n \cup V^-$. Observe that $A' \cup N'$ is a q+2-ball attached to ∂D^{q+2} along a hemisphere, and similarly for $A'' \cup N''$. Hence $\overline{D^{q+2} - A' \cup A'' \cup N' \cup N''}$ is a q+2-disc, which embeds in W, forming the core of a q+2-handle of V. This handlebody structure of V allows us to arrange F, rel. $M^+ \cup M^-$ so that it induces proper maps of degree 1 $V \to H$, $\partial^r V \to \partial^r H$, $W^0 \to Y^0$, where $W^0 \equiv \overline{W-V}$ (H may be smaller). This is the required Mayer-Vietoris decomposition of F. <u>Claim</u>: The operations III. 9 which provide the Lagrangian planes $K_q(\dot{M}_n^{0\pm})$ extend to Y^0, rel. V. The problem is to see if the square

$$\begin{array}{ccc} K_c^{q+2}(W_n^0, \partial W_n^0) & \longrightarrow & K_c^{q+2}(W_n^0, M_n^{0+} \cup M_n^{0-}) \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \bar{\psi}} \\ K_{q+1}(W_n^0) & \longrightarrow & K_{q+1}(W_n^0, \overset{\bullet}{W}{}_n^0) \end{array}$$

survives. For the first operation, which enlarge $X_n^{0\pm}$ and $\overset{\blacksquare}{X}{}_n^{0\pm}$ by a piece of $M^{0\pm}$, it suffices to enlarge $Y^0, \overset{\blacksquare}{Y}{}^0$ by a corresponding piece of W^0. We still get a square

$$\begin{array}{ccc} K_c^{q+2}(W_n^0, \partial W_n^0)' & \longrightarrow & K_c^{q+2}(W_n^0, M_n^{0+} \cup M_n^{0-})' \\ \uparrow{\scriptstyle \psi'} & & \uparrow{\scriptstyle \bar{\psi}'} \\ K_{q+1}(W_n^0)' & \longrightarrow & K_{q+1}(W_n^0, \overset{\blacksquare}{W}{}_n^0)' \end{array}$$

by taking the old one with extended coefficients. The second operation kills the kernel of $\bar{\psi}'^{\pm}: K_{q+1}(M_n^{0\pm}, \overset{\blacksquare}{M}{}_n^{0\pm})' \to K_c^{q+1}(M_n^{0\pm}, \partial^r V_n^{\pm})'$ by adding cells e^{q+2} to $\overset{\blacksquare}{X}{}_n^{0\pm} \cup \overline{M_n^{0\pm} - M_r^{0\pm}}$ inside $X_n^{0\pm}$. If we enlarge $\overset{\blacksquare}{Y}$ correspondingly (with e^{q+2}) then $\bar{\psi}'$ passes to the quotient, because of the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\pm} K_c^{q+1}(M_n^{0\pm}, \partial^r V_n^{\pm})' \to \bigoplus_{\pm} K_c^{q+1}(M_n^{0\pm})' & \to & K_c^{q+2}(W_n^0, M_n^{0+} \cup M_n^{0-}) \\ \uparrow{\scriptstyle \bar{\psi}'^{\pm}} & & \uparrow{\scriptstyle \bar{\psi}} \\ \bigoplus_{\pm} K_{q+1}(M_n^{0\pm}, \overset{\blacksquare}{M}{}_n^{0\pm})' & \longrightarrow & K_{q+1}(W_n^0, \overset{\blacksquare}{W}{}_n^0) \end{array} .$$

Then the diagram

$$\begin{array}{ccccccc}
0 \to \oplus K_c^{q+1}(M_n^{0\pm})'' & \longrightarrow & K_c^{q+2}(W_n^0, M_n^{0+} \cup M_n^{0-})'' & \longrightarrow & K_c^{q+2}(W_n^0)'' & & \\
\uparrow \bar{\psi}''^{\pm} & & \uparrow \bar{\psi}'' & & \uparrow (\text{dotted}) & & \\
\underset{\pm}{\oplus} K_{q+1}(M_n^{0\pm}, \overset{\blacksquare}{M}_n^{0\pm}) & \to & K_{q+1}(W_n^0, \overset{\blacksquare}{W}_n^0)'' & \longrightarrow & K_{q+1}(W_n^0, \partial W_n^0)'' & \longrightarrow & 0
\end{array}$$

