

# A Maslov cocycle for unitary groups

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## ABSTRACT

We introduce a 2-cocycle for symplectic and skew-hermitian hyperbolic groups over arbitrary fields and skew-fields, with values in the Witt group of hermitian forms. This cocycle has good functorial properties: it is natural under extension of scalars and stable, and so it can be viewed as a universal 2-dimensional characteristic class for these groups. Over  $\mathbb{R}$  and  $\mathbb{C}$ , it coincides with the first Chern class.

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## Introduction

We introduce a Maslov index and Maslov cocycle for symplectic and hyperbolic unitary groups over arbitrary fields and skew-fields. In the classical work of Lion and Vergne [13], this is done by associating to triples  $(X, Y, Z)$  of Lagrangians in a real symplectic vector space  $M$  a certain integral invariant, the Maslov index. This invariant is used to construct a  $\mathbb{Z}$ -valued cocycle for the symplectic group. The corresponding group extension of the symplectic group is the topological universal covering group of  $\mathbf{Sp}_{2n} \mathbb{R}$ .

In this approach, it is somewhat cumbersome that one has to deal with arbitrary triples of Lagrangians. Our starting point was the idea that the whole construction should also work if one considers only triples of Lagrangians in ‘general position’, that is, triples  $(X, Y, Z)$  in  $M$  that are pairwise opposite,

$$M = X + Y = Y + Z = Z + X.$$

Geometrically, such triples are much easier to classify. Moreover, these triples carry an interesting algebraic structure. To each pair  $(X, Y)$  of opposite Lagrangians one can associate a linear map  $[Y; X]$  that identifies  $X$  with the dual of  $Y$  and the dual of  $X$  with  $Y$ . In this way we obtain a graph, the opposition graph, whose vertices are the Lagrangians and whose edges join opposite Lagrangians. Concatenating the linear maps  $[Y; X]$  along closed paths in

this graph, we arrive at an interesting groupoid  $\mathcal{GM}$ , the projectivity groupoid. A minimal closed path has length 3, and the resulting element in the holonomy group turns out to be a complete geometric invariant for the triple consisting of the three Lagrangians along the path. This makes sense and works not just for symplectic forms, but for arbitrary hyperbolic skew-hermitian forms over fields or skew-fields.

In order to relate this invariant to group cohomology, we need a chain complex. A natural candidate is the flag complex of the opposition graph, whose simplices are the finite complete subgraphs (cliques). If the field is infinite, then this flag complex is contractible and the symplectic (or unitary) group acts on it, and thus its equivariant cohomology is isomorphic to the group cohomology.

The final ingredient is the observation that along a closed path of length 3, the element in the holonomy group determines a nondegenerate hermitian form, which may be viewed as an element in a Witt group. In this way we associate to every triangle in the opposition graph an element in the Witt group of hermitian forms. We verify that this map is indeed an invariant cocycle, which gives us a 2-cocycle for the unitary group.

This cocycle, which we call the Maslov cocycle, has good functorial properties. It is stable under direct sums of hermitian spaces and well behaved under extension of scalars. Furthermore, it coincides in the symplectic setting over fields of characteristic not equal to 2 with the classical Maslov cocycle. Our cocycle, however, exists over arbitrary fields and skew-fields of any characteristic. Furthermore, the cocycle can be reduced to a subgroup of the Witt group, the kernel of the signed discriminant.

The classical Maslov cocycle is important, as it yields a central extension of the symplectic group. The question which extension is defined by our general Maslov cocycle can by and large be reduced to a map in algebraic  $K$ -theory. In the smallest case  $\mathbf{Sp}_2 D = \mathbf{SL}_2 D$  this is due to Barge [1] and Nekovar [19]. Nevertheless, even in the classical situation of a symplectic group  $\mathbf{Sp}_{2n} D$  over a field  $D \neq \mathbb{R}$ , our result appears to be the first complete proof for this. In general, the cocycle is related to certain symbols and depends on algebraic properties of the field. We carry this out in some detail for local fields. For  $\mathbb{R}$  and  $\mathbb{C}$  the Maslov cocycle ‘is’ the first Chern class  $c_1$  and gives the universal covering groups of  $\mathbf{Sp}_{2n} \mathbb{R}$  and  $\mathbf{SU}(n, n)$ . Over nonarchimedean local fields, we obtain a covering of degree at most 2.

A Witt group-valued Maslov cocycle appears already in [13]. Besides this, our paper is influenced from [19, 21] (but see the remarks after Theorem 23). The idea of a ‘partially defined cocycle’ seems to go back to Weil and appears also in a topological context in [16]. The opposition graph is used (in a different way) in [20]. The Maslov index itself has been generalized in several ways [3, 18]. Buildings [10, 27] are not mentioned in this paper, although the motivation for our approach is the opposition relation in spherical buildings. Lurking behind the linear algebra is the projectivity groupoid for spherical buildings, which was first studied systematically by Knarr [8] for spherical buildings of rank 2.

We assume that the reader is familiar with basic homological algebra, as well as hermitian forms and unitary groups. Apart from this, we tried to make the paper self-contained and accessible to nonexperts.

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### 1. Lagrangians and hyperbolic modules

In this section we introduce some standard terminology from the theory of hermitian forms. Everything we need can be found in [6, 9, 23]. We work over a field or division ring  $D$  of

arbitrary characteristic. The modules we consider are finite-dimensional right  $D$ -modules. We assume that  $J$  is an involution of  $D$ , that is, an antiautomorphism whose square is the identity (we allow  $J = \text{id}$ ). The involution extends naturally to an involution of the matrix ring  $D^{n \times n}$ , which we also denote by  $J$ . For  $\varepsilon = \pm 1$  we consider

$$D^\varepsilon = \{a \in D \mid a - a^J \varepsilon = 0\}.$$

### 1.1. Forms

A *form* on a right  $D$ -module  $M$  is a biadditive map  $f : M \times M$  with the property that

$$f(ua, vb) = a^J f(u, v)b$$

for all  $u, v \in M$  and all  $a, b \in D$ . An  $\varepsilon$ -*hermitian form*  $h$  is a form with the additional property that

$$h(u, v) = h(v, u)^J \varepsilon,$$

and  $(M, h)$  is called a *hermitian module*. If  $f$  is any form, then

$$h_f(u, v) = f(u, v) + f(v, u)^J \varepsilon$$

is  $\varepsilon$ -hermitian. The hermitian forms that arise in this way are called *trace  $\varepsilon$ -hermitian* or *even*. If  $\text{char}(D) \neq 2$ , then every  $\varepsilon$ -hermitian form is automatically trace  $\varepsilon$ -hermitian; this is also true in characteristic 2 if  $J$  is an involution of the second kind, that is, if  $J|_{\text{Cen}(D)} \neq \text{id}$ , but may fail otherwise [6, 6.1.2]. Note also that  $h_f(u, u) = 0$  is equivalent to  $f(u, u) \in D^{-\varepsilon}$ .

1.2. The dual  $M^\vee$  of  $M$  (which is a left  $D$ -module) can be made into a right  $D$ -module  $M^J$  by twisting the scalar multiplication with  $J$ , that is, by setting

$$\xi a = [v \mapsto a^J \xi(v)]$$

(where  $a \in D$ ,  $\xi \in M^\vee$  and  $v \in M$ ). Thus forms are just linear maps  $M \rightarrow M^J$ . A form is called *nondegenerate* if the associated linear map is injective (and hence bijective). There is a natural notion of an isomorphism (or isometry) of forms; the automorphism group of a nondegenerate  $\varepsilon$ -hermitian form is the *unitary group*

$$\mathbf{U}(M, h) = \mathbf{U}(M) = \{g \in \mathbf{GL}(V) \mid h(u, v) = h(g(u), g(v)) \text{ for all } u, v \in M\}.$$

### 1.3. Lagrangians

For any subset  $X \subseteq V$  we have the subspace  $X^\perp = \{u \in M \mid h(x, u) = 0 \text{ for all } x \in X\}$ , the *perp*. A subspace that is contained in its own perp is called *totally isotropic* and a subspace that coincides with its perp is called a *Lagrangian*. A nondegenerate hermitian form that admits Lagrangians is called *metabolic*.

### 1.4. The hyperbolic functor

Given a right  $D$ -module  $X$ , there is a natural form  $f$  on  $M = X \oplus X^J$ , given by  $f((x, \xi), (y, \eta)) = \xi(y)$ . The associated trace  $\varepsilon$ -hermitian form

$$h_X((x, \xi), (y, \eta)) = \xi(y) + \eta(x)^J \varepsilon$$

(and every isometric hermitian module) is called *hyperbolic*. Obviously,  $X$  is a Lagrangian, and so hyperbolic modules are metabolic. The converse is true for trace-valued hermitian forms and hence in particular in characteristic not equal to 2 (see [9, I 3.7.3]). The *rank* of a hyperbolic module is the dimension of  $X$  (that is half the dimension of the hyperbolic module). We note that the assignment

$$\text{hyp} : X \mapsto (X \oplus X^J, h_X)$$

is a functor from  $D$ -modules to hermitian modules, and that  $\text{hyp}$  induces an injection  $\mathbf{GL}(X) \rightarrow \mathbf{U}(X \oplus X^J)$ .

### 1.5. Special cases and Lie groups

Every hyperbolic form  $(M, h)$  can be reduced to one of the following three types.

*Symplectic groups:* If  $(J, \varepsilon) = (\text{id}, -1)$ , then  $D$  is necessarily commutative and  $\mathbf{U}(M) = \mathbf{Sp}(M)$  is the *symplectic group*. For  $M = \mathbb{R}^{2n}$  and  $M = \mathbb{C}^{2n}$ , these Lie groups are often denoted by  $\mathbf{Sp}(n, \mathbb{R})$  and  $\mathbf{Sp}(n, \mathbb{C})$ , respectively

*Hyperbolic orthogonal groups:* If  $J = \text{id}$  and  $\varepsilon = 1 \neq -1$ , then  $D$  is commutative and of characteristic different from 2. The group  $\mathbf{U}(M) = \mathbf{O}(M)$  is the hyperbolic *orthogonal group*; for  $\mathbb{R}$  and  $\mathbb{C}$ , these Lie groups are often denoted by  $\mathbf{O}(n, n)$  and  $\mathbf{O}(2n, \mathbb{C})$ . We shall see in Subsection 2.3 below that the Maslov cocycle is uninteresting in this situation.

*Standard hyperbolic unitary groups:* If  $J \neq \text{id}$ , then  $\mathbf{U}(M)$  is the *standard hyperbolic unitary group*. Scaling the hermitian form by a suitable constant and changing the involution, we can assume that  $\varepsilon = -1$  ('Hilbert 90'; see [6, p. 211]). The  $-1$ -hermitian forms are also called *skew-hermitian*. Examples of involutions are the standard conjugation  $z \mapsto \bar{z}$  on  $\mathbb{C}$  and on the real quaternion division algebra  $\mathbb{H}$ . Note that there is also the 'nonstandard' involution  $z^\alpha = -i\bar{z}i$  on  $\mathbb{H}$ . The skew-hyperbolic unitary groups corresponding to  $(\mathbb{C}^n, z \mapsto \bar{z})$ ,  $(\mathbb{H}^n, z \mapsto \bar{z})$  and  $(\mathbb{H}^n, z \mapsto z^\alpha)$  are the Lie groups denoted by  $\mathbf{U}(n, n)$ ,  $\mathbf{SO}^*(4n)$  and  $\mathbf{Sp}(n, n)$  in [7, X, Table V].

## 2. The opposition graph and triples of Lagrangians

In this section we construct an invariant  $\kappa$  that classifies triples of pairwise opposite Lagrangians in a  $-\varepsilon$ -hermitian hyperbolic module up to isometry. The invariant is a nondegenerate  $\varepsilon$ -hermitian form. In particular, we will have to work simultaneously with  $\varepsilon$ - and  $-\varepsilon$ -hermitian forms. We assume throughout that  $M$  is a  $-\varepsilon$ -hermitian hyperbolic module and we let

$$\mathcal{L} = \mathcal{L}(M) = \{X \in M \mid X = X^\perp\}$$

denote its set of Lagrangians.

**DEFINITION 1.** We call two Lagrangians  $X$  and  $Y$  *opposite* if  $X \cap Y = 0$  or, equivalently, if  $M = X + Y$ . If the rank of  $M$  is 1, then Lagrangians are 1-dimensional, and  $X$  is opposite  $Y$  if and only if  $X \neq Y$ .

