Lectures on the Theorem of Browder and Novikov

and

Siebenmann's Thesis

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PART I

THEOREM OF BROWDER AND NOVIKOV

§ 1. PRELIMINARIES.

1.1. THE CAP-PRODUCT.

The homology and the cohomology groups we use are the singular ones. Let $\mathbb{Z}$ denote the ring of integers and Λ an arbitrary commutative ring with $1 \neq 0$. For any topological space X and any integer $n \geqslant 0$ the set of singular n-simplices of X is denoted by $S_n(X)$. For any $s \in S_n(X)$ and any integer i satisfying $0 \leqslant i \leqslant n$ let $s(0,..i)$ (resp. $s(i,..,n)$) denote the element of $S_i(X)$ (resp. $S_{n-i}(X)$) got by restricting s to the front i-dimensional (resp. the rear (n-i)-dimensional) face of the standard n-simplex Δ_n. Let $C(X)$ denote the singular chain complex of X over $\mathbb{Z}$ and $C = C(X) \otimes_{\mathbb{Z}} \Lambda$ the chain complex of X over Λ. The cochain complex of X over Λ which is defined as $\mathrm{Hom}_{\mathbb{Z}}(C(X), \Lambda)$ is canonically isomorphic to $\mathrm{Hom}_{\Lambda}(C(X) \otimes_{\mathbb{Z}} \Lambda, \Lambda)$. The boundary homomorphism δ in $C^* = \mathrm{Hom}_{\Lambda}(C, \Lambda)$ is given by $f = (-1)^{n-1} f \circ \partial$ for every $f \in C^n(X, \Lambda) = \mathrm{Hom}\,(C_n, \Lambda)$ where $\partial : C_n \longrightarrow C_{n-1}$ is the boundary homomorphism in C. As usual C^* is considered as a chain complex with $C^*_{-n} = C^n(X, \Lambda)$. The evaluation map $e : C^* \otimes_{\Lambda} C \longrightarrow \Lambda$ is defined by $e(f \otimes c) = f(c) \; \forall f \in C^*_{-n}$ and $c \in C_n$ and $e \mid C^*_{-p} \otimes C_q = 0$ whenever $p \neq q$. Considering Λ as a chain complex (with all its elements of degree zero) it is easily seen that $e : C^* \otimes_{\Lambda} C \longrightarrow \Lambda$ is a chain homomorphism.

For any two chain complexes A and B over Λ let $\alpha : H(A)\otimes_\Lambda H(B) \longrightarrow H(A\otimes_\Lambda B)$ be the natural map. If $x\in H_p(A)$ and $y\in H_q(B)$ and if z and z' are respectively cycles of A and B representing x and y, then $z\otimes z'$ is a cycle of $A\otimes B$ and the homology class of $z\otimes z'$ is by definition $\alpha(x\otimes y)$. Let $T : A\otimes_\Lambda B \longrightarrow B\otimes_\Lambda A$ be the chain isomorphism given by $T(a\otimes b) = (-1)^{pq}\, b\otimes a \;\forall a \in A_p,\ b \in B_q$.

The Alexander-Whitney diagonal map $m_o : C \longrightarrow C\otimes_\Lambda C$ is defined to be the unique Λ-homomorphism satisfying

$$m_o(s) = \sum_{i=0}^{n} s(0,..,i)\otimes_\Lambda s(i,..,n) \;\forall s \in S_n(X).$$

It is well-known and is not hard to check that m_o is a chain map. We denote the composition of the chain homomorphisms indicated in the following diagram

$$C^*\otimes_\Lambda C \xrightarrow{Id_{C^*}\otimes m_o} C^*\otimes_\Lambda C\otimes_\Lambda C \xrightarrow{T\otimes Id_C} C\otimes_\Lambda C^*\otimes_\Lambda C \xrightarrow{Id_C\otimes e} C\otimes_\Lambda \Lambda = C$$

by $\cap : C^*\otimes_\Lambda C \longrightarrow C$. More explicitly this map is given by

$$\cap(f\otimes s) = f\cap s = \begin{cases} (-1)^{q(n-q)} f\,(s(n-q,\ldots,n)).s(o,..,n-q) & \text{if } n\geq q \\ o \text{ if } n\ \ q \end{cases}$$

for every $f \in C^q(X,\Lambda)$ and $s \in S_n(X)$. Let $H(\cap) : H(C^*\otimes_\Lambda C) \longrightarrow H(C)$ be the homomorphism induced by '$\cap$'.

For any $a \in H^q(C^*) = H_{-q}(C^*) = H^q(X, \Lambda)$ and $u \in H_n(C) = H_n(X, \Lambda)$ the element $H(\cap) \circ \alpha(a \otimes u)$ is called the cap-product of a by u and is denoted by $a \cap u$.

The chain map $e : C^* \hat{\otimes} C \longrightarrow \Lambda$ induces a homomorphism $H(e) : H(C^* \hat{\otimes} C) \longrightarrow \Lambda$. For any $a \in H^q(X, \Lambda)$ and $u \in H_q(X, \Lambda)$ the image $H(e) \circ \alpha(a \otimes u)$ is known as the value of the cohomology class a on the homology class u and is denoted by $a(u)$.

1.2. The following properties of the cap-product will be needed later.

① $(a \cup b) \cap u = a \cap (b \cap u) \; \forall a \in H^p(X, \Lambda)$, $b \in H^q(X, \Lambda)$ and $u \in H_n(X, \Lambda)$ with p, q, n arbitrary integers. Here $a \cup b$ denotes the Cup product of a and b.

② For any continuous map $f : Y \longrightarrow X$, if the induced homomorphisms in homology and cohomology are denoted by $f_* : H(Y, \Lambda) \longrightarrow H(X, \Lambda)$ and $f^* : H^*(X, \Lambda) \longrightarrow H^*(Y, \Lambda)$, then for any $a \in H^q(X, \Lambda)$ and $v \in H_n(Y, \Lambda)$

$$f_*(f^* a \cap v) = a \cap f_*(v).$$

1.3. POINCARÉ DUALITY.

When we refer to homology and cohomology groups without mentioning the coefficients we mean integer coefficients. Let M be a compact, connected, orientable manifold (without boundary) of dimension n. Then it is known that $H_n(M) \cong \mathbb{Z}$. A choice of a generator u for $H_n(M)$ is known as an orientation for M. M together with a chosen orientation is called an oriented manifold and the distinguished element of $H_n(M)$ is called the fundamental class of M and is denoted by $[M]$.

Let $h : \mathbb{Z} \to \Lambda$ be the obvious ring homomorphism (which sends 1 of $\mathbb{Z}$ into 1 of Λ). Let $v = h_*([M])$ where $h_* : H_n(M) \longrightarrow H_n(M, \Lambda)$ is the homomorphism induced by h. Then Poincaré duality can be stated as follows:

The map $\Delta : H^q(M,\Lambda) \to H_{n-q}(M, \Lambda)$ given by $\Delta(x) = x \cap v$ is an isomorphism for all q.

In case M is not necessarily orientable it is true that $H_n(M; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and if v denotes the non zero element of $H_n(M; \mathbb{Z}_2)$ then $\cap v : H^q(M; \mathbb{Z}_2) \to H_{n-q}(M; \mathbb{Z}_2)$ is an isomorphism for all q.

When M is compact and not necessarily connected M is orientable if and only if each of its connected components is orientable. M being compact, the number of connected components is finite and denoting them by $\{M_j\}_{j=1}^{r}$ we have $H_n(M) \simeq \bigoplus_{j=1}^{r} H_n(M_j)$. If each M_j is oriented and if $[M_j]$ is the fundamental class of M_j then $[M] = \sum_{j=1}^{r} [M_j] \in H_n(M) = \bigoplus_{j=1}^{r} H_n(M_j)$ is defined to be the fundamental class of M.

1.4. All the vector bundles we consider are real vector bundles. For any X the trivial vector bundle of rank l over X will be denoted by θ_X^l. The total space and the base space of any vector bundle ξ will be denoted by $E(\xi)$ and B_ξ respectively. To denote that ξ is of rank k we just write ξ^k. If $f : Y \to X$ is a continuous map and ξ any vector bundle over X the pull back bundle on Y is denoted by $f^!(\xi)$. If ξ carries a Riemannian metric, for any $\varepsilon > 0$ the subspace of $E(\xi)$ consisting of vectors of length $\leqslant \varepsilon$ is denoted by $E_\varepsilon(\xi)$ and the boundary consisting of vectors of length ε is denoted by $\dot{E}_\varepsilon(\xi)$. When B_ξ is compact the Thom space of ξ denoted by $T(\xi)$ is defined to be the one point compactification of $E(\xi)$. Let '∞' denote the point at infinity of $T(\xi)$. When ξ carries a Riemannian metric we can describe the Thom space alternatively as follows. Let $T_\varepsilon(\xi)$ be the quotient space got from $E_\varepsilon(\xi)$ by collapsing $\dot{E}_\varepsilon(\xi)$ to a point. The map $\beta : E_\varepsilon(\xi) \to T(\xi)$ defined by $\beta(\vec{v}) = \dfrac{\vec{v}}{\varepsilon - \|\vec{v}\|}$ for $\vec{v} \in E_\varepsilon(\xi) - \dot{E}_\varepsilon(\xi)$ and $\beta(\vec{v}) = \infty$ for $\vec{v} \in \dot{E}_\varepsilon(\xi)$ passes down to a homeomorphism $\theta : T_\varepsilon(\xi) \to T(\xi)$. Compactness of B_ξ is essential for θ to be a homeomorphism.

For any differential ($=C^\infty$) manifold M the tangent bundle of M will be denoted by τM. The word differentiable will always mean differentiable of class C^∞ for us. For the rest of this section M denotes a compact, connected, oriented differential manifold of dimension $n \geq 0$ with $[M]$ as the fundamental class. By Whitney's imbedding theorem M can be differentiably imbedded in $\mathbb{R}^{n+k}$. Except when $n = 0$ the compactness of M automatically implies that $k \geq 1$. Even when $n = 0$ we can assume $k \geq 1$. Let ν be the normal bundle of this imbedding. Then $\tau_M \oplus \nu \simeq \vartheta_M^{n+k}$. Since τ_M and ϑ_M^{n+k} are both orientable it follows that ν is an orientable vector bundle. Identifying the tangent space to $\mathbb{R}^{n+k}$ at any point with $\mathbb{R}^{n+k}$ in the usual way and taking the usual Riemannian metric on $\tau_{\mathbb{R}^{n+k}} \simeq \vartheta^{2n+2k}_{\mathbb{R}^{n+k}}$ any element of $E(\nu)$ can be thought of as a pair $(x, \vec{v})$ with $x \in M$ and $\vec{v} \in \mathbb{R}^{n+k}$ in a direction normal to M at x. Let $e : E(\nu) \longrightarrow \mathbb{R}^{n+k}$ be defined by $e(x,v) = x+v$. $\exists$ an $\varepsilon > 0$ such that e is a diffeomorphism of the set $E_\varepsilon(\nu)$ on to a neighbourhood A of M. A is called a closed tubular neighbourhood of M. Let $\dot{A} = e(\dot{E}_\varepsilon(\nu))$. Considering S^{n+k} as the one point compactification of $\mathbb{R}^{n+k}$ we can define a map $C : S^{n+k} \longrightarrow T(\nu)$. This is the map got by collapsing the complement of $A - \dot{A}$ in S^{n+k} to a point. More precisely, $C \mid A = \beta \circ e^{-1}$ and $C \mid (S^{n+k} - A) = \infty$.

Let $\Phi : H_n(M) \longrightarrow H_{n+k}(T(\nu))$ be the Thom isomorphism [5].

<u>Proposition 1.5</u>. $\Phi([M]) = C_*(\zeta)$ for a generator ζ of $H_{n+k}(S^{n+k})$.

<u>Proof</u>. We have only to show that $C_* : H_{n+k}(S^{n+k}) \longrightarrow H_{n+k}(T(\nu))$ is an isomorphism. We abbreviate $E_\varepsilon(\nu)$ by E_ε etc. Let $A_{\frac{1}{2}} = e(E_{\varepsilon/2})$. Clearly $\beta \mid E_{\frac{\varepsilon}{2}}$ is a homeomorphism of $E_{\frac{\varepsilon}{2}}$ onto

the image Γ (say). Let x be any point in M (such a point exists because $\dim M \geqslant 0$ by assumption) and

$$i_x : S^{n+k} \longrightarrow (S^{n+k}, S^{n+k} - x) \quad \text{and} \quad j_x : (S^{n+k}, S^{n+k} - M) \longrightarrow (S^{n+k}, S^{n+k} - x)$$

the respective inclusions. Consider the following commutative diagram.

$$\begin{tikzcd}
 & H_{n+k}(S^{n+k}, S^{n+k} - x) & & \\
H_{n+k}(S^{n+k}) \arrow[ur, "(i_x)_*"] \arrow[r, "j_*"'] \arrow[d, "C_*"'] & H_{n+k}(S^{n+k}, S^{n+k} - M) \arrow[u, "(j_x)_*"'] \arrow[d, "C_*", "1"'] & & H_{n+k}(A_{\frac{1}{2}}, A_{\frac{1}{2}} - M) \arrow[ll, "\text{Excision}"', "\approx"] \arrow[d, "(e^{-1})_*"'] \\
H_{n+k}(T(\nu)) \arrow[r, "\approx"] & H_{n+k}(T(\nu), T(\nu) - M) & H_{n+k}(\Gamma, \Gamma - M) \arrow[l, "\approx", "\text{Excision}"'] & H_{n+k}(E_{\frac{\varepsilon}{2}}, E_{\frac{\varepsilon}{2}} - M) \arrow[l, "\approx", "\beta_*"']
\end{tikzcd}$$

Diagram 1

The homomorphism indicated as β_* is an isomorphism since $\beta : E_{\frac{\varepsilon}{2}} \to \Gamma$ is a homeomorphism. It follows that the homomorphism numbered 1 is an isomorphism. The space $T(\nu) - M$ is contractible in itself to ∞. Hence the map $H_{n+k}(T(\nu)) \rightarrow H_{n+k}(T(\nu), T(\nu)-M)$ is an isomorphism. (The assumption $k \geqslant 1$ is used here). Since $H_{n+k}(T(\nu)) \approx H_n(M) \approx \mathbb{Z}$ we have $H_{n+k}(S^{n+k}, S^{n+k}-M)$. Since $(i_x)_*$ is an isomorphism it follows that j_* is a monomorphism and that image of j_* is a direct summand of $H_{n+k}(S^{n+k}, S^{n+k}-M)$. The groups $H_{n+k}(S^{n+k})$ and $H_{n+k}(S^{n+k}, S^{n+k}-M)$ being both isomorphic to $\mathbb{Z}$ it follows that j_* is an isomorphism. It now follows that $c_* : H_{n+k}(S^{n+k}) \longrightarrow H_{n+k}(T(\nu))$ is an isomorphism.

1.6. THE INDEX OF A 4d-DIMENSIONAL MANIFOLD.

Let M be a compact, connected, oriented manifold of dimension $4d$ with d an integer $\geqslant 0$ and let $[M]$ be the fundamental class of M. The image $h_*([M])$ of the fundamental class of M under the inclusion $h : \mathbb{Z} \rightarrow \mathbb{Q}$ is called the fundamental class with coefficients in $\mathbb{Q}$ and is also denoted by $[M]$. The map $(x,y) \rightsquigarrow (x \cup y)[M]$ of $H^{2d}(M, \mathbb{Q}) \times H^{2d}(M, \mathbb{Q}) \rightarrow \mathbb{Q}$ gives a symmetric, non degenerate bilinear form $H^{2d}(M, \mathbb{Q})$. Symmetry is clear from $x \cup y = (-1)^{2d.2d} y \cup x = y \cup x$. That it is non degenerate is a consequence of Poincaré duality together with the fact that $(a, u) \rightsquigarrow a(u)$ is a bilinear non degenerate pairing of $H^{2d}(M, \mathbb{Q}) \times H_{2d}(M, \mathbb{Q}) \rightarrow \mathbb{Q}$. This latter fact is embodied in the universal coefficient theorem $H^{2d}(M,\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(H_{2d}(M,\mathbb{Q}),\mathbb{Q})$. The signature (i.e. the number of +ve diagonal elements minus the number of −ve diagonal elements when diagonalised over $\mathbb{Q}$) of the bilinear form $(x,y) \rightsquigarrow (x \cup y)[M]$ on $H^{2d}(M,\mathbb{Q})$ is defined to be the index of M and is denoted by $I(M)$.

In case M is also differentiable we have the following Theorem of Hirzebruch [1].

<u>Theorem 1.7</u>. Let $L_k(p_1,\dots,p_k)$ be the multiplicative sequence of polynomials corresponding to the power series

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1+\frac{1}{3}t - \frac{1}{45}t^2 + \dots + (-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k t^k + \dots$$

(Here B_k is the k^{th} Bernouilli number). There the index $I(M)$ is equal to the L-genus of M defined as

$\left\{L_d(p_1(\tau_M),\dots,p_d(\tau_M)\right\}([M])$, where $p_i(\tau_M)$ is the i^{th} Pontrjagin class of τ_M.

For more information about the formalism of multiplicative sequences and the correspondence between power series and multiplicative sequences the reader is referred to [1], [5].

We just content ourselves with the remark that $L_k(p_1,\ldots,p_k)$ are universally defined polynomials (i.e. independent of M) with coefficients in the indeterminates $p_1, p_2, \ldots$. The total weight of each term of $L_k(p_1,\ldots,p_k)$ is $4k$ when p_j is alloted the weight $4j$. The first two of these polynomials are

$$L_1(p_1) = \frac{1}{3}p_1; \; L_2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2) .$$

1.8. We will be mainly concerned with a space X which is a finite simplicial complex. Given any vector bundle ξ^k over X there exists a vector bundle η over X with $\xi \oplus \eta \simeq \vartheta_X$ (of some rank). In fact $\exists$ a map $f : X \longrightarrow G_{k+\ell,k}$ (the Grassmann-manifold of k-planes in $\mathbb{R}^{k+\ell}$) for some ℓ such that $f^!(\gamma^k) = \xi$. Here γ^k is the universal bundle on $G_{k+\ell,k}$. The space $E(\gamma^k)$ is the subspace of $G_{k+\ell,k} \times \mathbb{R}^{k+\ell}$ consisting of elements $(y, \vec{v})$ with $\vec{v} \in y$. Let $\tilde{\gamma}^\ell$ be the vector bundle on $G_{k+\ell,k}$ consisting of elements $(y, \vec{w})$ with $\vec{w} \in \mathbb{R}^{k+\ell}$ orthogonal to y. Then $\eta = f^!(\tilde{\gamma}^\ell)$ satisfies $\xi \oplus \eta \simeq \vartheta_X^{k+\ell}$. Two vector bundles ξ and ξ' over X are said to be stably equivalent if $\xi \oplus \vartheta_X^\ell \simeq \xi' \oplus \vartheta_X^{\ell'}$ for some ℓ and ℓ'. The stable class of ξ is denoted by $[\xi]$. If ξ and ξ' are stably equivalent and if η and η' are such that $\xi \oplus \eta \simeq \vartheta^n$ and $\xi' \oplus \eta' \simeq \vartheta^{n'}$ for some n and n' it is easy to see that

η and η' are stably equivalent. The class of η is denoted by $-[\xi]$. It is known that the Pontrjagin classes of a vector bundle depend only on the stable class of the bundle.

If $\overline{p_1}(\xi), \overline{p_2}(\xi), \ldots$ denote the Pontrjagin classes of some η belonging to the class $-[\xi]$ it follows that the elements $L_{k'}(\overline{p_1}(\xi), \ldots, \overline{p_k}(\xi))$ depend only on the class $[\xi]$ of ξ.

Referring to the situation where M^{4d} is differentiably imbedded in $\mathbb{R}^{4d+k}$ with normal bundle ν we see that $L_{k'}(\overline{p_1}(\nu), \ldots, \overline{p_k}(\nu)) = L_{k'}(p_1(\tau_M), \ldots, p_k(\tau_M)) \in H^{4k'}(M, \mathbb{Q})$.

Thus Hirzebruch's theorem can be rephrased in terms of the normal bundle ν as $\left\{L_d(\overline{p_1}(\nu), \ldots, \overline{p_d}(\nu))\right\}([M]) = I(M)$.

§ 2. THE MAIN THEOREM.

Let X be a connected finite simplicial complex with $\pi_1(X) = 0$. The theorem of Browder and Novikov deals with conditions under which X will be of the same homotopy type as a compact differentiable manifold M without boundary. Since X is simply connected if such an M exists it has to be orientable. We first state the theorem, which actually consists of two parts.

Theorem 2.1. Let X be a connected finite simplicial complex with $\pi_1(X) = 0$. Suppose that the following two conditions are satisfied.

i) X satisfies Poincaré duality i.e. to say $\exists$ some integer n with $H_n(X) \simeq \mathbb{Z}$ and if u is a generator, $\cap u : H^q(X) \to H_{n-q}(X)$ is an isomorphism for all q.

ii) $\exists$ an oriented vector bundle ξ^k over X such that $\Phi(u) \in H_{n+k}(T(\xi))$ is spherical, $\Phi : H_n(X) \to H_{n+k}(T(\xi))$ being the Thom isomorphism.

Then if n is odd X is of the same homotopy type as a compact differentiable manifold M of dimension n under a homotopy equivalence $f : M \to X$ satisfying $[f!(\xi)] = -[\tau_M]$.

The second part of the theorem is concerned with the case $n = 4d$ with d an integer > 1.

X being a finite complex we have $H^q(X, \mathbb{Q}) = H^q(X) \otimes \mathbb{Q}$ and $H_i(X, \mathbb{Q}) = H_i(X) \otimes \mathbb{Q}$. Denoting the image of u in $H_n(X, \mathbb{Q})$ under $h_* : H_n(X) \to H_n(X, \mathbb{Q})$ where $h : \mathbb{Z} \to \mathbb{Q}$ is the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$ by v we have $\cap v : H^q(X, \mathbb{Q}) \to H_{n-q}(X, \mathbb{Q})$ an isomorphism for all q. Actually $\cap v$ can be identified with $(\cap u) \otimes \mathbb{Q}$. Thus assumption i) actually implies Poincaré duality for coefficients in $\mathbb{Q}$. Actually, it is true that assumption i) implies Poincaré duality for any arbitrary commutative coefficient ring Λ (with $1 \neq 0$). The procedure adopted to define the index $I(M^{4d})$ in § 1.6 can now be used to define the index $I(X)$ of X.

Assume in addition to i) and ii) we have the following valid for ξ .

iii) $I(X) = \left\{ L_d(\overline{p_1}(\xi), \ldots, \overline{p_d}(\xi)) \right\}(v)$.

Then X is of the same homotopy type as a compact differentiable manifold M of dimension 4d under an equivalence $f : M \to X$ satisfying $[f!(\xi)] = -[\tau_M]$.

Part I of these lectures is devoted to the proof of this theorem. From § 1 it actually follows that the conditions i), ii), and iii) when $n = 4d$, are necessary for the validity of the Theorem.

From the assumption $\pi_1(X) = 0$ it follows that the integer n satisfying condition i) of Theorem 2.1 has to be ≥ 3 whenever n is odd. But for $n = 3$ the condition i) itself implies that X is of the same homotopy type as S^3. Moreover every vector bundle on S^3 is trivial since $\pi_2(SO(k)) = 0$ for every integer $k \geq 0$. Thus for any vector bundle ξ over X and any homotopy equivalence $f : S^3 \to X$ we have $[f!(\xi)] = -[\tau_{S^3}]$. This shows that Theorem 2.1 is trivially valid for $n = 3$ and hence it only remains to prove the Theorem for $n \geq 5$. But some of the Lemmas and propositions that will be proved here are valid for $n \geq 4$, and it will be clear later when exactly we need the assumption $n > 4$.

2.2. Realizing X as a subcomplex of a simplex Δ_N for some integer N and imbedding Δ_N affinely in $\mathbb{R}^N$ we get an open set $U \supset X$ of $\mathbb{R}^N$ such that X is a deformation retract of U. Let $j : X \longrightarrow U$ be the inclusion and $r : U \longrightarrow X$ the retraction (i.e. $r \circ j = Id_X$) with $j \circ r \sim Id_U$ ($\sim$ = 'homotopic to'). Let ξ be a vector bundle on X satisfying condition ii) of Theorem 2.1. Let $\xi' = r!(\xi)$. It is easy to see that ξ' can be made into a differentiable vector bundle. Actually ξ' is induced by a certain map $g : U \rightarrow G_{k+\ell,k}$ for some integer ℓ, from the universal bundle γ^k on $G_{k+\ell,k}$. Since the map g can be approximated by a differentiable map $g : U \longrightarrow G_{k+\ell,k}$ with $g \sim g'$, it follows that ξ' can be made into a differentiable vector bundle. The Thom space $T(\xi')$ of ξ' is defined as follows. Introducing a fixed C^∞ Riemannian metric on ξ', let $E_1(\xi')$ be the subspace of $E(\xi')$ consisting of vectors of length $\leqslant 1$ and $\dot{E}_1(\xi')$ the boundary of $E_1(\xi')$ consisting precisely of vectors of length 1. The space $T(\xi')$ is defined as the quotient space $E_1(\xi')/\dot{E}_1(\xi')$. In this case $T(\xi')$ is not the one point compactification of $E(\xi')$. Still we denote the point of $T(\xi')$ to which $\dot{E}_1(\xi')$ is collapsed by "∞". Clearly $T(\xi') - \infty$ is a differentiable manifold.

Since $r \circ j = Id_X$ we have $\xi'/X = \xi$. Taking the restriction to ξ of the Riemannian metric on ξ', and realizing

$T(\xi)$ as $E_1(\xi)/\dot{E}_1(\xi)$ we see that the inclusion map $h : E(\xi) \longrightarrow E(\xi')$ induces a map $T(h) : T(\xi) \dashrightarrow T(\xi')$. The symbol Φ denotes throughout the Thom isomorphism. Let $f : S^{n+k} \longrightarrow T(\nu)$ be a map such that $f^*(\iota) = \Phi(u)$, ι being a generator of $H_{n+k}(S^{n+k})$. By condition ii) such a map exists. The naturality of the Thom isomorphism yields $(T(h) \circ f)_*(\iota) = \Phi(j_*(u))$. Denoting $T(h) \circ f$ by f' we see that $f' : S^{n+k} \longrightarrow T(\xi')$ is a map satisfying $f'_*(\iota) = \Phi(j_*(u))$. By the transverse regular approximation theorem [4], $\exists$ a differentiable map $f'' : S^{n+k} \longrightarrow T(\xi')$ (whenever it makes sense i.e. on $f''^{-1}(T(\xi') - \infty)$) with $f'' \sim f'$ and f'' transverse regular on U. Clearly $f''^{-1}(U) \neq \emptyset$ for if $f''(S^{n+k}) \cap U = \emptyset$ the map $f''_* : H_{n+k}(S^{n+k}) \longrightarrow H_{n+k}(T(\xi'))$ would factor through $H_{n+k}(T(\xi') - U) = 0$ (since $T(\xi') - U$ is contractible to "∞"). But $f''_*(\iota) = f'_*(\iota) = \Phi(j_*(u)) \neq 0$. Hence $M = f''^{-1}(U)$ is a differentiable manifold of codimension k in S^{n+k} with normal bundle $\nu_M \simeq f''!(\xi')$. But M need not necessarily be connected. Since $f''(\xi')$ and $\tau_{S^{n+k}}$ are orientable and since $\tau_{S^{n+k}} \big| M \simeq \tau_M \oplus f''!(\xi')$ we see that τ_M is orientable. Since U is closed in $T(\xi)$ we have $M = f''^{-1}(U)$ closed in S^{n+k} and hence M is a compact, orientable differentiable manifold of dimension n. Choose some C^∞ Riemannian metric for ν_M. It is known that $\exists$ a tubular neighbourhood i.e. a diffeomorphism D of $E_\varepsilon(\nu)$ for some $\varepsilon > 0$ onto a closed neighbourhood B of M in S^{n+k}, and a map $\overline{f} : S^{n+k} \dashrightarrow T(\xi')$ satisfying the following conditions:

1) $\overline{f}$ is differentiable on $\overline{f}^{-1}(T(\xi') - \infty)$ and transverse regular on U

2) $\overline{f} = f''$ on M and $\overline{f}^{-1}(U) = f''^{-1}(U) = M$

3) $\overline{f} \circ D$ is a bundle map of $E_\varepsilon(\nu)$ onto the image (i.e. maps the fibre of $E_\varepsilon(\nu)$ at $x \in M$ homeomorphically onto the image portion of the fibre at $f(x)$ in $E(\xi)$)

4) $\overline{f} \sim f'' : S^{n+k} \longrightarrow T(\xi')$.

For a proof refer to steps 1 and 2 of the proof of Theorem 3.16 in [4].

From the compactness of M it follows that $\exists$ a $\delta > 0$ with $\overline{f} \circ D(E_\varepsilon(\nu)) \supset E_\delta(\xi') \,\big|\, \overline{f}(M)$. Let $\{M_i\}_{i=1,\dots r}$ be the connected components of M and let $A_i = \overline{f}^{-1}(E_\delta(\xi'))\,\big|\, M_i$ and $\dot{A}_i = \overline{f}^{-1}(\dot{E}_\delta(\xi'))\,\big|\,M_i$. We will write the same symbols A_i, $\dot{A}_i$ to denote $D^{-1}(Ai)$, $D^{-1}(\dot{A}i)$ etc. In otherwords we identify $E_\varepsilon(\nu)$ and the tubular neighbourhood B .

We now introduce the following changes in notation. We write ξ , f and u for ξ , $\overline{f}$ and $j_*(u)$. With this altered notation $f : S^{n+k} \longrightarrow T(\xi)$ is a map satisfying $\Phi(u) = f_*(\zeta)$, differentiable on $f^{-1}(T(\xi) - \infty)$, transverse regular on U

Let $f_*[M] = du$. We have to show that $d = 1$. We have $(e_*^{-1})j_*[S^{n+k}] = \sum_i (j_i)_*([A_i, \dot{A}_i])$. To show that $d = 1$ it suffices to show that $\mathcal{V} f_*[M] = \mathcal{V}(u)$. From Diagram 2 we have

$$\mathcal{V} f_*[M] = f_*(\sum \mathcal{V}_i [M_i]) = f_*(\sum (j_i)_*([A_i, \dot{A}_i]))$$

$$= f_*(e_*)^{-1} j_*[S^{n+k}] = (e_{\xi *})^{-1}(j_\xi)_*(f_*[S^{n+k}])$$

$$= (e_{\xi *})^{-1}(j_\xi)_*(\Phi(u)) .$$

But by the definition of $\mathcal{V}$ we have $\mathcal{V}(u) = (e_{\xi *})^{-1}(j_\xi)_* \Phi(u)$.

We change our notations again and write $f : M \longrightarrow X$ for the map $r \circ f$ where $r : U \longrightarrow X$ is the homotopy equivalence chosen already and write u for the original generator of $H_n(X)$. Then f is of degree 1. The homomorphism $H_q(M) \longrightarrow H_q(X)$ induced by f is denoted by f_q.

<u>Lemma 2.5</u>. There exist homomorphisms $g_q : H_q(X) \longrightarrow H_q(M)$ with $f_q \circ g_q = Id_{H_q(X)}$ and hence $H_q(M) = \text{Kar } f_q \oplus g_q(H_q(X))$.

<u>Proof</u>. For any $x \in H_q(X)$ let $\gamma \in H^{n-q}(X)$ be the element $\Delta^{-1}(x)$ where $\Delta : H^{n-q}(X) \longrightarrow H_q(X)$ is the Poincaré isomorphism. Setting $g_q(x) = f^*(\gamma) \cap [M]$ we have $f_q\, g_q(x) = f_*(f^*(\gamma) \cap [M]) = \gamma \cap f_*[M] = \gamma \cap u = x$.

The proof of this lemma uses only two facts : (a) X satisfies

Poincaré duality and (b) $f : M \longrightarrow X$ is a map of degree 1.