provides an induced dotted map, which by duality gives a ψ''. For the last operation, which enlarges both $X_n^{0\pm}$ and $\overset{\blacksquare}{X}_n^{0\pm}$ by the same cells e^{q+2} representing generators of the kernel of $\psi'': K_{q+1}(M_n^{0\pm})'' \to K_c^{q+1}(M_n^{0\pm}, \overset{\blacksquare}{M}_n^{0\pm})''$, it suffices to enlarge Y_n $\overset{\blacksquare}{Y}_n$ correspondingly. The verification that the above square survives runs as above. Now, we can do the operations III.5 on $W^0 \overset{F}{\to} Y^0$ rel. ∂W^0. This will provide the diagram

$$\begin{array}{ccccc}
0 \to K^{q+1}(\overset{\blacksquare}{W}_n)^{\#} & \longrightarrow & K_c^{q+2}(W_n, \overset{\blacksquare}{W}_n) & \longrightarrow & K_c^{q+2}(W_n) \\
\uparrow & & \uparrow \approx & & \uparrow \text{inj.} \\
K_{q+1}(\overset{\blacksquare}{W}_n, \overset{\blacksquare}{M}_n^{+} \cup \overset{\blacksquare}{M}_n^{-})^{\#} & \to & K_{q+1}(W_n, M_n^{+} \cup M_n^{-}) & \to & K_{q+1}(W_n, \partial W_n)
\end{array}$$

which implies that the image t of the left bottom map is dual to $K_{q+1}(\overset{\blacksquare}{W}_n)$. The dual diagram gives a reciprocal duality $K_{q+1}(\overset{\blacksquare}{W}_n)^{\#} \cong t^*$. <u>Claim</u>: Via the Lagrangian transformation associated to the canonical maps $\phi: K_{q+1}(\overset{\blacksquare}{W}_n)^{\#} \to t$ and $\gamma: \underset{\pm}{\oplus} K_q(U_n^{\pm})^{\#} \overset{\iota^*}{\to} \underset{\pm}{\oplus} K_{q+1}(\overset{\blacksquare}{W}_n, \overset{\blacksquare}{M}_n^{\pm}) \to t$, where ι is $\underset{\pm}{\oplus} K_{q+1}(\overset{\blacksquare}{W}_n, \overset{\blacksquare}{M}_n^{\pm}) \to$ $\to K_{q+1}(\overset{\blacksquare}{W}_n, \overset{\blacksquare}{W}_n^0 \cup \overset{\blacksquare}{M}_n^{+} \cup \overset{\blacksquare}{M}_n^{-}) \overset{\text{exc}}{=} K_{q+1}(U_n, \partial U_n)$ the Lagrangian plane

$K_q(\partial^r U_n)^\#$ in $(\langle K_q(\partial U_n^+)\rangle \oplus \langle K_q(\partial U^-)\rangle ')^\#$ is trivial. The dual γ^* is such that the composition

$$\underset{\pm}{\oplus} K_{q+1}(\overset{\blacksquare}{M}{}_n^{\pm})^\# \longrightarrow K_{q+1}(\overset{\blacksquare}{W}_n)^\# \xrightarrow{\gamma^*} \underset{\pm}{\oplus} K_{q+1}(U_n^{\pm}, \partial U_n^{\pm})^\#$$

is, via excision, the first map of the exact sequence $K_{q+1}(\overset{\blacksquare}{M}{}_n^{\pm})^\# \to K_{q+1}(\overset{\blacksquare}{M}{}_n^{\pm}, \overset{\blacksquare}{M}{}_n^{0\pm})^\# \to K_q(\overset{\blacksquare}{M}{}_n^{0\pm})^\# \to K_q(\overset{\blacksquare}{M}{}_n^{\pm})$. Note that ϕ vanishes on the image of $\underset{\pm}{\oplus} K_{q+1}(\overset{\blacksquare}{M}{}_n^{\pm})^\# \to K_{q+1}(\overset{\blacksquare}{W}_n)$. The composition $K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^\# \longrightarrow \underset{\pm}{\oplus} K_q(\partial U_n^{\pm})^\# \longrightarrow \underset{\pm}{\oplus} K_q(U_n^{\pm})^\# \overset{\gamma}{\to} t$ vanishes because of the exact sequence

$$K_{q+2}(\overset{\blacksquare}{W}{}_n^0, \partial)^\# \oplus K_{q+2}(U_n, \partial)^\# \xrightarrow{\text{surj.}} K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^\# \to K_{q+1}(\overset{\blacksquare}{W}_n, \overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^\#.$$