**LEMMA 2.** *If  $M$  has rank 1, then  $\mathcal{L}$  has  $|D^\varepsilon| + 1$  elements.*

*Proof.* Let  $x$  be a nonzero vector in the 1-dimensional space  $X$  and let  $\xi \in X^J$  be its dual, that is,  $\xi(x) = 1$ . Then  $x$  and  $\xi$  span  $X \oplus X^J \cong M$ . The vector  $v = (xa, \xi)$  spans a Lagrangian if and only if  $\xi(xa) = a \in D^\varepsilon$ . There is precisely one additional Lagrangian, spanned by  $(x, 0)$ .  $\square$

Later it will be important that there are enough Lagrangians. We note that  $D^\varepsilon$  is infinite if  $D$  is an infinite field, unless  $J = \text{id}$  and  $\varepsilon = -1 \neq 1$ . If  $D$  is not commutative, then  $D^\varepsilon$  is always infinite [6, 6.1.3].

**PROPOSITION 3.** *If  $|D^\varepsilon| \geq k$ , then there exists for every finite collection  $X_1, \dots, X_k$  of Lagrangians a Lagrangian  $Y$  opposite to  $X_1, \dots, X_k$ .*

*Proof.* Let  $n$  denote the rank of  $M$ . We proceed by induction on  $k \geq 1$ , modifying the proof in [27, 3.30]. Let  $X_1, \dots, X_k$  be  $k$  Lagrangians. We choose a Lagrangian  $Y$  such that  $\ell = \dim(Y \cap X_1)$  is as small as possible, and (by the induction hypothesis) such that  $Y$  is opposite  $X_2, \dots, X_k$ . We claim that  $\ell = 0$ . Otherwise, we can choose a subspace  $Q \subseteq X_1$  of dimension  $n - 1$ , such that  $X_1 = Q + (Y \cap X_1)$ . Now  $Q^\perp$  can be split as  $Q \perp H$ , with  $H$  hyperbolic of rank 1. The 1-dimensional Lagrangians  $P$  of  $H$  parametrize the Lagrangians of  $M$  containing  $Q$  bijectively via  $P \mapsto Q \oplus P$ . Let  $P_1 = Y \cap H$ . For  $\nu = 2, \dots, k$ , each  $X_k$  determines a unique 1-dimensional Lagrangian  $P_k \subseteq H$  with  $\dim((Q + P_k) \cap X_k) \neq 0$ . By Lemma 2 we may choose a 1-dimensional Lagrangian  $P' \subseteq H$  different from  $P_1, \dots, P_k$ . Then  $Y' = P' \oplus Q$  is a Lagrangian opposite  $X_2, \dots, X_k$  with  $\dim(Y' \cap X_1) = \ell - 1$ , which is a contradiction.  $\square$

In particular, there exists always a Lagrangian  $Y$  opposite a given Lagrangian  $X$ . The map  $y \mapsto h(y, -)|_X$  is an isomorphism  $Y \xrightarrow{\cong} X^J$  and we have thus a unique isomorphism of hyperbolic modules  $X \oplus X^J \xrightarrow{\cong} X \oplus Y = M$  extending the inclusion  $X \hookrightarrow M$ . If  $(X', Y')$  is another such pair, then we can choose a linear isomorphism  $X \cong X'$  and obtain isomorphisms

$$X \oplus Y \xrightarrow{\cong} X \oplus X^J \cong X' \oplus X'^J \xleftarrow{\cong} X' \oplus Y'.$$

Hence we have established the following result (which also follows from Witt's theorem [6, 6.2.12]).

LEMMA 4. *The unitary group  $\mathbf{U}(M)$  acts transitively on ordered pairs of opposite Lagrangians.*

2.1. We now study this  $\mathbf{U}(M)$ -action in more detail. We fix a  $D$ -module  $X$  of dimension  $n$ , with basis  $\mathbf{x}$ . We consider  $Y = X^J$  and we let  $\mathbf{y}$  denote the dual basis. Then  $M = X \oplus Y$  is hyperbolic of rank  $n$ , with basis  $\mathbf{x}, \mathbf{y}$ , and we may work with  $2 \times 2$  block matrices. The hermitian form  $h = h_X$  on  $M$  is represented by the matrix

$$h = \begin{pmatrix} 0 & -\varepsilon \\ 1 & 0 \end{pmatrix}.$$

We find that the  $\mathbf{U}(M)$ -stabilizer  $L$  of the ordered pair  $(X, Y)$  consists of matrices of the form

$$\ell_a = \begin{pmatrix} a^{-J} & 0 \\ 0 & a \end{pmatrix},$$

with  $a \in \mathbf{GL}_n D$  and  $\ell_a \ell_{a'} = \ell_{aa'}$ , while the  $\mathbf{U}(M)$ -stabilizer  $U$  of  $(X, \mathbf{x})$  consists of matrices of the form

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

with  $t - t^J \varepsilon = 0$ , that is,  $t \in D^{n \times n}$  has to be  $\varepsilon$ -hermitian. Note also that  $u_t u_{t'} = u_{t+t'}$ ,  $u_t^{-1} = u_{-t}$ , and that

$$\ell_a u_t \ell_a^{-1} = u_{a^{-J} t a^{-1}}.$$

The  $\mathbf{U}(M)$ -stabilizer  $P$  of  $X$  splits therefore as a semidirect product  $P = LU$ , with *Levi factor*  $L$  and *unipotent radical*  $U \trianglelefteq P$ .

Next, we note that if  $Z$  is another Lagrangian opposite  $X$ , then we have a unique isomorphism  $X \oplus Y \rightarrow X \oplus Z$  fixing the basis  $\mathbf{x}$ . This isomorphism is therefore given by an element of the group  $U$ , and we have the following result.

LEMMA 5. *The group  $U$  acts regularly on the set  $X^{\text{opp}}$  of all Lagrangians opposite  $X$ .*

Let  $u_t \in U$ . Then the Lagrangian  $Z = u_t(Y)$  is opposite  $Y$  if and only if  $M$  is spanned by  $\mathbf{y}, u_t(\mathbf{y})$ . With the matrix notations we established before, we have

$$u(\mathbf{y}_\nu) = \mathbf{y}_\nu + \sum_{\mu} \mathbf{x}_\mu t_{\mu,\nu}.$$

A necessary and sufficient condition for  $Z = u_t(Y)$  being opposite  $Y$  is thus that the matrix  $t$  be invertible.

2.2. We let  $H = \{t \in D^{n \times n} \mid t - t^J \varepsilon = 0\}$  denote the set of all  $\varepsilon$ -hermitian  $n \times n$ -matrices. There is a natural left action  $(a, t) \mapsto a^{-J} t a^{-1}$  of  $\mathbf{GL}_n D$  on  $H$ , and we denote the orbit of  $t$  by  $\langle t \rangle$ . The orbit space

$$\text{Herm}_\varepsilon(n) = \{\langle t \rangle \mid t \in H\} = L \backslash H$$

consists thus of the isomorphism classes of  $\varepsilon$ -hermitian forms on  $D^n$ . We denote the subset corresponding to the nonsingular hermitian forms by  $\text{Herm}_\varepsilon^\circ(n)$ . Then we have an  $L$ -equivariant bijection given by

$$H \longrightarrow X^{\text{opp}} \quad t \longmapsto u_t(Y).$$

Factoring out the  $L$ -action, we get bijections as follows:

$$\text{Herm}_\varepsilon(n) \longrightarrow L \backslash X^{\text{opp}} \quad \text{and} \quad \text{Herm}_\varepsilon^\circ(n) \longrightarrow L \backslash (X^{\text{opp}} \cap Y^{\text{opp}})$$

While the isomorphism  $H \rightarrow U$  depends on the chosen basis  $\mathbf{x}$ , these two maps are base-independent as one can easily check (this will also follow from Subsection 4.3). Summarizing these results, we have the following theorem.

**THEOREM 6.** *Let  $\mathcal{L}^{(3)} \subseteq \mathcal{L} \times \mathcal{L} \times \mathcal{L}$  denote the set of all triples of pairwise opposite Lagrangians. Then we have a  $\mathbf{U}(M)$ -invariant surjective map*

$$\mathcal{L}^{(3)} \xrightarrow{\kappa} \text{Herm}_\varepsilon^\circ(n)$$

whose fibers are the  $\mathbf{U}(M)$ -orbits in  $\mathcal{L}^{(3)}$ . The map  $\kappa$  is given by

$$\kappa(g(X), g(Y), g u_t(Y)) = \langle t \rangle,$$

where  $X, Y$  is our fixed pair of opposite Lagrangians as in Subsection 2.1.

The result will be refined in Proposition 9.

2.3. According to Subsection 1.5, we have the following cases.

*Symplectic groups:* The triples are classified by isomorphism classes of nondegenerate symmetric matrices.

*Hyperbolic orthogonal groups:* The triples are classified by isomorphism classes of nondegenerate skew-symmetric matrices. There is one such class if  $n$  is even, and  $\mathcal{L}^{(3)} = \emptyset$  if  $n$  is odd.

*Standard hyperbolic unitary groups:* We may assume that  $\varepsilon = 1$  (thus the form is skew-hermitian), and then the triples are classified by isomorphism classes of  $n$ -dimensional nondegenerate hermitian forms.

### 3. Flag complexes of graphs

We continue to assume that  $M$  is a  $-\varepsilon$ -hermitian hyperbolic module. Now we consider the simplicial complex whose  $k$ -simplices are  $k + 1$ -sets of pairwise opposite Lagrangians. It will be convenient to do this in the general setting of graphs, flag complexes and simplicial sets.

### 3.1. The opposition graph

By a *graph*  $\Gamma = (V, E)$  we understand an undirected graph without loops or multiple edges;  $V$  is its set of *vertices*,  $E$  its set of *edges*, and edges are unordered pairs of vertices. If  $\{u, v\}$  is an edge, then we call  $u, v$  *adjacent*. For the hyperbolic module  $M$ , we put  $V = \mathcal{L}$  and  $\mathcal{O} = \{\{X, Y\} \mid X, Y \in \mathcal{L} \text{ and } M = X + Y\}$ . The resulting graph  $\Gamma = (\mathcal{L}, \mathcal{O})$  is called the *opposition graph* of  $M$ .

### 3.2. Flag complexes

The *flag complex*  $\text{Fl}(\Gamma)$  of a graph  $\Gamma$  is the simplicial set whose  $k$ -simplices are tuples  $(x_0, \dots, x_k)$  of vertices, such that for all  $0 \leq \mu < \nu \leq k$  we have either  $x_\mu = x_\nu$  or  $\{x_\mu, x_\nu\} \in E$ . We have the standard  $\mathbb{Z}$ -free chain complex  $C_*(\text{Fl}(\Gamma))$  with the usual boundary operator

$$\partial(x_0, \dots, x_k) = \sum_{\nu} (-1)^\nu (x_0, \dots, \hat{x}_\nu, \dots, x_k)$$

and the resulting homology and cohomology groups [14].

We will also use *alternating chains*, which are defined as follows [5]. Let  $N_k$  denote the submodule of  $C_k(\text{Fl}(\Gamma))$  generated by all elements  $(x_0, \dots, x_k)$  with  $x_\mu = x_\nu$  for some  $\mu < \nu$ , and all elements of the form  $(x_0, \dots, x_k) - \text{sign}(\pi)(x_{\pi_0}, \dots, x_{\pi_k})$  for  $\pi \in \text{Sym}(k+1)$ . The alternating chain complex is defined as the quotient chain complex given by

$$\tilde{C}_*(\text{Fl}(\Gamma)) = C_*(\text{Fl}(\Gamma))/N_*.$$

The natural projection  $C_*(\text{Fl}(\Gamma)) \rightarrow \tilde{C}_*(\text{Fl}(\Gamma))$  is a chain equivalence, that is, induces an isomorphism in homology and cohomology; see [5, VI.6]. The coset of  $(x_0, \dots, x_k)$  is denoted by  $\langle x_0, \dots, x_k \rangle$ , with the relations  $\langle x_0, \dots, x_k \rangle = 0$  if  $x_\mu = x_\nu$  for some  $\mu < \nu$ , and

$$\langle x_0, \dots, x_k \rangle = \text{sign}(\pi) \langle x_{\pi_0}, \dots, x_{\pi_k} \rangle.$$

### 3.3. Equivariant cohomology

The unitary group  $\mathbf{U}(M)$  acts in a natural way on the opposition graph and its flag complex. In general, when a group  $G$  acts (from the left, say) on a chain complex  $C_*$ , then we may consider the *equivariant homology* of  $C_*$ , which is defined as follows. If  $P_* \rightarrow \mathbb{Z}$  is a projective resolution of  $G$  over  $\mathbb{Z}$ , then the equivariant homology  $H_*^G(C_*)$  is defined as the total homology of the double complex  $P_* \otimes_G C_*$ ; see [2, Chapter VII.5]. The two canonical filtrations on the double complex yield two spectral sequences  $'E$  and  $''E$  converging to  $H_*^G(C_*)$  and the first one has on its second page

$$'E_{pq}^2 = H_p(G; H_q(C_*)).$$

If  $C_*$  is acyclic (for example, if  $C_* = \mathbb{Z}$  is concentrated in dimension 0), then  $'E$  collapses on the second page, and there is a natural isomorphism  $H_*^G(C_*) \cong H_*(G)$ .