Let η' be a bundle over X (of rank ℓ' say) such that $\xi \oplus \eta' \simeq \theta_X^{k+\ell'}$. Let $\eta = \eta' \oplus \theta_X^{k+n}$. Then

$[\eta] = [\eta'] = -[\xi]$ and

$$f!(\eta) = f!(\eta') \oplus \theta_M^{k+n} \simeq f!(\eta') \oplus \tau_M^n + \nu_M^k \simeq \tau_M^n \oplus f!(\eta') \oplus f!(\xi)$$

$$\simeq \tau_M^n \oplus f!(\eta' \oplus \xi) \simeq \tau_M^n \oplus \theta_M^{k+\ell'} .$$

Denoting $k+\ell'$ by ℓ we have the following situation:

$\exists$ a vector bundle η of rank $n+\ell$ on X with $[\eta] = -[\xi]$ and a map $f : M \longrightarrow X$ of degree 1 satisfying $f!(\eta) \approx \tau_M^n \oplus \theta_M^\ell$.

Without loss of generality we can assume $\ell' \geqslant 1$. Our aim is to surgerize M finitely many times and obtain a connected simply connected manifold M' together with a map $f' : M' \longrightarrow X$ inducing isomorphisms in homology and further satisfying $f'!(\xi) \approx \tau_{M'}^n \oplus \theta_{M'}^\ell$. If this is done the theorem is proved since f' will then be a homotopy equivalence by a theorem of J.H.C. Whitehead and the relation $f'!(\xi) = \tau_{M'}^n \oplus \theta_{M'}^\ell$ implies $[f'!(\xi)] = -[\tau_{M'}^n]$. In case n is odd and $\geqslant 5$ we will be able to achieve this using conditions i) and ii) and when $n = 4d$ with d an integer > 1 we will also need condition iii) to do the same.

§ 3. SURGERY OR SPHERICAL MODIFICATION.

The unit disk $\{(x_1,..,x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \leq 1\}$ in $\mathbb{R}^n$ is denoted by D^n and the unit open ball $\{(x_1,..,x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 < 1\}$ by B^n. For any real number $t > 0$ the closed disk and the open ball of radius t are denoted by tD^n and tB^n respectively. All the manifolds we consider are oriented C^∞ manifolds. We use the letter V to denote a compact manifold without boundary, of dimension $n \geq 1$.

Definition 3.1. Given an orientation preserving differentiable imbedding $\varphi: S^q \times \frac{3}{2} D^{n-q} \to V$ with $n > q \geq 0$ let $\chi(V, \varphi)$ denote the quotient manifold obtained from the disjoint union $V - \varphi(S^q \times \frac{1}{2} D^{n-q}) \cup \frac{3}{2} B^{q+1} \times S^{n-q-1}$ by identifying $\varphi(x, t\,y)$ with (tx, y) $\forall\, x \in S^q$, $y \in S^{n-q-1}$ and $\frac{1}{2} < t < 3/2$.

It is easy to check that $\chi(V, \varphi)$ is Hausdorff. Since $\varphi(x,ty) \rightsquigarrow (tx,y)$ is a diffeomorphism for $x \in S^q, y \in S^{n-q-1}$ and $\frac{1}{2} < t < 3/2$ it follows that $\chi(V,\varphi)$ is a C^∞-manifold. It is clearly compact and oriented. The manifold $\chi(V, \varphi)$ is said to be got from V by a surgery of type $(q+1, n-q)$.

Two compact oriented manifolds V and V' are said to be χ-equivalent if $\exists$ a finite sequence of manifolds $V_1 = V_1, V_2, \ldots, V_r = V'$ such that V_{i+1} is got from V_i by a surgery.

Lemma 3.2. Suppose V has s connected components with $s \geqslant 2$ and $\varphi : S^0 \times D^n \longrightarrow V$ an orientation preserving imbedding which carries the two components of $S^0 \times D^n$ into distinct components of V. Then $\chi(V, \varphi)$ has exactly $(s-1)$ connected components.

Proof. Trivial for $n \geqslant 2$. For $n = 1$ we have to use the fact that every component of V is diffeomorphic to S^1.

Using conditions i) and ii) of Theorem 2.1. we obtained a compact oriented manifold M of dimension n, a vector bundle η of rank $(n+\ell)$ on X with $[\eta] = -[\xi]$ and a map $f : M \longrightarrow X$ of degree 1 satisfying $f^!(\eta) \simeq \tau^n_M \oplus \varepsilon^\ell_M$. Let $\varphi : S^q \times \frac{3}{2} D^{n-q} \longrightarrow M$ be an orientation preserving imbedding with $n > q \geqslant 0$. Assume further that $f \circ \varphi(S^q \times \frac{3}{2} D^{n-q}) = x^*$, a chosen base point for X. Let $M' = \chi(M, \varphi)$ and let $f' : M' \longrightarrow X$ be defined as follows. Setting $M_0 = M - \varphi(S^q \times B^{n-q})$ the map f' is given by $f' | M_0 = f | M_0$ and $f' | \varphi'(D^{q+1} \times S^{n-q-1}) = x^*$ where $\varphi' : D^{q+1} \times S^{n-q-1} \longrightarrow M'$ denotes the imbedding induced by the inclusion $D^{q+1} \times S^{n-q-1} \longrightarrow \frac{3}{2} B^{q+1} \times S^{n-q-1}$. Clearly f' is well defined and continuous.

Lemma 3.3. The map $f' : M' \longrightarrow X$ is of degree 1.

<u>Proof</u>. Consider the following commutative diagram.

$$\begin{tikzcd}
H_n(M) \arrow[r, "j_*"] & H_n(M, \varphi(S^q \times D^{n-q})) \arrow[r, "f_*"] & H_n(X, x^*) \\
& H_n(M_o, \varphi(S^q \times S^{n-q-1})) \arrow[u, "e_*"', "\wr"] \arrow[d, "e'_*"', "\wr"] \arrow[ur, "f_*"] & \\
H_n(M') \arrow[r, "j'_*"] & H_n(M', \varphi'(D^{q+1} \times S^{n-q-1})) \arrow[uur, "f'_*"'] &
\end{tikzcd}$$

<u>Diagram 3</u>.

Here j_*, j'_*, e_* and e'_* are homomorphisms induced by the respective inclusions. The maps e_* and e'_* are isomorphisms by excision and homotopy. That f' is of degree 1 now follows from $e'^{-1}_* \, j'_* [M'] = e^{-1}_* \, j_* [M]$.

Suppose M is not connected. Choosing $\varphi : S^o \times \frac{3}{2} D^n$ such that the two components of $S^o \times \frac{3}{2} D^n$ go into distinct components of M let $M' = \chi(M, \varphi)$. Since X is connected it follows that $f \circ \varphi : S^o \times \frac{3}{2} D^n \longrightarrow X$ is homotopic to the constant map. By homotopy extension property we can choose a map $g : M \longrightarrow X$ with $g \sim f$ and $g | \varphi(S^o \times \frac{3}{2} D^n) = x^*$. Then clearly g is of degree 1 and $g^!(\eta) \simeq \tau^n_M \oplus \epsilon^\ell_M$. Thus we can without loss of generality assume that f itself satisfies the condition $f\varphi(S^o \times \frac{3}{2} D^n) = x^*$. Let $f' : M' \longrightarrow X$ be the associated map i.e. $f'|M_o = f|M_o$ and $f'|\varphi'(D^1 \times S^{n-1}) = x^*$.

<u>Lemma 3.4</u>. $f' : M' \longrightarrow X$ is of degree 1 and $f'^!(\eta) \simeq \tau^n_{M'} \oplus \vartheta^\ell_{M'}$.

<u>Proof</u>. That f' is of degree 1 follows from Lemma 3.3. Let $T_M = \tau^n_M \oplus \vartheta^\ell_M$ and $T_{M'} = \tau^n_{M'} \oplus \vartheta^\ell_{M'}$ and $\psi : T_M \longrightarrow f^!(\eta)$ a bundle isomorphism. Our aim is to get a bundle isomorphism $\psi' : T_{M'} \longrightarrow f'^!(\eta)$. Since $T_{M'}|M_o = T_M|M_o$ and $f'|M_o = f|M_o$ we can take $\psi' = \psi$ on $T_{M'}|M_o$. We denote the image of $S^o \times D^n$ by φ in M by $\mathrm{Im}\,\varphi$ and the image of $D^1 \times S^{n-1}$ under φ' in M' by $\mathrm{Im}\,\varphi'$. We identify $T_{M'}|\mathrm{Im}\,\varphi' = \tau_{\varphi'(D^1 \times S^{n-1})}$ with $(\tau_{\frac{3}{2}B^1 \times \frac{3}{2}B^2}|D^1 \times S^{n-1}) \oplus \vartheta^{\ell-1}_{D^1 \times S^{n-1}}$. Let $w_1, \ldots, w_{n+\ell}$ be a trivialization of $\tau_{\frac{3}{2}B^1 \times \frac{3}{2}B^n} \oplus \vartheta^{\ell-1}_{\frac{3}{2}B^1 \times \frac{3}{2}B^n}$ and take the induced trivialization of $T'_M|\mathrm{Im}\,\varphi'$ to identify it with $D^1 \times S^{n-1} \times \mathbb{R}^{n+\ell}$. Let $e_1, \ldots, e_{n+\ell}$ be a basis of the fibre of η at x and let $u_1, \ldots, u_{n+\ell}$ be the pull back trivialisation of $f'^!(\eta)|\mathrm{Im}\,\varphi'$. Using this trivialization we identify $f'^!(\eta)|\mathrm{Im}\,\varphi'$ with $D^1 \times S^{n-1} \times \mathbb{R}^{n+\ell}$. The map $\psi : T_{M'}|\mathrm{Bdry}\ M_o \longrightarrow f'^!(\eta)|\mathrm{Bdry}\ M_o$ then corresponds to an orientation preserving bundle map

$\psi : S^o \times S^{n-1} \times \mathbb{R}^{n+\ell} \longrightarrow S^o \times S^{n-1} \times \mathbb{R}^{n+\ell}$ and thus to a continuous map $\Theta: S^0 \times S^{n-1} \longrightarrow GL_+(n+\ell, \mathbb{R})$ given by

$\mathcal{K}(x, \vec{v}) = (x, \Theta(x)\vec{v}) \ \forall v \in \mathbb{R}^{n+\ell}$. To get a bundle map

$T_{M'} \longrightarrow f'!(\eta)$ extending $\mathcal{K}' : T_{M'} | M_o \longrightarrow f'!(\eta) | M_o$

it suffices to get a continuous extension of Θ into a map

$D^1 \times S^{n-1} \longrightarrow GL_+(n+\ell, \mathbb{R})$. But we know that $\mathcal{K}$ comes from a

bundle map $T_M | \operatorname{Im}\varphi \longrightarrow f!(\eta) | \operatorname{Im}\varphi$. Since $f | \varphi(S^0 \times D^n) = x^*$

the trivialization $u_1, \ldots, u_{n+\ell}$ of $T'_M | \text{Bdry } M_o = T_M | \text{Bdry } M_o$

extends to a trivialization of $f!(\eta) | \operatorname{Im}\varphi$. Also

$T_M | \operatorname{Im}\varphi = \tau_{\varphi(S^0 \times D^n)} \oplus \vartheta^{\ell}_{\varphi(S^0 \times D^n)}$ can be identified with

$\left(\tau_{\frac{3}{2} B^1 \times \frac{3}{2} B^n} \oplus \vartheta^{\ell-1}_{\frac{3}{2} B^1 \times \frac{3}{2} B^n}\right) \Big| S^0 \times D^n$. Thus the trivialization

$w_1, \ldots, w_{m+\ell}$ extends to a trivialization of $T_M | \operatorname{Im}\varphi$. Using

these trivializations we see that $\mathcal{K}$ corresponds to a bundle map

$S^0 \times D^n \times \mathbb{R}^{n+\ell} \longrightarrow S^0 \times D^n \times \mathbb{R}^{n+\ell}$. In otherwords $\exists$ an extension

$\bar{\Theta}$ of Θ into a map $S^0 \times D^n \longrightarrow GL_+(n+\ell, \mathbb{R})$. Since $GL_+(n+\ell, \mathbb{R})$

is connected and D^n contractible it follows that $\exists$ a map

$D^1 \times D^n \longrightarrow GL_+(n+\ell, \mathbb{R})$ extending $\bar{\Theta}$. This completes the proof

of Lemma 3.4.

As an immediate consequence of Lemmas 3.2 and 3.4 we get the

following:

<u>Proposition 3.5</u>. There exists a connected, compact, oriented C^{∞} manifold M' which is χ-equivalent to M and a map $f' : M' \longrightarrow X$ of degree 1 with $f'!(\eta) \simeq T_{M'} = \tau^{n}_{M'} \oplus \theta^{\ell}_{M'}$.

We now change our notations. We replace M' by M and f' by f. Thus M is connected and $f : M \to X$ is of degree 1 with $f!(\eta) \approx \tau^{n}_{M} \oplus \theta^{\ell}_{M}$.

Let $\varphi : S^{q} \times \frac{3}{2} D^{n-q} \longrightarrow M$ be an orientation preserving imbedding where $n > q \geqslant 1$ and let us assume $f\varphi(S^{q} \times \frac{3}{2} D^{n-q}) = x^{*}$. Let $f' : M' = \chi'(M, \varphi) \longrightarrow X$ be the associated map. In general $f'!(\eta)$ need not be isomorphic to $\tau^{n}_{M'} \oplus \theta^{\ell}_{M'}$. Consider the following alteration of the map φ. Let $\alpha : S^{q} \longrightarrow SO(n-q)$ be a C^{∞} map and let $\varphi_{\alpha} : S^{q} \times \frac{3}{2} D^{n-q} \longrightarrow M$ be given by $\varphi_{\alpha}(x,y) = \varphi(x, \alpha(x)\, y) \;\forall (x,y) \in S^{q} \times \frac{3}{2} D^{n-q}$. Clearly φ_{α} is an imbedding, also satisfying $f\varphi_{\alpha}(S^{q} \times \frac{3}{2} D^{n-q}) = x^{*}$. Let $f'_{\alpha} : M'_{\alpha} = \chi'(M, \varphi_{\alpha}) \longrightarrow X$ be the associated map. The sets $\varphi(S^{q} \times D^{n-q})$ and $\varphi'(D^{q+1} \times S^{n-q-1})$ (and similarly $\varphi_{\alpha}(S^{q} \times D^{n-q})$ and $\varphi'_{\alpha}(D^{q+1} \times S^{n-q-1})$) are denoted by $\operatorname{Im} \varphi$ and $\operatorname{Im} \varphi'$ respectively (similarly by $\operatorname{Im} \varphi_{\alpha}$ and $\operatorname{Im} \varphi'_{\alpha}$ respectively). Let ψ' be defined to be ψ on $T_{M'} \mid M_{o} = T_{M} \mid M_{o}$ into $f'!(\eta) \mid M_{o} = f!(\eta) \mid M_{o}$. Let $e_{1}, \ldots, e_{n+\ell}$ be a fixed admits of an extension $D^{q+1} \times D^{n-q} \longrightarrow GL_+(n+\ell, \mathbb{R})$. Choosing a fixed point $y_0 \in S^{n-q-1}$ the obstruction to the existence of such an extension is given by the homotopy class of the map $\gamma : S^q \longrightarrow GL_+(n+\ell, \mathbb{R})$ where $\gamma(x) = \theta(x,y_0)$. Let us denote this obstruction class by $\gamma(\varphi) \in \pi_q(GL_+(n+\ell, \mathbb{R}))$. Let the obstruction class for the imbedding φ_α be denoted by $\gamma(\varphi_\alpha)$.

Lemma 3.6. The obstruction $\gamma(\varphi_\alpha)$ depends only on $\gamma(\varphi)$ and the homotopy class (α) of α in $\pi_q(SO(n-q))$. More precisely identifying $\pi_q(SO(n-q))$ with $\pi_q(GL_+(n-q), \mathbb{R})$ we have $\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$ where $s_* : \pi_q(GL_+(n-q, \mathbb{R})) \longrightarrow \pi_q(GL_+(n+\ell, \mathbb{R}))$ is the map induced by the inclusion $s : GL_+((n-q), \mathbb{R}) \longrightarrow GL_+(n+\ell, \mathbb{R})$.

Proof. Suppose $\varepsilon_1, \ldots, \varepsilon_{n+\ell}$ is any trivialisation of $T_{M'}|\operatorname{Im}\varphi'$ and suppose $\lambda : S^q \times S^{n-q-1} \longrightarrow GL_+(n+\ell, \mathbb{R})$ the map given by $v(x,y) = \lambda(x,y)\,\varepsilon(x,y)\ \forall (x,y) \in S^q \times S^{n-q-1}$. Then $\exists$ a conts map $P : D^{q+1} \times S^{n-q-1} \longrightarrow GL_+(n+\ell, \mathbb{R})$ such that $\theta(x,y) = \lambda(x,y)\,P(x,y)$. Actually P is the transformation relating the frame $\varepsilon(x,y)$ to $v'(x,y)$. Hence the homotopy class of $\theta | S^q \times y_0$ is the same as that of $\lambda | S^q \times y_0$. Now let $\overline{\Phi}' : D^{q+1} \times (D^{n-q} - \{0\}) \longrightarrow M' \times \mathbb{R}$ be the map given by $\overline{\Phi}'(x,y) = \left(\varphi'\left(x, \frac{y}{\|y\|}\right), \|y\| - 1\right)$. Choosing some trivialisation $C_0, C_1, .., C_{\ell-1}$ of $\mathcal{V}^{\ell}_{\mathrm{Im}\,\varphi}$, we see that

$$\frac{\partial \Phi'}{\partial \xi} = \left(\frac{\partial \Phi'}{\partial x_1}, .., \frac{\partial \Phi'}{\partial x_{q+1}}, \frac{\partial \Phi'}{\partial y_1}, .., \frac{\partial \Phi'}{\partial y_{n-q}}, \; C_1, .., C_{\ell-1} \right)$$

can be chosen as a trivialization for $T_{M'} \mid \mathrm{Im}\,\varphi'$. Thus the obstruction $\gamma(\varphi)$ is the class of the continuous map $\gamma(x)$ given by $\gamma(x) = \left\langle \frac{\partial \Phi'}{\partial \xi}, v \right\rangle (x)$, the matrix of v w.r.t. the basis $\frac{\partial \Phi'}{\partial \xi}$. The obstruction $\gamma(\varphi_\alpha)$ is the homotopy class of the map $\gamma_\alpha(x) = \left\langle \frac{\partial \Phi'_\alpha}{\partial \xi}, v \right\rangle (x)$ where Φ'_α is defined similar to Φ' using φ. It is easily seen that we have $\frac{\partial \Phi'_\alpha}{\partial x_i} = \frac{\partial \Phi'}{\partial x_i} + \sum_k \frac{\partial \Phi'}{\partial y_k} a_{ki}$ (for some a_{ki})

$\frac{\partial \Phi'_\alpha}{\partial y_j} = \sum_k \frac{\partial \Phi'}{\partial y_k} A_{kj}$ where $(A_{kj}(x)) = \alpha(x)$. If, for every $0 \leq t \leq 1$ the frame $\left(\frac{\partial \Phi'_\alpha}{\partial \xi}\right)_t$ is defined by

$$\left(\frac{\partial \Phi'_\alpha}{\partial x_i}\right)_t = \frac{\partial \Phi'}{\partial x_i} + t \sum_k \frac{\partial \Phi'}{\partial y_k} a_{ki} \quad (i = 1,2,..\ q+1)$$

$$\left(\frac{\partial \Phi'_\alpha}{\partial y_j}\right)_t = \frac{\partial \Phi'_\alpha}{\partial y_j} \quad (j = 1,2,..\ n-q) \text{ and}$$

$$(C_\mu)_t = C_\mu \; (= 1,2,..\ \ell-1) .$$

We see that $\gamma^t_\alpha(x) = \left\langle \left(\frac{\partial \Phi'_\alpha}{\partial \xi}\right)_t, v \right\rangle (x)$ gives a homotopy between the map $\gamma^0_\alpha(x) = \gamma(x) \cdot s(x)$ where $s : GL_+(n-q, \mathbb{R}) \longrightarrow GL_+(n+\ell, \mathbb{R})$ is the inclusion and

$\gamma^1_\alpha(x) = \gamma_\alpha(x)$. Thus the homotopy class $[\gamma_\alpha]$ is the same as $[\gamma] + s_*(\alpha)$. That is to say $\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$.

Perhaps we should have remarked earlier that while dealing with oriented bundles the trivializations are supposed to be those belonging to the orientation class. Since $s_* : \pi_q(SO(n-q)) \longrightarrow \pi_q(SO(n+\ell))$ is surjective for $q < n-q$ we have the following:

<u>Proposition 3.7</u>. If $q < \frac{n}{2}$ $\exists$ a C^∞ map $\alpha : S^q \longrightarrow SO(n-q)$ such that $f'_\alpha : M'_\alpha = \chi(M, \varphi_\alpha) \longrightarrow X$ satisfies $f'_\alpha{}^!(\eta) \simeq \tau^n_{M'_\alpha} \oplus \delta^\ell_{M'_\alpha}$.

Let now V be connected of dimension $n \geq 4$ and v^* some chosen base point in V. Choose some base point p^* in S^1 and let $\varphi : S^1 \times \frac{3}{2} D^{n-1} \longrightarrow V$ be an orientation preserving imbedding such that $\varphi(p^*, 0) = v^*$ and $\varphi \mid S^1 \times 0$ represents $\lambda \in \pi_1(V, v^*)$. Let $V' = \chi(V, \varphi)$ and let V_o and $\varphi' : D^2 \times S^{n-2} \longrightarrow V'$ have their usual meanings i.e. $V_o = V - \varphi(S^1 \times B^{n-1})$ and φ' is the imbedding of $D^2 \times S^{n-2}$ into V' induced by the inclusion of $D^2 \times S^{n-2}$ in $\frac{3}{2} B^2 \times S^{n-2}$. Choose some fixed $z^* \in S^{n-2}$ and choose $v'^* = \varphi(p^*, z^*) = \varphi'(p^*, z^*)$ as the base point of V'. Let σ be the path in V given by $\sigma(t) = \varphi(p^*, tz^*)$; it is a path joining v^* to v'^* in V and let $\sigma_* : \pi_1(V,v^*) \to \pi_1(V,v'^*)$ be the isomorphism induced by σ.

Lemma 3.8. Let $N(\lambda)$ be the normal subgroup of $\pi_1(V, v'^*)$ generated by $\sigma_*(\lambda)$. Then $\pi_1(V',v'^*)$ is isomorphic to $\pi_1(V,v'^*)/N(\lambda)$.

Proof. Let $j : (V_0,v'^*) \to (V,v'^*)$ be the inclusion. We claim that $j_* : \pi_1(V_0,v'^*) \to \pi_1(V,v'^*)$ is an isomorphism. In fact if $\theta : (S^1,p^*) \to (V,v'^*)$ is any map and $\bar\theta : (S^1,p^*) \to (V,v'^*)$ a map homotopic to θ and transverse regular on $\varphi(S^1 \times 0)$ (such a map exists since $v'^* \notin \varphi(S^1 \times 0)$), since Codim $\varphi(S^1 \times 0)$ in V is ≥ 2 (actually Codim $\varphi(S^1 \times 0)$ in $V \geq 3$). We see that $\bar\theta(S^1) \cap \varphi(S^1 \times 0) = \emptyset$. Choosing a deformation retraction $r : S^1 \times (D^{n-1} - 0) \to S^1 \times S^{n-2}$ we see that $r' = \varphi r \varphi^{-1} : \varphi(S^1 \times (D^{n-1} - 0)) \to \varphi(S^1 \times S^{n-2})$ is a deformation retraction and that $r'\bar\theta$ is a map homotopic to $\bar\theta$ and satisfying $r'\bar\theta(S^1) \subset V_0$. Thus j_* is onto. Also if $\psi : (S^1,p^*) \to (V_0,v'^*)$ is a map such that $j\psi$ is homotopic to a constant map then $\exists$ an extension (also denoted by ψ) of ψ into a map $\psi : D^2 \to V$ with $\psi(0) = v'^*$. We can get a map $\bar\psi$ with $\bar\psi \mid S^1 \cup 0 = \psi \mid S^1 \cup 0$ and $\bar\psi$ transverse regular on $\varphi(S^1 \times 0)$. Since Codim of $\varphi(S^1 \times 0)$ in $V \geq 3$ we see that $\bar\psi(D^2) \cap \varphi(S^1 \times 0) = \emptyset$ and an argument similar to the one above yields a homotopy of $\mathcal{V}: (S^1,p^*) \rightarrow (V_o,v^*)$ with the constant map, taking place on V_o itself. This shows that j_* is a monomorphism.

We have $V' = V_o \cup \operatorname{Im} \varphi'$ (as usual $\operatorname{Im} \varphi' = \varphi'(D^2 \times S^{n-2}))$ with $V_o \cap \operatorname{Im} \varphi' =$

$= \varphi(S^1 \times S^{n-2}) = \varphi'(S^1 \times S^{n-2})$. Clearly $V_o, \operatorname{Im} \varphi'$ and $V_o \cap \operatorname{Im} \varphi'$ are connected. Lemma 3.8 follows immediately from Van Kampen theorem. Also, clearly V' is connected.

As already remarked earlier by us Theorem 2.1 needs to be proved only when $n \geqslant 5$. We have already obtained a compact, connected, oriented C^{∞} manifold M of dimension n and a map $f : M \rightarrow X$ of degree 1 with $f!(\eta) \simeq \tau_M^n \oplus \varepsilon_M^{\ell}$. (Refer Proposition 3.5.)

<u>Proposition 3.9</u>. There exists a connected simply connected manifold M' which is χ-equivalent to M and map $f' : M' \rightarrow X$ of degree 1 satisfying $f'!(\eta) \simeq \tau_{M'}^n \oplus \varepsilon_{M'}^{\ell}$.

<u>Proof</u>. Choose some base point $m^* \in M$. We can without loss of generality assume that $f(m^*) = x^*$ for otherwise we can change f to a homotopic map satisfying this condition. Since M is a compact manifold $\pi_1(M,m^*)$ is finitely generated. Let $\lambda_1, \ldots, \lambda_r$ be generators for $\pi_1(M,m^*)$. We can get an imbedding $\varphi: S^1 \rightarrow M$ representing λ_1 (for this $n \geqslant 3$ is sufficient). Since M is oriented the normal bundle of φ in M is trivial and hence it can

be extended into an orientation preserving diffeomorphism $\varphi : S^1 \times \frac{3}{2} D^{n-1} \longrightarrow M$. Since X is simply connected we have $f \circ \varphi$ homotopic to the constant map. By changing f if necessary to a homotopic map we can assume $f\varphi(S^1 \times \frac{3}{2} D^{n-1}) = x^*$. Now let $M'_\varphi = \chi(M, \varphi)$ and $f'_\varphi : M'_\varphi \longrightarrow X$ be the map associated to f. By proposition 3.7 $\exists$ a C^∞ map $\alpha : S^1 \to SO(n-1)$ such that $f'_\alpha : M'_\alpha = M'_{\varphi_\alpha} = \chi(M, \varphi_\alpha) \longrightarrow X$ satisfies $f'^{!}_\alpha(\eta) \simeq \tau^n_{M'_\alpha} \oplus \varepsilon^\ell_{M'_\alpha}$ and is of degree 1. The map $\varphi_\alpha \big| S^1 \times 0$ is the same as $\varphi \big| S^1 \times 0 = \varphi : S^1 \longrightarrow M$. Hence $\varphi_\alpha \big| S^1$ represents the same element as φ i.e. λ_1. By Lemma 3.8 it follows that $\pi_1(M'_\alpha)$ is isomorphic to $\pi_1(M) \big/$(Normal s.g generated by λ_1) and hence $\pi_1(M'_\alpha)$ is generated by $(r-1)$ elements. It now follows that after a finite number of surgeries we can get a connected, simply connected manifold M' and a map $f' : M' \longrightarrow X$ satisfying the requirements of the proposition.

Remark. For applying Lemma 3.8 we only need that dim $M = n \geq 4$. Moreover we have so far used only conditions i) and ii) of Theorem 2.1.

§ 4. EFFECT OF SURGERY ON HOMOLOGY.

Let A and B be any two connected, simply connected topological spaces and q an integer ≥ 2. Suppose $h : A \longrightarrow B$ is a continuous map such that $h_* : H_i(A) \longrightarrow H_i(B)$ is an isomorphism for $i < q$ and an epimorphism for $i = q$. Denote the Kernel of $h_q : H_q(A) \longrightarrow H_q(B)$ by K_q.

Lemma 4.1. Any $x \in K_q$ can be represented by a map $\theta : S^q \longrightarrow A$ (i.e. $\theta_*(i_q) = x$ where i_q is a generator of $\pi_q(S^q)$) with $h \circ \theta$ homotopic to a constant map.

Proof. Without loss of generality we can assume h to be an inclusion map, for otherwise, we replace h by the inclusion of A into the mapping cylinder of h. For the proof of Lemma 4.1 we use the Relative Hurewicz Theorem. Since $h_* : H_i(A) \longrightarrow H_i(B)$ is an isomorphism for $i < q$ and an epimorphism for $i = q$ it follows from the exact homology sequence of the pair (B,A) that $H_i(B,A) = 0$ for $i \leq q$. Hence by the relative Hurewicz Theorem $\pi_i(B,A) = 0$ for $i \leq q$ and $\rho : \pi_{q+1}(B,A) \xrightarrow{\approx} H_{q+1}(B,A)$ where ρ is the Hurewicz homomorphism. Now consider the following diagram.

$$\begin{tikzcd}
\pi_{q+1}(B,A) \arrow[r, "\partial"] \arrow[d, "\rho", "\wr"'] & \pi_q(A) \arrow[r, "h_*"] \arrow[d, "\rho"] & \pi_q(B) \arrow[r] \arrow[d, "\rho"] & \pi_q(B,A) = 0 \arrow[d, "\rho"] \\
H_{q+1}(B,A) \arrow[r, "\partial"] & H_q(A) \arrow[r, "h_*"] & H_q(B) \arrow[r] & H_q(B,A) = 0
\end{tikzcd}$$

Diagram 4

The maps indicated by ρ are the Hurewicz homomorphisms. If $x \in K_q$ then $\exists\ y \in H_{q+1}(B,A)$ such that $\partial y = x$.

Let $y^1 \in \pi_{q+1}(B,A)$ be given by $\rho^{-1}(y)$. The element $z \in \pi_q(A)$ given by $z = \partial y^1$ satisfies $\rho(z) = x$ and $h_*(z) = h_*(\partial y^1) = 0$. Hence if $\Theta: S^q \longrightarrow A$ represents $z \in \pi_q(A)$ then Θ satisfies the requirements of the Lemma.

Lemma 4.2. Suppose ν is a vector bundle of rank $(n-q)$ over S^q which is stably trivial. If $2q < n$ then ν itself is trivial.

Proof. Let ν be determined by the element μ of $\pi_{q-1}(SO(n-q))$. Stable triviality of ν implies that $\exists$ an integer $r \geq n-q$ such that $s_*(\mu) = 0$ where $s_*: \pi_{q-1}(SO(n-q)) \to \pi_{q-1}(SO(r))$ is the homomorphism induced by the inclusion $SO(n-q) \longrightarrow (SO(r))$. But if $2q < n$ the map s_* is an isomorphism. Hence $\mu = 0$.

Let V be a compact, connected, oriented C^∞ manifold with $\pi_1(V) = 0$ of dimension n and let B be any connected, simply connected space. Let $h : V \longrightarrow B$ be a continuous map with $h_* : H_i(V) \longrightarrow H_i(B)$ an isomorphism for $i < q$ and an epimorphism for $i = q$ where $q \geq 2$. Further assume $\exists$ a vector bundle ζ on B with $[h!(\zeta)] = [\tau_V]$. Denote the Kernel of h_q by K_q.