We consider the Lagrangian plane $K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^\# \oplus t$ in $(\langle K_q(\partial U_n^+)\rangle \oplus \langle K_q(\partial U_n^-)\rangle ')^\# \oplus \langle t \oplus t^*\rangle$. Let us parametrize

$\underset{\pm}{\oplus} K_q(U_n^{\pm})^\#$ by x, $\underset{\pm}{\oplus} K_{q+1}(U_n^{\pm}, \partial U_n^{\pm})^\#$ by p, t by t, and t^*

by h. Our Lagrangian plane becomes $\{(x(u), p(u), t, 0)\}$, where u describes $K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^\#$. It projects along

$\ell_1 \equiv \{(x, \pm\gamma^* t, t, -\gamma x - \phi t)\}$ to a plane which projects along $\{(x,0,0,0)\}$ isomorphically to $\ell_0 \equiv \{(0,p,0,h)\}$ by diagram chasing as follows. If $u = 0$ and t comes form $K_{q+1}(\overset{\blacksquare}{M}{}_n^{\pm})^\#$, then $\phi t = 0$ and we hit $(0, \gamma^*(K_{q+1}\overset{\blacksquare}{M}{}_n^{\pm})^\#, 0, 0)$. By the latter exact sequence, the obstruction to hit all values of $(0,p,0,0)$

is then the kernel of $\oplus_{\pm} K_q(\overset{\blacksquare}{M}{}_n^{0\pm})^{\#} \to \oplus_{\pm} K_q(\overset{\blacksquare}{M}{}_n^{\pm})^{\#}$. But we shall see that $K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^{\#}$ projects onto $\oplus_{\pm} K_q(\overset{\blacksquare}{M}{}_n^{0\pm})^{\#}$ hence we can hit all $(0,p,0,0)$. Next, a section of $\oplus_{\pm} K_q(U_n^{\pm})^{\#} \to \oplus_{\pm} K_q(\overset{\blacksquare}{M}{}_n^{\pm})$ provides a map $K_{q+1}(\overset{\blacksquare}{W}_n, \overset{\blacksquare}{M}{}_n^+ \cup \overset{\blacksquare}{M}{}_n^-)^{\#} \overset{\rho}{\to} \oplus_{\pm} K_q(U_n^{\pm})^{\#}$ such that $h-\gamma(\rho h)$ is of the form ϕt. So we hit all $(\rho h, p, 0, h)$, i.e. the graph of $(p,0,h) \mapsto \rho h$. This projects along $\{(x,0,0,0\}$ ismorphically to ℓ_0. <u>Claim</u>: In $\langle K_q(\partial U_n^+)^{\#}\rangle \oplus \langle K_q(\partial U_n^-)^{\#}\rangle'$ the Lagrangian plane $K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#}$ projects along $K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^{\#}$ to 0, while the Lagrangian plane $K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-}, \partial U_n^-)^{\#}$ projects injectively onto a direct summand of $K_q(\partial^r U_n)^{\#}$. First, the composition

$K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^{\#} \to K_q(\partial U_n^+ \cup \partial U_n^-)^{\#} \to K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#}$ is surjective, because in the exact sequence

$$K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^{\#} \to K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#} \to K_q(\overset{\blacksquare}{M}{}_n^{0+} \cup \partial^r U_n, \partial U_n^-)^{\#}$$

the right term vanishes by Mayer-Vietoris argument:

$$K_q(\partial U_n^+)^{\#} \longrightarrow K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#} \oplus K_q(\partial^r U_n, \partial U_n^-)^{\#} \longrightarrow K_q(\overset{\blacksquare}{M}{}_n^{0+} \cup \partial^r U_n, \partial U_n^-)^{\#} \longrightarrow 0$$

surj.

surj.