Similar remarks hold for cohomology; here, one looks at the double complex  $\text{Hom}_G(P_*, C^*)$ . Note also that if  $c : C_* \rightarrow A$  is a  $G$ -invariant cochain (thus  $G$  acts trivially on the coefficient module  $A$ ) and if  $\eta : P_1 \rightarrow \mathbb{Z}$  is the augmentation map, then  $c$  may be viewed in a natural way as a cochain in  $\text{Hom}_G(P_*, \text{Hom}_{\mathbb{Z}}(C_*, A)) \cong \text{Hom}_{\mathbb{Z}}(P_* \otimes_G C_*, A)$  via

$$c(p \otimes z) = \eta(p)c(z).$$

It is well known that for a complete graph (that is, for  $E = \binom{V}{2}$ ) the simplicial set  $\text{Fl}(\Gamma)$  is acyclic. The following concept is a weakening of (infinite) complete graphs.

### 3.4. The star property

A (nonempty) graph  $\Gamma = (V, E)$  has the *star property* if for every finite set  $x_0, \dots, x_k$  of vertices, there exists a vertex  $y$  that is adjacent to the  $x_\nu$  for  $\nu = 0, \dots, k$ .

Note that we require that  $y \neq x_0, \dots, x_k$ . A graph with the star property is obviously infinite. Note also that the opposition graph of a hyperbolic module has by Proposition 3 the star property if  $D^\varepsilon$  is infinite.

LEMMA 7. *If  $\Gamma$  has the star property, then  $\text{Fl}(\Gamma)$  is acyclic.*

*Proof.* If  $(x_0, \dots, x_k)$  is a  $k$ -simplex in  $C_k(\text{Fl}(\Gamma))$  and if  $y$  is adjacent to  $x_0, \dots, x_k$  put  $y\#(x_0, \dots, x_k) = (y, x_0, \dots, x_k)$ . Suppose that  $c$  is a  $k$ -cycle, that is,  $c$  is a finite linear combination of  $k$ -simplices and  $\partial c = 0$ . Let  $y$  be a vertex adjacent to all vertices appearing in the simplices of  $c$ . Then  $\partial(y\#c) = c - y\#\partial c = c$ , and thus  $c$  is a boundary.  $\square$

The geometric realization  $|\text{Fl}(\Gamma)|$  of the flag complex of a graph with the star property is in fact contractible. To see this, it suffices by Hurewicz's Theorem to show that  $\pi_1|\text{Fl}(\Gamma)| = 0$ ; see [26, Chapter 7.6.24 and 7.6.25]. However, from the star property, any simplicial path in  $|\text{Fl}(\Gamma)|$  is contained in a contractible subcomplex, and every path is homotopic to a simplicial path [26, 3.6].

PROPOSITION 8. *If  $D^\varepsilon$  is infinite, then the flag complex of the opposition graph of a  $-\varepsilon$ -hermitian hyperbolic module is acyclic and its geometric realization is contractible. Consequently, we have in equivariant homology a natural isomorphism*

$$H_*^{\mathbf{U}(M)}(\text{Fl}(\Gamma)) \xrightarrow{\cong} H_*(\mathbf{U}(M))$$

*induced by the constant map  $\text{Fl}(\Gamma) \rightarrow \text{Fl}(\{pt\})$ , and similarly for cohomology.*

Using this natural isomorphism, we often identify these two (co)homology groups.

## 4. The projectivity groupoid

If  $X$  and  $Y$  are opposite Lagrangians in the hyperbolic module  $M$ , then we have canonical isomorphisms  $X \oplus X^J \cong X \oplus Y \cong Y \oplus Y^J$ , such that the first isomorphism is the identity on  $X$  and the second isomorphism is the identity on  $Y$ . In this way, we associate an isomorphism  $X \oplus X^J \rightarrow Y \oplus Y^J$  to every *oriented* edge  $(X \rightarrow Y)$  of the opposition graph  $\Gamma$ .

### 4.1. The projectivity groupoid

Recall that a groupoid is a small category where every arrow is an isomorphism. The *projectivity groupoid*  $\mathcal{GM}$  of  $M$  is defined as follows. The objects of  $\mathcal{GM}$  are 2-graded vector spaces  $X_*$  with  $X_1 = X$  and  $X_{-1} = X^J$ , where  $X \in \mathcal{L}$  is a Lagrangian. To each oriented edge  $(X \rightarrow Y)$  we associate an isomorphism  $[Y; X] : X_* \rightarrow Y_*$  of degree  $-1$ , the composite  $[X, Y]$  of which is given by

$$[Y; X] : X \oplus X^J \xrightarrow{\cong} X \oplus Y \xrightarrow{\cong} Y \oplus Y^J.$$

These maps generate the morphisms of  $\mathcal{GM}$ . We note that each object  $X_*$  in  $\mathcal{G}$  carries a natural structure of a hyperbolic module with  $-\varepsilon$ -hermitian form  $h_X$ , and that the morphisms preserve

this structure. Furthermore

$$[X; Y][Y; X] = \text{id}_{X^*},$$

and hence a morphism along a simplicial path depends only on the homotopy class of the path in  $\Gamma$  (that is, we have a natural transformation from the *fundamental groupoid*  $\pi_1\Gamma$  to  $\mathcal{GM}$ ). Finally, we note that  $\mathcal{GM}$  is in a natural way 2-graded: the paths of even length induce maps of degree 1, and the paths of odd length maps of degree  $-1$ .

4.2. Now we determine the morphism corresponding to a closed path of length 3. Let  $X, Y$  be opposite Lagrangians with bases  $\mathbf{x}, \mathbf{y}$  as in Subsection 2.1 and let  $Z = u_t(Y)$ . We write  $[Z; Y; X] = [Z; Y][Y; X]$  and so on. Then

$$[Z; X; Y](\mathbf{y}_\nu) = u_t(\mathbf{y}_\nu) \quad \text{and} \quad [Z; X; Y](\mathbf{x}_\nu) = u_t(\mathbf{x}_\nu) = \mathbf{x}_\nu.$$

Now, we have

$$h(\mathbf{y}_\lambda, u_t(\mathbf{y}_\nu)) = h\left(\mathbf{y}_\lambda, \mathbf{y}_\nu + \sum_{\mu} \mathbf{x}_\mu t_{\mu, \nu}\right) = h\left(\mathbf{y}_\lambda, \sum_{\mu} \mathbf{x}_\mu t_{\mu, \nu}\right).$$

The dual basis of  $\mathbf{y}$  is  $h(-, \mathbf{x})^J$ . With respect to the graded basis  $(\mathbf{y}, h(-, \mathbf{x})^J)$  for  $Y_* = Y \oplus Y^J$ , the morphism  $\varphi = [Y; Z; X; Y]$  is therefore given by a block matrix of the form  $\varphi = \begin{pmatrix} * & * \\ t & * \end{pmatrix}$ .

As this matrix has to be unitary and of degree  $-1$ , and because  $t^J = t\varepsilon$ , we obtain

$$\varphi = \begin{pmatrix} 0 & -t^{-1} \\ t & 0 \end{pmatrix}.$$

4.3. If  $h_Y$  denotes the canonical  $-\varepsilon$ -hermitian form on  $Y_*$ , then

$$h_\varphi(-, -) = h_Y(-, \varphi(-))(-\varepsilon)$$

is the  $\varepsilon$ -hermitian form  $h_\varphi = \begin{pmatrix} t & 0 \\ 0 & t^{-J} \end{pmatrix}$ . We note that  $t^J t^{-J} t = t$ , therefore both blocks represent the same isomorphism type  $\langle t \rangle$  in  $\text{Herm}_\varepsilon^\circ(n)$ , and we define

$$\tilde{\kappa}(Z, X, Y) = \langle t \rangle.$$

Note also that this class does not depend on the basis  $\mathbf{y}$  and that  $\tilde{\kappa}$  is  $\mathbf{U}(M)$ -invariant. Furthermore, we have  $\tilde{\kappa}(Z, X, Y) = \kappa(X, Y, Z)$ , where  $\kappa$  is the invariant from Theorem 2.6. We shall see shortly that both invariants agree completely.

4.4. From  $\varphi^{-1} = \begin{pmatrix} 0 & t^{-1} \\ -t & 0 \end{pmatrix}$  we see that

$$\tilde{\kappa}(X, Z, Y) = \langle -t \rangle.$$

Next we note that for  $y_1, y_2 \in Y_*$  we have

$$\begin{aligned} h_Y(y_1, \varphi(y_2)) &= h_X([X; Y]y_1, [X; Y]\varphi[Y; X][X; Y]y_2) \\ &= h_X([X; Y]y_1, [X; Y][Y, Z, X, Y][Y; X][X; Y]y_2) \\ &= h_X([X; Y]y_1, [X; Y, Z, X][X; Y]y_2), \end{aligned}$$

whence

$$\tilde{\kappa}(Y, Z, X) = \tilde{\kappa}(Z, X, Y),$$

that is,  $\tilde{\kappa}$  is invariant under cyclic permutations of the arguments. In particular, we have

$$\tilde{\kappa} = \kappa$$

Since  $\kappa$  classifies by Theorem 2.6 triples of pairwise opposite Lagrangians, we have the following sharpening of Theorem 2.6.

**PROPOSITION 9.** *The setwise  $\mathbf{U}(M)$ -stabilizer of a triple  $X, Y, Z$  of pairwise opposite Lagrangians induces (at least) the cyclic group  $\mathbb{Z}/3$  on this set. It induces the full symmetric group  $\text{Sym}(3)$  if and only if  $a^J t a = -t$  for some  $a \in \mathbf{GL}_n D$ , where  $\kappa(X, Y, Z) = \langle t \rangle$ .*

### 5. The Maslov cocycle

We want to turn the invariant  $\kappa : \mathcal{L}^{(3)} \rightarrow \text{Herm}_\varepsilon^\circ(n)$  into a 2-cocycle for the flag complex  $\text{Fl}(\Gamma)$  of the opposition graph. Suppose that  $A$  is an abelian group and that  $\alpha : \text{Herm}_\varepsilon^\circ(n) \rightarrow A$  is a map. By the properties of  $\kappa$  derived in Subsection 4.4, we see that

$$c : \langle X, Y, Z \rangle \mapsto \alpha(\kappa(X, Y, Z))$$

is a 2-cochain on the alternating chain complex  $\tilde{C}_2(\text{Fl}(\Gamma))$ , provided that we have the relation  $\alpha(\langle -t \rangle) = -\alpha(\langle t \rangle)$  for all  $t \in \text{Herm}_\varepsilon^\circ(n)$ . Now we investigate under what conditions this map is a cocycle, that is, under what conditions  $c(\partial\langle X, Y, Z, Z' \rangle) = 0$ , that is, when

$$c(\langle Y, Z, Z' \rangle - \langle X, Z, Z' \rangle + \langle X, Y, Z' \rangle - \langle X, Y, Z \rangle) = 0.$$

5.1. We fix again  $(X, \mathbf{x}), (Y, \mathbf{y})$  as in Subsection 2.1. Suppose that  $Z = u_t(Y)$  and  $Z' = u_{t'}(Y)$ , and that  $X, Y, Z, Z'$  are pairwise opposite. Thus we have

$$\kappa(Z, X, Y) = \langle t \rangle \quad \text{and} \quad \kappa(Z', X, Y) = \langle t' \rangle.$$

As  $u_t^{-1}(Z) = Y$  and  $u_t^{-1}u_{t'} = u_{-t+t'}$ , we obtain

$$\kappa(Z', X, Z) = \kappa(u_t^{-1}u_{t'}(Y), X, Y) = \langle t' - t \rangle.$$

It remains to determine  $\kappa(Z', Y, Z)$ . Let  $w = \begin{pmatrix} 0 & 1 \\ -\varepsilon & 0 \end{pmatrix}$ . Then  $w$  is unitary and interchanges  $X$  and  $Y$ . We have  $w(Z) = wu_t(Y) = wu_t w^{-1}(X)$  and we consider  $v_t = wu_t w^{-1} = \begin{pmatrix} 1 & 0 \\ -t\varepsilon & 1 \end{pmatrix}$ . Then we have

$$u_r v_t = \begin{pmatrix} 1 - rt\varepsilon & r \\ -t\varepsilon & 1 \end{pmatrix},$$

whence  $u_r w(Z) = u_r v_t(X) = Y$  for  $r = t^{-1}\varepsilon$ . Thus far we have achieved

$$u_r w(Y) = X \quad \text{and} \quad u_r w(Z) = Y.$$

We seek  $t''$  such that  $u_{t''}(Y) = u_r w(Z') = u_r w u_{t'}(Y)$ , or  $Y = u_{r-t''} w u_{t'}(Y)$ . Now we have

$$u_{r-t''} w u_{t'} = \begin{pmatrix} 1 & r - t'' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\varepsilon & -t'\varepsilon \end{pmatrix} = \begin{pmatrix} (t'' - r)\varepsilon & 1 + (t'' - r)t'\varepsilon \\ -\varepsilon & -t'\varepsilon \end{pmatrix},$$

whence  $1 = (r - t'')t'\varepsilon$ , which gives  $t'' = r - t'^{-1}\varepsilon = (t^{-1} - t'^{-1})\varepsilon$ , and thus we have

$$\kappa(Z', Y, Z) = h(u_{t''}(Y), X, Y) = \langle t^{-J} - t'^{-J} \rangle.$$

Plugging this into the boundary formula, we have the next result.