Lemma 4.3. If $2q < n$ any $x \in K_q$ can be represented by a C^∞ imbedding $\varphi: S^q \longrightarrow V$ whose normal bundle ν_φ is trivial and which further satisfies $h \circ \varphi \sim$ constant map.

Proof. By Lemma 4.1 $\exists$ a map $\Theta: S^q \to V$ representing x such that $h \circ \Theta$ is homotopically tirivial. If $2q < n$ $\exists$ a C^∞ imbedding $\varphi: S^q \to V$ with $\Theta \sim \varphi$. We have

$\tau_V | \varphi(S^q) \simeq \tau_{\varphi(S^q)} \oplus \nu_\varphi$ where ν_φ is the normal bundle of the imbedding φ. Since $\tau_{\varphi(S^q)} \oplus \theta_{\varphi(S^q)} \simeq \theta^{q+1}_{\varphi(S^q)}$, we see that $[\tau_V | \varphi(S^q)] = [\nu_\varphi]$. But

$[\tau_V | \varphi(S^q)] = [h!(\xi) | \varphi(S^q)]$. Since $h \circ \varphi$ is homotopically trivial by construction we see that ν_φ is stably trivial. Now Lemma 4.2 yields that ν_φ itself is trivial.

Assume $2q < n$. Let $x \in K_q$ and let $\varphi: S^q \to V$ be a C^∞ imbedding representing x. Since the normal bundle ν_φ is trivial we can extend φ into an orientation preserving imbedding $\varphi: S^q \times \frac{3}{2} D^{n-q} \to V$. Since $h \circ \varphi$ is homotopic to the constant map, changing h in its homotopy class we may assume $h \circ \varphi = \text{Const } b^*$. Let $V' = \chi(V, \varphi)$ and $h': V' \to B$ the associated map i.e. to say $h' | V_0 = h | V_0$ and $h' | \operatorname{Im} \varphi' = b^*$ where $V_0, \operatorname{Im} \varphi$ and $\operatorname{Im} \varphi'$ have their customary meanings.

Proposition 4.4. $h'_*: H_i(V') \to H_i(B)$ is an isomorphism for $i < q$ and the Kernel K'_q of $h'_q = H_q(V') \to H_q(B)$ is isomorphic to $K_q | (x)$, whenever $2q < n-1$.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
H_i(S^q \times D^{n-q}) \arrow[r, "\varphi_*"] & H_i(V) \arrow[r, "j_*"] \arrow[d, "h_*"] & H_i(V, \mathrm{Im}\,\varphi) \arrow[r, "\partial"] \arrow[d, "e", "\wr"'] & H_{i-1}(S^q \times D^{n-q}) \\
 & H_i(B) & H_i(Vo, \mathrm{Bdry}\ Vo) \arrow[l] & \\
H_i(D^{q+1} \times S^{n-q-1}) \arrow[r] & H_i(V') \arrow[r, "j'_*"'] \arrow[u, "h'_*"] & H_i(V', \mathrm{Im}\,\varphi') \arrow[r, "\partial"] \arrow[u, "e'"', "\wr"] & H_{i-1}(D^{q+1} \times S^{n-q-1})
\end{tikzcd}$$

Diagram 5.

Since by assumption $2q < n-1$, whenever $1 \leqslant i \leqslant q$ we have

$H_i(S^q \times D^{n-q}) = 0 = H_i(D^{q+1} \times S^{n-q-1})$ and hence

$j_*^{-1} \circ e^{-1} \circ e' \circ j'_* : H_i(V') \longrightarrow H_i(V)$ will then be an isomorphism

satisfying commutativity in

$$\begin{tikzcd}
H_i(V') \arrow[rr, "\approx"] \arrow[dr, "h'"'] & & H_i(V) \arrow[dl, "h_*"] \\
 & H_i(B) &
\end{tikzcd}$$

This shows that $h'*$ is an isomorphism for $i < q$.

When $i = q$ Diagram 5 yields the following diagram.

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, "\varphi_*"] & H_q(V) \arrow[r, "j_*"] \arrow[d, "h_q"] & H_q(V, \operatorname{Im}\varphi) \arrow[r] \arrow[d, "e", "\wr"'] & 0 \\
1 \arrow[r, squiggly] & H_q(B) & H_q(V_0, \operatorname{Bdry} V_0) & \\
0 \arrow[r] & H_q(V') \arrow[u, "h'_q"] \arrow[r, "j'_*"'] & H_q(V', \operatorname{Im}\varphi') \arrow[u, "e'"', "\wr"] \arrow[r] & 0
\end{tikzcd}
$$

Diagram 6.

The map φ_* is given by $\varphi_*(1) = x$. We get an isomorphism of $H_q(V)/(x)$ (induced by j_*) with $H_q(V, \operatorname{Im}\varphi)$ and then we see that $\exists$ an isomorphism $H_q(V)/(x) \xrightarrow{\approx} H_q(V')$ making

$$
\begin{tikzcd}
H_q(V)/(x) \arrow[rr, "\approx"] \arrow[dr, "\text{got from } h_*"'] & & H_q(V') \arrow[dl, "h'_*"] \\
& H_q(B) &
\end{tikzcd}
\quad \text{commutative}
$$

This proves that $K'_q \approx K_q/(x)$.

Assuming conditions i) and ii) of Theorem 2.1 with $n \geqslant 4$ we have obtained a compact, connected oriented C^{∞} manifold M of dimension n with $\pi_1(M) = 0$ and a map $f : M \longrightarrow X$ of degree 1 satisfying $f!(\eta) \approx \tau^n_M \oplus \varepsilon^{\ell}_M$.

<u>Proposition 4.5</u>. There exists a connected, simply connected manifold M' which is χ equivalent to M and a map $f' = M' \longrightarrow X$ of degree 1 such that $f'!(\eta) \approx \tau^n_{M'} \oplus \varepsilon^{\ell}_{M'}$ and $f'_*: H_i(M') \rightarrow H_i(X)$ an isomorphism for $i < \frac{n}{2}$.

<u>Proof</u>. For $n = 4$ there is nothing to prove for $f : M \longrightarrow X$ already satisfies the requirements of the proposition. Since M is compact the homology groups $H_i(M)$ are all finitely generated. For $n \geqslant 5$ Proposition 4.5 is a consequence of this fact, Lemma 4.3 and Propositions 4.4 and 3.7.

<u>Remark 4.5'</u>: If $f'_* : H_q(M') \longrightarrow H_q(X)$ also is an isomorphism for $q = \left[\frac{n}{2}\right]$ then $f' : M' \longrightarrow X$ will be a homotopy equivalence. To show this we have only to show that $f'_* : H_i(M') \longrightarrow H_i(X)$ is an isomorphism for every i. As already proved (Lemma 2.5) the fact that f' is of degree 1 implies that $f'_*: H_i(M') \longrightarrow H_i(X)$ is onto for every i. Let $a \in H_i(M')$ be such that $f'_*(a) = 0$ $(i > q)$. Let $\alpha = \Delta^{-1}(a) \in H^{n-i}(M')$. Since $i > q$ we have $n - i \leqq q$. Since $f'_* : H_j(M') \longrightarrow H_j(X)$ is an isomorphism

for $j \leq q$ we have $f'^{*}: H^{j}(X) \longrightarrow H^{j}(M')$ an isomorphism for $j \leq q$ by the Universal Coefficient Theorem. Hence α can be written as $f'^{*}(\beta)$ for a unique $\beta \in H^{n-i}(X)$. Then if $x = \beta \cap u \in H_i(X)$ by the definition of g given in Lemma 2.5, we have $g(x) = a$. But $H_i(M') = \operatorname{Ker} f'_{*} \oplus g_i\, H_i(X)$ (direct sum). This implies $a = 0$ and hence f'_{*} an isomorphism for all i.

Let A be any connected topological space satisfying Poincaré duality with $u \in H_n(A) \simeq \mathbb{Z}$ as the fundamental class.

Definition 4.6. Let $a \in H_i(A)$ and $b \in H_{n-i}(A)$. The homology intersection of a and b, denoted by $a \cdot b$ is defined as follows: We identify $H_0(A)$ with $\mathbb{Z}$ with any element (i.e. pt) w of A as a generator. Let $\alpha = \Delta^{-1}(a)$ and $\beta = \Delta^{-1}(b)$ where Δ is the Poincaré isomorphism. Then $\alpha \cup \beta \in H^n(A)$. The homology intersection $a \cdot b$ is that integer which satisfies $(\alpha \cup \beta) \cap u = (a.b)w$. Because of (1) § 1.2 we see that $a \cdot b$ can also be defined as the value $(\alpha \cup \beta)[u]$ of $\alpha \cup \beta$ on the homology class u.

Let V be a compact, connected, simply connected C^{∞} manifold of dimension $n \geq 4$ and let $q = \left[\frac{n}{2}\right]$.

Lemma 4.7. Let $a \in H_q(V)$ and suppose $\exists\, b \in H_{n-q}(V)$ such that $a \cdot b = 1$. Suppose also that a is represented by an imbedding $\varphi: S^q \times \frac{3}{2} D^{n-q} \longrightarrow V$ (i.e. $\varphi \mid S^q \times 0$ represents a).

Let $V' = \chi(V, \varphi)$. Then Rank $H_q(V') <$ Rank $H_q(V)$ and $H_i(V') \approx H_i(V)$ for $i < q$.

Proof. Let V_o, Im φ and Im φ' have their customary meanings. By excision and homotopy we have $H_i(V,V_o) \underset{\approx}{\overset{\varphi_*}{\longleftarrow}} H_i(S^q \times D^{n-q}, S^q \times S^{n-q})$.

Also $H_i(S^q \times D^{n-q}, S^q \times S^{n-q-1}) = \begin{cases} \mathbb{Z} \text{ if } i = n-q \text{ or } n \\ 0 \text{ otherwise} \end{cases}$.

From the homology exact sequence of the pair (V,V_o) we see that $H_i(V_o) \xrightarrow{(i_o)_*} H_i(V)$ is an isomorphism whenever $i \neq n-q$ and n. (Here $i_o : V_o \longrightarrow V$ denotes the inclusion). Also we have the following exact sequence:

$$0 \longrightarrow H_{n-q}(V_o) \longrightarrow H_{n-q}(V) \xrightarrow{j_*} H_{n-q}(V,V_o) \simeq \mathbb{Z} \xrightarrow{\partial} H_{n-q-1}(V_o) \longrightarrow \cdots$$

The homomorphism $j_* : H_{n-q}(V) \longrightarrow H_{n-q}(V,V_o)$ can more explicitly be described as follows. Identifying $H_{n-q}(V,V_o)$ with $H_{n-q}(S^q \times D^{n-q}, S^q \times S^{n-q-1})$ we see that $\varphi(x_o \times D^{n-q})$ with x_o some fixed base point in S^q, is a generator for the group $H_{n-q}(V,V_o) \simeq \mathbb{Z}$. Denoting this generator by 1 we have $j_*(y) = \pm a.y1$. In fact the intersection number of $\varphi(S^q \times 0)$ with $\varphi(x_o \times D^{n-q})$ being clearly ± 1 we have $j_*(y) = \pm a.y1$.

The existence of an element $b \in H_{n-q}(V)$ with $a \cdot b = 1$ ensures that $j_* : H_{n-q}(V) \longrightarrow \mathbb{Z}$ is an epimorphism and hence we have the exact sequence

$$0 \to H_{n-q}(V_o) \longrightarrow H_{n-q}(V) \xrightarrow{j_*} \mathbb{Z} \to 0 \quad .$$

In particular Rank $H_{n-q}(V_o) <$ Rank $H_{n-q}(V)$.

We have $V' = V_o \cup D^{q+1} \times S^{n-q-1}$ with $V_o \cap D^{q+1} \times S^{n-q-1} = S^q \times S^{n-q-1}$.

Letting $j_1 : S^q \times S^{n-q-1} \longrightarrow D^{q+1} \times S^{n-q-1}$ and $i' = V_o \longrightarrow V'$ denote the respective inclusions we have the Mayer-Vietais sequence.

$$H_i(S^q \times S^{n-q-1}) \xrightarrow{(-j_1)_* \oplus \varphi_*} H_i(D^{q+1} \times S^{n-q-1}) \oplus H_i(V_o) \xrightarrow{\varphi'_* + i'_*} H_i(V') \to H_{i-1}(S^q \times S^{n-q-1})$$

It follows that if $1 < i < n - q - 1$ we have

$$H_i(V_o) \xrightarrow{i'_*} H_i(V') \ .$$

Also if $i = 1$ and $i < n-q-1$ we have the exact sequence

$$0 \to 0 \oplus H_1(V_o) \xrightarrow{i'_*} H_1(V') \longrightarrow \mathbb{Z} \xrightarrow{(-j_1)_* \oplus \varphi_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi'_* + i'_*} \mathbb{Z} .$$

The map $(-j_1)_* \oplus \varphi_*$ carries $1 \in \mathbb{Z} = H_o(S^q \times S^{n-q-1})$ into $(-1,1)$ of $\mathbb{Z} \oplus \mathbb{Z}$ and hence a monomorphism. Therefore $H_1(V_o) \xrightarrow{i'_*} H_1(V')$ is also an isomorphism in this case. Thus we see that if $i < n-q-1$ then $H_i(V_o) \xrightarrow{i'_*} H_i(V')$ is an isomorphism. We now consider the two cases $n = 2q + 1$ and $n = 2q$ separately.

<u>Case (1)</u>. $n = 2q + 1$. Then $q = n-q-1$. We have already proved that $H_i(V_o) \xrightarrow{(i_o)_*} H_i(V)$ is an isomorphism for $i \neq n-q$ and n.

The Mayer-Victoris sequence for $i = q$ yields the exact sequence

$$H_q(S^q \times S^q) \xrightarrow{(-j_1)_* \oplus \varphi_*} H_q(D^{q+1} \times S^q) \oplus H_q(V_o) \longrightarrow H_q(V') \longrightarrow 0 .$$

Writing $H_q(S^q \times S^q)$ as $\mathbb{Z} \oplus \mathbb{Z}$ we see that $(-j_1)_* \oplus \varphi_*$ carries $(1,0)$ of $\mathbb{Z} \oplus \mathbb{Z}$ into $(0,a)$ of $H_q(D^{q+1} \times S^q) \oplus H_q(V_o)$ and $(0,1)$ into $(-1,0)$. Since the intersection number $a \cdot b = 1$ we see that a has to be of infinite order and the above sequence now yields $H_q(V') \simeq H_q(V_o)/(a)$. Observing that $(i_o)_* : H_q(V_o) \to H_q(V)$ is an isomorphism we see that Rank $H_q(V') <$ Rank $H_q(V)$. Actually $H_q(V') \simeq H_q(V)/(a)$.

<u>Case (2)</u>. $n = 2q$. As already verified $H_i(V_o) \xrightarrow{i'_*} H_i(V')$ is an isomorphism for $i < n - q - 1 = q - 1$. Also $H_i(V_o) \xrightarrow{(i_o)_*} H_i(V)$ is an isomorphism for $i \neq q$ and n. Combining these

$H_i(V) \xrightarrow{i'_* \circ (i_o)_*^{-1}} H_i(V')$ is an isomorphism for $i < q - 1$.

For $i = q - 1$ the Mayer-Victoris sequence yields the exact sequence

$$H_{q-1}(S^q \times S^{q-1}) \xrightarrow{(-j_1)_* \oplus \varphi_*} H_{q-1}(D^{q+1} \times S^{q-1}) \oplus H_{q-1}(V_o) \to H_{q-1}(V') \to 0.$$

But $H_{q-1}(S^q \times S^{q-1}) \simeq \mathbb{Z}$, $H_{q-1}(D^{q+1} \times S^{q-1}) \simeq \mathbb{Z}$ and the map $(-j_1)_* \oplus \varphi_*$ carries 1 of $H_{q-1}(S^q \times S^{q-1})$ into $(-1,0)$. Hence $i'_* : H_{q-1}(V_o) \longrightarrow H_{q-1}(V')$ is an isomorphism. Since $(i_o)_* : H_{q-1}(V_o) \longrightarrow H_{q-1}(V)$ is also an isomorphism we have $H_{q-1}(V) \xrightarrow{i'_* \cdot (i_o)_*^{-1}} H_{q-1}(V')$ an isomorphism. For $i = q$ the Mayer-Victoris sequence yields

$$H_q(S^q \times S^{q-1}) \to 0 \oplus H_q(V_o) \to H_q(V') \to H_{q-1}(S^q \times S^{q-1}) \xrightarrow{\text{'mono'}} H_{q-1}(D^{q+1} \times S^{q-1}) \oplus H_{q-1}(V_o).$$

The map $H_{q-1}(S^q \times S^{q-1}) \xrightarrow{(-j_1)_* \oplus \varphi_*} H_{q-1}(D^{q+1} \times S^{q-1}) \oplus H_{q-1}(V_o)$ which carries the generator 1 of $H_{q-1}(S^q \times S^{q-1})$ into $(-1,0)$ is clearly a monomorphism. Hence $H_q(S^q \times S^{q-1}) \longrightarrow H_q(V_o) \xrightarrow{i'_*} H_q(V') \to 0$ is exact. It follows that Rank $H_q(V') <$ Rank $H_q(V_o)$. The map $H_q(S^q \times S^{q-1}) \longrightarrow H_q(V_o)$ carries the generator of $H_q(S^q \times S^{q-1})$ into a', an element of infinite order. As already verified Rank $H_q(V_o) <$ Rank $H_q(V)$ (since $q = n - q$, and we actually verified Rank $H_{n-q}(V_o) <$ Rank $H_{n-q}(V)$).

This completes the proof of Lemma 4.7.

§ 5. PROOF OF THE MAIN THEOREM FOR $n = 4d > 4$.

We have already obtained a compact, connected, simply connected C^∞ manifold M of dimension $4d$ and a map $f : M \longrightarrow X$ of degree 1 satisfying $f^!(\eta) \simeq \tau_M^n \oplus \theta_M^\ell$ and $f_*: H_i(M) \to H_i(X)$ an isomorphism $\forall\ i < 2d$. (Proposition 4.5).

Let $K_{2d} = \operatorname{Ker} f_{2d} : H_{2d}(M) \longrightarrow H_{2d}(X)$.

Lemma 5.1. K_{2d} is a free abelian group.

Proof. Since $H_{2d}(M)$ is finitely generated and K_{2d} a direct summand of $H_{2d}(M)$ (Lemma 2.5) it follows that K_{2d} is finitely generated. To prove that K_{2d} is free it therefore suffices to prove that K_{2d} is torsion free. We write q for $2d$ for simplicity. If possible let $x \in K_q$ be any torsion element and let $x^1 \in H^q(M)$ correspond to x under Poincaré duality i.e. $x^1 \cap [M] = x$. x^1 is then a torsion element of $H^q(M)$. By the Universal Coefficient Theorem for cohomology we have the following commutative diagram.

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_{q-1}(M), \mathbb{Z}) \arrow[r, "\beta"] & H^q(M) \arrow[r, "\alpha"] & \operatorname{Hom}(H_q(M), \mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ext}(H_{q-1}(X), \mathbb{Z}) \arrow[r, "\beta"] \arrow[u, "\operatorname{Ext}(f_*, \mathrm{Id}_{\mathbb{Z}})", "\wr"'] & H^q(X) \arrow[r, "\alpha"] \arrow[u, "f^*"'] & \operatorname{Hom}(H_q(X), \mathbb{Z}) \arrow[r] \arrow[u, "\operatorname{Hom}(f_*, \mathrm{Id}_{\mathbb{Z}})"'] & 0
\end{tikzcd}
$$

Diagram 7.

Clearly, $\mathrm{Hom}(H_q(M),\mathbb{Z})$ is torsion free. Also for any finitely generated abelian group A the group $\mathrm{Ext}(A,\mathbb{Z})$ is a torsion group. It follows that $\beta(\mathrm{Ext}\,(H_{q-1}(M),\mathbb{Z}))$ is precisely the torsion subgroup of $H^q(M)$. Hence $\exists$ an element $y^1 \in \mathrm{Ext}\,(H_{q-1}(M),\mathbb{Z})$ with $\beta(y^1) = x^1$. Since $f_* : H_i(M) \to H_i(X)$ is an isomorphism for $i \leq q-1$ we have

$\mathrm{Ext}(f_*, \mathrm{Id}_{\mathbb{Z}}) : \mathrm{Ext}(H_{q-1}(X),\mathbb{Z}) \to \mathrm{Ext}(H_{q-1}(M),\mathbb{Z})$ an isomorphism. Let $z^1 \in H^q(X)$ be given by $z^1 = \beta \circ (\mathrm{Ext}(f_*,\mathrm{Id}_{\mathbb{Z}})^{-1}(y'))$. Then clearly $f^*(z^1) = x^1$. Our aim is to show that K_q has no torsion, or that $x = 0$. For this it suffices to show that $x^1 = 0$ since $\cap[M] = \Delta : H^q(M) \to H_q(M)$ is an isomorphism. Now consider the element $z^1 \cap u \in H_q(X)$. Since f is of degree 1 we have $f_*([M]) = u$. We have

$0 = f_*(x) = f_*(x^1 \cap [M]) = f_*(f^*(z^1) \cap [M]) = z^1 \cap f_*[M] = z^1 \cap u.$

But by assumption $\cap u : H^q(X) \to H_q(X)$ is an isomorphism. Hence $z^1 = 0$ and therefore $x^1 = f^*(z^1) = 0$. This completes the proof of Lemma 5.1.

For the rest of § 5 we denote 2d by q.

Let $H_q(M) = K_q \oplus gH_q(X)$ be the splitting given by Lemma 2.5.

Lemma 5.2. For any $a \in K_q$ and any $b \in gH_q(X)$ the intersection number $a \cdot b = 0$. Also if $b_1 = g(c_1)$ and $b_2 = g(c_2)$ with $c_1, c_2 \in H_q(X)$ then the intersection number $b_1 \cdot b_2$ is the same as $c_1 \cdot c_2$.

Proof. Let $b = g(c)$ with $c \in H_q(X)$ (c is unique since g is a mono). Let $\gamma \in H^q(X)$ be such that $\gamma \cap u = c$. Then by the very definition of g we have $b = f^*(\gamma) \cap [M]$. To prove that $a \cdot b = 0$ it suffices to verify that $f_*((\alpha \cup f^*(\gamma)) \cap [M]) = 0$ with $\alpha \in H^q(M)$ satisfying $\alpha \cap [M] = a$. Since $q = 2d$ we have $\alpha \cup f^*(\gamma) = f^*(\gamma) \cup \alpha$. Hence

$$f_*((\alpha \cup f^*\gamma) \cap [M]) = (-1)^{q \cdot q} f_*((f^*\gamma \cup \alpha) [M]) =$$

$$f_*(f^*\gamma \cap (\alpha \cap [M]))(\text{since } q = 2d) = f_*(f^*\gamma \cap a) = \gamma \cap f_*(a) = 0$$

since $f_*(a) = 0$. Choosing γ_1, γ_2 in $H^q(X)$ with $\gamma_1 \cap u = c_1$, $\gamma_2 \cap u = c_2$ we have $b_1 = f^*(\gamma_1) \cap [M]$ and $b_2 = f^*(\gamma_2) \cap [M]$. Now

$$f_*((f^*\gamma_1 \cup f^*\gamma_2) \cap [M]) = f_*(f^*(\gamma_1 \cup \gamma_2) \cap [M]) = (\gamma_1 \cup \gamma_2) \cap f_*([M])$$
$$= (\gamma_1 \cup \gamma_2) \cap u .$$

From this the equality $b_1 \cdot b_2 = c_1 \cdot c_2$ follows.

Denoting by $T_q(M)$ and $T_q(X)$ respectively the torsion subgroups of $H_q(M)$ and $H_q(X)$ we have $H_q(M)/T_q(M) \simeq K_q \oplus \dfrac{H_q(X)}{T_q(X)}$ (because of Lemma 5.1). Lemma 5.2 precisely states that we can find bases for K_q and $\dfrac{H_q(X)}{T_q(X)}$ such that the matrix A_M of the intersection bilinear form on $H_q(M)/T_q(M)$ takes the form

$\begin{pmatrix} A_K & 0 \\ 0 & A_X \end{pmatrix}$ where A_K and A_X are the matrices of the form restricted to K_q and $H_q(X)/T_q(X)$. Also the lemma asserts that the restriction of the intersection bilinear form on $H_q(M)/T_q(M)$ to $H_q(X)/T_q(X)$ agrees with the intersection bilinear form on $H_q(X)/T_q(X)$ got from the fact that X satisfies Poincaré duality. Since intersection by definition corresponds to cup-product under Poincaré duality we see that the signature of A_M is the same as the index of the manifold $I(M)$ defined in 1.6 and similarly signature of A_X is $I(X)$. Let us denote the signature of A_K by $I(K)$. Then we have

$I(X) + I(K) = I(M)$.

Lemma 5.3. $I(K)$ is zero.

Proof. The assumption iii) of Theorem 2.1 is actually used in concluding that $I(K) = 0$. We have a map $f : M \longrightarrow X$ of degree 1 with $f!(\eta) = \tau_M^n \oplus \varepsilon_M^\ell$. Also $[\eta] = -[\xi]$. By Hirzobruch's Index Theorem $I(M) = \{L_d(p_1(\tau_M^n), \ldots, p_d(\tau_M^n))\}[M]$.

But $L_d(p_1(\tau_M^n), \ldots, p_d(\tau_M^n)) = L_d(p_1(f!(\eta)), \ldots, p_d(f!(\eta)))$ (since $L_k(p_1(\lambda), \ldots, p_k(\lambda))$ for any vector bundle λ depends only on the stable class of λ). Hence

$$\begin{aligned} I(M) &= \{L_d(p_1(f!(\eta)), \ldots, p_d(f!(\eta)))\}[M] \\ &= \{L_d(p_1(\eta), \ldots, p_d(\eta))\}(f_*[M]) \\ &= \{L_d(\overline{p_1(\xi)}, \ldots, \overline{p_d(\xi)})\}(u) \\ &= I(X) \text{ by assumption (iii).} \end{aligned}$$

This proves that $I(K) = 0$.

Denote the group $H^q(M)/T^q(M)$ (where $T^q(M)$ is the torsion of $H^q(M)$) by $B^q(M)$ and similarly the group $H_q(M)/T_q(M)$ by $B_q(M)$. Choosing any basis $x_1,..,x_r$ for B^q we see that $y_i = x_i \cap [M]$ (actually $\cap[M] : H^q(M) \longrightarrow H_q(M)$ gives a well determined isomorphism also denoted by $\cap M$ of $B^q(M)$ onto $B_q(M)$) form a basis for $B_q(M)$. Since $B^q(M) \simeq \mathrm{Hom}(B_q(M), \mathbb{Z})$ we can get elements $y_1^1,..,y_r^1$ in B^q such that $y_i^1(y_j) = \delta_{ij}$. The bilinear form $(x,y) \rightsquigarrow (x \cup y)[M]$ on B^q is easily seen to have determinant ± 1, for $(y_j^1 \cup x_i)[M] = y_j^1(x_i \cap [M]) = y_j^1(y_i) = \delta_{ij}$. It follows that A_M has determinant ± 1. Similarly A_X has determinant ± 1. It follows that A_K has determinant ± 1.

Lemma 5.4. If B is a symmetric non-degenerate bilinear form on a finitely generated free abelian group H, with determinant ± 1 and if the signature of B is Zero then $\exists\, x \neq 0$ in H such that $B(x,x) = 0$.

A proof of this can be found in [6]. As a corollary we see that if $K_q \neq 0$ $\exists$ an element $a \neq 0$ in K_q such that $a.a = 0$. Moreover we can choose 'a' to be indivisible in K_q. Then $K_q/(a)$ is free and hence we can find a basis of the form $a, b_2,..,b_r$ for K_q. Since A_K has determinant ± 1 and $a.a = 0$ we cannot have $a.b_j = 0 \ \forall j$. If $j_1,..,j_{r'}$ are the indices in $(2,..,r)$ with $a.b_j \neq 0$ then $\underset{i=1,..r'}{\mathrm{g.c.d}}\,(a.b_{j_i})$ has to be 1 for otherwise this greatest common divisor will divide determinant of A_K.

Hence $\exists$ integers m_{j_i} such that $\sum_{i=1}^{r'} m_{j_i}(a \cdot b_{j_i}) = 1$. The element $b \in K_q$ given by $b = \sum_{i=1}^{r'} m_{j_i} b_{j_i}$ satisfies $a \cdot b = 1$.

Lemma 5.5. If $d > 1$ there exists an imbedding $\varphi : S^q \longrightarrow M^{4d} (q = 2d)$ representing a and further satisfying $f \circ \varphi \simeq \tilde{x}^*$ (where $\tilde{x}^*$ is the constant map $S^q \longrightarrow X$ carrying the whole of S^q into x^*.)

Proof. It is for the proof of this lemma that we need d to be 1. By Lemma 4.1 $\exists$ a continuous map $\theta : S^q \longrightarrow M$ representing 'a' and satisfying $f \circ \theta \simeq \tilde{x}^*$. We use the fact that M is simply connected. Also since M is of dimension 4d with d an integer > 1 it follows from Lemma 6 of [6] that $\exists$ a C^∞ imbedding $\varphi : S^q \longrightarrow M$ with $\varphi \simeq \theta$. This proves Lemma 5.5.

Remark. It is not true that a continuous map $\theta : S^2 \longrightarrow V^4$ is homotopic to a C^∞ imbedding even if V^4 is a compact, simply connected C^∞ manifold (of dimension 4). An example is given by Kervaire and Milnor in [3].

Lemma 5.6. For any C^∞ imbedding $\varphi : S^q \longrightarrow M$ representing 'a' and satisfying $f \circ \varphi \simeq \tilde{x}^*$ the normal bundle ν_φ is trivial.

Proof. We have $\tau_M | \varphi(S^q) \simeq \tau^q_{\varphi(S^q)} \oplus \nu^q_\varphi$. Since M and S^q are orientable it follows that ν_φ is orientable. Also from $f!(\eta)|\varphi(S^q) \simeq (\tau_M^n \oplus \theta_M^\ell)|\varphi(S^q)$, we have

$$f!(\eta)|\varphi(S^q) \simeq \tau^q_{\varphi(S^q)} \oplus \nu^q_\varphi \oplus \theta^\ell_{\varphi(S^q)} \approx \tau_{\varphi(S^q)} \oplus \theta^1_{\varphi(S^q)} \oplus \nu^q_\varphi \oplus \theta^{\ell-1}_{\varphi(S^q)}$$

$$\simeq \theta^{q+1}_{\varphi(S^q)} \oplus \nu_\varphi \oplus \theta^{\ell-1}_{\varphi(S^q)} \simeq \nu_\varphi \oplus \theta^{q+\ell}_{\varphi(S^q)}.$$

But since $f \circ \varphi \sim \tilde{x}^*$ we have $f!(\eta)|\varphi(S^q) \simeq \theta^{2q+\ell}_{\varphi(S^q)}$.

Thus $\nu_\varphi \oplus \theta^{q+\ell}_{\varphi(S^q)} \simeq \theta^{2q+\ell}_{\varphi(S^q)}$. Thus ν_φ is stably trivial.