Hence a section of $K_q(\partial U_n^+ \cup \partial U_n^-)^{\#} \to K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#}$ can be obtained by using a section of $K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-) \to K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#}$. Then $K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#}$ projects to 0 along $K_{q+1}(\partial^r U_n, \partial U_n^+ \cup \partial U_n^-)^{\#}$. Note that the same argument would apply to $K_q(\overset{\blacksquare}{M}{}_n^{0-})^{\#}$. Next, by replacing $K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-}, \partial U_n^-)^{\#}$ by $K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-} \cup \partial^r U_n, \partial^r U_n)^{\#}$, the exact sequence $K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-} \cup \partial^r U_n)^{\#} \to K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-} \cup \partial^r U_n, \partial^r U_n)^{\#} \to K_q(\partial^r U_n)^{\#}$

starts with 0 by the Mayer-Vietoris argument

$$\underbrace{K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-})^{\#}\oplus K_{q+1}(\partial^r U_n)^{\#}}_{0}\to K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-}\cup\partial^r U_n)^{\#}\to K_q(\partial U_n^-)^{\#}\to K_q(\overset{\blacksquare}{M}{}_n^{0-})^{\#}\oplus K_q(\partial^r U_n)^{\#}$$

$$K_q(\partial U_n^-)^{\#}\xrightarrow{\text{inj}} K_q(\partial^r U_n)^{\#}$$

where $K_{q+1}(\partial^r U_n,\partial U_n^-)^{\#}\cong K^q(\partial^r U_n,\partial U_n^+)^{\#}=0$ because of the pipes induced in $\overset{\bullet}{W}_n$. As a result, the Lagrangian plane $K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#}\oplus K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-},\partial U_n^-)^{\#}$ in $\langle K_q(\partial U_n^+)^{\#}\rangle\oplus\langle K_q(\partial U_n^-)^{\#}\rangle'$ is trivial (see former claim about $K_q(\partial^r U_n)^{\#}$). As $K_{q+1}(\overset{\blacksquare}{M}{}_n^{0-},\partial U_n^-)=K_q(\overset{\blacksquare}{M}{}_n^{0-})^{*}$, this means that $K_q(\overset{\blacksquare}{M}{}_n^{0+})^{\#}$ and $K_q(\overset{\blacksquare}{M}{}_n^{0-})^{\#}$ are equivalent.

4. <u>Theorem</u>. Let M be an open manifold of dim $2q+2\geq 6$ and $f\colon M\to X$ be a proper normal map of degree 1. Then, to the cobordism class [f] f are associated a sequence $(\ell_n)\in\varprojlim L_{2q+1}(\pi_1 X_n)$, and , if all ℓ_n vanish, an element $(\sigma_n)\in\varprojlim{}^1 L_{2q+2}(\pi_1 X_n)$ such that [f] contains a proper homotopy equivalence at ∞ iff all $\ell_n=0$ and $(\sigma_n)=0$.

<u>Proof</u>. With the notations of Proposition 2, define ℓ_n by the Lagrangian plane $K_q(\overset{\blacksquare}{M}{}_n^{0})^{\#}$ in $K_q(\partial U_n)$, considering the latter as a standard free hyperbolic form by the exact sequence

$$0\to K_{q+1}(U_n,\partial U_n)\to K_q(\partial U_n)\to K_q(U_n)\to 0.$$

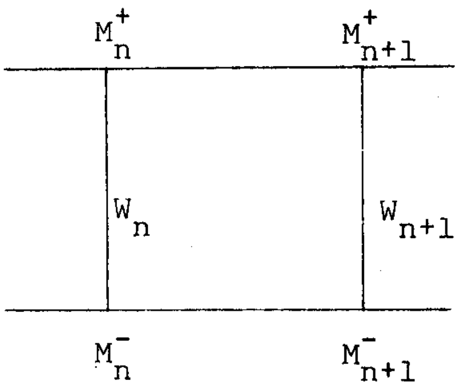
A "canonical" equivalence between $\ell_{n+1}^{\#}$ and ℓ_n is obtained similarly as in §3 above, as follows in two steps: first, the Lagrangian plane $K_q(\overline{\partial^r V_n-\partial^r V_{n+1}})^{\#}$ in $\langle K_q(\partial U_n)^{\#}\rangle\oplus\langle K_q(\partial U_{n+1})^{\#}\rangle'$

is trivialized by a "canonical" Lagrangian transformation, secondly the Lagrangian plane $(K_q(\dot{M}^0_n) \oplus K_q(\dot{M}^0_{n+1})^*)^\#$ in the above hyperbolic module projects along $K_q(\overline{\partial^r V_n - \partial^r V_{n+1}})^* =$