**PROPOSITION 10.** *Let  $A$  be an abelian group. A function  $\alpha : \text{Herm}_\varepsilon^\circ(n) \rightarrow A$  determines a  $\mathbf{U}(M)$ -invariant 2-cocycle  $c$  on the alternating 2-chains of  $\text{Fl}(\Gamma)$  if and only if the following*

two relations hold for all  $r, s, t \in \text{Herm}_\varepsilon^\circ(n)$ :

$$\begin{aligned} r + s = 0 & \text{ implies } \alpha\langle r \rangle + \alpha\langle s \rangle = 0, \\ r + s + t = 0 & \text{ implies } \alpha\langle r \rangle + \alpha\langle s \rangle + \alpha\langle t \rangle + \langle -r^{-J} - s^{-J} \rangle = 0. \end{aligned}$$

Recall that the Grothendieck–Witt group  $KU_0^\varepsilon(D, J)$  of hermitian forms is defined as the abelian group completion of the commutative monoid consisting of the isomorphism classes of nondegenerate  $\varepsilon$ -hermitian forms [23, p. 239]. The Witt group  $W^\varepsilon(D, J)$  is the factor group of  $KU_0^\varepsilon(D, J)$  by the subgroup generated by the  $\varepsilon$ -hermitian hyperbolic modules. We let  $[t]$  denote the image of  $\langle t \rangle$  in  $KU_0^\varepsilon(D, J)$  and  $W^\varepsilon(D, J)$ .

**THEOREM 11.** *Let  $\alpha\langle t \rangle = [t] \in W^\varepsilon(D, J)$ . Then  $\alpha$  satisfies the two conditions of Proposition 10 and therefore*

$$\mathfrak{m} : \langle X, Y, Z \rangle \longmapsto [\kappa(X, Y, Z)]$$

defines a  $W^\varepsilon(D, J)$ -valued  $\mathbf{U}(M)$ -invariant 2-cocycle on the alternating chain complex  $\tilde{\mathcal{C}}_2(\text{Fl}(\Gamma))$ .

*Proof.* We proceed similar to [21, Proposition 1.2] and use the fact that metabolic forms vanish in the Witt group  $W^\varepsilon(D, J)$ , see [23, 7.3.7], and that a  $2k$ -dimensional nondegenerate hermitian form is metabolic if it admits a totally isotropic subspace of dimension  $k$ .

For the  $2n$ -dimensional  $\varepsilon$ -hermitian form  $(r) \oplus (-r)$ , the vectors  $(x, x)$ , with  $x \in D^n$ , span an  $n$ -dimensional totally isotropic subspace, and thus this form is metabolic and  $[r] + [-r] = 0$ .

Similarly we find for  $r + s + t = 0$  and the  $4n$ -dimensional form  $(r) \oplus (s) \oplus (t) \oplus (-r^J - s^J)$  that the vectors  $(x, x, x, 0)$  and  $(r^{-J}x, s^{-J}x, 0, x)$ , with  $x \in D^n$ , span a totally isotropic  $2n$ -dimensional subspace, and thus this form is also metabolic and  $[r] + [s] + [t] + [-r^J - s^J] = 0$ .  $\square$

## 5.2. The Maslov cocycle

We call the  $W^\varepsilon(D, J)$ -valued cocycle

$$\mathfrak{m} : \langle X, Y, Z \rangle \longmapsto [\kappa(X, Y, Z)]$$

(and the corresponding cocycle for the equivariant homology of  $\text{Fl}(\Gamma)$ ) the *Maslov cocycle*.

## 6. Naturality of the Maslov cocycle

We now study naturality of the Maslov cocycle under restriction maps. There are two obvious types, coming from field and from vector space inclusions. We start with field inclusions, which are easier.

### 6.1. Extension of scalars

Suppose that  $D$  and  $E$  are division rings with involutions  $J$  and  $K$ , respectively, and that  $\varphi : D \rightarrow E$  is a homomorphism commuting with these involutions. If  $M$  is a hyperbolic module over  $D$ , then  $M \otimes_\varphi E$  is hyperbolic over  $E$ . The map sending a Lagrangian  $X \subseteq M$  to  $X \otimes_\varphi E$  induces an injection  $\mathcal{L}(M) \rightarrow \mathcal{L}(M \otimes_\varphi E)$  and an injection  $\Gamma(M) \rightarrow \Gamma(M \otimes_\varphi E)$  on the respective opposition graphs. There is a natural map  $W_E^D : W^\varepsilon(D, J) \rightarrow W^\varepsilon(E, K)$  and, obviously, this map takes the Maslov cocycle  $\mathfrak{m}_D$  of  $M$  to the Maslov cocycle  $\mathfrak{m}_E$  of  $M \otimes_\varphi E$ .

as follows:

$$\begin{array}{ccc}
 \tilde{C}_2 \text{ Fl}(\Gamma(M)) & \longrightarrow & \tilde{C}_2 \text{ Fl}(\Gamma(M \otimes_\varphi E)) \\
 \downarrow \mathfrak{m}_D & & \downarrow \mathfrak{m}_E \\
 W^\varepsilon(D, J) & \xrightarrow{W_E^D} & W^\varepsilon(E, K).
 \end{array}$$

This gives the following result.

**THEOREM 12.** *Let  $\varphi : (D, J) \rightarrow (E, K)$  be a homomorphism of skew-fields with involutions and assume that  $D^\varepsilon$  is infinite. Consider the natural group monomorphism*

$$\Phi : \mathbf{U}(M) \longrightarrow \mathbf{U}(M \otimes_\varphi E).$$

Then  $(W_E^D)_* \mathfrak{m}_D = \Phi^* \mathfrak{m}_E$  in the diagram

$$\begin{array}{ccc}
 H^2(\mathbf{U}(M); W^\varepsilon(D, J)) & \xrightarrow{(W_E^D)_*} & H^2(\mathbf{U}(M); W^\varepsilon(E, K)) \\
 & & \uparrow \Phi^* \\
 & & H^2(\mathbf{U}(M \otimes_\varphi E); W^\varepsilon(E, K)).
 \end{array}$$

6.2. Suppose now that  $M_1$  and  $M_2$  are hyperbolic modules (both over  $D$ ) with corresponding sets  $\mathcal{L}_1, \mathcal{L}_2$  of Lagrangians. Then their direct sum  $M = M_1 \oplus M_2$  is in a natural way a hyperbolic module. There is an obvious map

$$\mathbf{U}(M_1) \longrightarrow \mathbf{U}(M)$$

and the question is what happens with the Maslov cocycle under this map. The problem is that the opposition graph  $\Gamma_1$  of  $M_1$  is not a subgraph of the opposition graph  $\Gamma$  of  $M$ . However, there is a natural subgraph of  $\Gamma$  that projects  $\mathbf{U}(M_1)$ -equivariantly onto  $\Gamma_1$  and that yields a good comparison map. The construction is as follows.

If  $X_1 \subseteq M_1$  and  $X_2 \subseteq M_2$  are Lagrangians, then  $X_1 \oplus X_2$  is Lagrangian in  $M$ , and thus we have a natural injection  $\mathcal{L}_1 \times \mathcal{L}_2 \rightarrow \mathcal{L}$ . Moreover,  $X_1 \oplus X_2$  is opposite  $Y_1 \oplus Y_2$  in  $M$  if and only if  $X_\nu$  is opposite  $Y_\nu$  for  $\nu = 1, 2$ . This leads us to the following notion.

**DEFINITION 13.** The *categorical product*  $\Gamma_1 \times \Gamma_2$  of two graphs has  $V_1 \times V_2$  as its set of vertices and  $(x_1, x_1)$  and  $(y_1, y_2)$  are adjacent if and only if  $\{x_1, y_1\} \in E_1$  and  $\{x_2, y_2\} \in E_2$ . There are natural maps  $\Gamma_1 \leftarrow \Gamma_1 \times \Gamma_2 \rightarrow \Gamma_2$  with the usual universal properties.

The next result is immediate.

**LEMMA 14.** *The categorical product of two graphs having the star property has again the star property. In particular, its flag complex is acyclic.*

Note that the categorical product of the graph consisting of one single edge with itself is not even connected:

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \times \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{ccc} \bullet & & \bullet \\ & \diagdown & / \\ & \bullet & \\ & / & \diagdown \\ \bullet & & \bullet \end{array}$$

The fact that  $y \neq x_0, \dots, x_k$  in the star property is crucial for the lemma.

6.3. Thus far we have for  $\nu = 1, 2$  a diagram of  $\mathbf{U}(M_1)$ -equivariant maps

$$\begin{array}{ccc} \mathrm{Fl}(\Gamma_1) & & \\ \swarrow \mathrm{pr}_1 & & \\ & \mathrm{Fl}(\Gamma_1 \times \Gamma_2) \longrightarrow \mathrm{Fl}(\Gamma) & \\ \searrow \mathrm{pr}_2 & & \\ \mathrm{Fl}(\Gamma_2) & & \end{array}$$

and if  $D^\varepsilon$  is infinite, these three complexes are acyclic. Next we note that if we have a triangle  $(X_1 \oplus X_2, Y_1 \oplus Y_2, Z_1 \oplus Z_1)$  in  $\Gamma_1 \times \Gamma_2$  and if we choose bases  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$  for  $X_1, X_2, Y_1, Y_2$ , then

$$[\kappa(X_1 \oplus X_2, Y_1 \oplus Y_2, Z_1 \oplus Z_1)] = [t_1 \oplus t_2] = [t_1] + [t_2],$$

with  $\kappa(X_\nu, Y_\nu, Z_\nu) = \langle t_\nu \rangle$ . Thus we have a commutative diagram

$$\begin{array}{ccccc} \tilde{C}_2 \mathrm{Fl}(\Gamma_1) & \xrightarrow{i_1} & \tilde{C}_2 \mathrm{Fl}(\Gamma_1) \oplus \tilde{C}_2 \mathrm{Fl}(\Gamma_2) & \xleftarrow{(\mathrm{pr}_1, \mathrm{pr}_2)} & \tilde{C}_2 \mathrm{Fl}(\Gamma_2 \times \Gamma_2) \\ & \searrow m_1 & \downarrow m_1 + m_2 & & \swarrow m \\ & & W^\varepsilon(D, J) & & \end{array}$$

that yields in cohomology:

$$H_{\mathbf{U}(M_1)}^2(\mathrm{Fl}(\Gamma_1)) \longleftarrow H_{\mathbf{U}(M_1)}^2(\mathrm{Fl}(\Gamma_1)) \oplus H_{\mathbf{U}(M_1)}^2(\mathrm{Fl}(\Gamma_2)) \longrightarrow H_{\mathbf{U}(M_1)}^2(\mathrm{Fl}(\Gamma_1 \times \Gamma_2))$$

$$[\mathbf{m}_1] \longleftarrow [\mathbf{m}_1] + [\mathbf{m}_2] \longrightarrow [\mathbf{m}]$$

(we omit here the coefficient group  $W^\varepsilon(D, J)$ ). Note that  $[\mathbf{m}_2] = 0$  in  $H_{\mathbf{U}(M_1)}^2(\mathrm{Fl}(\Gamma_2))$ , as  $\mathbf{U}(M_1)$  acts trivially on  $\mathrm{Fl}(\Gamma_2)$ . Mapping to the one-point space  $\{pt\}$ , we see that  $[\mathbf{m}_1]$  and  $[\mathbf{m}]$  have the same image in  $H_{\mathbf{U}(M_1)}^2(\{pt\}) = H^2(\mathbf{U}(M_1))$ , and from

$$\begin{array}{ccccccc} H_{\mathbf{U}(M_1)}^2(\mathrm{Fl}(\Gamma_1)) & & H_{\mathbf{U}(M_1)}^2(\mathrm{Fl}(\Gamma_1 \times \Gamma_2)) & \xleftarrow{\cong} & H_{\mathbf{U}(M_1)}^2(\mathrm{Fl}(\Gamma)) & \longleftarrow & H_{\mathbf{U}(M)}^2(\mathrm{Fl}(\Gamma)) \\ \cong \uparrow & & \cong \uparrow & & \cong \uparrow & & \cong \uparrow \\ H^2(\mathbf{U}(M_1)) & \longleftarrow & H^2(\mathbf{U}(M_1)) & \longleftarrow & H^2(\mathbf{U}(M_1)) & \longleftarrow & H^2(\mathbf{U}(M)) \end{array}$$

we obtain the following stability result.