If $\nu \in \pi_{q-1}(SO_q)$ is the element corresponding to the bundle ν_φ on S^q we have $s_*(\nu) = 0$ where $s_* : \pi_{q-1}(SO_q) \to \pi_{q-1}(SO_{2q+\ell})$ is the homomorphism induced by the inclusion. Since $\pi_{q-1}(SO_{q+1}) \to \pi_{q-1}(SO_{2q+\ell})$ is an isomorphism it follows that $i_*(\nu) = 0$ where $i_* : \pi_{q-1}(SO_q) \to \pi_{q-1}(SO_{q+1})$ is induced by the inclusion. Since $SO_{q+1}/SO_q = S^q$ we have a fibration of SO_{q+1} by SO_q as the fibre and S^q as the base. Consider the corresponding exact sequence

$$\pi_q(S^q) \xrightarrow{\partial} \pi_{q-1}(SO_q) \xrightarrow{i_*} \pi_{q-1}(SO_{q+1}).$$

∂ carries a generator of $\pi_q(S^q)$ into the element τ of $\pi_{q-1}(SO_q)$

corresponding to the tangent bundle of S^q. Since $i_*(\nu) = 0$ it follows that $\nu = k\tau$ for some integer k. The map which assigns to an isomorphism class λ of an orientable vector bundle of rank q over S^q its Euler class $\chi(\lambda)$ defines a homomorphism $\chi: \pi_{q-1}(SO_q) \longrightarrow H^q(S^q)$. For the tangent bundle τ of S^q the class $\chi(\tau)$ is known to be twice a generator of $H^q(S^q)$. (That $q = 2d$ is even, we use here). Thus the composition $\pi_q(S^q) \xrightarrow{\partial} \pi_{q-1}(SO_q) \xrightarrow{\chi} H^q(S^q)$ is a monomorphism and any element in the image of ∂ is zero if and only if its Euler class is zero. The Euler class of the normal bundle of the imbedding φ representing 'a' can be identified with $a \cdot a$ times a generator of $H^q(S^q)$. For, given a normal vector field with a finite number of zeros on $\varphi(S^q)$ we can deform $\varphi(S^q)$ along these vectors to obtain a new imbedding which intersects $\varphi(S^q)$ at only finitely many places. The multiplicity of each such intersection is equal to the index of the corresponding zero of the normal vector field.

<u>Remark</u>. A more 'formal' proof for the fact that $\chi(\nu_\varphi) = a.a$ times a generator of $H^q(S^q)$ can be given as follows.

Denoting the imbedded manifold $\varphi(S^q)$ by S^q itself, let $\Phi: H^i(S^q) \longrightarrow H^{q+i}(T(\nu))$ be the Thom isomorphism. If $U = \Phi(1) \in H^q(T(\nu))$ then the Euler class of ν can be defined by $\chi(\nu) = \Phi^{-1}(U \cup U)$ [5]. Taking a tubular neighbourhood A of S^q in M and collapsing the exterior of A to a point we get a map $C : M \longrightarrow T(\nu)$. If $\gamma \in H^q(M)$ is the class which corresponds to 'a' under Poincaré duality (i.e. $\gamma \cap [M] = a$) it is known that $C^*(U) = \gamma[9]$. Hence $C^*(U \cup U) = \gamma \cup \gamma = a \cdot a\,[M]$ by the definition of the intersection number. But from the diagram

$$\begin{tikzcd}
H^q(S^q) \arrow[r, "\Phi", "\approx"'] & H^{2q}(T(\nu)) \arrow[dl, "C^*"'] & H^{2q}(T(\nu),T(\nu) - S^q) \arrow[l, "\approx"'] \arrow[d, "\wr"', "\text{excision}"] \\
H^{2q}(M) & H^{2q}(M,M-S^q) \arrow[l, "j_*"] \arrow[r, "\text{excision}", "\approx"'] & H^{2q}(A, A - S^q)
\end{tikzcd}$$

We see that $H^{2q}(M,M-S^q) \simeq \mathbb{Z}$. Taking any pt $x \in S^q$ we have the triangle:

$$\begin{tikzcd}
\mathbb{Z} \cong H^{2q}(M) & H^{2q}(M,M-S^q) \arrow[l, "j^*"'] \\
 & H^{2q}(M,M-x) \arrow[ul, "\wr"] \arrow[u]
\end{tikzcd}$$

Hence $H^{2q}(M,M-x) \simeq \mathbb{Z}$ has to be a direct summand of $H^{2q}(M,M-S^q)$ which is also $\simeq \mathbb{Z}$. It follows that $j^* : H^{2q}(M,M-S^q) \approx H^{2q}(M)$. Examining the diagram again we see that $C^* = H^{2q}(T(\nu)) \approx H^{2q}(M)$. Hence $U \cup U = a \cdot a$ times a generator of $H^{2q}(T(\nu))$ and $\Phi^{-1}(U \cup U) = a \cdot a$ times a generator of $H^q(S^q)$.

We are now almost at the end of the proof of Theorem 2.1 for the case $n = 4d$. Choosing an indivisible $a \neq 0$ in K_q with

$a.a = 0$ we saw that $\exists$ $b \in K_q$ with $a.b = 1$. The existence of such an 'a' was guaranteed by Lemma 5.4. From Lemmas 5.5 and 5.6 we see that $\exists$ an orientation preserving imbedding $\varphi : S^q \times \frac{3}{2} D^q \longrightarrow M$ with $f \circ \varphi \sim \tilde{x}^*$ and representing 'a'. Let now $M' = \chi(M, \varphi)$ and $f' : M' \longrightarrow X$ the associated map which is constructed after altering f in its homotopy class so as to satisfy $f \circ \varphi = x^*$. By Lemma 3.3 f' is of degree 1. To get an isomorphism $\tau^n_{M'} \oplus \theta^{\ell}_{M'} \longrightarrow f'!(\eta)$ we had an obstruction $\gamma \in \pi_q(SO_{n+\ell})$ and when φ was replaced by φ_α given by $\varphi_\alpha(x, y) = \varphi(x, \alpha(x)y)$ with $\alpha : S^q \longrightarrow SO_q$ a C^∞ map then the new obstruction γ_α satisfied the relation $\gamma_\alpha = \gamma + s_*(\alpha)$ where $s_* : \pi_q(SO_q) \longrightarrow \pi_q(SO_{n+\ell})$ is the homomorphism induced by the inclusion. (Lemma 3.6). Since q is even the homomorphism $\pi_q(SO_q) \longrightarrow \pi_q(SO_{q+1})$ is onto. [8]. Also $\pi_q(SO_{q+1}) \longrightarrow \pi_q(SO_{n+\ell})$ is onto. Thus there exists an α such that $f'_\alpha : M' = \chi(M, \varphi_\alpha) \longrightarrow X$ satisfies the condition $f'_\alpha!(\eta) \simeq \tau^n_{M'} \oplus \theta^{\ell}_{M'}$ in addition to being of degree 1. Thus without loss of generality we can assume that f' itself was 'good' in the sense that $f'!(\eta) \simeq \tau^n_{M'} \oplus \theta^{\ell}_{M'}$. Denoting the inclusions of M_0 in M and M' respectively by i and i' we have the following commutative diagram for every integer j.

$$\begin{tikzcd} H_j(M) \arrow[drr, "f_*"] & & \\ & & H_j(X) \\ H_j(M_o) \arrow[uu, "i_*"] \arrow[dd, "i'_*"'] & & \\ & & \\ H_j(M') \arrow[uurr, "f'_*"'] & & \end{tikzcd}$$

By Case 2 of Lemma 4.7 we have $i_*: H_j(M_o) \longrightarrow H_j(M)$ and $i'_* : H_j(M_o) \longrightarrow H_j(M')$ to be isomorphisms for $j < q$. Since $f_* : H_j(M) \longrightarrow H_j(X)$ is an isomorphism for $j < q$ it follows that $f'_* : H_j(M') \longrightarrow H_j(X)$ is an isomorphism for $j < q$. Also by the same lemma RK $H_q(M') <$ RK $H_q(M)$. If K'_q denotes the Kernel of $f'_q = H_q(M') \longrightarrow H_q(X)$ we have K'_q free and of rank $<$ rank of K_q. It follows that after a finite number of spherical modications we can obtain a manifold M'' and a map $f'' : M'' \longrightarrow X$ with $\deg f'' = 1$, $f''!(\eta) \simeq \tau^n_{M''} \oplus \sigma^{\ell}_{M''}$ and $K''_q = \operatorname{Ker} f''_q = 0$. It follows from the Remark 4.5' that $f'' : M'' \longrightarrow X$ is a homotopy equivalence. This completes the proof of the main theorem for $n = 4d > 4$.

§ 6. PROOF OF THE MAIN THEOREM FOR $n = 2q+1$.

Throughout § 6 we will assume $n = 2q + 1$ with q an integer $\geqslant 2$. Let $W = W^{2q+2}$ be a compact orientable topological manifold of dimension $2q+2$ with boundary bW. Let F be any fixed field. The semi-characteristic $e^*(bW;F)$ of bW with respect to F is defined to the residue class $\sum_{i=0}^{q}$ Rank $H_i(bW;F)$ modulo 2. Let ρ_F be the rank of the bilinear pairing $H_{q+1}(W;F) \otimes H_{q+1}(W;F) \longrightarrow F$ given by the intersection number and $e(W)$ the Euler characteristic of W,

<u>Lemma 6.1</u>. We have $e^*(bW;F) + e(W) \equiv \rho_F \pmod 2$.

<u>Proof</u>. Consider the homology exact sequence of the pair (W,bW) with coefficients in F,

$$H_{q+1}(W;F) \xrightarrow{j_*} H_{q+1}(W,bW;F) \xrightarrow{\partial} H_q(bW;F) \longrightarrow \cdots \longrightarrow H_0(W;bW;F) \longrightarrow 0.$$

By Poincaré-Lefschetz duality if $z \in H_{q+1}(W,bW;F)$ is such that $x \cdot Z = 0 \;\forall\; x \in H_{q+1}(W;F)$ then $Z = 0$.. It follows from this remark and the relation $x \cdot y = x \cdot j_*(y)$ for any $x, y \in H_{q+1}(W;F)$ that $\operatorname{Ker} j_*$ is precisely the nullity of the intersection bilinear form on $H_{q+1}(W;F)$. Hence

$$\rho_F = \dim H_{q+1}(W;F) - \dim \operatorname{Ker} j_* = \dim \operatorname{Im} j_* = \dim \operatorname{Ker} \partial$$

$$= \dim H_{q+1}(W,\ bW;F) - \dim \operatorname{Im} \partial$$

Denoting the dimensions of $H_j(W;F)$ and $H_j(W,bW;F)$ by $b_j(W;F)$ and $b_j(W,\ bW;\ F)$ respectively we have

$$\rho_F = b_{q+1}(W,bW;F) - b_q(bW;F) + b_q(W;F) - b_q(W,\ bW;\ F) + \cdots\ .$$

But $b_i(W,bW;F) = b_{2q+2-i}(W;F)$ by Poincaré-Lefschetz duality.

Thus $\rho_F \equiv e^*(bW;F) + e(W) \pmod 2$.

Let V be a compact connected oriented C^∞ manifold of dimension $n = 2q+1$ and let $a \in H_q(V)$ be any torsion element $\neq 0$. Suppose further $\varphi: S^q \times \frac{3}{2} D^{n-q} \longrightarrow V$ is an orientation preserving imbedding representing the homology class 'a'.

Let $V' = \chi(V, \varphi)$.

<u>Lemma 6.2</u>. If q is even we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_q(V') \longrightarrow H_q(V)/(a) \longrightarrow 0$$

where (a) is the subgroup generated by a in $H_q(V)$.

<u>Proof</u>. As usual let $V_o = V - \varphi(S^q \times B^{q+1})$ and let $\varphi' : D^{q+1} \times S^q \longrightarrow V'$ be the imbedding induced by the inclusion of $D^{q+1} \times S^q$ in $\frac{3}{2} B^{q+1} \times S^q$. We then have the following commutative diagram with exact horizontal rows.

$$\begin{tikzcd}
\mathbb{Z} \simeq H_q(S^q \times D^{q+1}) \arrow[r, "\varphi_*"] & H_q(V) \arrow[r] & H_q(V, \varphi(S^q \times D^{q+1})) \arrow[r] & 0 \\
 & & H_q(V_o, \varphi(S^q \times S^q)) \arrow[u, "\wr"'] & \\
\mathbb{Z} \simeq H_q(D^{q+1} \times S^q) \arrow[r, "\varphi'_*"] & H_q(V') \arrow[r] & H_q(V', \varphi'(D^{q+1} \times S^q)) \arrow[r] & 0
\end{tikzcd}$$

<u>Diagram 8</u>

Since $\varphi_*(1) = a$ by assumption it follows that $H_q(V', \varphi'(D^{q+1}\times S^q))$ $H_q(V, \varphi(S^q\times D^{q+1})) \simeq H_q(V)/(a)$. To prove Lemma 6.2 we have only to show that $\varphi'_*: \mathbb{Z} \longrightarrow H_q(V')$ is a monomorphism. Since 'a' is a torsion element to show that φ'_* is a monomorphism we have only to prove that $b_q(V', \mathbb{Q}) \not\equiv b_q(V, \mathbb{Q})(\text{mod } 2)$ where $b_q(V, \mathbb{Q})$ is the q^{th} Bettinumber of V i.e. the rank of $H_q(V, \mathbb{Q})$. Since $H_i(V) \simeq H_i(V, \varphi(S^q\times D^{q+1})) \simeq H_i(V_o, \varphi(S^q\times S^q)) \simeq H_i(V', \varphi'(D^{q+1}\times S^q)) \simeq H_i(V')$ for $i < q$ the statement $b_q(V', \mathbb{Q}) \not\equiv b_q(V, \mathbb{Q})(\text{mod } 2)$ will follow if we show that $\sum_{i=0}^{q} b_i(V', \mathbb{Q}) + \sum_{i=0}^{q} b_i(V, \mathbb{Q}) \not\equiv 0 \ (\text{mod } 2)$.

Let $W = I\times V \bigcup_{\varphi} D^{q+1}\times D^{q+1}$ be the topological manifold got as follows. We take the disjoint union of $I\times V$ and $D^{q+1}\times D^{q+1}$ and identify the points of $S^q\times D^{q+1}$ with their images under φ in $V\times 1$. Then W is a compact orientable manifold of dimension $2q+2$ with boundary consisting of the disjoint union of V and V'. Hence by Lemma 6.1 we have $e^*(bW; \mathbb{Q}) + e(W) \equiv \rho(\text{mod } 2)$ where ρ is the rank of the intersection bilinear pai ring $H_{q+1}(W, \mathbb{Q})\times H_{q+1}(W, \mathbb{Q}) \longrightarrow \mathbb{Q}$. Since q is even, this intersection bilinear pairing is skew symmetric and hence ρ is even. But
$$e^*(bW; \mathbb{Q}) \equiv \sum_{i=0}^{q} b_i(V', \mathbb{Q}) + \sum_{i=0}^{q} b_i(V, \mathbb{Q})(\text{md } 3).$$
Also W is of the same homotopy type as the space got from V by attaching D^{q+1} by means of $\varphi | S^q\times 0$ and hence $e(W) = e(V) + (-1)^{q+1}$. Since V

is of dimension $2q+1$ by Poincaré duality we have $e(V) \equiv 0 \pmod 2$ and hence the relation $e^*(bW; \mathbb{Q}) + e(W) \equiv 0 \pmod 2$ yields

$$\sum_{i=0}^{q} b_i(V', \mathbb{Q}) + \sum_{i=0}^{q} b_i(V, \mathbb{Q}) + (-1)^{q+1} \equiv 0 \pmod 2 \quad \text{or}$$

$$\sum_{i=0}^{q} b_i(V', \mathbb{Q}) + \sum_{i=0}^{q} b_i(V, \mathbb{Q}) \not\equiv 0 \pmod 2.$$ This completes the proof of Lemma 6.2.

We now consider the case when q is odd. Let d be the order of 'a.' Since $a \neq 0$ and is a torsion element of $H_q(V)$, d is an integer > 1. Now suppose the imbedding $\varphi : S^q \times \frac{3}{2} D^{q+1} \longrightarrow V$ representing 'a' is replaced by φ_α given by $\varphi_\alpha(x,y) = \varphi(x, \alpha(x) \cdot y)$ with $\alpha : S^q \longrightarrow SO_{q+1}$ a C^∞ map satisfying $s_*(\alpha) = 0$ where $s_* : \pi_q(SO_{q+1}) \longrightarrow \pi_q(SO_{2q+1+\ell})$ is the homomorphism induced by the inclusion $s : SO_{q+1} \longrightarrow SO_{2q+1+\ell}$. Let y^* be a base point chosen once for all and let $j : SO_{q+1} \longrightarrow S^q$ be the map given by $j(w) = y^* \cdot w$. (We consider y^* as a row vector in $\mathbb{R}^{q+1}$ and the matrix w operates on the right on y^*). We want to study the q^{th} homology of $V'_\alpha = \chi(V, \varphi_\alpha)$. Clearly the manifold $V_0 = V - \varphi_\alpha(S^q \times B^{q+1})$ is independent of α and the meridiam $\varphi_\alpha(y^* \times S^q)$ of the torus $\varphi_\alpha(S^q \times S^q) = \text{Bdry } V_0$ as a point set does not depend on α, hence its homology class ε' in $H_q(V_0)$ does not depend on α. On the other hand the homology class ε_α of $\varphi_\alpha(S^q \times y^*)$ in $H_q(V_0)$ does depend on α. Let ε be the homology class of $\varphi(S^q \times y^*)$ in $H_q(V_0)$. Then we have

$\mathcal{E}_\alpha = \mathcal{E} + j_*(\alpha)\ \mathcal{E}'$ where $j_* : \pi_q(SO_{q+1}) \longrightarrow \pi_q(S^q) \simeq \mathbb{Z}$

is the homomorphism induced by j.

We claim that $\exists$ an integer d'_α such that $d\,\mathcal{E}_\alpha = d'_\alpha \mathcal{E}'$

in $H_q(V_o)$. Actually in the homology exact sequence

$$\longrightarrow H_{q+1}(V_o) \xrightarrow{i_*} H_{q+1}(V) \longrightarrow H_{q+1}(V,V_o) \xrightarrow{\partial} H_q(V_o) \xrightarrow{i_*} H_q(V) \longrightarrow$$

identifying $H_{q+1}(V,V_o)$ with $\mathbb{Z} \simeq H_{q+1}(S^q \times D^{q+1}, S^q \times S^q)$ by excision

we saw that the homomorphism $H_{q+1}(V) \longrightarrow H_{q+1}(V,V_o)$ was given by

$x \leadsto \pm\, a\cdot x$ (Refer to the proof of Lemma 4.7). Since 'a' is a

torsion element we have $a\cdot x = 0$ and hence

$$0 \longrightarrow \mathbb{Z} \simeq H_{q+1}(V,V_o) \xrightarrow{\partial} H_q(V_o) \xrightarrow{i_*} H_q(V) \longrightarrow \cdots$$

is exact. ∂ carries the generator $\varphi(y^* \times D^{q+1})$ of the relative

group $H_{q+1}(V,V_o)$ into $\mathcal{E}'$ in $H_q(V_o)$. The element $d\,\mathcal{E}$ of

$H_q(V_o)$ gets mapped into $da = 0$ by i_* and hence $\exists$ an integer

d' such that $d\mathcal{E} = d'\,\mathcal{E}'$. From $\mathcal{E}_\alpha = \mathcal{E} + j_*(\alpha)\ \mathcal{E}'$ we have

$d\,\mathcal{E}_\alpha = d\mathcal{E} + dj_*(\alpha)\ \mathcal{E}' = (d' + dj_*(\alpha))\ \mathcal{E}'$. Thus

$d'_\alpha = d' + dj_*(\alpha)$ satisfies the requirement $d\,\mathcal{E}_\alpha = d'_\alpha\, \mathcal{E}'$.

Let a'_α be the element $(i'_\alpha)_*(\mathcal{E}') \in H_q(V'_\alpha)$ where $i'_\alpha : V_o \to V'_\alpha$

is the inclusion. Then from the exact sequence

$$H_{q+1}(V', V_o) \xrightarrow{\partial} H_q(V_o) \xrightarrow{(i'_\alpha)_*} H_q(V'_\alpha) \longrightarrow 0$$

we see that $(i'_\alpha)_*(d_\alpha \mathcal{E}') = (i'_\alpha)_*(d\, \mathcal{E}_\alpha) = 0$ since ∂ carries the generator $\varphi'_\alpha(D^{q+1} \times y^*)$ of the relative group $H_{q+1}(V', V_o)$ into the element $\mathcal{E}_\alpha \in H_q(V_o)$ represented by $\varphi_\alpha(S^q \times y^*)$. It follows that a'_α is of order $|d' + dj_*(\alpha)|$ with d' = the order of $a' \in H_q(V')$ represented by $\varphi'(y^* \times S^q)$.

Identifying the stable group $\pi_q(SO_{2q+1+\ell})$ with $\pi_q(SO_{q+2})$ there is an exact sequence associated with the fibration $SO_{q+2}/SO_{q+1} = S^{q+1}$:

$$\pi_{q+1}(S^{q+1}) \xrightarrow{\partial} \pi_q(SO_{q+1}) \xrightarrow{s_*} \pi_q(SO_{q+2}).$$

The composition $\pi_{q+1}(S^{q+1}) \xrightarrow{\partial} \pi_q(SO_{q+1}) \xrightarrow{j_*} \pi_q(S^q)$ (for q odd) carries a generator of $\pi_{q+1}(S^{q+1})$ into twice a generator of $\pi_q(S^q)$. It follows that $j_*(\alpha)$ with $\alpha \in \ker s_*$ can take any even value. (+ ve or - ve). Thus if d' is not divisible by d we can choose an $\alpha \in \operatorname{Ker} s_*$ such that the order $|d'_\alpha|$ of a'_α satisfies $|d'_\alpha| < d$. Thus we have proved the following

Lemma 6.3. Let q be odd and > 1 and $\varphi: S^q \times \frac{3}{2} D^{q+1} \longrightarrow V$ an orientation preserving imbedding representing a torsion element $a \in H_q(V)$ of order $d > 1$. Then the element $a' \in H_q(V')$ represented by $\varphi'(y^* \times S^q)$ is of finite order; moreover if d' is the order of a' and if d' is not divisible by d then $\exists$ an $\alpha \in \operatorname{Ker} s_*$ such that the element a'_α in $H_q(V'_\alpha) = H_q(\chi(V, \varphi_\alpha))$ represented by $\varphi'_\alpha(y^* \times S^q)$ has order strictly less than that of a in $H_q(V)$.

Next we deal with the case when d' is divisible by d. We recall the definition of linking numbers [Siefert-Threlfall[7]]. Let $\lambda \in H_p(V)$ and $\mu \in H_{n-p-1}(V)$ be torsion classes in the respective groups. Associated with the coefficient sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{h} \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

we have the exact homology sequence

$$\longrightarrow H_{p+1}(V;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} H_p(V) \xrightarrow{h_*} H_p(V;\mathbb{Q}) \longrightarrow \cdots$$

(h is the inclusion of $\mathbb{Z}$ in $\mathbb{Q}$). Since λ is a torsion element we have $h_*(\lambda) = 0$. Therefore $\exists\, \nu \in H_{p+1}(V;\mathbb{Q}/\mathbb{Z})$ such that $\partial(\nu) = \lambda$. The pairing $(\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ defined by multiplication gives an intersection pairing $H_{p+1}(V;\mathbb{Q}/\mathbb{Z}) \otimes H_{n-p-1}(V) \longrightarrow \mathbb{Q}/\mathbb{Z}$. We denote this pairing by a dot '.' .

<u>Definition 6.4</u>. The linking number $L(\lambda, \mu)$ is the rational number modulo 1 defined by $L(\lambda, \mu) = \nu . \mu$. This linking number is well-defined and satisfies the relation

$L(\mu, \lambda) + (-1)^{p(n-p-1)} L(\lambda, \mu) = 0$ [Ref: Siefert-Threlfall [7]].

Lemma 6.5. $L(a,a) = \pm d'd \pmod 1$. (This lemma is valid even if d' is not divisible by d. In fact when d' is divisible by d this lemma asserts that $L(a,a) = 0$).

Proof. We have $d\epsilon - d'\epsilon' = 0$ in $H_q(V_0)$. Therefore the cycle $d\varphi(S^q \times y^*) - d'\varphi'(y^* \times S^q)$ bounds a chain C in V_0. Let $C_1 = \varphi(y^* \times D^{q+1})$ be the cycle in $\varphi(S^k \times D^{k+1}) \subset V$ with boundary $\varphi(y^* \times S^q)$. The chain $C + d'C_1$ has boundary $d\varphi(S^q \times y^*)$. Hence $\frac{C+d'C_1}{d}$ has boundary $\varphi(S^q \times y^*)$. Also $\varphi(S^q \times 0)$ represents the same class $a \in H_q(V)$ as $\varphi(S^q \times y^*)$. Taking the intersection of $\frac{C+d'C_1}{d}$ with $\varphi(S^k \times 0)$ we get $\pm d'/d$ since C is disjoint from $\varphi(S^k \times 0)$ and C_1 has intersection number ± 1 with $\varphi(S^k \times 0)$. Thus $L(a,a) = \pm d'/d \pmod 1$.

Lemma 6.6. Let $V = V^{2q+1}$ be a compact oriented C^∞ manifold with $q > 1$ odd, and $f = V \to X$ a map of degree 1 satisfying the following conditions.

① $f_* : H_i(V) \to H_i(X)$ is an isomorphism for $i < q$

② $K_q = \operatorname{Ker} f_* : H_q(V) \to H_q(X)$ is a torsion group. Suppose further that $L(a,a) = 0 \ \forall a \in K_q$. Then K_q is a direct sum of a finite number of copies of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

<u>Remark.</u> When stating this lemma we have a complex X satisfying the conditions of Theorem 2.1 in our mind. In particular X satisfies Poincaré duality and it is only this that is needed for the validity of Lemma 6.6.

<u>Proof</u>. Since X satisfies Poincaré duality for integer coefficients it follows that X satisfies Poincaré duality for coefficients in any arbitrary commutative ring. Using the fact that f is of degree 1, monomorphisms $g_j : H_j(X) \longrightarrow H_j(V)$ were constructed satisfying $H_j(V) = \operatorname{Ker} f_j \oplus g_j(H_j(x))$ for every j [Lemma 2.5]. The same procedure can be adopted to define monomorphisms $g_{j,\Lambda} : H_j(X, \Lambda) \longrightarrow H_j(V, \Lambda)$ for any commutative coefficient ring and we still have $H_j(V, \Lambda) = \operatorname{Ker} f_{j,\Lambda} \oplus g_{j,\Lambda}(H_j(X, \Lambda))$. Also the exact sequences in homology corresponding to the exact coefficient sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ give rise to a commutative diagram.

$$\begin{tikzcd}
H_{q+1}(V, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\partial"] & H_q(V; \mathbb{Z}) \arrow[r, "h_*"] & H_q(V, \mathbb{Q}) \arrow[r] & {} \\
H_{q+1}(X, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[u, "g_q"'] & H_q(X, \mathbb{Z}) \arrow[r, "h_*"] \arrow[u, "g_q"'] & H_q(X, \mathbb{Q}) \arrow[r] \arrow[u, "g_q"'] & {}
\end{tikzcd}$$

Let $T_q(V)$ and $T_q(X)$ denote the torsion subgroups of $H_q(V)$ and $H_q(X)'$ respectively. Then from assumption (2) we have $T_q(V) = K_q \oplus g\, T_q(X)$. For any $b, b^1\ T_q(V)$ let $L(b,b^1)$ denote their linking number. Then since q is odd we have $L(b,b^1) = L(b^1,b)$. According to Poincaré duality theorem for torsion group [7, p. 245] L defines a non degenerate pairing $T_q(V) \otimes T_q(V) \longrightarrow \mathbb{Q}/\mathbb{Z}$ We claim that $L \mid K_q \otimes K_q$ gives a non degenerate pairing $K_q \otimes K_q \longrightarrow \mathbb{Q}/\mathbb{Z}$. Let $b \in K_q$ satisfy $L(b,b^1) = 0\ \forall\ b^1 \in K_q$. We have to show that $L(b,c) = 0\ \forall c \in T_q(V)$. Since $T_q(V) = K_q \oplus g\, T_q(X)$ we have only to prove that $L(b,y) = 0\ \forall y \in g\ T_q(X)$. Let $y^1 \in T_q(X)$ be such that $g(y^1) = y$. Then $h_*(y^1) = 0$ (since y^1 is a torsion element) and therefore $\exists\ Z^1 \in H_{q+1}(X,\mathbb{Q}/\mathbb{Z})$ such that $\partial Z^1 = y^1$. The element $Z \in H_{q+1}(V,\mathbb{Q}/\mathbb{Z})$ given by $Z = g(Z^1)$ satisfies $\partial Z = y$. Now $L(b,y) = L(y,b) = Z.b$ (this intersection is the one corresponding to the pairing $(\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$). Thus we have only to verify $K_q \cdot g(H_{q+1}(X,\mathbb{Q}/\mathbb{Z}) = 0$. This can be proved in a way similar to Lemma 5.2. Thus $L \mid K_q \otimes K_q \longrightarrow \mathbb{Q}/\mathbb{Z}$ gives a non-degenerate pairing.

We now claim that every element $a \in K_q$ is of order 2. In fact for any $b \in K_q$ we have

$0 = L(a+b,\ a+b) = L(a,b) + L(b,a) = L(2a,b)$. Hence $2a = 0$.

This completes the proof of Lemma 6.6.

Lemma 6.7. Let $f : V \to X$ be of degree 1 satisfying the following conditions.

1) $f_* : H_i(V) \to H_i(X)$ an isomorphism for every $i < q$

2) $K_q = \text{Ker } f_q : H_q(V) \to H_q(X)$ a direct sum of a finite number of copies of $\mathbb{Z}_2$ and that $\forall\, a \in K_q$ the linking number $L(a,a) = 0$.

Suppose $\varphi : S^q \times \frac{3}{2} D^{q+1} \to V$ is an imbedding representing $a \neq 0$ in K_q. Then for the manifold $V' = \chi(V, \varphi)$ the Bettinumber $b_q(V'; \mathbb{Z}_2)$ (i.e. the dimension of $H_q(V'; \mathbb{Z}_2)$) satisfies
$b_q(V'; \mathbb{Z}_2) \not\equiv b_q(V; \mathbb{Z}_2) (\text{mod } 2)$.

Proof. Let $W = 1 \times V \cup_\varphi D^{q+1} \times D^{q+1}$ as in the proof of Lemma 6.2. By Lemma 6.1 we have $e^*(V'; \mathbb{Z}_2) + e^*(V; \mathbb{Z}_2) + e(W) \equiv \rho \ (\text{mod } 2)$ where ρ is the rank of the intersection bilinear $H_{q+1}(W; \mathbb{Z}_2)$. If we show that ρ is even then as in the proof of Lemma 6.2 it will follow that $b_q(V'; \mathbb{Z}_2) \not\equiv b_q(V; \mathbb{Z}_2) (\text{mod } 2)$. Thus we have only to show that ρ is even. If for every $x \in H_{q+1}(W; \mathbb{Z}_2)$ the intersection $x \cdot x$ is zero then ρ will be even. Thus we have only to show that $x \cdot x = 0\ \forall x \in H_{q+1}(W; \mathbb{Z}_2)$. In the homology exact sequence for the pair (W,V) with $\mathbb{Z}_2$ coefficients

$$H_{q+1}(V; \mathbb{Z}_2) \xrightarrow{j_*} H_{q+1}(W; \mathbb{Z}_2) \longrightarrow H_{q+1}(W,V; \mathbb{Z}_2) \xrightarrow{\partial} H_q(V; \mathbb{Z}_2)$$

the group $H_{q+1}(W,V; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ with $\varphi(D^{q+1} \times y^*)$ as generator and ∂ carries it into $a \neq 0$ in $H_q(V;\mathbb{Z}_2)$. Actually if we use $\mathbb{Z}_2$ coefficients and take the kernel $K_q(\mathbb{Z}_2)$ of $f_* : H_q(V;\mathbb{Z}_2) \longrightarrow H_q(X,\mathbb{Z}_2)$ it will be isomorphic to K_q since K_q is a direct sum of a finite number of copies of $\mathbb{Z}_2$ and $f_* : H_j(V) \longrightarrow H_j(X)$ is an isomorphism for $j < q$. Hence $\partial : H_{q+1}(W,V;\mathbb{Z}_2) \longrightarrow H_q(V;\mathbb{Z}_2)$ is a monomorphism and therefore $j_* : H_{q+1}(V;\mathbb{Z}_2) \longrightarrow H_{q+1}(W;\mathbb{Z}_2)$ is onto. It is clear that $x \cdot x = 0$ for elements of the form $x = j_*(y)$ with $y \in H_{q+1}(V;\mathbb{Z}_2)$ because a cycle representing y can be deformed in W so as not to intersect V. This completes the proof of Lemma 6.7.