$= K_{q+1}(\overline{\partial^r V_n - \partial^r V_{n+1}}, \partial U_n \cup \partial U_{n+1})^\#$ onto a direct summand of $K_q(\overline{\partial^r V_n - \partial^r V_{n+1}})^\#$. From the diagram

$$\begin{array}{ccccccc}
0 \to K^{q+1}(\overline{M_n - M_{n+1}}, \dot{M}_n \cup \dot{M}_{n+1})^\# & \to & K_c^{q+1}(M_n, \dot{M}_n) & \cup & K_c^{q+1}(M_{n+1}) & & \\
\uparrow & & \uparrow \approx & & \uparrow \text{inj.} & & \\
K_{q+1}(\overline{M_n - M_{n+1}})^\# & \longrightarrow & K_{q+1}(M_n) & \longrightarrow & K_{q+1}(M_{n+1}, \dot{M}_{n+1}) & \to & 0
\end{array}$$

and its dual, we get a reciprocal duality between $t \equiv K_{q+1}(\overline{M_n - M_{n+1}}, \dot{M}_n \cup \dot{M}_{n+1})^\#$ and the image t^* of $K_{q+1}(\overline{M_n - M_{n+1}})^\#$. Let $\phi : t^* \to t$ be the canonical map, and $\gamma : K_q(U_n)^\# \oplus K_q(U_{n+1})^\# \to t$ be a lifting of the canonical map $K_q(U_n)^\# \oplus K_q(U_{n+1})^\# \to K_q(\dot{M}_n)^\# \oplus K_q(\dot{M}_{n+1})^\#$, which exists because $t \to K_q(\dot{M}_n)^\# \oplus K_q(\dot{M}_{n+1})^\#$ is surjective. Now the whole argument of § 3 goes through, with ∂U_n instead of ∂U_n^+ and ∂U_{n+1} instand of ∂U_n^-. The above Lagrangian transformation is canonical in the sense that ϕ is canonical. By § 3, the element $(\ell_n) \in \varprojlim_n L_{2q+1}(\pi_1 X_n)$ is independent of all choices and invariant by cobordism. Suppose that all $\ell_n = 0$, i.e. for each n there is a Lagrangian transformation α_n of $K_q(\partial U_n)^\#$ which trivilizes $K_q(\dot{M}^0_n)^\#$. By superposition, we get a Lagrangian transformation $\beta_0(\alpha_n \oplus \alpha_{n+1}^{-1})$ of $\langle K_q(\partial U_n)^{\#}\rangle \oplus \langle K_q(\partial U_{n+1})^{\#}\rangle'$, where β is the above canonical transformation. As $K_q(\overset{\bullet}{M}{}^0_n)^{\#} \oplus K_q(\overset{\bullet}{M}{}^0_{n+1})^{\#*}$ projects onto a direct summand of $K_q(\overline{\partial^r V_n - \partial^r V_{n+1}})^{\#}$, the above composite Lagrangian transformation carries the standard Lagrangian plane $K_q(U_n)^{\#} \oplus K_q(U_{n+1})^{\#}$ to a trivial plane. According to [5], this transformation determines a non-singular quadratic module $\sigma_n \in L_{2q+2}(\pi_1 X_n)$. In the case $\alpha_n = \mathrm{id}$ this is nothing but the non-singular part of ϕ, a direct summand of t on which the intersection pairing is non-singular. If one changes the choice of the Lagrangian transformations α_n, then the sequence (σ_n) is altered by a sequence in the image of S (see III.10 for definition). The same is true if one replaces $f: M \to X$ by a cobordant map. Actually, if $F: W^{2q+3} \to Y$ is a cobordism between $f^{\pm}: M^{\pm} \to X^{\pm}$, then as in Chapter III one produces singular quadratic modules $K_{q+1}(\overset{\bullet}{W}_n)^{\#}$ whose non-singular part determines an element $\omega_n \in L_{2q+2}(\pi_1 X_n)$



Then from the exact sequence

$$K_{q+2}(\overline{W_n - W_{n+1}}, \partial)^{\#} \to K_{q+1}(\overset{\bullet}{W}_n)^{\#} \underset{\pm}{\oplus} K_{q+1}(\overline{M_n^{\pm} - M_{n+1}^{\pm}})^{\#} \underset{\pm}{\oplus} K_{q+1}(\overset{\bullet}{W}_{n+1})^{\#} \to K_{q+1}(\overline{W_n - W_{n+1}})^{\#} \to 0$$

one deduces a trivialization of the non-singular part of the middle quadratic module, i.e. $\mathcal{O}_n^+ - \mathcal{O}_n^- = \omega_n - \omega_{n+1}^\#$. In this way we get a well-defined element $(\mathcal{O}_n) \in \varprojlim^1 L_{2q+2}(\pi_1 X_n)$. Note that by [11] one can arrange to get $\alpha_n = id$. Suppose then that $(\mathcal{O}_n) = 0$. This means that the intersection pairing on $K_{q+1}(\overline{M_n - M_{n+1}})$ has the following property:

$K_{q+1}(\overline{M_n - M_{n+1}}) = \ker \phi \oplus H_n$, where the form H_n is hyperbolic. By Mayer-Vietoris argument, the H_r for $r \geqslant n$ do not match up in $K_{q+1}(M_n)$, so one sees that there is a subsystem $Q_n = \bigoplus_{r \geqslant n} H_n \subset K_{q+1}(M_n)$, such that the inclusion is an equivalence and Q_n is a projective hyperbolic form which can be assumed free of countable rank: $Q_n \equiv U_n \oplus (U_n)_c^*$ (the second factor is the dual with compact support). Now, each basis element u can be represented by an embedded sphere $S^{q+1} \subset M_n^{2q+2}$ (because $\langle u,u \rangle = 0$). By piping each S^{q+1} to ∞ and carving out the result (as in Chapter II), one verifies easily that Q_n is killed, and the new inverse system $\{K_{q+1}(M_n)\}$ becomes equivalent to 0. In other words, we have found a cobordism to a proper homotopy equivalence at ∞.

5. <u>Corollary</u> (see III.11). We have an exact sequence

$$0 \to \varprojlim^1 L_{2q+2}(\pi_1 X_n) \to L_{2q+2}(\varepsilon X) \to \varprojlim L_{2q+1}(\pi_1 X_n) \to L_{2q+1}(\pi_1 X).$$

The proof is analog to III. 11. This can also be globalized as in III.12, to form an exact sequence

$$\Pi_{2q+2} \xrightarrow{1-S} L_{2q+2}(\pi_1 X) \oplus \Pi_{2q+2} \to L_{2q+2}(X) \to \Pi_{2q+1} \xrightarrow{1-S} L^p_{2q+1}(\pi_1 X) \oplus \Pi_{2q+1}.$$

Together with III. 12, this provides a long exact sequence.

CHAPTER V. THE ALGEBRA OF INVERSE AND DIRECT SYSTEMS

1. An inverse system of groups $\{G_n\}$ is a sequence of homomorphisms $G_1 \longleftarrow G_2 \longleftarrow \dots$ and an inverse system of modules $\{A_n\}$, where A_n is a G_n-module, is a sequence of pseudo-linear maps $A_1 \longleftarrow A_2 \longleftarrow \dots$. A morphism $\{\alpha\} : \{A_n\} \longrightarrow \{A'_n\}$ is a class of compatible pseudo-linear maps $A_{r_n} \longrightarrow A'_{r'_n}$, for some subsequences r_n, r'_n where $\{\alpha\} \sim \{\beta\}$ if the diagram

$$\begin{array}{ccccccc} & & A_{r_n} & \xrightarrow{\alpha} & A'_{r'_n} & & \\ & \nearrow & & & & \searrow & \\ A_{u_n} & & & & & & A'_{u'_n} \\ & \searrow & & & & \nearrow & \\ & & A_{s_n} & \xrightarrow{\beta} & A'_{s'_n} & & \end{array}$$

commutes for some subsequences u_n, u'_n. Two morphisms $\{A_n\} \longrightarrow \{A'_n\}$, $\{A'_n\} \longrightarrow \{A''_n\}$ may be composed in a well-defined class. In particular, there are defined canonical isomorphisms $\{A_n\} \longrightarrow \{A^{\#}_n\}$ and $\{A^{\#}_{n+1}\} \longrightarrow \{A_n\}$. By reversing all the arrows, we get the notion of a direct system. The following progressive assertions are easy to prove (for both direct and inverse systems).

2. A system $\{A_n\}$ is equivalent to O iff, for some subsequence r_n, the maps $A_{r_{n+1}} \longrightarrow A_{r_n}$ are O.

3. Let $\alpha : \{A_n\} \longrightarrow \{B_n\}$ be an equivalence of systems given by $\alpha_n : A_n \longrightarrow B_n$. Then the systems $\{\ker\alpha_n\}$ and $\{\operatorname{coker}\alpha_n\}$ are equivalent to $\{O\}$.