**THEOREM 15.** *Assume that  $D^\varepsilon$  is infinite, let  $M_1, M_2$  be hyperbolic modules and consider  $M = M_1 \oplus M_2$ . Then the restriction map*

$$H^2(\mathbf{U}(M_1); W^\varepsilon(D, J)) \longleftarrow H^2(\mathbf{U}(M); W^\varepsilon(D, J))$$

*maps the Maslov cocycle  $[\mathbf{m}]$  for  $\mathbf{U}(M)$  onto the Maslov cocycle  $[\mathbf{m}_1]$  for  $\mathbf{U}(M_1)$ .*

### 7. Reduction of the cocycle

Our next aim is to show that the Maslov cocycle can be reduced to a subgroup of the Witt group. For this, we need a refinement of the Lagrangians and the opposition graph. We noted in Subsection 2.3 that the Maslov cocycle is trivial in the hyperbolic orthogonal situation, where  $J = \text{id}$  and  $\varepsilon = -1 \neq 1$ , and hence we may disregard this case. By Subsection 1.5 there is no loss of generality in assuming that

$$\varepsilon = 1$$

in the remaining cases, and we shall do this in this section.

#### 7.1. Based Lagrangians

Let  $\Gamma = (V, E)$  be a graph and  $f : X \rightarrow V$  a map. The *induced graph*  $f^*\Gamma$  on  $X$  is the graph whose vertices are the elements of  $X$ , and  $\{x, x'\}$  is an edge if and only if  $\{f(x), f(x')\}$  is an edge of  $\Gamma$ . If  $f$  is surjective and if  $\Gamma$  has the star property, then  $f^*\Gamma$  also has the star property. In what follows, we consider the set  $\widehat{\mathcal{L}}$  of *based Lagrangians*, that is, pairs  $(X, \mathbf{x})$  where  $X \subseteq M$  is a Lagrangian and  $\mathbf{x}$  is a basis for  $X$ . There is a forgetful surjection  $F : \widehat{\mathcal{L}} \rightarrow \mathcal{L}$  and we let

$$\widehat{\Gamma} = F^*\Gamma$$

denote the induced graph on this vertex set. We call  $\widehat{\Gamma}$  the *based opposition graph*. Because the  $\mathbf{U}(M)$ -stabilizer  $P$  induces the full group  $\mathbf{GL}(X)$  on  $X$ , we see that  $\mathbf{U}(M)$  acts transitively on  $\widehat{\mathcal{L}}$ . With the notation of Subsection 2.1, the stabilizer of  $(X, \mathbf{x})$  is the group  $U$ . The map  $\widehat{\Gamma} \rightarrow \Gamma$  is equivariant, and  $\text{Fl}(\widehat{\Gamma})$  is acyclic if  $D^1$  is infinite. In particular, we may use  $\text{Fl}(\widehat{\Gamma})$  to compute the group cohomology of  $\mathbf{U}(M)$ .

We also have a based version  $\widehat{\mathcal{G}}M$  of the projectivity groupoid. The objects are again the 2-graded spaces  $X \oplus X^J$ , but now with a preferred graded basis consisting of  $\mathbf{x}$  and the dual basis of  $\mathbf{x}$ . The morphisms in  $\widehat{\mathcal{G}}M$  are thus given by unitary matrices.

7.2. We recalculate the Maslov cocycle in terms of the based spaces. In Subsection 4.2 we have seen that we have in terms of our standard basis  $\mathbf{x}, \mathbf{y}$  the matrices

$$(Y_*, \mathbf{y}) \xrightarrow{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} (X_*, \mathbf{x}) \xleftarrow{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} (Z_*, u_t(\mathbf{y})) \xrightarrow{\begin{pmatrix} 0 & -t^{-1} \\ t & 0 \end{pmatrix}} (Y_*, \mathbf{y}).$$

If we add base changes through matrices  $a, b, c \in \mathbf{GL}_n D$  for  $X, Y$  and  $Z$  and reverse the middle arrow, we arrive at the diagram

$$(Y_*, b\mathbf{y}) \xrightarrow{\begin{pmatrix} 0 & -ab^J \\ a^{-J}b^{-1} & 0 \end{pmatrix}} (X_*, a\mathbf{x}) \xrightarrow{\begin{pmatrix} 0 & ca^J \\ -c^{-J}a^{-1} & 0 \end{pmatrix}} (Z_*, cu_t(\mathbf{y})) \xrightarrow{\begin{pmatrix} 0 & -bt^{-1}c^J \\ b^{-J}tc^{-1} & 0 \end{pmatrix}} (Y_*, b\mathbf{y})$$

(and  $cu_t(\mathbf{y}) = u_{c^{-J}tc^{-1}}(c\mathbf{y})$ ). With respect to the basis  $b\mathbf{y}$ , we have

$$[Y; Z; X; Y] = \begin{pmatrix} 0 & -bt^{-1}b^J \\ b^{-J}tb^{-1} & 0 \end{pmatrix}.$$

Using invariants of these matrices, we now construct a refined cocycle.

#### 7.3. Invariants of hermitian forms

The dimension induces a natural homomorphism  $\dim : KU_0^1(D, J) \rightarrow \mathbb{Z}$ . Since the dimension of any hyperbolic module is even, there is an induced map  $W^1(D, J) \rightarrow \{\pm 1\}$  mapping the class  $[t]$  to  $(-1)^{\dim(t)}$ . We denote its kernel by  $I(D, J)$ ; its elements are represented by even-dimensional hermitian forms. In the quadratic case ( $J = \text{id}$  and  $\varepsilon = 1 \neq -1$ ),  $ID = I(D, \text{id})$  is called the *fundamental ideal* in the Witt ring  $WD = W^1(D, \text{id})$  (see [12, Chapter II.1]).

Recall that the determinant is a homomorphism from  $\mathbf{GL}_n D$  to  $K_1(D)$ , the abelianization of  $D^* = \mathbf{GL}_1 D$ . The involution  $J$  induces an automorphism  $J$  on  $K_1(D)$ . We let  $N$  denote the subgroup of  $K_1(D)$  consisting of elements of the form  $x^J x$  and put  $S = K_1(D)/N$ . Since  $\det(g^J t g) = \det(g^J g) \det(t)$ , we have a well-defined homomorphism  $[t] \mapsto \det(t)N$  from  $KU_0^1(D, J)$  to  $S$ . However, this map cannot be factored through  $W^1(D, J)$ . Similarly as in [12, Chapter II.2] we introduce therefore the abelian group

$$\widehat{S} = S \times \{\pm 1\},$$

endowed with the commutative group law

$$(x, (-1)^m) + (y, (-1)^n) = (xy(-1)^{mn}, (-1)^{m+n}),$$

and we define the *signed discriminant* as

$$\text{disc}(t) = (\det(t)N(-1)^{n(n-1)/2}, (-1)^n),$$

where  $n = \dim(t)$ . This map vanishes on hyperbolic forms and induces therefore a homomorphism  $\text{disc} : W^1(D, J) \rightarrow \widehat{S}$ . We let  $II(D, J) \subseteq W^1(D, J)$  denote the subgroup generated by all elements  $[t]$ , where  $\dim(t) \in 4\mathbb{Z}$  and  $\det(t) = 1$ . Obviously,  $II(D, J) \subseteq \ker(\text{disc})$ .

LEMMA 16. *The sequence*

$$0 \longrightarrow II(D, J) \longrightarrow W^1(D, J) \longrightarrow \widehat{S}$$

is exact.

*Proof.* Let  $[t]$  be a form in the kernel of  $\text{disc}$ . Then  $\dim(t)$  is even and we distinguish two cases. If  $\dim(t) = 4$ , then  $\det(t) = x^J x \in N$ . Choose  $g \in \mathbf{GL}_n D$  with  $\det(g) = x^{-1}$ , then  $\det(g^J t g) = 1$  and  $[t] = [g^J t g] \in II(D, J)$ . For  $\dim(t) = 4\ell + 2$  we have  $\det(t) = -x^J x$  and we consider the  $4\ell + 4$ -dimensional form  $t \oplus h$ , for  $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Then  $\det(t \oplus h) = x^J x$ . By the previous remark,  $[t \oplus h] = [t] \in II(D, J)$ .  $\square$

In the quadratic case,  $II(D, J)$  is the square  $I^2 D$  of the fundamental ideal [12, Chapter II, 2.1]

7.4. We define an  $\widehat{S}$ -valued equivariant 1-cochain  $f$  on  $\tilde{C}_1 \text{Fl}(\widehat{\Gamma})$  by

$$f\langle (X, \mathbf{ax}), (Y, \mathbf{by}) \rangle = (\det(-ab^J)(-1)^{n(n-1)/2} N, (-1)^n) \in \widehat{S},$$

where the notation is as in Subsection 7.2. Note that this is indeed an alternating cochain:

$$(\det(g)N, (-1)^n) + (\det(-g^J)N, (-1)^n) = (\det(-gg^J)(-1)^{n^2} N, (-1)^{2n}) = (N, 1).$$

Then  $df = f\partial$  is an  $\widehat{S}$ -valued 2-coboundary, and we have

$$\begin{aligned} df\langle (Z, cu_t(\mathbf{y})), (X, \mathbf{ax}), (Y, \mathbf{by}) \rangle &= f\langle (X, \mathbf{ax}), (Y, \mathbf{by}) \rangle - f\langle (Z, cu_t \mathbf{y}), (Y, \mathbf{by}) \rangle \\ &\quad + f\langle Z, cu_t(\mathbf{y})(X, \mathbf{ax}) \rangle \\ &= (\det(-ab^J)(-1)^{n(n-1)/2} N, (-1)^n) \\ &\quad + (\det(-bt^{-1}c^J)(-1)^{n(n-1)/2} N, (-1)^n) \\ &\quad + (\det(ca^J)(-1)^{n(n-1)/2} N, (-1)^n) \\ &= (\det(aa^J bb^J cc^J t)(-1)^{n(n-1)/2} N, (-1)^n) \\ &= (\det(t)(-1)^{n(n-1)/2} N, (-1)^n), \end{aligned}$$

whence

$$\text{disc}_* \mathfrak{m} + df = 0,$$

where  $\text{disc}_*$  denotes the coefficient homomorphism induced by  $\text{disc} : W^1(D, J) \rightarrow \widehat{S}$ . Consequently, the image of  $\mathfrak{m}$  vanishes in  $H^2(\mathbf{U}(M); \widehat{S})$ .

7.5. Recall that  $\mathbf{EU}(M) \subseteq \mathbf{U}(M)$  is the invariant subgroup generated by the Eichler transformations. This group is perfect if  $D^1$  is infinite [6, 6.3.15] and consequently  $H^1(\mathbf{EU}(M); A) = \text{Hom}(\mathbf{EU}(M), A) = 0$  for any coefficient group  $A$  with trivial  $\mathbf{EU}(M)$ -action. We consider  $\widehat{S}_0 = \text{disc}(W^1(D, J)) \subseteq \widehat{S}$ . The long exact cohomology sequences for the coefficient maps

$$0 \longrightarrow \widehat{S}_0 \longrightarrow \widehat{S} \longrightarrow \widehat{S}/\widehat{S}_0 \longrightarrow 1$$

$$0 \longrightarrow H(D, J) \longrightarrow W^1(D, J) \longrightarrow \widehat{S}_0 \longrightarrow 0$$

yield therefore monomorphisms

$$0 \longrightarrow H^2(\mathbf{EU}(M); \widehat{S}_0) \longrightarrow H^2(\mathbf{EU}(M); \widehat{S})$$

$$0 \longrightarrow H^2(\mathbf{EU}(M); H(D, J)) \longrightarrow H^2(\mathbf{EU}(M); W^1(D, J)).$$

This gives us the next Theorem. To keep the notation simple, we denote the restriction of  $\mathfrak{m}$  to the subgroup  $\mathbf{EU}(M)$  also by  $\mathfrak{m}$ .