Now we go to the proof of Theorem 2.1 when $n = 2q+1$ with $q \geqslant 2$. We have already obtained a connected simply connected, compact oriented C^∞ manifold M of dimension n and a map $f : M \longrightarrow X$ of degree 1 satisfying $f!(\eta) \simeq \tau_M^n \oplus \epsilon_M^{\ell}$ and $f_* : H_j(M) \longrightarrow H_j(X)$ isomorphism for $j < q$. Let K_q be the Kernel of $f_q : H_q(M) \longrightarrow H_q(X)$. Let $K_q = F_q \oplus T(K_q)$ with F_q free and $T(K_q)$ the torsion subgroup of K_q. Choose an element 'a' forming part of a basis for F_q. As an easy consequence of Poincaré duality we get an element $b \in H_{q+1}(M)$ such that $a \cdot b = 1$. By Lemma 4.3 $\exists$ a C^∞ imbedding $\varphi : S^q \longrightarrow M$ representing 'a' with trivial normal bundle ν_φ and further satisfying $f \circ \varphi \sim x^*$ (the constant map). Extending φ to an orientation preserving imbedding $\varphi : S^q \times \frac{3}{2} D^{q+1} \longrightarrow M$ and performing surgery we get a manifold

$\chi(M, \varphi) = M'$ and a map $f' : M' \longrightarrow X$ of degree 1 with $f'_* : H_j(M') \longrightarrow H_j(X)$ isomorphisms for $j < q$ and $K'_q = \operatorname{Ker} f'_q : H_q(M') \longrightarrow H_q(X)$ isomorphic to $K_q/(a)$. (Refer to case (i) of Lemma 4.7). Changing φ to φ_α if necessary for a suitable C^∞ map $\alpha : S^q \longrightarrow SO_{q+1}$ we may assume $f'!(\eta) \simeq \tau^n_{M'} \oplus \varepsilon^\ell_{M'}$ (Proposition 3.7). Applying surgery successively to 'kill' elements of a basis of F_q we get a connected, simply connected compact oriented C^∞ manifold M'' and a map $f'' : M'' \longrightarrow X$ of degree 1 satisfying the following conditions:

1) $f''_* : H_j(M'') \longrightarrow H_j(X)$ is an isomorphism $\forall\, j < q$ and $K''_q = \operatorname{Ker} f''_q : H_q(M'') \longrightarrow H_q(X)$ is precisely the torsion subgroup of K_q.

2) $f''!(\eta) \simeq \tau^n_{M''} \oplus \varepsilon^\ell_{M''}$.

Thus changing notations we may assume that the original $f : M \longrightarrow X$ itself satisfied the conditio that K_q is a torsion group. Now assume q even. Choosing an element $a \neq 0$ in K_q and applying surgery to 'kill' a' (this is possible because of Lemma 4.3) we introduce an additional $\mathbb{Z}$ to the Kernel, but the torsion subgroup of the Kernel becomes $K_q/(a)$. (Refer to Lemma 6.2) But by our earlier remarks we can successfully apply surgery to kill $\mathbb{Z}$. In other words by two suitable surgeries on M we can get a compact, oriented, connected, simply connected C^{∞} manifold M^1 and a map $f^1 : M^1 \longrightarrow X$ of degree 1 with $f^1!(\eta) \approx \tau^n_{M^1} \oplus \theta^{\ell}_{M^1}$, $f^1_* : H_j(M) \longrightarrow H_j(X)$ isomorphism for $j < q$ and $K'_q = \operatorname{Ker} f'_q : H_q(M^1) \longrightarrow H_q(X)$ definitely smaller than K_q. Iteration of this procedure a finite number of times proves Theorem 2.1 for $n = 2q + 1$ with q even.

We have still to consider the case q odd. If $a \neq 0$ in K_q is of order d when we perform surgery by means of an imbedding $\varphi : S^q \times \frac{3}{2} D^{q+1} \longrightarrow M$ representing 'a' and get $f^1 : M^1 = \chi(M, \varphi) \longrightarrow X$ we introduce a new element of finite order in the Kernel of f^1. To get $f^1!(\eta) \simeq \tau^n_{M^1} \oplus \theta^{\ell}_{M^1}$ we may have to alter φ into φ_α for a suitable $\alpha : S^q \longrightarrow SO_{q+1}$ and this can be done by Proposition 3.7. We can assume that φ itself satisfied this requirement also. However if we change again φ to φ_α with $\alpha \in \operatorname{Ker} s_*$ there is no obstruction to getting an isomorphism of $f^1_\alpha!(\eta)$ with $\tau^n_{M^1_\alpha} \oplus \theta^{\ell}_{M^1_\alpha}$. It is this freedom of choice of α in $\operatorname{Ker} s_*$ that helps in proving Theorem 2.1 for $n = 2q + 1$ with q odd > 1. If the order d^1 of $a^1 \in H_q(M^1)$ represented by $\varphi^1(y^* \times S^q)$ is not divisible by d then for a suitable $\alpha \in \operatorname{Ker} s_*$ the element $a^1_\alpha \in H_q(M^1_\alpha)$ will have order strictly less than d (Lemma 6.3). It follows now from Lemmas 6.5 and 6.6 that we can get a manifold M'' which is $\mathcal{X}$-equivalent to M and a map $f'' : M'' \to X$ satisfying the following conditions.

1. M'' is connected, simply connected and f'' is of degree 1.
2. $f''_* : H_j(M'') \to H_j(X)$ is an isomorphism for $j < q$; the Kernel K''_q of $f''_q : H_q(M'') \to H_q(X)$ is a direct sum of a finite number of copies of $\mathbb{Z}_2$.
3. $f''!(\eta) \simeq \tau^n_{M''} \oplus \nu^{\ell}_{M''}$.

Lemma 6.7 coupled with the observations made above helps in getting a manifold M''' which is connected and simply connected and $\mathcal{X}$-equivalent to M'' and a map $f''' : M''' \to X$ with $f'''_* : H_j(M''') \to H_j(X)$ isomorphism for $j \quad q$ and $f'''!(\eta) \simeq \tau^n_{M'''} \oplus \nu^{\ell}_{M'''}$. From the remark 4.5 it follows that f''' is a homotopy equivalence. This completes the proof of Theorems 2.1.

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PART II

SIEBENMANN'S THEOREM

§ 1. THE ASSUMPTION OF SIMPLE-CONNECTEDNESS IN THE BROWDER-NOVIKOV THEOREM.

In this section we will illustrate by examples that simple connectedness of X and condition (iii) are essential for the validity of Theorems 2.1 of Part I. We first construct a compact, connected combinational manifold Y of dimension 12 with $\pi_1(Y) = 0$ and satisfying condition (ii) of Theorem 2.1 which however is not of homotopy type of any closed C^{∞} manifold. Since Y is an orientable ($\pi_1(Y) = 0$) compact manifold condition (i) is automatically satisfied. This example thus illustrates that condition (iii) of Theorem 2.1 (Part I) is not redundant. Let k be any integer ≥ 1 and $\prod^k S^1$ the cartesian product of k copies of the circle. We will show that $X = Y \times \prod^k S^1$ satisfies condition (ii), and in case k is divisible by 4 satisfies condition (iii) as well. However from Siebenmann's Theorem (which will be stated later) it follows that X is not of the homotopy type of any closed C^{∞} manifold.

1.1. The symmetric 8×8 matrix given below is a unimodular matrix of signature 8.

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Denote the (i,j) th entry of this matrix by C_{ij}. It is known that one can choose C^∞ imbeddings $f_i : S^5 \times 0 \longrightarrow b\,D^{12} = S^{11}$ $(i = 1,..,8)$ with disjoint images such that the linking numbers $L(f_i(S^5 \times 0), f_j(S^5 \times 0))$ of $f_i(S^5 \times 0)$ and $f_j(S^5 \times 0)$ in $b\,D^{12}$ for $i \neq j$ are C_{ij}. Moreover, for each i we can choose f_i so that $\exists$ a differentiably imbedded disk $D_i'^6$ in D^{12} which bounds $f_i(S^5 \times 0)$. A tubular neighbourhood of $f_i(S^5)$ in $b\,D^{12}$ can be got as the restriction of a tubular neighbourhood of $D_i'^6$ in D^{12}. In otherwords $\exists$ C^∞ imbeddings $g_i : D^6 \times D^6 \longrightarrow D^{12}$ such that $g_i(S^5 \times D^6) \subset b\,D^{12}$, $g_i | S^5 \times 0 = f_i$ and $D_i'^6 = g_i(D^6 \times 0)$. We can choose these g_i such that $g_i(S^5 \times D^6)$ are pair-wise disjoint in $b\,D^{12}$. Let $\alpha : S^5 \longrightarrow SO_6$ be a C^∞ map representing the element $\partial \iota_6 \in \pi_5(SO_6)$ where $\iota_6 \in \pi_6(S^6)$ is a generator and ∂ is the boundary homomorphism in the exact sequence $\pi_6(S^6) \longrightarrow \pi_5(SO_6) \longrightarrow \pi_5(SO_7)$ corresponding to the fibration $SO_7/SO_6 = S^6$. Let $\varphi_i : S^5 \times D^6 \longrightarrow b\,D^{12}$ be defined by $\varphi_i(x,y) = g_i(x, \alpha(x)\,y)$. Let $D_i^6 \times D_i^6$ $(i = 1,.. 8)$ be eight disjoint copies of $D^6 \times D^6$ and let $S_i^5 \times D_i^6$ be the submanifold $S^5 \times D^6$ of $D_i^6 \times D_i^6$. Let $W^{12} = D^{12} + (\varphi_1^6) + \ldots + (\varphi_8^6)$ be the compact C^∞ manifold with boundary got from the disjoint union $D^{12} \cup (\cup_i D_i^6 \times D_i^6)$ by identifying points of $S_i^5 \times D_i^6$ with their images under φ_i and then rounding off the corners. We claim that W^{12} is a manifold with boundary, with $H_6(W^{12})$ free of rank 8 and having the given matrix as intersection matrix for a suitable choice of a basis for $H_6(W^{12})$. In W^{12} the image of $D_i^6 \times 0$ is also a **disk** bounding $f_i(S^5 \times 0)$ and

$\Sigma_i^6 = D_i'^6 \cup (D_i^6 \times 0)$ is a differentiably imbedded sphere in W^{12} whose normal bundle corresponds to the element $\partial \iota_6 \in \pi_5(SO_6)$. The classes corresponding to Σ_i^6 form a basis for $H_6(W^{12})$ since the classes corresponding to $D_i^6 \times 0$ form a basis for $H_6(W^{12}, D^{12})$. The intersections of Σ_i^6 and Σ_j^6 in W^{12} are precisely those of $D_i'^6$ and $D_j'^6$ in D^{12} which by definition are the linking numbers $L(f_i(S^5 \times 0), f_j(S^5 \times 0))$. Hence $\Sigma_i^6 \cdot \Sigma_j^6 = C_{ij}$ for $i \neq j$. Also if $k_* : \pi_5(SO_6) \longrightarrow \pi_5(S^5)$ is the map induced by $\rho \xrightarrow{k} x_0 \cdot \rho$ (x_0 a fixed element in S^5) of SO_6 in S^5 then it is known that $k_* \partial \iota_6 = \pm 2 \iota_5$ (i_5 a generator for $\pi_5(S^5)$). Also $k_*(\partial \iota_6)$ is precisely the Euler class of the normal bundle of each Σ_i^6 in W^{12}, and this as we have seen already (Refer to proof of Lemma 5.6, Part I) is the self intersection $\Sigma_i^6 \cdot \Sigma_i^6$ times a generator of $\pi_5(S^5)$. Thus by proper choice of $\iota_6 \in \pi_6(S^6)$ we see that $\Sigma_i^6 \cdot \Sigma_i^6$ can be made equal to 2. Since the matrix we started with is a unimodular matrix it follows that the boundary ∂W is a homotopy sphere [12]. Hence by Smale [10] W is actually a combinatorial S^{11}. By attaching the cone over S^{11} to W by a PL-isomorphism we get a closed combinatorial manifold Y^{12}. Clearly W is 5-connected and since Y^{12} is got by attaching a 12-cell to W it follows that Y is also 5-connected and that $H_6(W) \cong H_6(Y^{12})$ under the map induced by the inclusion $W \longrightarrow Y$. It follows that Y is a 5-connected combinatorial manifold of dimension 12, having the given matrix as intersection matrix for a suitable choice of basis for $H_6(Y)$.

Lemma 1.2. Y is not of the homotopy type of any compact C^∞ manifold.

Proof. For if Y were of the homotopy type of a compact C^∞ manifold there should exist classes $p_i \in H^{4i}(Y;\mathbb{Z})$ $(i = 1,2,3)$ such that $\{L_3(p_1, p_2, p_3)\}[Y] = \left\{\frac{1}{3^3.5.7}(62p_3 - 13p_2 p_1 + 2p_1^3)\right\}[Y] = 8.$

Since $H^4(Y;\mathbb{Z}) = 0$ and $H^8(Y;\mathbb{Z}) = 0$ the above implies that $\exists$ a class $p_3 \in H^{12}(Y;\mathbb{Z}) \simeq \mathbb{Z}$ such that $\frac{1}{3^3.5.7} 62p_3 [Y] = 8$. This in turn means the existence of an integer ℓ_3 such that $62\ell_3 = 3^3.5.7.8$. This is impossible since the prime 31 does not divide $3^3.5.7.8$.

Lemma 1.3. Let ξ be the trivial line bundle over Y. Then for the Thom space $T(\xi)$ of ξ the homology $H_{13}(T(\xi))$ has a spherical generator.

(This observation is due to A. Vasquez.)

Proof. Y is a 5-connected polyhedron with $H_6(Y)$ free abelian of rank 8, $H_{12}(Y) \simeq \mathbb{Z}$; $H_j(Y) = 0$ for all other $j \geq 1$. Thus a 'homology decomposition' [2] for Y will be $(S^6 \vee .. \vee S^6) \cup_h e^{12}$ where the wedge is a 8 fold wedge and to it is attached a 12-cell by means of a map $h : S^{11} \longrightarrow S^6 \vee .. \vee S^6$ representing the so called k-invariant or the dual Postnikov invariant. The Thom space $T(\xi)$ of ξ is homotopy equivalent to the suspension $\Sigma(Y \cup \text{'a'})$ of the disjoint union of Y and a point 'a'. Hence $T(\xi) \sim S^1 \vee (S^7 \vee .. \vee S^7) \cup_g e^{13}$ (we use ' ' to mean homotopy equivalence) where $g : S^{12} \longrightarrow S^1 \vee S^7 \vee .. \vee S^7$ is some map. It is know that $\pi_{12}(S^7) = 0$ [4].

By a theorem of Hilton [3] it follows that $\pi_{12}(S^1 V S^7 V..V S^7) = 0$. This shows that g is homotopically trivial and hence $T(\xi) \sim S^1 V (S^7 V..V S^7) V S^{13}$. The inclusion of S^{13} in $S^1 V(S^7 V..V S^7) V S^{13}$ followed by a homotopy equivalence $f : S^1 V(S^7 V..V S^7) V S^{13} \to T(\xi)$ represents a generator of $H_{13}(T(\xi))$.

Lemma 1.4. Let V be a closed, connected, orientable combinatorial manifold satisfying condition (ii) of Theorem 2.1 (Part I). Then $V \times S^1$ also satisfies condition (ii). If $\dim. V = 4d - 1$ then $V \ S^1$ also satisfies condition (iii).

Proof. Let $\dim V = n$ and let ξ^k be an orientable vector bundle of rank k on V with $H_{n+k}(T(\xi)) \simeq \mathbb{Z}$ with a spherical generator, say represented by the map $f : S^{n+k} \to T(\xi)$. Choose any orientable vector bundle η of rank ℓ over S^1 with a spherical generator for $H_{\ell+1}(T(\eta)) \simeq \mathbb{Z}$ represented by $g : S^{\ell+1} \to T(\eta)$. Such a bundle exists since S^1 is a C^∞ manifold. (In fact the trivial line bundle itself satisfies this condition). Let $\xi \times \eta$ be the cartesian product bundle on $V \times S^1$. Choosing fixed Riemannian metrics for ξ and η denote the associated unit disk bundles by A_ξ and A_η and let $\dot{A}_\xi$ and $\dot{A}_\eta$ be the boundaries of A_ξ and A_η respectively. Then $T(\xi) = A_\xi / \dot{A}_\xi$ and $T(\eta) = A_\eta / \dot{A}_\eta$. For the bundle $\xi \times \eta$ with the cartesian product Riemannian metric we have $A_{\xi\times\eta} = A_\xi \times A_\eta$ and $\dot{A}_{\xi\times\eta} = \dot{A}_\xi \times A_\eta \cup A_\xi \times \dot{A}_\eta$. Choosing the respective points at ∞ as base points in $T(\xi)$ and $T(\eta)$ let $T(\xi) \# T(\eta) = \dfrac{T(\xi) \times T(\eta)}{T(\xi) \vee T(\eta)}$. The canonical projections $\sigma_\xi : A_\xi \to T(\xi)$ and $\sigma_\eta : A_\eta \to T(\eta)$ yield the map

$\epsilon_\xi \times \epsilon_\eta : A_\xi \times A_\eta \rightarrow T(\xi) \times T(\eta)$. If $p : T(\xi) \times T(\eta) \rightarrow T(\xi) \# T(\eta)$ is the canonical map then $po(\epsilon_\xi \times \epsilon_\eta) : A_\xi \times A_\eta \rightarrow T(\xi) \# T(\eta)$ yields a (1 - 1) onto map of $\frac{A_\xi \times A_\eta}{A_\xi \times \dot{A}_\eta \cup \dot{A}_\xi \times A_\eta} \rightarrow T(\xi) \# T(\eta)$. The compactness of the spaces involved shows that the map $T(\xi \times \eta) \rightarrow T(\xi) \# T(\eta)$ thus obtained is a homeomorphism. Clearly the map $f \# g : S^{n+k} \# S^{\ell+1} = S^{n+1+k+\ell} \longrightarrow T(\xi) \# T(\eta)$ represents a generator of $H_{n+1+k+\ell}(T(\xi \times \eta))$.

Suppose $n = 4d - 1$. Choose a basis $X_1, \ldots, X_r$ for $H^{2d-1}(V; \mathbb{Q})$. By Poincare duality $\exists$ a basis $Y_1, \ldots, Y_r$ for $H^{2d}(V; \mathbb{Q})$ such that $X_i . Y_j = \delta_{ij}$. Then for $H^{2d}(V \times S^1; \mathbb{Q})$ the elements $X_1 \otimes s, \ldots, X_r \otimes s \,;\, Y_1 \otimes 1, \ldots, Y_r \otimes 1$ where $s \in H^1(S^1, \mathbb{Q})$ is a generator form a basis. With respect to this basis the intersection matrix is

$$2d\left\{ \overbrace{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}^{2d} \right. .$$

Hence the signature of the manifold $V \times S^1$ is 0. Choosing η to be the trivial line bundle on S^1 we have

$$L_d(\bar{p}_1(\xi \times \eta), \ldots, \bar{p}_d(\xi \times \eta)) [V \times S^1] = L_d(\bar{p}_1(\xi) \otimes 1, \ldots, \bar{p}_d(\xi) \otimes 1) [V \times S^1] = 0.$$

It follows from Lemmas 1.3 and 1.4 that $X^{12+k} = Y^{12} \times \frac{k}{\Pi} S^1$ satisfies conditions (i) and (ii) of Theorem 2.1 (Part I) and also (iii) in case $k \geq 1$ is divisible by 4. From Siebenmann's Theorem stated below and Lemma 1.2 it will follow that none of the manifolds $X^{12+k} (k \geq 0)$ is of the homotopy type of a compact C^∞ manifold.

Let π be any multiplicative group and $\mathbb{Z}(\pi)$ the group ring of π over $\mathbb{Z}$. Two finitely generated projective $\mathbb{Z}(\pi)$-modules P_1 and P_2 are said to be equivalent if $\exists$ finitely generated free

$\mathbb{Z}(\pi)$-modules F_1 and F_2 with $P_1 \oplus F_1 \cong P_2 \oplus F_2$. The set of equivalence classes of finitely generated projective modules is denoted by $\tilde{K}_0(\mathbb{Z}(\pi))$; it is an abelian group under the operation induced by the direct sum operation on projective modules.

<u>Theorem 1.5</u>. (Siebenmann). <u>Let</u> X <u>be a finite complex such that</u> $X \times S^1$ <u>is of the homotopy type of a compact, connected,</u> C^∞ <u>manifold</u> V^{n+1} <u>without boundary of dimension</u> $n+1$ <u>with</u> $n \geq 5$. <u>Suppose</u> $\mathbb{Z}(\pi)$ <u>is</u> Noetherian <u>and that</u> $\tilde{K}_0(\mathbb{Z}(\pi)) = 0$ <u>where</u> $\pi = \pi_1(X)$. Choosing <u>a homotopy equivalence</u> $\theta : V \longrightarrow X \times S^1$ <u>and denoting the projection onto the second factor</u> $X \times S^1 \longrightarrow S^1$ <u>by</u> p_2 <u>let</u> W <u>be the covering of</u> V <u>got as the pull back of the covering</u> $\mathbb{R} \xrightarrow{(\mathrm{Exp}\ 2\pi i)} S^1$ <u>by means of the map</u> $p_2 . \theta : V \longrightarrow S^1$. <u>Then</u> W <u>with the natural differential structure it acquires as a covering manifold of</u> V, <u>is diffeomorphic to</u> $N^n \times \mathbb{R}$ <u>with</u> $N = N^n$ <u>a compact</u> C^∞ <u>manifold without boundary, of dimension</u> n.

<u>Remark</u>: As W is of the homotopy type of $X \times \mathbb{R}$ or X it follows that X is of the homotopy type of N. If π is free abelian of rank $\ell < \infty$ we have $\mathbb{Z}(\pi) \cong \mathbb{Z}[x_1, \ldots, x_\ell, x_1^{-1}, \ldots, x_\ell^{-1}]$ where $x_1, x_2, \ldots x_\ell$ are ℓ-indeterminates over $\mathbb{Z}$ and in this case $\mathbb{Z}(\pi)$ is Noetherian and $\tilde{K}_0(\mathbb{Z}(\pi)) = 0$. It is now clear that none of the manifolds $X^{12+k} = Y^{12} \times \frac{k}{\pi} S^1$ is of the homotopy type of any compact C^∞ manifold without boundary.

The theorem remains true if we drop the assumption that $\mathbb{Z}(\pi)$ is Noetherian. We give some more details on this in §3. The assumption $\tilde{K}_0(\mathbb{Z}(\pi))$ is however essential. An example of a group with $\widetilde{K}_0(\mathbb{Z}/\pi) \neq 0$ is the cyclic group or order 23. (See D.S. Rim [9]).

The rest of Part II deals with the Proof of Theorem 1.5. Let $f : V \to S^1$ be a C^∞ approximation to $p_2 \circ \Theta$ with $f \sim p_2 \circ \Theta : V \longrightarrow S^1$ (we use '$\sim$' to mean 'homotopic'). We denote the map $\mathrm{Exp}(2\pi i) : \mathbb{R} \longrightarrow S^1$ by q and let $p : W \longrightarrow V$ denote the covering mapping. By definition W is the inverse image of the covering $q : \mathbb{R} \longrightarrow S^1$ by means of the map $p_2 \circ \Theta : V \longrightarrow S^1$. Since $f \sim p_2 \circ \Theta$ $\exists$ a map $F : W \to \mathbb{R}$ making the following diagram commutative. Moreover F is C^∞.

$$\begin{tikzcd} W \arrow[r, "F"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "q"] \\ V \arrow[r, "f"] & S^1 \end{tikzcd}$$

<u>Diagram 1</u>

By Sard's Theorem $\exists$ a regular value $a \in S^1$ for f and without loss of generality we can assume $1 \in S^1$ to be a regular value for f. Then any integer is a regular value of F.

§ 2. <u>THE EXISTENCE OF ARBITRARY SMALL 0 and 1-NEIGHBOURHOOD OF '∞' AND '$-\infty$'</u>.

<u>Definition 2.1</u>. A C^∞ sub-manifold $M = M^{n+1}$ of dimension $n+1$ with boundary $b\,M$, of W is said to be a 0-nbd of ∞ (respy "$-\infty$") if

(1) M is a closed subset of W

(2) $\exists$ integers $m_1 < m_2$ with $F^{-1}[m_1, \infty) \supset M \supset F^{-1}[m_2, \infty)$
$\left\{ \text{respy } F^{-1}(-\infty, m_1] \subset M \subset F^{-1}(-\infty, m_2] \right\}$

(3) $b\,M$ is compact; M and $b\,M$ are connected.

M is said to be a 1-nbd of ∞ (respy "$-\infty$") if it is already a 0-nbd of ∞ (respy "$-\infty$") and the maps $\pi_1(b\,M) \longrightarrow \pi_1(M)$; $\pi_1(M) \longrightarrow \pi_1(W)$ induced by the respective inclusions are isomorphisms.

Definition 2.2. By the statement " arbitrary small 0 (or 1)-nbds of ∞ (respy $-\infty$)" we mean that given any compact set $K \subset W \,\exists$ a 0 (or 1)-nbd M of ∞ (respy $-\infty$) with $M \subset W - K$.

Let J denote an infinite cyclic group and let x be a generator of J. The Deck transformation group of the covering $\mathbb{R} \xrightarrow{q} S^1$ can be identified with J with x acting as the homeomorphism $r \to r+1$ of $\mathbb{R}$ onto itself. Since $W \xrightarrow{p} V$ is the pull back of the covering space $\mathbb{R} \xrightarrow{q} S^1$ the Deck transformation group of the covering $W \xrightarrow{p} V$ is also J and we denote the homeomorphism of W which corresponds to the generator x by α.

Lemma 2.3. Let σ be any arc in V and $w_o \in W$ any point with $p(w_o) = \sigma(0)$. Let τ^{w_o} be the unique lift of σ such that $\tau^{w_o}(0) = w_o$. The variation $\max\limits_{t,\, t' \in [0,1]} \left| F\,\tau^{w_o}(t) - F\,\tau^{w_o}(t')\right|$ of F on τ^{w_o} depends only on σ and not on the lift w_o of $\sigma(0)$.

This quantity which depends only on σ we refer to as the "variation of F on σ" and denote it by $V_F(\sigma)$.

Proof. Suppose w_o' is any other element of W with $p(w_o') = \sigma(o)$, then $w_o' = \alpha^k w_o$ for some integer k. The unique lift $\tau^{w_o'}$ of such that $\tau^{w_o'}(0) = w_o'$ is given by $\tau^{w_o'}(t) = \alpha^k\,\tau^{w_o}(t)$. Because of the commutativity of diagram 1 we have

$$F\ \tau^{w_0'}(t) = k + F\ \tau^{w_0}(t)$$

for all $t \in [0,1]$. The lemma follows.

Lemma 2.4. There exists a constant $C > 0$ such that any two points of V can be joined by means of an arc σ such that the variation $V_F(\sigma)$ of F on σ is less than C.

Proof. For any $v \in V$ $\exists$ an arcwise connected open nbd U_v of v in V such that $p^{-1}(U_v)$ decomposes into a disjoint union of open sets $\{W_v^j\}$ each of which gets mapped homeomorphically onto U_v by the restriction of p. We can choose another arcwise connected open set U_v' containing v such that $\bar{U}_v' \subset U_v$. Then each of the sets $W_v'^{\,j} = W_v^j \cap p^{-1}(U_v')$ gets mapped homeomorphically by p onto U_v' and $\overline{W_v'^{\,j}} = p^{-1}(\overline{U_v'}) \cap W_v^j$ is compact since $\bar{U}'$ is compact, being a closed subset of the compact space V. The argument used in Lemma 2.3 can be used to show that $\underset{w,\, w^1 \in \bar{W}'^{\,j}}{\mathrm{Max}} \left| F(w) - F(w^1) \right|$ is finite and depends only on U' (finiteness being a consequence of the compactness of $\bar{W}'^{\,j}$). We may call the above quantity the variation of F on U' or $\bar{U}'$. Compactness of V implies the existence of a finite number of sets $U_{v_1}', \ldots, U_{v_r}'$ covering V. Writing U_i' for U_{v_i}' and denoting the variation of F on U_i' by C_i let C be anyconstant $> C_1 + \ldots + C_r$. Then C satisfies the requirement of the Lemma. For if v_0, v_1 are any two points of V, since V is arcwise connected we can find distinct indices $j_1, \ldots j_\ell$ in $1, 2, \ldots, r$ such that $v_0 \in U_{j_1}'$ and

$v_1 \in U'_{j_\ell}$ and $U'_{j_\mu} \cap U'_{j_\mu+1} \neq \emptyset$. Choosing points $v'_\mu \in U'_{j_\mu} \cap U'_{j_\mu+1}$ and joining v_0 to v'_1 by an arc in U'_{j_1}; v'_1 to v'_2 by an arc in U'_{j_2} and so on we get an arc σ joining v_0 to v_1 such that $V_F(\sigma) \leq C_{j_1} + .. + C_{j} < C$.

<u>Lemma 2.5</u>. $\exists$ <u>a constant</u> $d > 0$ <u>with the following property:</u>
<u>For every</u> $v \in V$ $\exists$ <u>a loop</u> Θ_v <u>at</u> v <u>in</u> V <u>such that the loop</u> $f\Theta_v$ <u>represents the positive generator of</u> $\pi_1(S^1, f(v))$, <u>and</u> $V_F(\Theta_v) < d$.

<u>Proof</u>. Choose a point $v_0 \in V$ and any loop Θ_{v_0} at v_0 such that $f\Theta_{v_0}$ represents the positive generator of $\pi_1(S^1, f(v_0))$. Let e be the variation of F on Θ_{v_0} and $C > 0$ the constant of Lemma 2.4. Then $d = 2C+e$ satisfies the requirement of Lemma 2.5. For given any $v \in V$ $\exists$ a path σ^v in V such that $\sigma^v(0) = v$, $\sigma^v(1) = v_0$ and $V_F(\sigma^v) < C$. If we define Θ_v for any $v \neq v_0$ by $\Theta_v = \sigma^v \Theta_{v_0} (\sigma^v)^{-1}$ then clearly $f\Theta_v$ represents the positive generator of $\pi_1((S^1), f(v))$ and $V_F(\Theta_v) < C+e+C = 2C+e = d$.

According to our choice of d we have $d > C$.

<u>Lemma 2.6</u>. <u>Let</u> w <u>be any element of</u> $F^{-1}[\ell+d, \infty)$ <u>with</u> ℓ <u>any real number and</u> $v = p(w)$. <u>For any integer</u> $k \geq 0$ <u>let</u> τ_k <u>be the unique lift of</u> Θ_v^k <u>satisfying</u> $\tau_k(0) = w$. <u>Then the path</u> τ_k <u>lies in</u> $F^{-1}[\ell, \infty)$ <u>and</u> $F(\tau_k(1)) = k + F(w)$.

<u>Proof</u>. That $F(\tau_k(1)) = k+F(w)$ follows from the fact that $f \circ \Theta_v^k$ represents the element $k.$(+ ve generator) of $\pi_1(S^1, f(v))$. That τ_k lies in $F^{-1}[\ell, \infty)$ is proved by induction on k. For $k = 0$ there is

nothing to prove. Assume $k \geq 1$ and the lemma valid for $(k-1)$ instead of k. Let μ be the lift of θ_v with initial point $\mu(0) = \tau_{k-1}(1)$. Then $F\mu(0) = (k-1) + F(w) \geq (k-1) + \ell + d$. Since the variation of F on $\theta_v < d$ we have $F\mu(t) \geq (k-1) + \ell \; \forall \; t \in [0,1]$. Since $k \geq 1$ this implies $F\mu(t) \geq \ell$. Now τ_k is precisely the product $\tau_{k-1} \cdot \mu$ and whenever $t \leq \frac{1}{2}$, $F\tau_k(t) = F\tau_{k-1}(2t) \geq \ell$ (by induction hypothesis) and if $t \geq \frac{1}{2}$, $F\tau_k(t) = F\mu(2t-1) \geq (k-1) + \ell$ (by what is proved above). This shows that τ_k lies in $F^{-1}[\ell, \infty)$.