4. Let $O \longrightarrow \{A_n\} \xrightarrow{\alpha} \{B_n\} \xrightarrow{\beta} \{C_n\} \longrightarrow O$ be an exact sequence of systems, i.e. for some subsequence $r_n \leq s_n \leq t_n < r_{n+1}$ the sequences $O \longrightarrow A^{\#}_{r_n} \xrightarrow{\alpha_n} B^{\#}_{s_n} \xrightarrow{\beta_n} C_{t_n} \longrightarrow O$ are exact. Then α,

res. β, is an equivalence iff $\{C_n\}$, resp. $\{A_n\}$, is equivalent to $\{0\}$.

5. A morphism of systems $\alpha:\{A_n\} \to \{B_n\}$ is an equivalence iff the systems $\{\ker \alpha_n\}$ and $\{\mathrm{coker}\ \alpha_n\}$ are equivalent to $\{0\}$.

6. Let
$$\begin{array}{ccc} \{A_n\} & \xrightarrow{\alpha} & \{B_n\} \\ \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi'} \\ \{A_n'\} & \xrightarrow{\alpha'} & \{B_n'\} \end{array}$$

be a commutative square of systems, i.e. for some subsequences r_n, s_n, t_n, u_n, the squares

$$\begin{array}{ccc} A^{\#}_{r_n} & \xrightarrow{\alpha_n} & B^{\#}_{s_n} \\ \uparrow{\scriptstyle \phi_n} & & \uparrow{\scriptstyle \phi_n'} \\ A'^{\#}_{t_n} & \xrightarrow{\alpha_n'} & B'^{\#}_{u_n} \end{array}$$

are commutative. If ϕ and ϕ' are equivalences, then so are the induced morphisms $\{\ker \alpha_n'\} \to \{\ker \alpha_n\}$ and $\{\mathrm{coker}\ \alpha_n'\} \to \{\mathrm{coker}\ \alpha_n\}$.

7. Let
$$\begin{array}{ccccccccc} \{0\} & \to & \{A_n\} & \to & \{B_n\} & \to & \{C_n\} & \to & \{0\} \\ & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \\ \{0\} & \to & \{A_n'\} & \to & \{B_n'\} & \to & \{C_n'\} & \to & \{0\} \end{array}$$

be a commutative exact ladder of systems. Then, if two of the morphisms α, β, γ are equivalences so is the third.

8. The five lemma holds for systems.

9. <u>Proposition</u>. Let $\{C(n)\}$ be a system of chain complexes. Assume that each $C(n)$ has the form

$$0 \to C_L(n) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_1(n) \xrightarrow{\partial} C_0(n) \to 0$$

where $L > 0$ is independent of n, each $C_k(n)$ is free of countable (resp. finite) rank, moreover, that the associated homology system $\{H_k(n)\}$ are equivalent to $\{0\}$ for $k < L$. Then $\{H_L(n)\}$ is equivalent by injections $H_L(n) \to P_n$ to a system of countably (resp. finitely) generated projective module P_n.

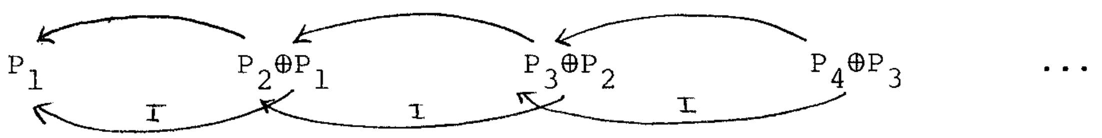
<u>Proof</u>. To fix the idea, suppose the system is inverse. By induction on $r \leqslant L$, we can factorize $C(n) \to C(n-r)$ through a free chain complex $E(n)$ of the above form, such that $H_k E(n) = 0$ for $k < r$. For $r = 0$, take $E(n) \equiv C(n)$. Suppose we are done for $r-1$. By the folding trick (see [6]), $E(n)$ is chain homotopy equivalent to a similar chain complex nul in dimension $< r-1$. Hence $H_{r-1}E(n)$ is countably (resp. finitely) generated. Let (z_i) be a countable (resp. finite) set of $(r-1)$-cycles in $E_{r-1}(n)$ generating $H_{r-1}E(n)$, and F the free module on (z_i). Define a chain complex $\overline{E}(n)$ by