**THEOREM 17.** *Assume that  $\varepsilon = 1$  and that  $D^1$  is infinite. There exists a unique cohomology class  $[\tilde{\mathfrak{m}}] \in H^2(\mathbf{EU}(M); H(D, J))$  that maps under the coefficient homomorphism  $H(D, J) \rightarrow W^1(D, J)$  onto  $[\mathfrak{m}]$ . We call this class the reduced Maslov cocycle.*

*Proof.* As we have proved in Subsection 7.4, we see that  $\text{disc}_*[\mathfrak{m}] + [df] = 0$  in  $H^2(\mathbf{EU}(M); \widehat{S})$ , whence  $\text{disc}_*[\mathfrak{m}] = 0$  in  $H^2(\mathbf{EU}(M); \widehat{S}_0)$ . Therefore  $[\mathfrak{m}]$  has a preimage  $[\tilde{\mathfrak{m}}]$  in  $H^2(\mathbf{EU}(M); H(D, J))$ . The map  $H^2(\mathbf{EU}(M); H(D, J)) \rightarrow H^2(\mathbf{EU}(M); W^1(D, J))$  is injective, and so the preimage is unique.  $\square$

7.6. In the symplectic situation  $(J, \varepsilon) = (\text{id}, 1)$  it is possible to give an explicit formula for the reduced cocycle  $\tilde{\mathfrak{m}}$ . Then  $\widehat{S} = \widehat{S}_0$  and  $\mathbf{EU}(M) = \mathbf{U}(M) = \mathbf{Sp}_{2n} D$  and we can directly define a  $W^1(D, J)$ -valued 1-cochain on  $\text{Fl}(\widehat{\Gamma})$  by

$$\tilde{f}(\langle X, \mathbf{ax} \rangle, \langle Y, \mathbf{by} \rangle) = \langle \det(-ab), 1, \dots, 1 \rangle,$$

where the right-hand side denotes as usual the  $n$ -dimensional symmetric bilinear form with the given entries on the diagonal. Under the map  $p : \text{Fl}(\widehat{\Gamma}) \rightarrow \text{Fl}(\Gamma)$ , this is a lift of  $f$  and we have  $\text{disc}_* d\tilde{f} = p^* df$ . Thus

$$\tilde{\mathfrak{m}} = p^* \mathfrak{m} + d\tilde{f}$$

is the reduced Maslov cocycle on  $\text{Fl}(\widehat{\Gamma})$  in the symplectic case. Explicitly, it reads as

$$\begin{aligned} \tilde{\mathfrak{m}}(\langle X, \mathbf{ax} \rangle, \langle Y, \mathbf{by} \rangle, \langle Z, c\mathbf{u}_t \mathbf{y} \rangle) &= \langle \det(-ab), 1, \dots, 1 \rangle + \langle \det(ca), 1, \dots, 1 \rangle \\ &\quad + \langle \det(-btc), 1, \dots, 1 \rangle - \langle t \rangle. \end{aligned}$$

### 8. Kashiwara's Maslov cocycle

In the symplectic situation over a field  $D$  of characteristic not equal to 2, the Maslov index is classically defined through a different quadratic form [13]. (A variant is used in [21], while a

topological generalization for bounded symmetric domains of tube type is given in [18]. See [3] for a survey of topological Maslov indices.)

8.1. *Kashiwara’s Maslov index*

Let  $D$  be a field of characteristic not equal to 2. We assume that we are in the symplectic situation  $\varepsilon = 1$ ,  $J = \text{id}$ . Given three Lagrangian  $X, Y, Z$  (not necessarily pairwise opposite), we consider the following  $3n$ -dimensional quadratic form  $q_{X,Y,Z}$  on the direct sum  $X \oplus Y \oplus Z$  as follows:

$$q_{X,Y,Z}(x, y, z) = h(x, y) + h(y, z) + h(z, x).$$

If the Lagrangians are not pairwise opposite, then the quadratic form is going to have a radical. The *Kashiwara–Maslov index* of  $(X, Y, Z)$  is the class in the Witt group  $WD$  that is represented by the nondegenerate part  $q_{X,Y,Z}^+$  of  $q_{X,Y,Z}$ .

For  $D = \mathbb{R}$ , the Witt group  $W\mathbb{R}$  is isomorphic to  $\mathbb{Z}$  via the signature and the Maslov index can directly be defined as the signature of  $q_{X,Y,Z}$  (even if the form is degenerate). This is essentially Kashiwara’s definition of the symplectic Maslov index as developed in [13, 1.5.1].

If  $X, Y, Z$  are pairwise opposite, then we find that with respect to our standard basis  $\mathbf{x}, \mathbf{y}, u_t \mathbf{y}$  for  $X \oplus Y \oplus Z$  the quadratic form is represented by the matrix

$$q_{X,Y,Z} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & t \\ 1 & 0 & 0 \end{pmatrix}.$$

We note that  $X \oplus Y \oplus 0$  is a hyperbolic submodule in  $X \oplus Y \oplus Z$  whose orthogonal complement is spanned by vectors of the form  $(tz, z, z) \in D^{3n}$ . The restriction of  $q_{X,Y,Z}$  to this subspace is given by  $z \mapsto (z^T tz)$ , and thus  $q_{X,Y,Z} = q_{X,Y,Z}^+$  is represented by  $[t]$  in  $WD$ . This is our first result.

**PROPOSITION 18.** *If  $X, Y, Z$  are pairwise opposite Lagrangians, then the Kashiwara–Maslov index of  $(X, Y, Z)$  agrees with the image  $[t]$  of  $\langle t \rangle = \kappa(X, Y, Z)$  in the Witt group  $WD$ .*

Next we get to Kashiwara’s Maslov cocycle, which is defined as follows. We fix a Lagrangian  $X_0 \in \mathcal{L}$  and define  $\tau : \mathbf{Sp}_{2n} D \times \mathbf{Sp}_{2n} D \rightarrow WD$  via

$$\tau(g, h) = \langle q_{X_0, g(X_0), gh(X_0)}^+ \rangle.$$

We want to relate this group cocycle to our Maslov cocycle defined in terms of the flag complex of the opposition graph.

8.2. Recall the bar notation [2, I.5] for the standard free resolution of a group  $G$  over  $\mathbb{Z}$ . Its chain complex is given as

$$F_n = \mathbb{Z}G^{n+1}$$

and the generator  $(1, g_1, g_1g_2, g_1g_2g_3, \dots, g_1 \dots g_n) \otimes 1 \in F_n \otimes_G \mathbb{Z}$  is denoted by  $[g_1] \cdot \dots \cdot [g_n]$ . Then  $\tau$  can be viewed as the  $WD$ -valued 2-cochain  $[g|h] \mapsto \tau(g, h)$  for  $G = \mathbf{Sp}_{2n} D$  and one can verify the cocycle identity [13, 1.5.8].

In general, suppose that  $X$  is a set on which a group  $G$  acts, and that  $c : X \times X \times X \rightarrow A$  is a  $G$ -invariant map taking values in an abelian group  $A$ , such that  $c$  satisfies the cocycle identity  $c(x, y, z) - c(w, y, z) + c(w, x, z) - c(w, x, y) = 0$ . If we choose a base point  $o \in X$ , then it is not difficult to see that the cocycle  $(g_1, g_2, g_3) \mapsto c(g_1(o), g_2(o), g_3(o))$  defined on the standard free resolution  $F_*$  of  $G$  over  $\mathbb{Z}$  and the cocycle  $g \otimes (x, y, z) \mapsto c(x, y, z)$  defined on  $F_0 \otimes_G C_2 \subseteq F_* \otimes_G C_*$  are homologous ( $C_*$  is the standard complex of  $k + 1$ -tuples of elements of  $X$ ). However,

we cannot use this directly to compare our Maslov cocycle with its classical counterpart, since our cocycle is defined only on special triples of Lagrangians. We need to refine this idea using some elementary homological algebra. We do this in general, as we need it also in the next section.

8.3. Let  $\Gamma = (V, E)$  be a graph with the star property. Suppose that  $G$  is a group acting transitively on the vertices of  $\Gamma$ . Let  $o \in V$  be a base point and consider the induced graph  $\Gamma_G$  on  $G$  under the map  $G \rightarrow V$ ,  $g \mapsto g(o)$  and its flag complex given by

$$F'_* = C_*\text{Fl}(\Gamma_G) \subseteq F_*.$$

Obviously, this chain complex is a free resolution of  $G$  over  $\mathbb{Z}$  and a subcomplex of the standard free resolution  $F_*$  of  $G$ . Both chain complexes  $F_*$  and  $F'_*$  can be used to determine the group (co)homology of  $G$ .

Suppose now that  $c : C_2\text{Fl}(\Gamma) \rightarrow A$  is a  $G$ -invariant cocycle. Then we can construct two 2-cocycles for  $G$ , one via

$$\hat{c} : (g_0, g_1, g_2) \mapsto c(g_0(o), g_1(o), g_2(o))$$

on  $F'_2 \otimes_G \mathbb{Z}$ , and the other via

$$c : g \otimes (x, y, z) \mapsto c(x, y, z)$$

on  $F'_0 \otimes_G C_2\text{Fl}(\Gamma) \subseteq F'_* \otimes_G C_2\text{Fl}(\Gamma)$ . Our first aim is to prove that both cocycles are homologous. We consider  $C_* = C_*\text{Fl}(\Gamma)$  and we call a generator  $(g_0, \dots, g_m) \otimes (x_0, \dots, x_n) \in F'_m \otimes_G C_n$  *admissible* if  $\{g_0(o), \dots, g_m(o), x_0, \dots, x_n\}$  consists of pairwise adjacent elements in  $\Gamma$ . This is a well-defined notion, that is, invariant under the left diagonal action of  $G$ . Let  $D_{**} \subseteq F'_* \otimes_G C_*$  denote the submodule generated by the admissible elements. We note that this submodule is  $\mathbb{Z}$ -free and closed under the vertical and horizontal differentials, and so it is a double complex.

LEMMA 19. *The inclusion  $D_{**} \hookrightarrow F'_* \otimes_G C_*$  induces an isomorphism in homology and cohomology (for coefficient groups with trivial  $G$ -action).*

*Proof.* We show that the relative homology groups of the pair  $(F'_* \otimes_G C_*, D_{**})$  vanish. Let  $z \in \bigoplus_{m+n=k} F'_m \otimes_G C_n$  be a relative  $k$ -cycle and let  $\tilde{z} \in \bigoplus_{m+n=k} F'_m \otimes_{\mathbb{Z}} C_n$  be an element that maps onto  $z$ . We choose a group element  $j$  such that for all terms  $(g_0, \dots, g_m) \otimes (x_0, \dots, x_n)$  appearing in  $\tilde{z}$ , the vertex  $j(o)$  is adjacent to  $g_0(o), \dots, g_m(o), x_0, \dots, x_n$  (this is a well-defined condition as we work with  $\tilde{z} \in \bigoplus_{m+n=k} F'_m \otimes_{\mathbb{Z}} C_n$ , where the  $G$ -action is not factored out). Consider the  $k+1$ -chain  $j\#\tilde{z}$ , whose  $(m+1, n)$ -terms are of the form  $(j, g_0, \dots, g_m) \otimes (x_0, \dots, x_n)$ . The total differential is

$$\partial(j\#\tilde{z}) = \tilde{z} - j\#(\partial\tilde{z}).$$

Projecting this equation back to  $F'_m \otimes_G C_n$ , we see that the image of  $j\#\partial\tilde{z}$  is in  $D_{*+1, *}$ . Thus  $z$  is a relative boundary and  $H_*(F'_* \otimes_G C_*, D_{**}) = 0$ . From the long exact homology sequence we get an isomorphism  $H_*(D_{**}) \xrightarrow{\cong} H_*(F'_* \otimes_G C_*)$ . Since both  $F'_* \otimes_G C_*$  and  $D_{**}$  are  $\mathbb{Z}$ -free, the universal coefficient theorems and the 5-Lemma yield isomorphisms for homology and cohomology with arbitrary coefficient groups  $A$  (with trivial  $G$ -action); see [26, 5.3.15, 5.5.3].  $\square$

The remaining part of the comparison is routine. We denote elements of  $G$  by  $g, h, i$  and vertices of  $\Gamma$  by  $u, v, w$ . We define two 1-cochains  $f_1, f_2$  on  $D_{**}$  by

$$f_1((g) \otimes (u, v)) = c(g(o), u, v) \quad \text{and} \quad f_2((g, h) \otimes (u)) = c(g(o), h(o), u),$$

where  $c$  is the given  $G$ -invariant 2-cocycle on  $\text{Fl}(\Gamma)$ . Then  $df_\nu = f_\nu \partial$  and using the cocycle identity for  $c$ , we obtain

$$\begin{aligned} df_1((g) \otimes (u, v, w)) &= (c(g(o), v, w) - c(g(o), u, w) + c(g(o), u, v)) \\ &= c(u, v, w) \\ df_1((g, h) \otimes (u, v)) &= c(h(o), u, v) - c(g(o), u, v) \\ df_2((g, h) \otimes (u, v)) &= -c(g(o), h(o), v) + c(g(o), h(o), u) \\ &= df_1((h, i) \otimes (x, y)) \\ df_2((g, h, i) \otimes (u)) &= c(h(o), i(o), u) - c(g(o), i(o), u) + c(g(o), h(o), u) \\ &= c(g(o), h(o), i(o)), \end{aligned}$$

which shows that

$$df_1 - df_2 = c - \hat{c}.$$

**THEOREM 20.** *Let  $G$  be a group acting vertex-transitively on a graph  $\Gamma$  having the star property, and let  $c : C_2(\text{Fl}(\Gamma)) \rightarrow A$  be a  $G$ -invariant  $A$ -valued 2-cocycle (where  $G$  acts trivially on  $A$ ). Fix a vertex  $o$  of  $\Gamma$  and let  $F'_* \subseteq F_*$  and  $C_*$  be as in Subsection 8.3. Then the cocycles*

$$\hat{c} : F'_2 \otimes_G \mathbb{Z} \longrightarrow A, \quad (g_0, g_1, g_2) \otimes 1 \longmapsto c(g_0(o), g_1(o), g_2(o))$$

and

$$c : F_0 \otimes_G C_2 \longrightarrow A, \quad g \otimes (x, y, z) \longmapsto c(x, y, z)$$

are homologous under the isomorphism

$$H^2(G; A) \xrightarrow{\cong} H_G^2(C_*; A).$$

Moreover, there exists a cocycle  $\hat{c} : F_2 \otimes_G \mathbb{Z} \rightarrow A$  extending  $\hat{c}$ , i.e.  $\hat{c} = \hat{c}|_{F'_*}$ .