<u>Proposition 2.7</u>. <u>There exist arbitrary small</u> 0-neighbourhoods <u>of</u> '∞' (resp. $-\infty$) <u>in</u> W.

<u>Proof</u>. We prove the assertion for ∞, the proof for "$-\infty$" being similar is left out. Let K be any compact subset of W. $\exists$ an integer ℓ such that $F^{-1}[\ell, \infty) \subset W-K$. Since ℓ is a regular value of F we see that $F^{-1}[\ell, \infty)$ is a C^∞ submanifold of W, with boundary $F^{-1}(\ell)$. Let d be the constant of Lemma 2.5 (which as commented earlier has been chosen to be $> C$ the constant of Lemma 2.4).

<u>Claim</u>: Any two points w_o, w_1 of $F^{-1}[\ell + 2d, \infty)$ can be joined by means of a path in $F^{-1}[\ell, \infty)$.

Let $p(w_o) = v_o$, $p(w_1) = v_1$. By Lemma 2.4 $\exists$ an arc σ in V such that $\sigma(0) = v_o$, $\sigma(1) = v_1$ and $V_F(\sigma) < C$. Let τ be the unique lift of σ with initial point $\tau(0) = w_o$. Then $\tau(1)$ and w_1 are points on the same fibre of W and hence $F(w_1) = k + F(\tau(1))$ for a certain integer k. It follows that $\sigma^1 = \theta^k_{v_o} \cdot \sigma$ is a path joining v_o to v_1 in V whose lift τ^1 with initial point

$\tau^1(0) = w_0$ satisfies $\tau^1(1) = w_1$. We now consider the cases $k \geqslant 0$ and $k < 0$ separately. Case (i) $k \geqslant 0$. Since $V_F(\tau) < C < d$ and $F(\tau(0)) = F(w_0) \geqslant \ell + 2d$ it follows that $F(\tau(t)) > \ell + d$. From Lemma 2.6 we now have $F(\tau^1(t)) \geqslant \ell \ \forall\ t \in [0,1]$. Case (ii) $k < 0$. The path $(\tau^1)^{-1}$ is the composition $(\tau_{-k}).\tau^{-1}$ where τ_{-k} is the lift of $\theta_{v_0}^{-k}$ having as initial point $\tau_{-k}(0) = w_1$. Now, by assumption $F(w_1) \geqslant \ell + 2d$ and $-k > 0$. From Lemma 2.6 we see that τ_{-k} is an arc in $F^{-1}[\ell, \infty)$. Since τ (and hence τ^{-1} also) is an arc in $F^{-1}[\ell + d, \infty)$ we see that $(\tau^1)^{-1} = \tau_{-k}.\tau^{-1}$ is an arc in $F^{-1}[\ell, \infty)$ and hence τ^1 too is an arc $F^{-1}[\ell, \infty)$.

This completes the proof of the claim. Now it is clear that $F^{-1}[\ell, \infty)$ has only one non-compact connected component say M' and a finite number of compact connected components. Since $M' \supset F^{-1}[\ell + 2d, \infty)$ it follows that the boundary bM' of M' lies in $F^{-1}[\ell, \infty) - F^{-1}(\ell + 2d, \infty)$ and is therefore compact. If bM' were connected then M' itself would be a 0-nbd of ∞. Suppose bM' is not connected. Choosing a smooth path in M' from one component of bM' to another meeting bM' orthgonally and only at the end points and removing the interior of a tubular neighbourhood of the path we get a connected C^∞ submanifold M'' of W with

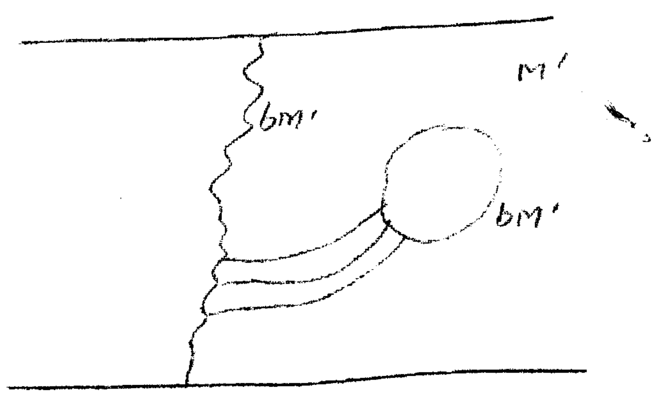

Diagram 2

bM'' compact and bM'' having one component less than bM'. Refer to Diagram 2. Since there are only a finite number of components after a finite number of such operations we get a connected C^{∞} cubmanifold M of W with bM compact and connected. Further $M \supset F^{-1}[m, \infty)$ for some integer m since the original M' contained $F^{-1}[\ell + 2d, \infty)$. Thus M is a 0-nbd of ∞.

Lemma 2.8. Let M^{n+1} be a C^{∞} submanifold of W^{n+1} with boundary $bM = N$ and let M and N be connected. Let M be a closed subset of W. Suppose the homomorphism $\pi_1(N) \longrightarrow \pi_1(W)$ induced by the inclusion is an isomorphism. Then $\pi_1(M) \longrightarrow \pi_1(W)$ induced by the inclusion of M in W is also an isomorphism.

Proof. Let $i : N \longrightarrow M$ and $j : M \longrightarrow W$ be the respective inclusions. Then $joi : N \longrightarrow W$ induced an isomorphism $(joi)_* : \pi_1(N) \longrightarrow \pi_1(W)$ by our hypothesis. Since $(joi)_* = j_* o i_*$ it follows that $j_* : \pi_1(M) \longrightarrow \pi_1(W)$ is an epimorphism. To show that $j_* : \pi_1(M) \longrightarrow \pi_1(W)$ is an isomorphism it therefore suffices to prove that j_* is a monomorphism. Since $\dim M = n+1$ and $n \geq 5$ any element of $\pi_1(M)$ can be represented by a C^{∞} imbedding $\varphi : S^1 \longrightarrow \operatorname{Int} M$ (in fact for this assertion to be valid it suffices that $n+1 \geq 3$). Suppose $\alpha \in \pi_1(M)$ is such that $j_*(\alpha) = 0$ and suppose $\varphi : S^1 \longrightarrow \operatorname{Int} M$ represents α. From $j_*(\alpha) = 0$ it follows that $\exists$ a map $h : D^2 \longrightarrow W$ extending φ. Since $\varphi(S^1) \cap N = \emptyset$ we can approximate h by a C^{∞} map $\theta : D^2 \longrightarrow W$ such that $\theta/S^1 = \varphi$

and θ is transverse regular on N. Then $D^2 \cap \theta^{-1}(N)$ consists of a finite number of disjoint simple closed curves (each one of them is a C^∞ imbedded S^1) in the interior of D^2. Take an inner most curve C. Now $\theta/C \longrightarrow W$ admits of an extension $\theta : \Delta \longrightarrow W$ where Δ is the closed region (inner most) bounded by C. Thus θ/C represents the trivial element of $\pi_1(W)$ and $\theta(C) \subset N$. Since $\pi_1(N) \longrightarrow \pi_1(W)$ is an isomorphism it follows that $\exists$ a map $\lambda : \Delta \longrightarrow N$ with $\lambda/C = \theta/C$. (Refer to diagram 3). Now using the fact that N is collared in M it is easy to get a map $\theta' : D^2 \longrightarrow W$ with the following properties:

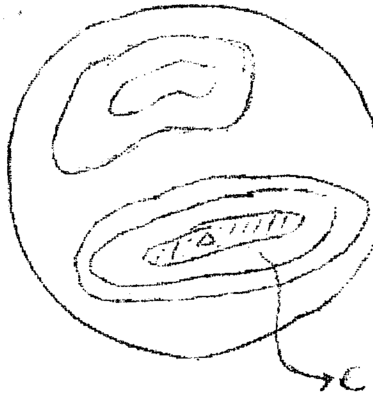

Diagram 3

(1) $\theta'/S^1 = \varphi$

(2) $\exists$ a nbd A of Δ in D^2 with A disjoint from the curves of $\theta^{-1}(N) \cap D^2$ different from C such that $\theta'(A) \cap N = \emptyset$ and $\theta'/D^2 - A = \theta/D^2 - A$.

For this θ' we have $\theta'^{-1}(N) \cap D^2$ consisting precisely of the curves in $\theta^{-1}(N) \cap D^2$ excepting C. Repeating this argument a finite number of times we finally get a map $\Phi : D^2 \longrightarrow W$ such that $\Phi / S^1 = \varphi$ and $\Phi^{-1}(N) \cap D^2 = \emptyset$. Since $\varphi(S^1) \subset \text{Int } M$ and since D^2 is connected we should have $\Phi(D^2) \subset \text{Int } M$, for otherwise $D^2 \cap \Phi^{-1}(\text{Int } M)$

and $D^2 \cap \Phi^{-1}(W - M)$ will be non void disjoint open sets of D^2. This means that $\alpha \in \pi_1(M)$ is the zero element and hence $\pi_1(M) \to \pi_1(W)$ is a monomorphism.

Proposition 2.9. There exist arbitrary small 1-neighbourhoods of "∞".

In the proof of this lemma we use a result in group theory which we state below without proof.

Lemma 2.10. Suppose G and H are finitely presentable groups and $G \xrightarrow{h} H \to 1$ is an exact sequence. Then the Kernel of h is the normal subgroup in G generated (as a normal subgroup) by a finite number of elements.

We now go to the proof of Proposition 2.9. We have $\pi_1(W) \simeq \pi_1(X)$ and by assumption X is a finite polyhedron. It follows that $\pi_1(W)$ is finitely presentable. Let M' with $N' = bM'$ be a zero neighbourhood of ∞ with $M' \subset W - K$. Choosing a base point $w_0 \in \mathrm{Int}\, M'$ and a small "contractible open set O" in $\mathrm{Int}\, M'$ as the "new base point" we can represent a finite system of generators $\alpha_1, .., \alpha_r$ of $\pi_1(W)$ by disjoint C^∞ imbeddings $\varphi_i : S^1 \to W$ $(i = 1, .. r)$ with the base point of S^1 going into O. To represent each α_i by a C^∞ imbedding we need that $\dim W \geq 3$ and also to get the imbeddings to have disjoint images we need $\dim W \geq 3$. But by hypothesis $\dim W \geq 6$. By choosing w_0 properly we can assume that $\varphi_i(S^1) \subset \mathrm{Int}\, M'$ for every i.

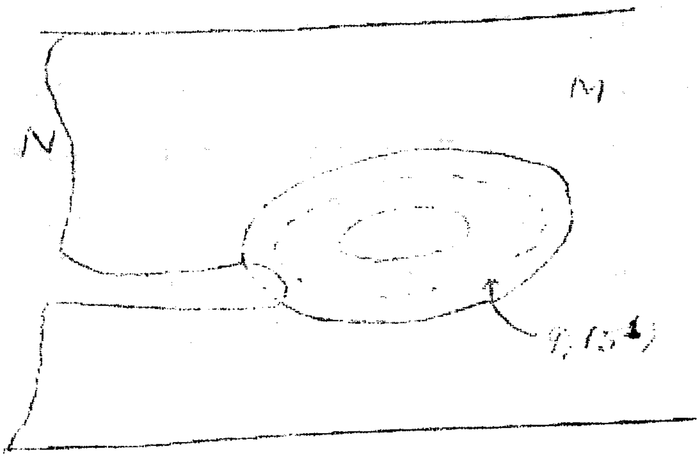

Diagram 4

The normal bundle of φ_i has a section for every i. Let U_i be an open tubular neighbourhood of $\varphi_i(S^1)$ for every i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Define $M'' = M' - \cup_i U_i$. Then M'' is still connected though $bM'' = N''$ is not in general. By choosing C^∞ paths in M'' meeting the components of bM'' only at the end points and orthogonally and removing the interiors of tubular neighbourhoods of these paths one gets a zero-neighbourhood $M''' \subset W - K$. Sections of the normal bundles $bU_i \longrightarrow \varphi_i(S^1)$ yield elements $\alpha_1',\ldots, \alpha_r' \in \pi_1(bM''')$ which map onto $\alpha_1,\ldots, \alpha_r \in \pi_1(W)$. (Refer to diagram 4). Thus $\pi_1(N''') \longrightarrow \pi_1(W)$ is onto, where $N''' = bM'''$. We denote (M''', N''') again by (M,N) and may assume (by Lemma 2.10) that $\pi_1 N \longrightarrow \pi_1 W$ is the normal closure in $\pi_1 N$ of a finite number of elements $\beta_1,\ldots, \beta_k$. Choose C^∞ imbeddings $\psi_i : S^1 \longrightarrow N$

with base point of S^1 going into some contractible open set B of N such that ψ_i represents β_i. $(i = 1,..,k)$. It is given that ψ_i represents the zero element in $\pi_1 W$. Hence there exists a map which can be assumed to be a C^∞ imbedding $\psi_i : D^2 \to W$ extending $\psi_i : S^1 \to N$. By translating M if necessary by a deck transformation we can assume that the images $\psi_i(D^2)$ all lie in W-K. We can get a tubular neighbourhood of $\psi_i(S^1)$ in N as the restriction to $\psi_i(S^1)$ of a tubular neighbourhood of $\psi_i(D^2)$ in W. We may assume that these tubular neighbourhoods are disjoint, and that their intersections with N are tubular neighbourhoods of $\psi_i(D^2) \cap N$. Let $C \subset D^2$ be an inner most simple closed component curve of $\psi_i^{-1}(N)$ for some i, and let D be the region of D^2 founded by C. Then $\psi_i(\text{int } D) \cap N = \emptyset$.

There are two cases:

If $\psi_i(\text{Int } D) \subset W$-$M$ then add the tubular neighbourhood of $\psi_i(D)$ to M. That is to say, a handle $D^2 \times D^{n-1}$ is attached to M. (Refer to diagram 5').

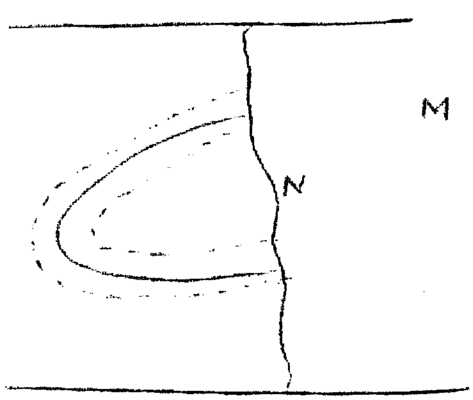

Diagram 5'

If $\varphi_i(\text{Int } D) \subset \text{int } M$ delete from M the tubular neighbourhood of $\varphi_i(D)$ (Refer to diagram 5'').

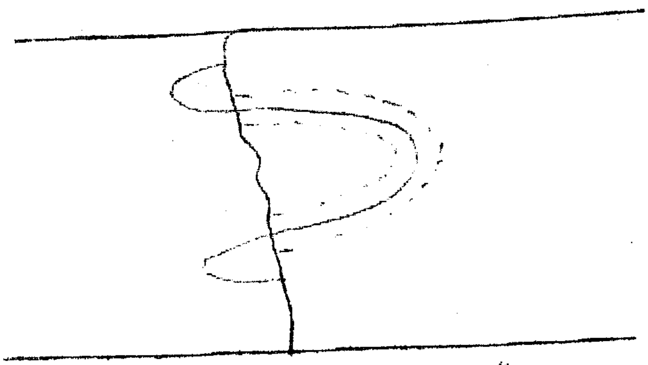

Diagram 5''

The new manifold M' with boundary N' is still a 0-neighbourhood of a. Moreover, $\pi_1 N'$ is a quotient of $\pi_1 N$ and the kernel of $\pi_1 N' \longrightarrow \pi_1 W$ is still (normally) generated by the classes of $\varphi_j(S^1)$, $j = 1,\ldots,k$ with $j \neq i$ if $C = bD^2$ for φ_i. But the φ_j extend to $\varphi_j : D^2 \longrightarrow M'$ with $\varphi_j(D^2) \cap N'$ consisting of one less component curve than the original intersection. After a finite number of such steps, one reaches a 0-neighbourhood M, $bM = N$, such that $\pi_1 N \longrightarrow \pi_1 W$ is an isomorphism. By Lemma 2.8, (M,N) is then a 1-neighbourhood.

§ 3. THE EXISTENCE OF ARBITRARY SMALL k. NEIGHBOURHOODS OF "∞" AND "$-\infty$" FOR $2 \leq k \leq n-2$.

Definition 3.1. Let k be an integer ≥ 2. A k-neighbourhood of ∞ (respy $-\infty$) in W is a 1-neighbourhood M of ∞ (respy $-\infty$) satisfying the following additional condition:

Denoting the universal covering of M by $\widetilde{M}$ with $p : \widetilde{M} \longrightarrow M$ the projection, let $\widetilde{N} = p^{-1}(N)$ where $N = bM$. The condition to be satisfied is : $H_i(\widetilde{M}, \widetilde{N}) = 0$ for $i \leq k$.

<u>Remark</u>: Since $\pi_1(N) \longrightarrow \pi_1(M)$ induced by the inclusion is an isomorphism it follows that $p : \widetilde{N} \longrightarrow N$ is the universal covering of N.

<u>Proposition 3.2</u>. <u>There exist arbitrary small</u> k-<u>neighbourhoods of</u> ∞ (respy '$-\infty$') <u>for any integer</u> k <u>such that</u> $2 \leq k \leq n-2$.

We prove this proposition for $k = 2$ first and then proceed by induction on k. It will be clear from the proof why we are forced to give a proof for $k = 2$ separately.

<u>Lemma 3.3</u>. <u>If</u> M <u>is a</u> 0 (respy 1) <u>neighbourhood of</u> '∞' <u>then</u> $M_0 = \overline{W - M}$ <u>is a</u> 0 (respy 1) <u>neighbourhood of</u> '$-\infty$'.

<u>Proof</u>. Clearly the boundary of M_0 is the same as that of M. Thus $bM_0 = bM = N$ is compact and connected. If $m_1 < m_2$ are integers such that $F^{-1}[m_1, \infty) \supset M \supset F^{-1}[m_2, \infty)$ then clearly $F^{-1}(-\infty, m_1] \subset M_0 \subset F^{-1}(-\infty, m_2]$. Let a,b be any two points in M_0. We will show that there is an arc in M_0 joining a and b. Since W is arcwise connected $\exists$ an arc σ in W with $\sigma(0) = a$ and $\sigma(1) = b$. If the arc σ lies in M_0 there is nothing to prove. If not $\exists$ real numbers to and t_1 such that $\sigma(t) \in M_0 \ \forall\, t \leq t_0$ and $\sigma(s) \in M_0 \ \forall\, s \geq t_1$ and $\sigma(t_0) \in N$, $\sigma(t_1) \in N$. Choosing an arc in N joining $\sigma(t_0)$ and $\sigma(t_1)$ we see that a and b can be joined by means of an arc in M_0. Thus M_0 is a 0-neighbourhood

of '$-\infty$'. If M is a 1-neighbourhood of ∞ then $\pi_1(bM) = \pi_1(bM_o) = \pi_1(N) \longrightarrow \pi_1(W)$ is an isomorphism and from Lemma 2.8 it follows that M_o is a 1-neighbourhood.

<u>Lemma 3.4</u>. <u>If</u> M <u>is a</u> 1-<u>neighbourhood of</u> ∞ <u>in</u> W, <u>then</u> $H_j(\widetilde{M})$ <u>is a finitely generated</u> $\mathbb{Z}(\pi)$-<u>module</u>.

For this we shall use the assumption that $\mathbb{Z}(\pi)$ is a noetherian ring. By an example of J. Stallings the above lemma is definitely false without this hypothesis. However, we really only need that if (M,N) is a $(k-1)$-neighbourhood, then $H_k(\widetilde{M},\widetilde{N})$ is finitely generated. In the general case ($\mathbb{Z}(\pi)$ not necessarily noetherian) one proves that (M,N) is dominated by a finite complex pair. It is then an exercise to deduce from this the finite generation of $H_k(\widetilde{M},\widetilde{N})$.

<u>Proof</u>. Let $N = bM$ and $M_o = \overline{W - M}$. By lemma 3.3, M_o is a 1-neighbourhood of "$-\infty$" If $\widetilde{W}$ is the universal covering of W with $p : \widetilde{W} \longrightarrow W$ the projection then $\widetilde{M} = p^{-1}(M)$, $\widetilde{M}_o = p^{-1}(M_o)$ and $\widetilde{N} = p^{-1}(N) = p^{-1}(M \cap M_o) = \widetilde{M} \cap \widetilde{M}_o$ are respectively the universal covering of M, M_o and N. This is so because $\pi_1(N) \to \pi_1(W)$, $\pi_1(M) \to \pi_1(W)$ and $\pi_1(M_o) \to \pi_1(W)$ induced by the respective inclusions are isomorphisms. From the Mayer-Vietoris sequence

$$H_j(\widetilde{N}) \longrightarrow H_j(\widetilde{M}_o) \oplus H_j(\widetilde{M}) \longrightarrow H_j(\widetilde{W})$$

which is a sequence of $\mathbb{Z}\pi$-modules it will follow that $H_j(\widetilde{M})$ is finitely generated over $\mathbb{Z}(\pi)$ if we show that $H_2(\widetilde{N})$ and $H_2(\widetilde{W})$ are finitely generated over $\mathbb{Z}(\pi)$. Since N is smooth and compact, choosing a triangulation of N of N we see that the chain groups of $\widetilde{N}$ with the lifted triangulation are finitely generated over $\mathbb{Z}\pi$. From the fact that $\mathbb{Z}\pi$ is noetherian again it follows that all the homology groups of $\widetilde{N}$ are finitely generated $\mathbb{Z}\pi$-modules. Also W is of the homotopy type of the finite polyhedron X and the same argument as above yields that all the homology groups of W are finitely generated $\mathbb{Z}\pi$-modules.

<u>Lemma 3.5</u>. <u>There exist arbitrary small</u> 2-<u>neighbourhoods of</u> "∞".

<u>Proof</u>. Let M' with $bM' = N'$ be a 1-neighbourhood of ∞ with $M' \subset W-K$. By Lemma 3.4, $H_2(M')$ is finitely generated over $\mathbb{Z}(\pi)$. Let $\alpha_1, \ldots, \alpha_r$ be a system of generators over $\mathbb{Z}(\pi)$ for $H_2(\widetilde{M}') \simeq \pi_2(\widetilde{M}') \simeq \pi_2(M')$. Choosing a small contractible open set in $\operatorname{Int} M'$ as the base point represent the elements α_i by C^∞ imbeddings $\varphi_i : S^2 \to \operatorname{Int} M'$, with disjoint images and the base point of S^2 going into the chosen contractible open set. For this to be possible we need that $\dim M' \geq 5$ but by assumption $\dim M' = n+1 \geq 6$. Let M be formed from M' as explained below: Choose closed tubular neighbourhoods T_i of $\varphi_i(S^2)$ in $\operatorname{Int} M'$ with $T_i \cap T_j = \emptyset$ whenever $i \neq j$. Choose C^∞ paths τ_i from N' to bT_i (the boundary of T_i) meeting N' and bT_i transversally and at the end points only. These paths can be chosen to be mutually disjoint, and tubular neighbourhoods Γ_i of τ_i can be chosen to be mutually disjoint. Let $M = M' - \bigcup_{i=1}^{r} \operatorname{Int} T_i \cup \operatorname{Int} \Gamma_i$. Then clearly M is

a 0-neighbourhood of ∞. We claim that M is a 2-neighbourhood of ∞. First of all, if $N = bM$ it is clear that $N = N' \# bT_1 \# \ldots \# bT_r$ (connected sum). Also bT_1 is an $(n-2)$-sphere bundle over S^2 with $n \geq 5$ and hence $\pi_1(bT_i) = 1$. By Van Kampen we see that $\pi_1(N) \simeq \pi_1(N')$, under an isomorphism making the following diagram commutative:

$$\begin{tikzcd}
\pi_1(N) \arrow[rr, "\simeq"] \arrow[d, "j_*"] & & \pi_1(N') \arrow[d, "j'_*"] \\
\pi_1(M) \arrow[rr, "\mu_*"] \arrow[dr, "i_*"'] & & \pi_1(M') \arrow[dl, "i'_*"] \\
& \pi_1(W) &
\end{tikzcd}$$

Diagram 6

Here the homomorphisms indicated by i_*, j_*, i'_*, j'_* and μ_* are all induced by inclusions and the isomorphism $\pi_1(N) \longrightarrow \pi_1(N')$ is got from Van Kampen's theorem. It follows that $i_* \circ j_*$ is an isomorphism since i'_* and j'_* are. Lemma 2.8 now implies that M is a 1-neighbourhood of ∞.

<u>Assertion</u>: $\pi_2(N) \xrightarrow{j_*} \pi_2(M)$ is an epimorphism.

To prove this it suffices to show that $\pi_2(N) \xrightarrow{\mu_* \circ j_*} \pi_2(M')$ is an epimorphism and that $\mu_* : \pi_2(M) \longrightarrow \pi_2(M')$ is an isomorphism.

Let $\nu_i \in \pi_1(SO(n-1))$ be the element corresponding to the normal bundle of $\varphi_i(S^2)$ in Int M'. As $s_* : \pi_1(SO(n-2)) \longrightarrow \pi_1(SO(n-1))$ is an isomorphism for $n \geq 5$ we see that $\hat{\nu}_i$ can be written as $\nu_i + \theta^1$ where θ^1 is a trivial line bundle. Hence there exists a non zero cross-section for the associated sphere bundle. Using this cross-section we see that $\exists$ an element in $\pi_2(bT_i)$ which represents the element $\alpha_i \in \pi_2(M')$ under the inclusion $bT_i \longrightarrow M'$. It now follows that $\pi_2(N) \xrightarrow{\mu_* \circ j_*} \pi_2(M')$ is an epimorphism.

This in particular gives: $\pi_2(M) \xrightarrow{\mu_*} \pi_2(M')$ is an epimorphism. To complete the proof of the assertion we have only to show that μ_* is a monomorphism. Let $x \in \pi_2(M)$ be such that $\mu_*(x) = 0$ and let $\Theta : S^2 \longrightarrow M$ be a C^∞ imbedding representing x. The fact that $\mu_*(x) = 0$ implies that $\exists$ a C^∞ map $\psi : D^3 \longrightarrow M'$ extending Θ. We can get ψ so as to be transverse regular on $\cup \varphi_i(S^2)$ (since $\Theta(S^2) \cap \varphi_i(S^2) = \emptyset$). The condition $n+1 \geq 6$ ($n+1 = \dim M'$) implies that $\psi(D^3)$ is then disjoint from $\cup \varphi_i(S^2)$. By a further deformation we can make $\psi(D^3)$ go into M.

Now, $\pi_2(N) \xrightarrow{j_*} \pi_2(M)$ being an epimorphism we have $\pi_2(\tilde{N}) \xrightarrow{j_*} \pi_2(\tilde{M})$ also an epimorphism and hence $\pi_2(\tilde{M},\tilde{N}) = 0$. The simply connectedness of $\tilde{M}$ and $\tilde{N}$ now yields by the Relative Hurewicz Theorem $H_2(\tilde{M},\tilde{N}) = \pi_2(\tilde{M},\tilde{N}) = 0$. This completes the proof that M is a 2-neighbourhood.

We now proceed to the proof of Proposition 3.2 for an arbitrary k satisfying $3 \leq k \leq n-2$. Assume by induction that arbitrary small $(k-1)$ neighbourhoods of ∞ exist.

Lemma 3.6. Suppose M is any $(k-1)$-neighbourhood of ∞. Let $N = bM$. Then

(1) $H_k(\widetilde{M},\widetilde{N})$ is a finitely generated $\mathbb{Z}(\pi)$-module.

(2) $\exists$ another $(k-1)$-neighbourhood M_1 of ∞ with $M_1 \subset M$ satisfying the following additional condition:

The homomorphism $H_k(\widetilde{U},\widetilde{N}) \longrightarrow H_k(\widetilde{M}.\widetilde{N})$ induced by the inclusion $(\widetilde{U},\widetilde{N}) \hookrightarrow (\widetilde{M},\widetilde{N})$ is an epimorphism, where $U = \overline{M-M_1}$ and $\widetilde{U}$ is the inverse image of U by the covering map $p : \widetilde{M} \longrightarrow M$.

Proof of (1). By Lemma 3.4 we have $H_j(\widetilde{M})$ finitely generated over $\mathbb{Z}(\pi)$ for every j. Also since N is compact $H_j(\widetilde{N})$ is finitely generated over $\mathbb{Z}(\pi)$. The exactness of $H_k(\widetilde{M}) \longrightarrow H_k(\widetilde{M},\widetilde{N}) \longrightarrow H_{k-1}(\widetilde{N})$ together with Noetherian nature of $\mathbb{Z}(\pi)$ now yield the finite generation of $H_k(\widetilde{M},\widetilde{N})$ over $\mathbb{Z}(\pi)$.

Proof of (2). Let $C_1,\ldots,C_\lambda$ be a finite set of generators for $H_k(\widetilde{M},\widetilde{N})$. There exists a compact set $\widetilde{K}_1$ in $\widetilde{M}$ such that $\exists$ integral singular cycles representing $C_1,\ldots,C_\lambda$ with their supports contained in $\widetilde{K}_1$. Let $K_1 = p(\widetilde{K}_1)$. By the inductive assumption regarding existence of arbitrary small $(k-1)$-neighbourhoods of ∞ we can find a $(k-1)$-neighbourhood M_1 of ∞ with $M_1 \subset W-K_1$ and $M_1 \subset M$. Then clearly $U = \overline{M-M_1}$ satisfies the condition $U \supset K_1$ and thus the chosen cycles representing $C_1,\ldots,C_\lambda$ are cycles of $(\widetilde{U},\widetilde{N})$. Hence

$H_k(\tilde{U},\tilde{N}) \longrightarrow H_k(\tilde{M},\tilde{N})$ is onto.

Remark A : *For the pair* $(\tilde{U},\tilde{N})$ *we have* $H_i(\tilde{U},\tilde{N}) = 0$ *for* $i < k-1$.

Proof. Let $N_1 = bM_1$. We have $H_i(\tilde{M},\tilde{U}) \underset{\simeq}{\longleftrightarrow} H_i(\tilde{M}_1,\tilde{N}_1)$ by excision. Now from the homology exact sequence of the triple (M,U,N) written below:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}(\tilde{M},\tilde{U}) \arrow[r] & H_i(\tilde{U},\tilde{N}) \arrow[r] & H_i(\tilde{M},\tilde{N}) \arrow[r] & H_i(\tilde{M},\tilde{U}) \arrow[r] & \cdots \\
& H_{i+1}(M_1,N_1) \arrow[u, "\text{excision}"', "\wr"] & & & H_i(\tilde{M}_1,\tilde{N}_1) \arrow[u, "\text{excision}"', "\wr"] &
\end{tikzcd}$$

and the fact that M_1 is a $(k-1)$-neighbourhood of ∞ we see that $H_i(\tilde{U},\tilde{N}) \to H_i(\tilde{M},\tilde{N})$ for $i < k-1$. Since M itself is a $(k-1)$-neighbourhood we have $H_i(\tilde{U},\tilde{N}) = 0$ for $i < k-1$.

Remark B : *The homomorphisms* $\pi_1(N) \longrightarrow \pi_1(U)$ *and* $\pi_1(N_1) \longrightarrow \pi_1(U)$ *induced by the inclusions are isomorphisms.*

The proof of this is similar to the proof of Lemma 2.8 and hence is omitted.

For completing the proof the Proposition 3.2 we need the following two propositions which we state without proof.