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\quad F \\ \qquad\qquad\qquad\qquad\qquad\quad \oplus \searrow^{\partial_F} \\ 0 \to E_L(n) \to \dots \to E_{r+1}(n) \to E_r(n) \to E_{r-1}(n) \to \dots \to E_0(n) \to 0 \end{array}$$

where $\partial_F(z_i) = z_i$. We can assume that the chain map $C(n-r+1) \longrightarrow C(n-r)$ induces 0 on homology in dimensions $< L$, hence so does the composite map $E(n) \longrightarrow C(n-r+1) \longrightarrow C(n-r)$. This implies that the map $F \xrightarrow{\partial_F} E_{r-1}(n) \longrightarrow C_{r-1}(n-r)$ has its image in $\partial C_r(n-r)$, so can be lifted to $C_r(n-r)$. This provides a factorization $C(n) \longrightarrow E(n) \xrightarrow{\text{incl.}} \bar{E}(n) \longrightarrow C(n-r)$ where $H_k\bar{E}(n) = 0$ for $k < r$. When we reach $r = L$, $E(n)$ has homology only in the top dimension L, hence $H_LE(n)$ is a direct summand P_n of $E_L(n)$ (ibid.) Finally, the injections $C(n) \longrightarrow E(n)$ induce the equivalence $H_L(n) \longrightarrow P_n$.

10. Addendum. There is a system of projective modules P_n, such that the image of $P_{n+1} \longrightarrow P_n$ is a retract (in particular projective), and an equivalence $H_L(n) \longrightarrow P_n$ which is injective for all n.

Proof: We can replace $\{P_n\}$ by the inverse system



which contains $\{P_n\}$ as an equivalent retract. This can also be done at chain level.

11. Addendum. If all $\{H_k(n)\}$ are equivalent to $\{0\}$, then $C(n) \longrightarrow C(n-L-i)$ is chain homotopic to 0.

Proof: As in the proof of Proposition 9, we can factorize this map through a projective acyclic chain complex.

12. Corollary. Let $\alpha: \{A(n)\} \to \{B(n)\}$ be a map of free chain systems (each $A(n)$, $B(n)$ is free and of finite dimension $\leq L$ independent of n) inducing an equivalence on the associated homology systems. Then so does the dual map $\alpha^*: \{B^*(n)\} \to \{A^*(n)\}$.

Proof. By applying the above addendum to the mapping cyclinders $M(n)$ of $A(n) \to B(n)$, we see that $\{M(n)\}$ is equivalent to a system of free acyclic chain complexes. Hence so is the dual system $\{M^*(n)\}$.

REFERENCES

[1] W.Browder Surgery on simply-connected manifolds
Springer (1972)

[2] F.T.Farrell and J.B.Wagoner
Algebraic torsion for infinite simple homotopy types
Comm. Math. Helv. 47, 502 - 513 (1972)

[3] Infinite matrices in algebraic K-theory and topology
ibid., 474 - 501 (1972)

[4] J.Milnor On axiomatic homology theory
Pac. J. Math. 12, 337 - 341 (1962)

[5] S.P.Novikov The algebraic construction and properties of hermitian analogues of K-theory for rings with involution, from the point of view of the hamiltonian formalism. Some applications to differential topology and the theory of characteristic classes
Izv. Akad. Nauk SSSR ser. mat. 34, I. 253 - 288, II. 478 - 500 (1970)

[6] G.deRham, S.Maumary and M.Kervaire
Torsion et type simple d'homotopie
Springer Lecture Notes 48 (1967)

[7] R.W.Sharpe Surgery on compact manifolds: the bounded even-dimensional case
Ann. of Maths. 98, 187 - 209 (1973)

[8] L.Siebenmann
The obstruction to finding a boundary for an open manifold of dimension greater than five
Princeton Ph.D. thesis (1965)

[9] Infinite simple homotopy types
Indag. Math. 32, 479 - 495 (1970)

[10] L.R.Taylor Surgery on paracompact manifolds
Berkeley Ph.D. thesis (1972)

[11] C.T.C.Wall Surgery on compact manifolds
Academic Press (1970)

[12] S.Maumary Proper surgery groups and Wall-Novikov groups
Springer Lecture Notes 343 (1973)

[13] J.L.Shaneson Wall's surgery obstruction groups for $G \times \mathbb{Z}$
Ann. of Maths. 90, 296 - 334 (1969)

[14] A.A.Ranicki Algebraic L-theory I. Foundations, II. Laurent extensions
Proc. Lond. Math. Soc. (3) 27, 101 - 125, 126 - 158 (1973)

[15] E.K.Pedersen and A.A.Ranicki
Projective surgery theory
Topology 19, 239 - 254 (1980)