*Proof.* Only the last claim remains to be proved. Since the inclusion  $F'_* \subseteq F_*$  induces an isomorphism in cohomology, we find a cocycle  $\tilde{c}$  on  $F_* \otimes_G \mathbb{Z}$  such that  $\hat{c} - \tilde{c}|_{F'_* \otimes_G \mathbb{Z}} = da$  is a coboundary. Now  $F'_* \otimes_G \mathbb{Z}$  is a direct summand in the  $\mathbb{Z}$ -free module  $F_* \otimes_G \mathbb{Z}$ , and thus we can extend  $a$  to a 1-cochain  $\tilde{a}$  on  $F_* \otimes_G \mathbb{Z}$ . Then  $(\tilde{c} + d\tilde{a})|_{F'_* \otimes_G \mathbb{Z}} = \hat{c}$ .  $\square$

**COROLLARY 21.** *For a field  $D$  of characteristic not equal to 2, Kashiwara's Maslov cocycle and our Maslov cocycle yield the same cohomology class in  $H^2(\mathbf{Sp}_{2n} D; WD)$ .*

We obtain also the following general result for unitary groups over arbitrary skew-fields.

**COROLLARY 22.** *If  $D^\varepsilon$  is infinite and  $o \in \mathcal{L}$  is a fixed Lagrangian, then there exists a group cocycle  $\tau : \mathbf{U}(M) \times \mathbf{U}(M) \rightarrow W^1(D, J)$  such that*

$$\tau(g, h) = \langle \kappa(o, g(o), gh(o)) \rangle$$

holds for all pairs  $g, h$  with  $o, g(o), gh(o)$  pairwise opposite.

### 9. The Maslov cocycle as a central extension

The reduced Maslov cocycle defines a central extension [2, IV.3; 6, 1.4C]

$$1 \longrightarrow \Pi(D, J) \longrightarrow \widehat{\mathbf{EU}(M)} \longrightarrow \mathbf{EU}(M) \longrightarrow 1$$

of  $\mathbf{EU}(M)$  by  $II(D, J)$ . This extension is uniquely determined by the homomorphism

$$[\tilde{m}] \in H^2(\mathbf{EU}(M); II(D, J)) \cong \text{Hom}(H_2(\mathbf{EU}(M)), II(D, J));$$

our aim is to determine this homomorphism  $H_2(\mathbf{EU}(M)) \rightarrow II(D, J)$  algebraically. In view of the naturality that we proved in Section 6, we begin with the smallest case  $\mathbf{Sp}_2 D = \mathbf{SL}_2 D$ , where  $D$  is an infinite field. We do allow fields of characteristic 2, as we rely on the results in [15, 17] which are valid over arbitrary (infinite) fields. Note, however, that in our setup the Witt group  $W^1(D, \text{id})$  is always the Witt group of symmetric bilinear forms (and not of quadratic forms).

9.1. *The Schur multiplier of  $\mathbf{SL}_2 D$  and the Steinberg cocycle*

We consider

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad a_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \quad b_r = \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix}$$

Since  $\mathbf{SL}_2 D$  is a two-transitive group, every element is either of the form  $a_r u_t$  or of the form  $u_s b_r u_t$ . We define  $K\text{Sp}_2 D$  as the abelian group generated by symbols  $\{x, y\}$ , for  $x, y \in D^*$ , (the *symplectic Steinberg symbols*), subject to the relations:

$$\begin{aligned} \{st, r\} + \{s, t\} &= \{s, tr\} + \{t, r\}, \\ \{s, 1\} &= \{1, s\} = 0, \\ \{s, t\} &= \{t^{-1}, s\}, \\ \{s, t\} &= \{s, -st\}, \\ \{s, t\} &= \{s, (1-s)t\} \quad \text{if } s \neq 1. \end{aligned}$$

According to [15, 5.11; 17, p. 199] the Schur multiplier of  $\mathbf{SL}_2 D$  is  $H_2(\mathbf{SL}_2 D) \cong K\text{Sp}_2 D$ . Moreover, the Steinberg normal form of the universal group cocycle

$$\text{stbg} : \mathbf{SL}_2 D \times \mathbf{SL}_2 D \longrightarrow H_2(\mathbf{SL}_2 D)$$

is given for ‘generic’ group elements by

$$\text{stbg}(g(s_1, r_1, t_1), g(s_2, r_2, t_2)) = \left\{ \frac{t}{r_1 r_2}, -\frac{r_1}{r_2} \right\} - \{-r_1, -r_2\},$$

where  $t = t_1 + s_2 \neq 0$  and  $g(s, r, t) = u_s b_r u_t$ ; cf. [11, 15, 5.12; 17, p. 198 (1)] in a more special situation. We note that the formula given in [17, p. 198 (1)] is incorrect. The formula above is due to Schwarze [24, 5.9] and agrees with Matsumoto’s calculations.

9.2. Given  $x, y \in D$ , we denote by  $(x, y)_D$  the 4-dimensional symmetric bilinear form:

$$(x, y)_D = \langle 1, -x, -y, xy \rangle.$$

If  $\text{char}(D) \neq 2$ , then this is the norm form of the quaternion algebra  $\left(\frac{x, y}{D}\right)$  (see [23, 2.§ 11]). Obviously,  $(x, y)_D \in II(D, \text{id})$ , and  $(x, y)_D = (y, x)_D = (xz^2, y)_D$ . Using the fact that the metabolic form  $\langle x, -x \rangle$  vanishes in  $W^1(D, \text{id})$ , it is routine to verify that these elements satisfy the first four defining relations of  $K\text{Sp}_2 D$ ; for example  $(s, -st)_D = \langle 1, -s, st, -s^2 t \rangle \cong \langle 1, -s, st, -t \rangle \cong (s, t)_D$ . For the last relation, it suffices to check that  $\langle -t, st \rangle \cong \langle -(1-s)t, (1-s)st \rangle$  for  $s \neq 1$ . This follows from

$$\begin{pmatrix} 1 & 1 \\ s & 1 \end{pmatrix} \begin{pmatrix} -t & 0 \\ 0 & st \end{pmatrix} \begin{pmatrix} 1 & s \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -(1-s)t & 0 \\ 0 & (1-s)st \end{pmatrix}$$

Thus we have a homomorphism

$$R : K\text{Sp}_2(D) \longrightarrow II(D, \text{id}) \subseteq W^1(D, \text{id}),$$

which maps the symplectic Steinberg symbol  $\{u, v\}$  to the 4-dimensional symmetric bilinear form  $R(\{u, v\}) = (u, v)_D$ .

Applying  $R$  to the Steinberg cocycle, for ‘generic’ group elements (with the same notation as before), we obtain

$$\begin{aligned} R \circ \text{stbg}(g(s_1, r_1, t_1), g(s_2, r_2, t_2)) &= \left( \frac{t}{r_1 r_2}, -\frac{r_1}{r_2} \right)_D - (-r_1, -r_2)_D \\ &= (r_1 r_2 t, r_1 r_2)_D - (-r_1, -r_2)_D \\ &= (-r_1 r_2, t)_D - (-r_1, -r_2)_D \\ &= \langle 1, r_1 r_2, -t, -r_1 r_2 t \rangle - \langle 1, r_1, r_2, r_1 r_2 \rangle \\ &= \langle 1, r_1 r_2, -t, -r_1 r_2, t, -1, -r_1, -r_2, -r_1 r_2 \rangle \\ &= \langle -t, -r_1 r_2 t, -r_1, -r_2 \rangle = -\langle t, r_1 r_2 t, r_1, r_2 \rangle. \end{aligned}$$

9.3. We compare this expression with the reduced Maslov cocycle. In  $M = D^2$  we consider  $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $X = \mathbf{x}D$  and  $o = (X, \mathbf{x})$ . Using the notation of Subsection 8.4, we have for  $F'_2$  the formula

$$\tau(g_1, g_2) = \tau([g_1|g_2]) = \tilde{\mathbf{m}}(o, g_1(o), g_1 g_2(o)),$$

where three vertices  $o, g_1(o), g_1 g_2(o)$  have to be pairwise opposite. For the first pair of vertices, this condition gives  $g_1 = u_{s_1} b_{r_1} u_{t_1}$ , and for the second pair  $g_2 = u_{s_2} b_{r_2} u_{t_2}$ . Then we have

$$\begin{aligned} \tau([g_1|g_2]) &= \tilde{\mathbf{m}}(o, g_1(o), g_1 g_2(o)) \\ &= \tilde{\mathbf{m}}(g_1^{-1}(o), o, g_2(o)) \\ &= -\tilde{\mathbf{m}}(o, g_1^{-1}(o), g_2(o)) \\ &= -\tilde{\mathbf{m}}(o, g(-t_1, -r_2, -s_1)(o), g(s_2, r_2, t_2)(o)) \\ &= -\tilde{\mathbf{m}}(o, u_{-t_1} b_{-r_1}(o), u_{s_2} b_{r_2}(o)) \\ &= -\tilde{\mathbf{m}}(o, b_{-r_1}(o), u_{t_1+s_2} b_{r_2}(o)), \end{aligned}$$

which yields the additional condition  $t = t_1 + s_2 \neq 0$  that ensures that the first and third vertices are opposite. Note that by Subsection 8.3 the class of any 2-cocycle is completely determined by its values on  $F'_2$ , and hence it suffices indeed to work with ‘generic’ elements. The explicit formula in Subsection 7.6 for the reduced Maslov cocycle now yields  $a = 1$ ,  $b = r_1^{-1}$  and  $c = -r_2^{-1}$ , whence

$$\tau([g_1|g_2]) = -\langle t, r_1, r_2, r_1 r_2 t \rangle = R_* \text{stbg}([g_1|g_2]).$$

For  $\mathbf{SL}_2 D$  over fields of characteristic not equal to 2, the following result was proved in [1, 19; Section 5].

**THEOREM 23.** *Let  $D$  be an infinite field. The central extension of  $\mathbf{Sp}_{2n} D$  determined by the reduced Maslov cocycle is given by the homomorphism  $R : K\mathbf{Sp}_2 D \rightarrow H(D, \text{id})$ .*

*Proof.* For  $n = 1$  we have shown this in Subsection 9.3. In general, the standard inclusion  $\mathbf{Sp}_{2n} D \hookrightarrow \mathbf{Sp}_{2n+2} D$  induces for all  $n \geq 1$  an isomorphism in 2-dimensional homology such that the universal Steinberg cocycle for  $\mathbf{Sp}_{2n+2} D$  restricts to the universal Steinberg cocycle for  $\mathbf{Sp}_{2n} D$  (see [15, 5.11]). The result now follows by induction on  $n$ .  $\square$

For fields of characteristic not equal to 2, this is stated in [21, 3.1]. However, the proof has a gap: the authors evaluate the reduced Maslov cocycle on the torus (the diagonal matrices) and compare it there with the universal Steinberg cocycle. However, they fail to show that the

reduced Maslov cocycle is a Steinberg cocycle, and therefore they cannot use the comparison theorem [15, 5.10].

In any case, this result settles the situation for symplectic groups over infinite fields of arbitrary characteristic. Note that for fields of characteristic not equal to 2, the map  $R$  is surjective [23, 4.5.5], and thus  $\widehat{\mathbf{Sp}}_{2n} D$  is an epimorphic image of the universal central extension.

#### 9.4. Local fields

By a local field we mean a locally compact (nondiscrete) field; the connected local fields are  $\mathbb{R}, \mathbb{C}$  and the totally disconnected ones are the finite extensions of the  $p$ -adic fields  $\mathbb{Q}_p$  and, in positive characteristic, the fields  $\mathbb{F}_q((X))$  of formal Laurent series over finite fields [29, 1.3]. Being a closed subgroup of the general linear group, a symplectic or unitary group over a local field is in a natural way a locally compact group.