Proposition 3.7. *Suppose* U *is a compact orientable* C^∞ *manifold of dimension* $n+1$ *with* $n \geq 5$ *and suppose* $bU = N \cup N_1$ *a disjoint union of two open and closed, connected submanifolds of* bU. *If the homomorphisms* $\pi_1(N) \longrightarrow \pi_1(U)$ *and* $\pi_1(N_1) \longrightarrow \pi_1(U)$ *induced by the inclusions are isomorphisms and if* $H_i(\tilde{U},\tilde{N}) = 0$ *for* $i \leq k-2 < n-2$

then (U,N) *has a handle decomposition with handles of type* $k-1, k, \ldots, n-1$.

In other words U *has a presentation of the form*

$$U = I \times N + \varphi_1^{k-1} + \ldots + \varphi_{\alpha_{k-1}}^{k-1} + \psi_1^k + \psi_{\alpha_k}^k + \ldots + \chi_1^{n-1} + \ldots + \chi_{\alpha_{n-1}}^{n-1} .$$

The proof is essentially given in [5], Lemma 1.

Proposition 3.8. Let X *and* Y *be closed* C^∞ *submanifolds of a* C^∞ *manifold* N, *where* $\dim X + \dim Y = \dim N > 4$, *and* $2 < \dim Y \leq \dim N-2$. *Suppose that* $\pi_1(N-Y) \to \pi_1 N$ *induced by the inclusion is an isomorphism. (This is a restriction only if* $\dim Y = \dim N-2$). *Suppose that* X *and* Y *can be lifted to closed submanifolds* $\tilde{X}$ *and* $\tilde{Y}$ *of* $\tilde{N}$, *the universal covering of* N, *and that*

$$\tilde{X}_i \cdot \tau \tilde{Y}_j = 0$$

(where $\cdot$ *denotes the homology intersection number) for all* $\tau \in \pi$ *and all connected components* $\tilde{X}_i, \tilde{Y}_j$ *of* $\tilde{X}$ *and* $\tilde{Y}$. *Then* X *is isotopic in* N *to a submanifold* X_1 *such that* $X_1 \cap Y = \emptyset$, *or equivalently* Y *is isotopic in* N *to a submanifold* Y_1 *such that* $X \cap Y_1 = \emptyset$.

This proposition is essentially due to Whitney.

As remarked already proposition 3.2 is proved by induction on k for k in the range $3 \leq k \leq n-2$. Assume arbitrary small $(k-1)$-neighbourhoods of ∞ exist. Let K be any compact subset of W and let M be any $(k-1)$-neighbourhood of ∞ with $M \subset W-K$.

By Lemma 3.6 $\exists$ a $(k-1)$-neighbourhood of ∞ say M_1 with $M_1 \subset M$ such that the homomorphism $H_k(\tilde{U},\tilde{N}) \longrightarrow H_k(\tilde{M},\tilde{N})$ induced by inclusion is onto, where $U = \overline{M-M_1}$ and $bM = N$, $bM_1 = N_1$. From Remark (A) following Lemma 3.6 we have $H_i(\tilde{U},\tilde{N}) = 0$ for $i < k-1$ and by Remark (B) the homomorphisms $\pi_1(N) \longrightarrow \pi_1(U)$, $\pi_1(N_1) \longrightarrow \pi_1(U)$ induced by the respective inclusions are isomorphisms. Hence by Proposition 3.7 we have a handle decomposition for (U,N) with handles of type $k-1, k, \ldots, n-1$. Let U_o be the union of $I \times N$ together with handles of type $k-1$ (Refer to diagram 7) and N_o the right hand boundary of U_o. Let $U_1 = \overline{U-U_o}$.

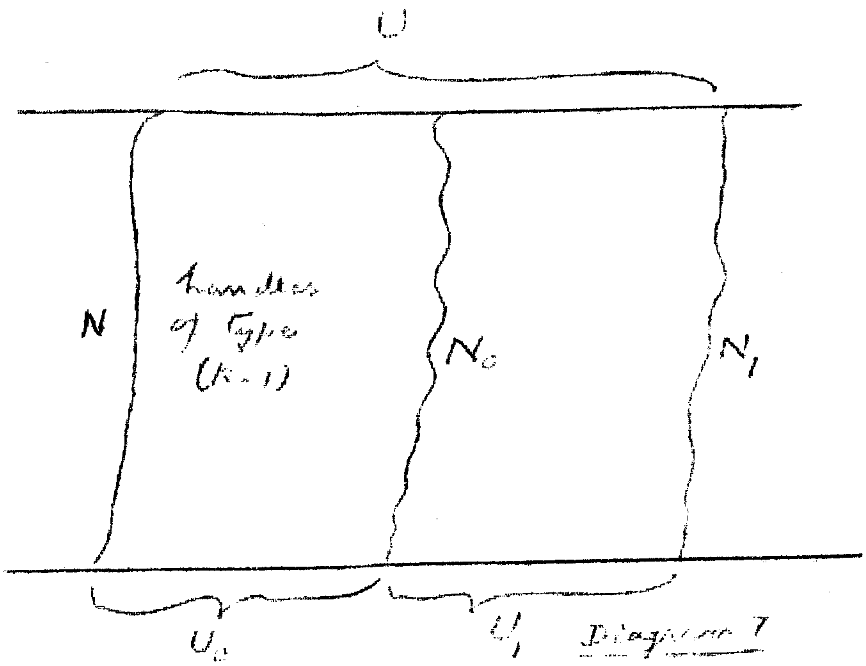

Diagram 7

<u>Convention</u>: In future when we are in a situation of the form $A \subset B$ or $(A,A') \subset (B,B')$ with A, A', B, B' topological spaces by the homomorphism $\pi_1(A) \longrightarrow \pi_1(B)$ or $H_j(A) \longrightarrow H_j(B)$ or $H_j(A,A') \longrightarrow H_j(B,B')$ we mean the one induced by the inclusion.

When $k > 3$ we see from Van Kampen theorem that $\pi_1(N) \longrightarrow \pi_1(U_o)$ is an isomorphism. When $k = 3$ we first observe that the 2-handles φ_i^2 are attached by means of trivial maps to $1 \times N$. In fact $\varphi_1^2(S^1 \times 0)$ bounds a disk in W and as M is a 1-neighbourhood we have $\pi_1(N) \longrightarrow \pi_1(W)$ an isomorphism. Now an application of Van Kampen immediately yields $\pi_1(N) \rightarrow \pi_1(U_o)$ is an isomorphism. Using the 'dual' handle decomposition for U_o and the fact that $k \leq n-2$ we see that $\pi_1(N_o) \longrightarrow \pi_1(U_o)$ is an isomorphism, again by applying Van Kampen. To get U_1 we attach handles of type $k,\ldots,n-1$ to U_o. It follows that whenever $k \geq 3$ the homomorphism $\pi_1(N_o) \longrightarrow \pi_1(U_1)$ is actually an isomorphism. Now choose any α in $H_k(\tilde{M},\tilde{N})$. By our choice of M_1 we have $H_k(\tilde{U},\tilde{N}) \longrightarrow H_k(\tilde{M},\tilde{N})$ epimorphism. Choose any $\beta \in H_k(\tilde{U},\tilde{N})$ getting mapped onto α. By excision $H_k(\tilde{U},\tilde{U}_o) \simeq H_k(\tilde{U}_1,\tilde{N}_o)$ the isomorphism being a $\mathbb{Z}(\pi)$-isomorphism since the maps induced by the various inclusions, namely $N \longrightarrow U_o$; $N_o \longrightarrow U_o$ and $N_o \longrightarrow U_1$ are isomorphisms on π_1. Let γ be the image of β under the composition of the maps

$$H_k(\tilde{U},\tilde{N}) \xrightarrow{(\text{incln})_*} H_k(\tilde{U},\tilde{U}_o) \xleftarrow[\text{excision}]{\simeq} H_k(\tilde{U}_1,\tilde{N}_o).$$

Since (U_1,N_o) has a handle decomposition with handles of type $k,\ldots,n-1$ we see that $H_i(\tilde{U}_1,\tilde{N}_o) = 0$ for $i \leq k-1$ and by Relative Hurewicz theorem $\pi_k(\tilde{U}_1,\tilde{N}_o) \simeq H_k(\tilde{U}_1,\tilde{N}_o)$. But $\pi_k(\tilde{U}_1,\tilde{N}_o) \simeq \pi_k(U_1,N_o)$. Thus $\pi_k(U_1,N_o) \simeq H_k(\tilde{U}_1,\tilde{N}_o)$.

<u>Claim</u>: The element γ can be represented by a C^∞ imbedding $\varphi : (D^k, S^{k-1}) \longrightarrow (U_1, N_o)$.

Now, γ is homologous to $\sum a_i D_i^k$ with $a_i \in \mathbb{Z}(\pi)$ and D_i^k the k-cell of the i-th handle of type k. D_i^k is a differentiably imbedded k-cell in U_1 with boundary S_i^{k-1} in N_o. Let $a_i = \sum_{\sigma \in \pi} a_i^{(\sigma)} \sigma$ with $a_i^{(\sigma)} \in \mathbb{Z}$ and $a_i^{(\sigma)} = 0$ for almost all σ. We can assume that all the S_i^{k-1} D^{n-k+1} intersect a contractible open set in N_o which can be chosen as the "base point" for homotopy considerations. Let $\ell_i = \sum_\sigma |a_i^\sigma|$. Let us take ℓ_i distinct points $x_1, \ldots, x_{\ell_i}$ in D_i^{n-k+1}. Form connected sum of $D_i^k \times x_1, \ldots, D_i^k \times x_{\ell_i}$ along paths in N_o representing the σ's for which $a_i^{(\sigma)} \neq 0$. This operation will give a C^∞ imbedding $\Theta_i : (D^k, S^{k-1}) \rightarrow (U_1, N_o)$ representing $a_i D_i^k$. Forming connected sum of the various $\Theta_i(D^k)$ along trivial arcs in N_o gives a C^∞ imbedding $\varphi : (D^k, S^{k-1}) \longrightarrow (U_1, N_o)$ representing γ.

Let S_j^{n-k+1} be the boundaries of the right hand disks D_j^{n-k+2} corresponding to the handles of type $(k-1)$.

<u>Claim</u>: Let $\tilde{\varphi}(S^{k-1})$ and $\tilde{S}_j^{n-k+1}$ be arbitrary lifts of $\varphi(S^{k-1})$ and S_j^{n-k+1} to N_o. Then for any $\tau \in \pi$ the homology intersection $\tilde{\varphi}(S^{k-1}) . \tau \tilde{S}_j^{n-k+1}$ in $\tilde{N}_o$ is zero.

Actually $\tilde{\varphi}(S^{k-1}) \underset{\tilde{N}_o}{.} \tau S_j^{n-k+1}$ is the same $\beta . \tau \{\tilde{S}_j^{n-k+1}\}$, this later intersection being the one associated to the pair $H_k(\widetilde{U},\widetilde{N})$ and $H_{n-k+1}(\widetilde{U})$. But $\{\widetilde{S}_j^{n-k+1}\} = 0$ in $H_{n-k+1}(\widetilde{U})$ since $\widetilde{S}_j^{n-k+1}$ bounds a disk in $\widetilde{U}$.

We now want to apply proposition 3.8 to $\varphi(S^{k-1}) = X$ and $Y = \cup S_j^{n-k+1}$ which are submanifolds of N_o. To be able to apply proposition 3.8 we need to have $n-k+1 \leq n-2$ and $\pi_1(N_o - Y) \to \pi_1(N_o)$ an isomorphism. The condition $n-k+1 \leq n-2$ gives $k \geq 3$. This is precisely the reason why we had to prove the existence of 2-neighbourhoods separately. We have already seen that $\pi_1(N) \to \pi_1(U_o)$ and $\pi_1(N_o) \to \pi_1(U_o)$ are isomorphisms. Since $\pi_1(N) \to \pi_1(W)$ is an isomorphism, it follows that $\pi_1(U_o) \to \pi_1(W)$ is an isomorphism and hence $\pi_1(N_o) \to \pi_1(W)$ an isomorphism. Let $\varphi_j(D^{k-1} \times D^{n-k+2})$ denote the handles of type $k-1$. Then the inclusion $N_o - \cup \varphi_j(B^{k-1} \times S^{n-k+1}) \to N_o - \cup S_j^{n-k+1}$ is a homotopy equivalence, and $N - \cup \varphi_j(S^{k-2} \times B^{n-k+2}) = N_o - \cup \varphi_j(B^{k-1} \times S^{n-k+1})$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(N - \cup \varphi_j(S^{k-2} \times B^{n-k+2})) & \longrightarrow & \pi_1(N) \\
& & \searrow \\
\| & & \pi_1(W) \\
& & \uparrow\approx \\
\pi_1(N - \cup \varphi_j(B^{k-1} \times S^{n-k+1})) & \longrightarrow & \pi_1(N_o - Y) \to \pi_1(N_o)
\end{array}$$

Diagram 8

The map

$$\pi_1(N - U_j\,\varphi_j(S^{k-2} \times B^{n-k+2})) \to \pi_1 N$$

is an isomorphism because it factors through

$$\pi_1(N - U_j\,\varphi_j(S^{k-2} \times B^{n-k+2})) \to \pi_1(N - U_j\,\varphi_j(S^{k-2} \times 0)) \to \pi_1 N,$$

where the first map is induced by a homotopy equivalence, and the second is also an isomorphism since codim $S^{k-2} = n-k+2 \geq 3$.

Thus Proposition 3.8 can be applied and it yields the following conclusion. The imbedding φ can be so chosen that $\varphi(S^{k-1}) \cap Y = \emptyset$. It now follows from Morse theory that $\varphi(S^{k-1})$ is diffeotopic in U_o to an imbedding $\varphi' : S^{k-1} \to N$. Actually one gets a C^∞ imbedding $\Phi : S^{k-1} \times I \to U_o$ extending φ i.e. $\Phi / S^{k-1} \times 0 = \varphi$ and satisfying $\Phi(S^{k-1} \times I) \subset N$. Taking the diffeotopy together with the imbedding $\varphi : (D^k, S^{k-1}) \to (U_1, N_o)$ we get an imbedding $\psi : (D^k, S^{k-1}) \to (U,N)$. (See diagram 9). The homology class in $H_k(\widetilde{U},\widetilde{N})$ represented by ψ clearly gets mapped into the homology class γ represented by φ in $H_k(\widetilde{U}_1,\widetilde{N}_o)$ under the composition

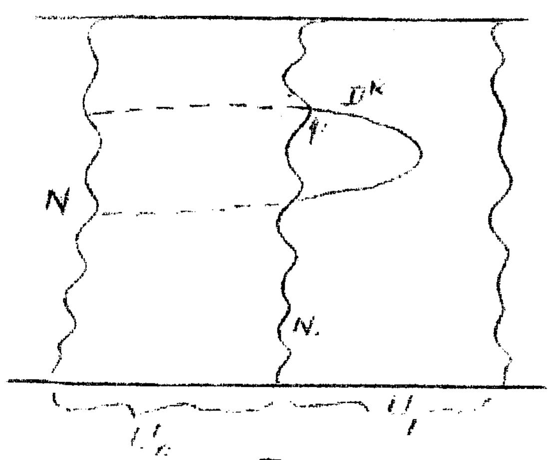

Diagram 9

$$H_k(\widetilde{U},\widetilde{N}) \to H_k(\widetilde{U},\widetilde{U}_o) \xleftarrow[\cong]{\text{excision}} H_k(\widetilde{U}_1,\widetilde{N}_o).$$ From the exact sequence of the triple $\widetilde{U}, \widetilde{U}_o, \widetilde{N}$ we have

$$H_k(\widetilde{U}_o,\widetilde{N}) \to H_k(\widetilde{U},\widetilde{N}) \to H_k(\widetilde{U},\widetilde{U}_o) \quad \text{exact.}$$

But $H_k(\tilde{U}_o,\tilde{N}) = 0$ since the handle decomposition of (U_o,N) we have, consists only of handles of type $(k-1)$. Thus $H_k(\tilde{U},\tilde{N}) \longrightarrow H_k(\tilde{U},\tilde{U}_o)$ is a monomorphism and hence β is the only element of $H_k(\tilde{U},\tilde{N})$ getting mapped into γ. It follows that the class in $H_k(\tilde{U},\tilde{N})$ represented by $\psi : (D^k,S^{k-1}) \longrightarrow (U,N)$ is β.

Let A be the union of a tubular neighbourhood of $\psi(D^k)$ in M together with a tubular neighbourhood of N in M. Define M' to be $\overline{M - A}$. Let $N' = bM'$.

<u>Claim</u>: M' <u>is a</u> $(k-1)$-<u>neighbourhood of</u> ∞ <u>with</u> $H_k(\tilde{M}',\tilde{N}') \simeq H_k(\tilde{M},\tilde{N})/(\alpha)$ <u>as a</u> $Z(\pi)$-<u>module</u>. Here (α) denotes the $Z(\pi)$-submodule of $H_k(\tilde{M},\tilde{N})$ generated by α.

Clearly M' is a 0-neighbourhood of ∞ and from Van Kampen's theorem we see that for k satisfying $3 \leq k \leq n-2$ $\pi_1(N') \xrightarrow{\simeq} \pi_1(N)$ and $\pi_1(M') \simeq \pi_1(M)$ where the latter isomorphism is induced by the inclusion. Also the isomorphism $\pi_1(N') \rightarrow \pi_1(N)$ makes the diagram

$$\begin{array}{ccc}
\pi_1(N') & \xrightarrow{\simeq} & \pi_1(N) \\
\downarrow (\text{incln})_* & & \wr\downarrow (\text{incln})_* \\
\pi_1(M') & \xrightarrow[(\text{incln})_*]{\simeq} & \pi_1(M)
\end{array}$$

commutative and hence $\pi_1(N') \longrightarrow \pi_1(M')$ is an isomorphism. It follows that M' is a 1-neighbourhood of ∞. From the homology sequence of the triple $(\tilde{M}, \tilde{A}, \tilde{N})$ where $\tilde{A} = p^{-1}(A)$ with $p : \tilde{M} \rightarrow M$

the covering map, we have the following diagram with the horizontal row exact.

$$\begin{tikzcd}
H_{i+1}(\widetilde{M},\widetilde{A}) \arrow[r] & H_i(\widetilde{A},\widetilde{N}) \arrow[r] & H_i(\widetilde{M},\widetilde{N}) \arrow[r] & H_i(\widetilde{M},\widetilde{A}) \arrow[r] & {} \\
H_{i+1}(\widetilde{M}',\widetilde{N}') \arrow[u, "\cong\ \text{excision}"'] & & & H_i(\widetilde{M}',\widetilde{N}') \arrow[u, "\cong\ \text{excision}"] &
\end{tikzcd}$$

<u>Diagram 10</u>

Now, $H_i(\widetilde{A},\widetilde{N}) = 0$ for $i \neq k$ and $H_k(\widetilde{A},\widetilde{N}) \cong \mathbb{Z}(\pi)$ and the map $H_i(\widetilde{A},\widetilde{N}) \longrightarrow H_i(\widetilde{M},\widetilde{N})$ carries 1 of $\mathbb{Z}(\pi)$ into α. It follows that $H_i(\widetilde{M}',\widetilde{N}') = 0$ for $i \leq k-1$ and that $H_k(\widetilde{M}',\widetilde{N}') \cong H_k(\widetilde{M},\widetilde{N})/(\alpha)$.

By Lemma 3.6 we have $H_k(\widetilde{M},\widetilde{N})$ finitely generated over $\mathbb{Z}(\pi)$. Choose a finite system of generators $\alpha_1,\ldots,\alpha_r$ and apply the above procedure to $\alpha = \alpha_1$. Then we get a (k-1)-neighbourhood M' such that $H_k(\widetilde{M}',\widetilde{N}')$ is generated by the images of $\alpha_2,\ldots,\alpha_r$ under the isomorphism $H_2(\widetilde{M}',\widetilde{N}') \cong H_2(M,N)/(\alpha_1)$. By interating this procedure a finite number of times we finally arrive at a k-neighbourhood M'' of ∞. Clearly $M'' \subset M \subset W-K$. This completes the proof of Proposition 3.2.

§ 4. <u>THE EXISTENCE OF ARBITRARY SMALL (n-1)-NEIGHBOURHOODS OF "∞"</u>.

So far we have not used the hypothesis $\widetilde{K}_0(\mathbb{Z}(\pi)) = 0$ any where. It is in the construction of arbitrary small (n-1)-neighbourhoods of ∞ that we use this hypothesis.

Lemma 4.1. *Let* M *be any* $(n-2)$-*neighbourhood of* ∞ *and let* $N = bM$. *Then the homology* $H_*(\widetilde{M},\widetilde{N})$ *is the homology of a* $\mathbb{Z}(\pi)$-*chain complex* of the form

$$0 \longrightarrow \widetilde{C}_{n-1} \xrightarrow{d} \widetilde{C}_{n-2} \longrightarrow 0$$

where $\widetilde{C}_{n-1}$ *and* $\widetilde{C}_{n-2}$ *are free but not necessarily finitely generated* $\mathbb{Z}(\pi)$-*modules*.

Proof. Pick a sequence of $(n-2)$-neighbourhoods

$$M = M_0 \supset M_1 \supset \dots \quad M_r \supset M_{r+1} \quad \dots .$$

such that $\bigcup_{r \geq 1} U_r = M$ where $U_r = \overline{M_{r-1} - M_r}$.

We know that $\exists$ Morse functions $\lambda_r : U_r \to [r-1,r]$ with critical points of index $(n-2)$ and $(n-1)$ only, having the components of bU_r for level manifolds $\lambda_r^{-1}(r-1)$ and $\lambda_r^{-1}(r)$ of λ_r. Thus U_1 is homotopically equivalent to a space of the form $N \bigcup_{\{f_i\}} e_i^{n-2}{}_{i \in I_1} \bigcup_{\{g_j\}} e_j^{n-1}{}_{j \in J_1}$ for
from N by
means of attaching a finite number of $(n-2)$ cells and then a finite number of $(n-1)$ cells, under a homotopy equivalence which is the identity on N. Choose a triangulation L of N. By the cellular approximation theorem to each of the characteristic maps f_i corresponds a homotopic cellular map $f_i' : S^{n-3} \to L^{n-3} \subsetneq L$.
Thus $N \bigcup_{\{f_i\}} e_i^{n-2}{}_{i \in I_1}$ is homotopy equivalent to the CW-complex

$F = N \bigcup_{\{f'_i\}} e_i^{n-2}{}_{i \in I_1}$ under an equivalence θ which is identity on N.

Replacing the maps $\Theta \circ g_j$ by cellular maps $g'_j : S^{n-2} \longrightarrow F$ we get a a CW-complex $K_1 = F \bigcup_{\{g'_j\}} e^{n-1}_j$, $j \in J_1$ and a homotopy equivalence $h_1 : U_1 \longrightarrow K_1$ which is identity on N. Also K_1 contains L as a subcomplex. Using the Morse function λ_2 we see that $U_1 \cup U_2$ is of the homotopy type of a space of the form $U_1 \bigcup_{\{f_i\}} e^{n-2}_i \bigcup_{\{g_j\}} e^{n-1}_j$, $i \in I_2$, $j \in J_2$ under an equivalence which is identity on U_1. Taking cellular approximations f'_i to $h_1 \circ f_i$ and attaching n-2 cells by means of f'_i to K_1 we get a CW-complex F_2 and a homotopy equivalence

$$U_1 \bigcup_{\{f_i\}} e^{n-2}_i \xrightarrow{\Theta_2} F_2 = K_1 \bigcup_{\{f'_i\}} e^{n-2}_i \quad i \in I_2$$

extending h_1. Taking cellular approximations g'_j to $\Theta_2 \circ g_j$ and attaching (n-1) cells to F_2 by means of the maps g'_j we get a CW-complex K_2 containing K_1 as a subcomplex and a homotopy equivalence $h_2 : U_1 \cup U_2 \longrightarrow K_2$ extending h_1. Proceeding thus we construct a sequence of Cw-complexes $L \subset K_1 \subset K_2 \subset K_3 \ldots\ldots$ and homotopy equivalences $h_r : \bigcup_{1}^{r} U_j \longrightarrow K_r$ such that h_r is an extension of h_{r-1} and $h_1 = \mathrm{Id}$ on $N = L$. Let $K = \bigcup_{r\,1} K_r$ be provided with the "union topology" i.e. to say a set in K is closed if and only if its intersection with each K_r is closed in K_r. Then $h : M \longrightarrow K$ defined by $h | U_1 \ldots U_r = h_r$ is seen to be a homotopy equivalence, because of J.H.C. Whitehead's theorem.

In fact it is easy to see that h induces isomorphisms of homotopy groups and Whitehead's theorem asserts that a map of CW-complexes inducing isomorphisms of homotopy groups is a homotopy equivalence. Since the cells of K that are not in L are either of dimension $n-2$ or of dimension $n-1$, we have proved Lemma 4.1.

Corollary 4.2. $H_{n-1}(\widetilde{M},\widetilde{N})$ is a finitely generated projective $\mathbb{Z}(\pi)$-module.

The proof for the finite generation of $H_{n-1}(\widetilde{M},\widetilde{N})$ over $\mathbb{Z}(\pi)$ is the same as that of (1) of Lemma 3.6. Since $H_i(\widetilde{M},\widetilde{N}) = 0$ for $i \leq n-2$ we see that $d : \widetilde{C}_{n-1} \longrightarrow \widetilde{C}_{n-2}$ has to be onto. The free nature of C_{n-2} implies $\widetilde{C}_{n-1} \simeq \text{Ker } d \oplus \widetilde{C}_{n-2}$. Now $H_{n-1}(\widetilde{M},\widetilde{N}) \simeq \text{Ker } d$ is a direct summand of the free module $\widetilde{C}_{n-1}$ hence projective.

For any integer $e \geq 0$ let $\sum_e \mathbb{Z}(\pi)$ denote the direct sum of e copies of $\mathbb{Z}(\pi)$. Since $\widetilde{K}_0(\mathbb{Z}(\pi)) = 0$ it follows that $\exists$ an integer $e \geq 0$ such that $H_{n-1}(\widetilde{M},\widetilde{N}) \oplus \sum_e \mathbb{Z}(\pi)$ is a free $(\mathbb{Z})$-module of finite rank. Let the rank of $H_{n-1}(\widetilde{M},\widetilde{N}) + \sum_e \mathbb{Z}(\pi)$ be r.

Lemma 4.3. Given any compact set K of W $\exists$ an $(n-2)$ neighbourhood M of ∞ with $M \subset W-K$ such that $H_{n-1}(\widetilde{M},\widetilde{N})$ is a free $\mathbb{Z}(\pi)$-module of finite rank, where $N = bM$.

Proof. Choose any $(n-2)$-neighbourhood M' of ∞ with $M' \subset W-K$, and let $N' = bM'$.

By corollary 4.2, $H_{n-1}(M',N')$ is a finitely generated projective $\mathbb{Z}(\pi)$-module and hence $\exists$ an integer $e \geq 0$ such that

$H_{n-1}(\widetilde{M}',\widetilde{N}') + \sum_e \mathbb{Z}(\pi)$ is free over $\mathbb{Z}(\pi)$ of finite rank say r. We can find an $(n-2)$-neighbourhood M'' of ∞ with $M'' \subset M'$ and $H_{n-1}(\widetilde{U},\widetilde{N}') \longrightarrow H_{n-1}(\widetilde{M}',\widetilde{N}')$ onto, where $U = \overline{M' - M''}$ (see 2, Lemma 3.6). By Proposition 3.7, (U,N') has a handle decomposition consisting of handles of type $(n-2)$ and $(n-1)$ only. Without even changing M' we can introduce e pairs of mutually cancelling handles of type $(n-2)$ and $(n-1)$. Let M be formed by removing from M' the union of the interiors of tubular neighbourhoods of the e newly introduced handles of type $n-2$ and a tubular neighbourhood of N', and let $N = bM$.

<u>Claim</u>: M <u>is an</u> $(n-2)$-<u>neighbourhood of</u> ∞ <u>such that</u> $H_{n-1}(\widetilde{M},\widetilde{N})$ <u>is a free</u> $\mathbb{Z}(\pi)$-<u>module of rank</u> r.

Let A be the union of the closures of the tubular neighbourhoods removed and let $\widetilde{A} = p^{-1}(A)$. Using Van Kampen and the fact that $n-2 \geq 3$ we see that M is a 1-neighbourhood of ∞. Also $H_i(\widetilde{A},\widetilde{N}') = 0$ for $i \neq n-2$ and $H_{n-2}(\widetilde{A},\widetilde{N}') = \sum_e \mathbb{Z}(\pi)$. From the homology exact sequence of the triple $(\widetilde{M}',\widetilde{A},\widetilde{N}')$,

$$\begin{tikzcd}
H_j(\widetilde{A},\widetilde{N}') \arrow[r] & H_j(\widetilde{M}',\widetilde{N}') \arrow[r] & H_j(\widetilde{M}',\widetilde{A}) \arrow[r] & H_{j-1}(\widetilde{A},\widetilde{N}') \arrow[r] & \cdots \\
 & & H_j(\widetilde{M},\widetilde{N}) \arrow[u, "\cong"', "\text{excision}"] & &
\end{tikzcd}$$

we see that $H_i(\widetilde{M},\widetilde{N}) = 0$ for $i \leq n-2$ and that $H_{n-1}(\widetilde{M},\widetilde{N}) = H_{n-1}(\widetilde{M}',\widetilde{N}') + \sum_e \mathbb{Z}(\pi)$. But by the choice of e, this is a free $\mathbb{Z}(\pi)$-module of rank r. This completes the proof of Lemma 4.3.

<u>Remark 4.4.</u>: If M is any $(n-2)$-neighbourhood of ∞ and if M_1 is another $(n-2)$-neighbourhood of ∞ with $M_1 \subset M$ and $H_{n-1}(\tilde{U},\tilde{N}) \longrightarrow H_{n-1}(\tilde{M},\tilde{N})$ onto, (where $U = \overline{M - M_1}$) then $H_{n-1}(\tilde{U},\tilde{N}) \rightarrow H_{n-1}(\tilde{M},\tilde{N})$ and $H_{n-1}(\tilde{M}_1,\tilde{N}_1) \simeq H_{n-2}(\tilde{U},\tilde{N})$.

<u>Proof</u>. In the homology exact sequence

$$\begin{array}{ccccccccc} H_n(\tilde{M},\tilde{U}) & \xrightarrow{\partial} & H_{n-1}(\tilde{U},\tilde{N}) & \longrightarrow & H_{n-1}(\tilde{M},\tilde{N}) & \longrightarrow & H_{n-1}(\tilde{M},\tilde{U}) & \longrightarrow & H_{n-2}(\tilde{U},\tilde{N}) \longrightarrow 0 \\ \uparrow \wr \text{ excision} & & & & & & \text{excision } \uparrow \wr & & \\ H_n(\tilde{M}_1,\tilde{N}_1) & & & & & & H_{n-1}(\tilde{M}_1,\tilde{N}_1) & & \end{array}$$

of the triple $(\tilde{M},\tilde{U},\tilde{N})$ we have $H_n(\tilde{M}_1,\tilde{N}_1) = 0$ by Lemma 4.1. By assumption $H_{n-1}(\tilde{U},\tilde{N}) \longrightarrow H_{n-1}(\tilde{M},\tilde{N})$ is an epimorphism. It is now immediate that $H_{n-1}(\tilde{U},\tilde{N}) \simeq H_{n-1}(\tilde{M},\tilde{N})$ and that $H_{n-1}(\tilde{M}_1,\tilde{N}_1) \simeq H_{n-2}(\tilde{U},\tilde{N})$.