#### 9.5. The Maslov cocycle over $\mathbb{R}$

For  $D = \mathbb{R}$ , the Witt group  $W\mathbb{R} = W^1(\mathbb{R}, \text{id})$  is isomorphic to  $\mathbb{Z}$  via the signature  $\text{sig} : W\mathbb{R} \rightarrow \mathbb{Z}$  (see [23, 2.4.8]); the fundamental ideal  $I\mathbb{R}$  has index 2, and  $II\mathbb{R} = I^2\mathbb{R}$  has index 4. We note that

$$\text{sig}((x, y)_D) = \begin{cases} 4 & \text{if } x, y < 0 \\ 0 & \text{else} \end{cases}$$

By [15, p. 51; 17, 10.4]; this  $4\mathbb{Z}$ -valued cocycle yields precisely the universal covering group  $\widehat{\mathbf{Sp}}_{2n} \mathbb{R}$  of  $\mathbf{Sp}_{2n} \mathbb{R}$ .

We compare the relevant classifying spaces. Let  $B\mathbf{Sp}_{2n} \mathbb{R}^\delta$  denote the classifying space for  $\mathbf{Sp}_{2n} \mathbb{R}$ , viewed as a discrete topological group and let  $B\mathbf{Sp}_{2n} \mathbb{R}$  be the classifying space for the Lie group  $\mathbf{Sp}_{2n} \mathbb{R}$ ; the latter is homotopy equivalent to  $B\mathbf{U}(n)$ , as  $\mathbf{U}(n) \subseteq \mathbf{Sp}_{2n} \mathbb{R}$  is from [7, X, Table V] and Iwasawa’s theorem [7, VI, § 2] a homotopy equivalence. The classifying space  $B\mathbf{Sp}_{2n} \mathbb{R}^\delta$  is an Eilenberg–MacLane space of type  $K(\mathbf{Sp}_{2n} \mathbb{R}, 1)$  whose cohomology is naturally isomorphic to the abstract group cohomology of  $\mathbf{Sp}_{2n} \mathbb{R}$  (see [2, II.4]). The identity map from the discrete group to the Lie group induces a continuous map between the classifying spaces

$$F : B\mathbf{Sp}_{2n} \mathbb{R}^\delta \longrightarrow B\mathbf{Sp}_{2n} \mathbb{R}.$$

On the right, the universal covering is classified by the first Chern class  $c_1$ . This shows that under the forgetful map  $F^*$  the first universal Chern class  $c_1 \in H^2(B\mathbf{SU}(n)) \cong H^2(B\mathbf{Sp}_{2n} \mathbb{R})$  pulls back to the Maslov cocycle as follows:

$$F^*(c_1) = [\tilde{m}]$$

(if the sign for the Chern classes is chosen appropriately). The real Maslov cocycle may be viewed therefore as a combinatorial description of the first Chern class; this was observed in [28].

**PROPOSITION 24.** *Under the forgetful functor from topological groups to abstract groups, the first Chern class for  $\mathbf{Sp}_{2n} \mathbb{R}$  maps to the reduced Maslov cocycle.*

As  $II(\mathbb{C}, \text{id}) = 0$  the Maslov cocycle for  $\mathbf{Sp}_{2n} \mathbb{C}$  vanishes. Now we turn to nonarchimedean local fields; cf. [13, p. 104–115].

9.6. We assume that  $D$  is a nonarchimedean and nondyadic local field (that is, the characteristic of the residue field of  $D$  is not equal to 2). The Witt group  $WD$  has 16 elements, the group  $\widehat{S}$  of extended square classes 8, and thus  $II(D, \text{id}) = I^2 D$  is cyclic of order 2 (see

[12, VI.2.2]). Its nontrivial element is represented by the norm form of the unique quaternion division algebra over  $D$ . Let  $S$  denote the group of square classes of  $D$  and let

$$(-, -)_H : S \times S \longrightarrow \{\pm 1\}$$

be the Hilbert symbol [12, p. 159]:  $(x, y)_H = -1$  if  $(x, y)_D$  is anisotropic, that is, the norm form of a quaternion division algebra. Consider  $e : I^2 D \xrightarrow{\cong} \{\pm 1\}$ , then we clearly have

$$e \circ R \circ \text{stbg}(x, y) = (x, y)_H$$

for  $\mathbf{SL}_2 D$ . Thus the reduced Maslov cocycle for  $\mathbf{SL}_2 D$  is the reduction of the universal cocycle  $\text{stbg}$  to  $\{\pm 1\}$  via the Hilbert symbol. As in the proof of Theorem 24, this carries over to  $\mathbf{Sp}_{2n} D$ . The following result is partially contained in [13, pp. 104–115].

**PROPOSITION 25.** *Let  $D$  be a nonarchimedean nondyadic local field. The reduced Maslov cocycle defines a twofold nontrivial covering of  $\mathbf{Sp}_{2n} D$  that is determined by the Hilbert symbol  $K\text{Sp}_2 D \rightarrow \{\pm 1\}$ . The corresponding covering group  $\widehat{\mathbf{Sp}}_{2n} D$  is a locally compact group; it is the unique nontrivial twofold covering of  $\mathbf{Sp}_{2n} D$  in the category of locally compact groups.*

*Proof.* Only the topological result remains to be proved. It is shown in [17, 10.4] that in the category of locally compact groups,  $\mathbf{Sp}_{2n} D$  admits a universal central extension  $\widehat{\mathbf{Sp}}_{2n} D$ ; the extending group is the group  $\mu(D)$  of all roots of unity in  $D$ . (See [22] for a modern account and a much more general result.) This group  $\mu(D)$  is a finite cyclic group [17, Ch. II] and of even order  $2n$ , as it contains the involution  $-1$ . The corresponding Steinberg cocycle is given by the norm residue symbol  $K\text{Sp}_2 D \rightarrow \mu(D)$  (see [17, Chapter II]). However, the  $n$ th power of the norm residue symbol is the Hilbert symbol. This shows that  $\widehat{\mathbf{Sp}}_{2n} D$  is a continuous quotient of  $\mathbf{Sp}_{2n} D$ . As the cyclic group  $\mu(D)$  has a unique subgroup of index 2, the extension is the unique nonsplit twofold topological extension.  $\square$

9.7. Finally, we consider unitary groups over fields. We assume that  $E$  is a field with an automorphism  $J \neq \text{id}$  of order 2; the fixed field is  $D \subseteq E$ . We denote the hyperbolic unitary group by  $\mathbf{U}_{2n} E$ ; then  $\mathbf{EU}_{2n} E = \mathbf{SU}_{2n} E = \mathbf{U}_{2n} E \cap \mathbf{SL}_{2n} E$  (see [6, 6.4.25, 6.4.27]). As we have noted in Section 6, there is a natural injection  $\Phi : \mathbf{Sp}_{2n} D \hookrightarrow \mathbf{SU}_{2n} E$  and we have a commutative diagram as follows:

$$\begin{array}{ccc} H_2(\mathbf{Sp}_{2n} D) & \xrightarrow{\Phi_*} & H_2(\mathbf{SU}_{2n} E) \\ \downarrow [\tilde{m}_D] & & \downarrow [\tilde{m}_E] \\ \Pi(D, \text{id}) & \xrightarrow{W_E^D} & \Pi(E, J). \end{array}$$

Unfortunately, the Schur multiplier  $H_2(\mathbf{SU}_{2n} E)$  seems to be less understood than its symplectic counterpart. However it is proved in [4, 2.1, 2.5] (and in a weaker form in [6, 6.5.12]) that the map  $\Phi_*$  is surjective, and thus  $H_2(\mathbf{SU}_{2n} E)$  is a quotient of  $K\text{Sp}_2 D$ . (Here  $\mathbf{SU}_{2n} E$  is the group of  $D$ -points of a quasisplit absolutely simple and simply connected algebraic group over  $D$ , and hence the results from [4] apply.)

The following facts concerning  $W_E^D$  were kindly pointed out by W. Scharlau. First, the map

$$W_E^D : W^1(D, \text{id}) \longrightarrow W^1(E, J)$$

is an epimorphism, because every hermitian form can be diagonalized (even in characteristic 2, see [9, I.6.2.4]) and thus is the image of a diagonal symmetric bilinear form over  $D$ .

Assume now that  $\text{char}(D) \neq 2$  and  $E = D(\sqrt{\delta})$ . Passing from a hermitian form  $h$  over  $E$  to its trace form  $b_h$  over  $D$  (see [23, p. 348]), we have an monomorphism  $\text{trf} : W^1(E, J) \rightarrow W^1(D, \text{id}) = WD$ ; explicitly, we hence

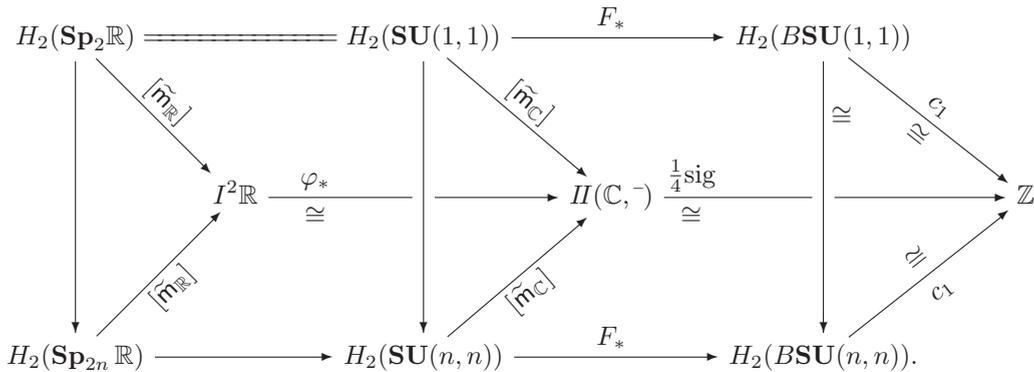
$$\text{trf}\langle a_1, \dots, a_n \rangle = \langle 1, -\delta \rangle \otimes \langle a_1, \dots, a_n \rangle.$$

In particular,  $\text{trf} \circ W_E^D((x, y)_D) = \langle 1, -\delta \rangle \otimes \langle 1, -x \rangle \otimes \langle 1, -y \rangle$ , and  $W_E^D(I^2D)$  is isomorphic to a subgroup of  $I^3D$ .

It follows that the Maslov cocycle for the unitary group over a nonarchimedean nondyadic local field  $E$  vanishes, because  $I^3D = 0$  (see [12, VI.2.15(3)]). The case of the complex numbers is more interesting.

9.8. Complex unitary groups

For  $E/D = \mathbb{C}/\mathbb{R}$  the map  $W_{\mathbb{C}}^{\mathbb{R}} : W\mathbb{R} \rightarrow W^1(\mathbb{C}, -)$  and its restriction  $I^2\mathbb{R} \rightarrow II(\mathbb{C}, -)$  is an isomorphism. We use the standard Lie group notation  $\mathbf{SU}_{2n}\mathbb{C} = \mathbf{SU}(n, n)$  (see [7]) (note that multiplication by  $i$  transforms skew-hermitian into hermitian matrices). The maximal compact subgroup is  $\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$ . As in Subsection 9.5 we compare the classifying space for the discrete group (whose homology is the abstract group homology) with the classifying space  $B\mathbf{SU}(n, n)$  for the Lie group. For  $n = 1$  we have an isomorphism  $\mathbf{Sp}_2\mathbb{R} = \mathbf{SU}_2\mathbb{C}$ , whence a big commutative diagram



PROPOSITION 26. If we identify the first Chern class  $c_1$  with the generator of  $H^2(B\mathbf{SU}(n, n))$ , it pulls under the forgetful map  $F$  back to the reduced Maslov cocycle for  $\mathbf{SU}(n, n)$ . Thus  $\mathbf{SU}(n, n)$  is the universal covering group of the Lie group  $\mathbf{SU}(n, n)$ .

9.9. Sharpe [25] (see also 6 5.6D\*) has constructed an exact sequence:

$$K_2(D) \longrightarrow KU_2^{-1}(D, J) \longrightarrow L_0^1(D, J) \longrightarrow 0$$

The  $L$ -group  $L_0^1(D, J)$  maps onto  $II(D, J)$  and we conjecture that the composite

$$KU_2^{-1}(D, J) \longrightarrow II(D, J)$$

‘is’ (in most cases) the reduced Maslov cocycle  $\tilde{m} : H_2(\mathbf{EU}(M)) \rightarrow II(D, J)$ . In the symplectic situation over fields of characteristic not equal to 2, this is indeed the case by Theorem 23 and [6, 5.6.8]. However, a proof would certainly require a different description of the relevant maps than the one in [25].

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