Let M be an $(n-2)$-neighbourhood of ∞ with $H_{n-1}(\tilde{M},\tilde{N})$ a free $\mathbb{Z}(\pi)$-module of finite rank (say r). We can find a translate M_1 of M by a Deck transformation such that $M_1 \subset M$ and $H_{n-1}(\tilde{U},\tilde{N}) \rightarrow H_{n-1}(\tilde{M},\tilde{N})$ onto, where $U = \overline{M - M_1}$. We have to only choose the translate M_1 so as not to intersect the compact set got as the projection by p of the union of supports of singular cycles (integral) representing a basis for $H_{n-1}(\tilde{M},\tilde{N})$ over $\mathbb{Z}(\pi)$ (See 2 of Lemma 3.6). Corresponding to any handle decomposition of (U,N) with only handles of type $n-2$ and $n-1$ we get a chain complex $0 \longrightarrow \tilde{C}_{n-1} \xrightarrow{d} \tilde{C}_{n-2} \longrightarrow 0$ whose homology will precisely be $H_*(\tilde{U},\tilde{N})$.

For the modules $\widetilde{C}_{n-1}$, $\widetilde{C}_{n-2}$ the cells corresponding to handles of type $(n-1)$ and $(n-2)$ respectively form a basis over $\mathbb{Z}(\pi)$.

Proposition 4.5. There exists a handle decomposition for (U,N) with $2m$ handles of type $(n-2)$ and $2m$ handles of type $(n-1)$ (where m is a certain integer $\geq r$) such that the boundary operator $C_{n-1} \xrightarrow{d} C_{n-2}$ with reference to the basis given by the handles has a matrix of the form $\begin{pmatrix} X & 0 \\ 0 & S^{-1}T \end{pmatrix}$, where S and T are $m \times m$ invertible matrices over $\mathbb{Z}(\pi)$ and X is the $m \times m$ matrix $\begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}$.

Proof. By Remark 4.4 we have $H_{n-1}(\widetilde{U},\widetilde{N}) \simeq H_{n-1}(\widetilde{M},\widetilde{N})$ and $H_{n-2}(\widetilde{U},\widetilde{N}) \simeq H_{n-1}(\widetilde{M}_1,\widetilde{N}_1)$. Since M_1 is a translate of M we have $H_{n-1}(\widetilde{M},\widetilde{N}) \simeq H_{n-1}(\widetilde{M}_1,\widetilde{N}_1)$ and by our choice of M, $H_{n-1}(\widetilde{M},\widetilde{N})$ is a free $\mathbb{Z}(\pi)$-module of rank r. The pair (U,N) has a handle decomposition with only handles of type $n-2$ and $n-1$. Choose one such and let $0 \to \widetilde{B}_{n-1} \xrightarrow{d} \widetilde{B}_{n-2} \to 0$ be the complex corresponding to the chosen handle decomposition, giving the homology of the pair $(\widetilde{U},\widetilde{N})$. Here $\widetilde{B}_{n-1}$ and $\widetilde{B}_{n-2}$ are free $\mathbb{Z}(\pi)$-modules of finite rank. Since the homology of the complex B is the same as $H_*(\widetilde{U},\widetilde{N})$ we get the following exact sequence.

$$0 \to \operatorname{Im} d \to \widetilde{B}_{n-2} \to H_{n-2}(\widetilde{U},\widetilde{N}) \simeq \sum_r \mathbb{Z}(\pi) \to 0.$$

It follows that $\operatorname{Im} d$ is finitely generated and $\mathbb{Z}(\pi)$-projective. Adding a finite number of pairs of mutually cancelling handles if necessary we can assume that $\operatorname{Im} d$ is a free $\mathbb{Z}(\pi)$-module. (Here we use the fact that $\operatorname{Im} d$ is stably free since $\widetilde{B}_{n-2}$ is free of finite rank). Also we have the exact sequence

$0 \to H_{n-1}(\widetilde{U},\widetilde{N}) \simeq \sum_{r} \mathbb{Z}(\pi) \to \widetilde{B}_{n-1} \xrightarrow{d} \mathrm{Im}\,d \to 0$. If the rank of the free $\mathbb{Z}(\pi)$-module $\mathrm{Im}\,d$ is k then it follows that both $\widetilde{B}_{n-1}$ and $\widetilde{B}_{n-2}$ have rank m where $m = k+r$ and that $\exists$ bases $u_1,\dots,u_m$ of $\widetilde{B}_{n-1}$ and $v_1,\dots,v_m$ of $\widetilde{B}_{n-2}$ satisfying

$du_1 = \dots = du_r = 0$; $du_{r+1} = v_{r+1},\dots,du_m = v_m$. Thus the matrix of d with reference to the bases $u_1,\dots,u_m$ and $v_1,\dots,v_m$ of $\widetilde{B}_{n-1}$ and $\widetilde{B}_{n-2}$ respectively is $X = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}$. Let $e_1^{n-1},\dots,e_m^{n-1}$ and $e_1^{n-2},\dots,e_m^{n-2}$ be the natural bases for $\widetilde{B}_{n-1}$ and $\widetilde{B}_{n-2}$ given by the handles and let the matrix of d with reference to this natural pair of bases be A. Now add m pairs of mutually cancelling handles of types $n-2$ and $n-1$. With respect to the handle decomposition of (U,N) thus obtained the chain modules $\widetilde{C}_{n-1}$ and $\widetilde{C}_{n-2}$ are both free $\mathbb{Z}(\pi)$-modules of rank $2m$ and the matrix of d with reference to the natural pair of bases constituted by the handles is $\begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix}$.

If $e_{m+1}^{n-1},\dots,e_{2m}^{n-1}$ and $e_{m+1}^{n-2},\dots,e_{2m}^{n-2}$ are the elements of $\widetilde{C}_{n-1}$ and $\widetilde{C}_{n-2}$ respectively, corresponding to the newly attached m pairs of mutually cancelling handles then $u_1,\dots,u_m$; $e_{m+1}^{n-1},\dots,e_{2m}^{n-1}$ and $v_1,\dots,v_m$; $e_{m+1}^{n-2},\dots,e_{2m}^{n-2}$ form bases for $\widetilde{C}_{n-1}$ and $\widetilde{C}_{n-2}$ with reference to which the matrix of d is $\begin{pmatrix} X & 0 \\ 0 & I_m \end{pmatrix}$. Now, there exist elements S,T $GL(m, \mathbb{Z}(\pi))$ such that $X = S\,A\,T^{-1}$. The matrices $\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix}$ and $\begin{pmatrix} T^{-1} & 0 \\ 0 & T \end{pmatrix}$ are products of elementary matrices in $GL(2m, \mathbb{Z}(\pi))$, and we have

$$\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} T^{-1} & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} X, & 0 \\ 0, & S^{-1}T \end{pmatrix}$$

Thus to prove Proposition 4.5 it suffices to prove the following.

<u>Lemma 4.6.</u> <u>One can change the matrix</u> $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ <u>of</u> d <u>by left or right multiplication by elementary matrices by performing an isotopy of the attaching map of the handles.</u>

<u>Proof.</u> Let $U = I \times N + \varphi_1^{n-2} + \dots + \varphi_{2m}^{n-2} + \psi_1^{n-1} + \dots + \psi_{2m}^{n-1}$ be the handle decomposition which gives the matrix $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ for d. For each i such that $1 \leq i \leq 2m$ let Y_i be the right hand boundary of $I \times N + \varphi_1^{n-2} + \dots + \varphi_{2m}^{n-2} +$ all the handles of type $(n-1)$ except the i th. First we prove the lemma for left multiplication by elementary matrices. We actually show that by an isotopy of ψ_i into Y_i one can change $d\, e_i^{n-1}$ by any $\sum_{j \neq i} x_j\, d\, e_j^{n-1}$ with arbitrary $x_j \in \mathbb{Z}(\pi)$. For this it suffices to prove the same assertion for $x_j\, d\, e_j^{n-1}$ for a particular $j \neq i$ and $x_j \in \pm \pi$. Now $\psi_j(S^{n-2} \times *)$ with $*$ any point on bD^2, is isotopic to the trivial imbedding in Y_i for $i \neq j$, because $\psi_j(S^{n-2} \times *)$ bounds a cell on the boundary of the handle ψ_j. Perform "connected sum" of ψ_i and ψ_j along an arc representing x_j and take it as the new ψ_i'. For proving the lemma for multiplication on the right by

an elementary matrix we look at the dual handle decomposition. Let $U = I \times N_1 + \varphi_1^{*2} + .. + \varphi_{2m}^{*2} + \psi_1^{*3} + ... + \psi_{2m}^{*3}$ be the dual handle decomposition. Let $0 \to \widetilde{C}_3 \xrightarrow{d^*} \widetilde{C}_2 \to 0$ be the chain complex corresponding to this handle decomposition. With respect to the canonical bases of $\widetilde{C}_3$ and $\widetilde{C}_2$ constituted by the handles of type 3 and 2 respectively, the matrix of d^* is the same as $\pm\begin{pmatrix} A^* & 0 \\ 0 & I_m \end{pmatrix}$ where $A^* = (a^*_{ij})$ with $a^*_{ij} = \overline{a_{ji}}$. Here (a_{ij}) is the matrix A and for each $a \in \mathbb{Z}(\ )$, $\bar{a}$ is the element which corresponds to a under the map which carries any $x \in \pi$ into the element $\pm x^{-1}$. (The sign depending on whether x preserves (+) or reverses (-) an orientation of $\widetilde{U}$). Choose liftings of 3 and 2 cells for the dual decomposition $\widetilde{\mathcal{E}}_1^3, .., \widetilde{\mathcal{E}}_{2m}^3 ; \widetilde{\mathcal{E}}_1^2, .., \widetilde{\mathcal{E}}_{2m}^2$ so as to satisfy $\widetilde{e}_i^{n-2} \cdot \widetilde{\mathcal{E}}_j^3 = \delta_{ij}$; $\widetilde{e}_i^{n-1} \cdot \widetilde{\mathcal{E}}_j^2 = \delta_{ij}$ and $\widetilde{e}_i^{n-1} \cdot \sigma\widetilde{\mathcal{E}}_j^2$

$= \delta_{ij} \ \delta_{\sigma,1}$ for every $\sigma \in \pi$. Using the formula

$\widetilde{\mathcal{E}}_k^3 \cdot d\,\widetilde{e}_i^{n-1} = d^* \widetilde{\mathcal{E}}_k^3 \cdot \widetilde{e}_i^{n-1}$ (up to a sign which depends only on n and not on i and k) it is easy to see that the matrix of d^* with reference to the pair of bases constituted by $\widetilde{\mathcal{E}}_1^3, .., \widetilde{\mathcal{E}}_{2m}^3$ and $\widetilde{\mathcal{E}}_1^2, .., \widetilde{\mathcal{E}}_{2m}^2$ is precisely $\begin{pmatrix} A^* & 0 \\ 0 & I \end{pmatrix}$ (up to sign). Now, by what we have proved already, this handle decomposition of (U, N_1) can be altered so as to alter the matrix $\begin{pmatrix} A^* & 0 \\ 0 & I \end{pmatrix}$ by left multiplication by an elementary matrix. Now, taking the dual of the altered handle decomposition we get a handle decomposition for (U,N) which alters the matrix $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ by right multiplication by an elementary matrix. This proves Lemma 4.6.

We choose a handle decomposition for (U,N) of the type mentioned in Proposition 4.5. Then the Kernel of $d : \tilde{C}_{n-1} \longrightarrow \tilde{C}_{n-2}$ is the free $\mathbb{Z}(\pi)$-module of rank r with the elements $\tilde{e}_1^{n-1}, \ldots, \tilde{e}_r^{n-1}$ corresponding to the first r handles of type $(n-1)$.

<u>Assertion</u>. <u>Any one of the elements</u> $\tilde{e}_i^{n-1}$ $(1 \leq i \leq r)$ <u>can be represented by a</u> C^∞ <u>imbedding</u> $\Theta_i : (D^{n-1}, S^{n-2}) \longrightarrow (U,N)$.

In fact $de_i^{n-1} = 0$ implies that any lifting $\tilde{\psi}_i(S^{n-2} \times *)$ of $\psi_i(S^{n-2} \times *)$ has trivial homology intersection in N_0 with any lifting $\tilde{\varphi}_j(* \times S^2)$ of any of the tranverse 2-spheres of the handles of type $n-2$. (Here N_0 is the right hand boundary of $I \times N + \sum_{j=1}^{2m} \varphi_j^{n-2}$. Now use Proposition 3.8 with $X = \sum_{j=1}^{2m} \varphi_j(* \times S^2)$ and $Y = \sum_{i=1}^{2m} \psi_i(S^{n-2} \times *)$. The condition $\pi(N_0 - Y) \rightarrow \pi_1 N_0$ an isomorphism is satisfied because of the following diagram (where as above N_1 is the right boundary of U):

$$\begin{array}{ccc}
\pi_1(N_0 - Y) \xrightarrow{\cong} \pi_1(N_1 - U_{i=1}^{2m} \psi_i^*(D^{n-1} \times S^1)) & \cong & \pi_1 N_1 \\
\downarrow & & \downarrow \cong \\
\pi_1 N_0 & \xrightarrow[\cong]{\hspace{6em}} & \pi_1 W
\end{array}$$

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The "upper" horizontal isomorphisms are obvious. The isomorphism $\pi_1 N_1 \longrightarrow \pi_1 W$ follows from the fact that (M_1, N_1) is a 1-neighbourhood. The "bottom" horizontal map is also an isomorphism because $\pi_1 N_0 \longrightarrow \pi_1 U_1$ is an isomorphism ($U_1 = I \times N_0$ + (handles of type n-1).) $\pi_1 U_1 \longrightarrow \pi_1 U$ is also an isomorphism since $U = U_1$ + (handles of type 3), and $\pi_1 U \longrightarrow \pi_1 W$ has been noted to be an isomorphism before. (Recall Lemma 2.8.) Using Proposition 3.8 as before we see that we can find C^∞ imbeddings $\Theta_i : (D^{n-1}, S^{n-2}) \longrightarrow (U,N)$ representing $\tilde{e}_i^{n-1} \in H_{n-1}(\tilde{U},\tilde{N})$. Let B be the union of tubular neighbourhoods of $\Theta_i(D^{n-1})$ and N in M and let $M' = \overline{M - B}$. By Van Kampen it is easy to see that $\exists$ an isomorphism $\pi_1(N) \longrightarrow \pi_1(N')$ where $N' = bM'$ and that the inclusion $M' \longrightarrow M$ induces an isomorphism $\pi_1(M') \longrightarrow \pi_1(M)$. Also the isomorphism $\pi_1(N) \longrightarrow \pi_1(N')$ makes the diagram.

$$\begin{tikzcd}
\pi_1(N) \arrow[r] \arrow[d, "\cong"] & \pi_1(M) \arrow[dr, "\cong"] \arrow[d, "\cong"'] & \\
 & & \pi_1(W) \\
\pi_1(N') \arrow[r] & {}_1(M') \arrow[ur] &
\end{tikzcd}$$

commutative. It follows that M' is a 1-neighbourhood. Now from the homology exact sequence of the triple $\tilde{M},\tilde{B},\tilde{N}$ it follows that $H_i(\tilde{M}',\tilde{N}') = 0$ for $i \leq n-2$ and $H_{n-1}(\tilde{M}',\tilde{N}') \simeq H_{n-1}(\tilde{M},\tilde{N})/(e_1,\ldots,e_r) = 0$. Thus starting from any $(n-2)$ neighbourhood M of ∞ with $H_{n-1}(\tilde{M},\tilde{N})$

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free of rank r over $\mathbb{Z}(\pi)$ we have constructed a $(n-1)$-neighbourhood M' of ∞ with $M' \subset M$.

Proposition 4.7. *There exist arbitrary small* $(n-1)$-*neighbourhoods of* ∞.

§ 5. COMPLETION OF THE PROOF OF SIEBENMANN'S THEOREM.

Lemma 5.1. *Suppose* M *and* M_1 *are two* $(n-1)$-*neighbourhoods of* ∞ *with* $M \supset M_1$ *and* $bM_1 = \emptyset$. *Then* $U = \overline{M - M_1}$ *is a* h-*cobordism between* bM *and* bM_1.

Proof. Denote bM and bM_1 by N and N_1 respectively. Then as already observed $\pi_1(N) \longrightarrow \pi_1(U)$, $\pi_1(N_1) \longrightarrow \pi_1(U)$ are isomorphisms. (Remark B after Lemma 3.6). Since M and M_1 are $(n-1)$-neighbourhoods we have $H_i(\tilde{M},\tilde{N}) = 0 = H_i(\tilde{M}_1,\tilde{N}_1)$ for all i. In fact by Lemma 4.1, $H_*(\tilde{M},\tilde{N})$ (or $H_*(\tilde{M}_1,\tilde{N}_1)$) is the homology of a complex of the form $0 \longrightarrow \tilde{B}_{n-1} \longrightarrow \tilde{B}_{n-2} \longrightarrow 0$. Thus $H_i(\tilde{M},\tilde{N}) = 0$ for $i > n$ and by definition of an $(n-1)$-neighbourhood of ∞ we have $H_i(M,N) = 0$ for $i \leq n-1$. From the homology exact sequence of the triple $(\tilde{M},\tilde{U},\tilde{N})$

$$\begin{tikzcd}
H_i(\tilde{U},\tilde{N}) \arrow[r] & H_i(\tilde{M},\tilde{N}) \arrow[r] & H_i(\tilde{M},\tilde{U}) \arrow[r] & H_{i-1}(\tilde{U},\tilde{N}) \arrow[r] & H_{i-1}(\tilde{M},\tilde{N}) \arrow[r] & \cdots \\
 & & H_i(\tilde{M}_1,\tilde{N}_1) \arrow[u, "\text{excision}"', "\cong"] & & &
\end{tikzcd}$$

we see immediately that $H_j(\tilde{U},\tilde{N}) = 0$ for every j. Thus to prove Lemma 5.1 it only remains to show that $H_j(\tilde{U},\tilde{N}_1) = 0$ for every j.

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For the pair (U,N) we have a handle decomposition with handles of type $n-2$ and $n-1$ only. If $0 \to \tilde{C}_{n-1} \xrightarrow{d} \tilde{C}_{n-2} \to 0$ is the corresponding complex giving the homology of $(\tilde{U},\tilde{N})$, from the fact that $H_i(\tilde{U},\tilde{N}) = 0 \ \forall i$ it follows that d is an isomorphism. If we use the dual handle decomposition for (U,N_1) the homology $H_*(\tilde{U},\tilde{N}_1)$ will be the homology of a complex of the form $0 \to \tilde{C}_3 \xrightarrow{d^*} \tilde{C}_2 \to 0$. If $A = (a_{ij})$ is the matrix of d with respect to the bases constituted by the handles of type $(n-2)$ and $(n-1)$, then as already seen the matrix of d^* with respect to the bases constituted by the handles of type 3 and 2 in the dual decomposition is $A^* = (a_{ij}^*)$ (up to sign) where $a_{ij}^* = \overline{a_{ji}}$. It follows that if d is an isomorphism so is d^*. Hence $H_*(\tilde{U}, \tilde{N}_1) = 0$.

Proposition 5.2. Let M be any $(n-1)$-neighbourhood of ∞ in W. Then M is diffeomorphic to $N \times [0,\infty)$ where $N = bM$.

The proof of this proposition uses the S-cobordism theorem of Barden - Mazur - Stallings [5], [6] or [8]. Let U be a h-cobordism between two compact, connected oriented C^∞ manifolds V^n and V'^n of dimension $n \geq 5$. Using the isomorphisms $\pi_1(V) \to \pi_1(U)$ and $\pi_1(V') \to \pi_1(U)$ we identify all the three groups $\pi_1(V)$, $\pi_1(U)$ and $\pi_1(V')$ and abstractly denote any one of them by π. Let $\tau(U,V) \in Wh(\pi)$ denote the torsion of the pair (U,V). We now state the S-cobordims theorem which actually consists of two parts.

S-Cobordism Theorem: (1) The inclusion of V in U can be extended into a diffeomorphism of $V \times I$ onto U if and only if $\tau(U,V) = 0$.
(2) Given a compact, connected C^{∞} manifold V^n of dimension $n \geq 5$ and any $\tau \in Wh(\pi)$ where $\pi = \pi_1(V)$, there exists a h-cobordism U between V and a certain V' such that $\tau(U,V) = \tau$.

For more information about torsion and the Whitehead group $Wh(\ )$ refer to [1], [5] or [13]. We list below some known properties of torsion that we need for the proof of Proposition 5.2.

The symbols V, V', V_1, V'_1 etc. are used to denote connected, compact, C^{∞} manifolds. Let U_1 be a h-cobordism between V_1 and V'_1, and U_2 a h-cobordism between V_2 and V'_2. Let $g : V_2 \longrightarrow V'_1$ be a diffeomorphism of V_2 onto V'_1. Let $U = U_1 \underset{g}{\cdot} U_2$ be the differential manifold got from the union of U_1 and U_2 by identifying V_2 with V'_1 by means of the diffeomorphism g. The groups $\pi_1(V_1)$, $\pi_1(U_1)$ and $\pi_1(V'_1)$ are all identified as explained already and let π_1 denote any one of them. Let π_2 have a similar meaning with respect to V_2, U_2 and V'_2 (i.e. $\pi_2 = {}_1(V_2)$ etc.). The diffeomorphism g induces an isomorphism $g_* : \pi_2 \longrightarrow \pi_1$. If $\tau_1 = \tau(U_1, V_1) \in Wh(\pi_1)$ and $\tau_2 = \tau(U_2, V_2) \in Wh(\pi_2)$ then $U = U_1 \underset{g}{\cdot} U_2$ is a h-cobordism between V_1 and V'_2 satisfying $\tau(U, V_1) = \tau_1 + g_*(\tau_2)$. In particular if U_1 is a h-cobordism between V and V' and if U_2 is a h-cobordism between V' and a certain V'' such that $\tau(U',V') = -\tau(U,V)$ then $U_1 \cdot U_2$ is diffeomorphic to $V \times I$ whenever $\dim V (= \dim V') \geq 5$. If U is a h-cobordism between V and V' with torsion $\tau(U,V)$, we can construct a h-cobordism U^{-1} from V' to some V'' with torsion $\tau(U^{-1},V') = -\tau(U,V)$. (Use part (2) of the S-cobordism theorem). Then, pasting U and U^{-1} along V' by the idendity mapping, the h-cobordism $U\ U^{-1}$ from V to V'' has torsion $\tau(U,V) + \tau(U^{-1},V') = 0$. It follows by part (1) of the S-cobordism theorem that $U \cup U^{-1}$ is diffeomorphic to $V \times I$ and in particular that V and V'' are diffeomorphic. The formation of products of h-cobordisms satisfies the following associativity rule. Let $U_i (i = 1,2,3)$ be a h-cobordism between V_i and V_i' and let $g : V_2 \longrightarrow V_1'$; $h : V_3 \longrightarrow V_2'$ be diffeomorphisms. Then $\exists$ a diffeomorphism $\alpha : (U_1 \underset{g}{\cdot} U_2) \underset{h}{\cdot} U_3 \longrightarrow U_1 \underset{g}{\cdot} (U_2 \underset{h}{\cdot} U_3)$ extending the identity map of V_1. Also if U is a h-cobordism between V and V' $\exists$ a diffeomorphism $\beta : U \rightarrow U.V' \times I$ with $\beta | V = \mathrm{Id}_V$ and $\beta(v') = (v',1) \forall v' V'$. (This is a consequence of the fact that V' is differentiably collared in U). For the proof of Proposition 5.2 we need the following Lemma on infinite products of h-cobordisms.

Lemma 5.3. *For every integer* $k \geq 1$ *let* U_k *be a h-cobordism between* V_k *and* V_k' *and let* $V_k' = V_{k+1}$. *If* $\dim V_1 \geq 5$ *then the infinite product* $U_1.U_2.U_3. ..$ *is diffeomorphic to* $V_1 \times [0,\infty)$.

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Proof. As observed already $\exists$ diffeomorphisms $\beta_k : U_k \to U_k . V_k' \; I$ with $\beta_k | V_k = Id_{V_k}$ and $\beta_k(v') = (v',1) \forall v' \in V_k'$. Hence the infinite product $U_1 . U_2 . U_3 . \ldots$ is also diffeomorphic to the infinite product $U_1 . V_1' \times I . U_2 . V_2' \times I . U_3 . V_3' \times I . \ldots$. For every integer $k \geq 1$ the product $U_k^{-1} . U_{k-1}^{-1} \ldots U_1^{-1} . U_1 \ldots U_k$ is a h-cobordism with torsion zero.

Therefore $\exists$ a diffeomorphism. $\theta_k : V_k' \times I \longrightarrow U_k^{-1} . \ldots . U_1^{-1} . U_1 . \ldots . U_k$ satisfying $\theta_k(v',0) = v'$ of the left hand boundary of $U_k^{-1} . \ldots U_1^{-1} . U_1 \ldots . U_k$. The map $v' \rightsquigarrow \theta_k(v',1)$ is a diffeomorphism g_k of V_k' onto the right hand boundary of $U_k^{-1} . \ldots . U_1^{-1} . U_1 . \ldots . U_k$.

Now it is clear that the product $U_1 . V_1' \times I . U_2 . V_2' \times I . U_3 . V_3' \times I . U_4 . \ldots$ is diffeomorphic to the product

$$U_1 . (U_1^{-1} . U_1) \underset{g_1}{.} U_2 . (U_2^{-1} . U_1^{-1} . U_1 . U_2) \underset{g_2}{.} U_3 . (U_3^{-1} . U_2^{-1} . U_1^{-1} . U_1 . U_2 . U_3) \underset{g_3}{.} U_4 \ldots$$

Also it is clear that the diffeomorphism $g_k : V_k' \longrightarrow V_k'$ is homotopic to the identity map of V_k' and hence $g_{k*} : \pi \longrightarrow \pi$ is the identity map. Since product formation of h-cobordisms is an associative operation we have

$$U_1 . (U_1^{-1} . U_1) \underset{g_1}{.} U_2 . (U_2^{-1} . U_1^{-1} . U_1 . U_2) \underset{g_2}{.} \ldots \quad \text{diffeomorphic to}$$

$$(U_1 . U_1^{-1}) . (U_1 \underset{g_1}{.} U_2 . U_2^{-1} . U_1^{-1}) . (U_1 . U_2 \underset{g_2}{.} U_3 . U_3^{-1} . U_2^{-1} . U_1^{-1}) . \ldots .$$

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Denoting the products $U_1 \cdot \ldots \cdot U_k$; $U_1 \cdot \ldots \cdot U_k \underset{g_k}{\cdot} U_{k+1} \cdot U_{k+1}^{-1} \cdot \ldots \cdot U_1^{-1}$ and $U_{k+1} \cdot U_{k+1}^{-1} \cdot U_k^{-1} \cdot \ldots \cdot U_1^{-1}$ by $A_k; B_k$ and C_k respectively we have

$\tau(B_k, V_1) = \tau(A_k, V_1) + (g_k)_* \; (\tau(C_k, V_{k+1})) = \tau(A_k, V_1) + \tau(C_k, V_{k+1})$

since g_{k*} is the identity map. But $\tau(A_k, V_1) + \tau(C_k, V_{k+1}) = 0$. Hence the inclusion of V_1 into B_k as the left hand boundary extends to a diffeomorphism of $V_1 \times I$ onto B. It follows that the product

$(U_1 \cdot U_1^{-1}) \cdot (U_1 \underset{g_1}{\cdot} U_2 \cdot U_2^{-1} \cdot U_1^{-1}) \cdot (U_1 \cdot U_2 \underset{g_2}{\cdot} U_3 \cdot U_3^{-1} \cdot U_2^{-1} \cdot U_1^{-1}) \cdot \ldots\ldots$

is diffeomorphic to $V_1 \times [0, \infty)$. This completes the proof of Lemma 5.3. We now take up the proof of Proposition 5.2. Let M be any (n-1)-neighbourhood of ∞ in W. The Deck transformation group of the covering $W \xrightarrow{p} V$ is the same as that of $\mathbb{R} \xrightarrow{q} S^1$. Let α denote the diffeomorphism of W which corresponds to translation by $+1$ of $\mathbb{R}$ on itself, under the isomorphism between the Deck transformation groups. Choose an integer $\ell > 1$ such that $N \cap \alpha^{\ell} N = \emptyset$ $(N = bM)$. Let $M_k = \alpha^{k\ell} M$ for each integer $k \geq 0$ and $N_k = bM_k$. We have $M_0 = M$, $M_k \supset M_{k+1}$ and $N_k \cap N_{k+1} = \emptyset$. Let $U_k = \overline{M_{k-1} - M_k}$ for any $k \geq 1$. We then have $\bigcup_{k \geq 1} U_k = M$. By Lemma 5.1, U_k is a h-cobordism between N_{k-1} and N_k. By Lemma 5.3 it now follows that M is diffeomorphic to $N \times [0, \infty)$. Actually the inclusion of N into M extends to a diffeomorphism of $N \times [0, \infty)$ onto M.

<u>Theorem 5.4</u>. <u>Let</u> M <u>be any</u> (n-1)-<u>neighbourhood of</u> ∞ <u>in</u> W. <u>Then</u> W <u>is diffeomorphic to</u> $N \times \mathbb{R}$ <u>where</u> $N = bM$.

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Proof. For the integer ℓ having the same meaning as above we see that $\alpha^{-\ell} N \cap N = \emptyset$. It follows that for every integer $k \geq 0$ if we define M_{-k} by $M_{-k} = \alpha^{-k\ell} M$ then $N_{-k} \cap N_{-k-1} = \emptyset \ \forall k \geq 0$, where $N_{-k} = bM_{-k}$. Also $M_{-k} \supset M_{-k-1}$. Now, if $U'_k = \overline{M_{-k+1} - M_{-k}}$ for each $k \geq 1$, by Lemma 5.1, U'_k is a h-cobordism between N_{-k} and N_{-k+1}. It is clear that if $M' = \overline{W - M}$, then M' is the infinite product of the h-cobordisms U'^{-1}_k and by arguments used in the proof of Lemma 5.3 we see that the inclusion map of N into M' can be extended into a diffeomorphism of $N \times (-\infty, 0]$ onto M'. This, combined with Proposition 5.2 gives Theorem 5.4.

This completes the proof of Siebenmann's Theorem.

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Errata

Page	line	for	read
5	3t	$\mathcal{O}'_X$	$\mathcal{O}^1_X$
6	9t	$\mathcal{O}^{2n+2k}_{\mathbb{R}^{n+k}}$	$\mathcal{O}^{n+k}_{\mathbb{R}^{n+k}}$
9	7b	there	then
25	2t	$\forall\ V\in\mathbb{R}^{n+\ell}$	$\forall\ \vec{V}\in\mathbb{R}^{n+\ell}$
28	10t	$(GL_+(n-q),\mathbb{R})$	$_q(GL_+(n-q,\mathbb{R}))$
41	3b	$\pm\ a.y\ \|$	$\pm(a.y)\ \|$
41	1b	$\pm\ a.y\ \|$	$\pm(a.y)\ \|$
43	7t	(1,0) of $\mathbb{Z}\oplus\mathbb{Z}$ into (0,a)	(1,0) of $\mathbb{Z}\oplus\mathbb{Z}$ into $(0,i_{o*}^{-1}(a))$
43	10t	$H_q(V')\simeq H_q(V_o)/(a)$	$H_q(V')\simeq H_q(V_o)/(i_{o*}^{-1}(a))$ $\simeq H_q(V)/(a)$
44	5b	The map $H_q(S^q\times S^{q-1})\to H_q(V_o)$	The composite map $H_q(S^q\times S^{q-1})\to H_q(V_o)\xrightarrow{i_{o*}} H_q(V)$
58	3b	$\sum_{i=0}^{q} b_i(V',\mathbb{Q})+\sum_{i=0}^{q} b_i(V,\mathbb{Q})\ (\text{mod } 3)$	$\sum_{i=0}^{q} b_i(V',\mathbb{Q})+\sum_{i=0}^{q} b_i(V,\mathbb{Q})\ (\text{mod } 2)$
59	9b	$j(w) = y^*w$	$j(w) = wy^*$
59	9b	y^* as a row vector	y^* as a column vector
59	8b	w operates on the right on y^*	w operates on the left on y^*
70	1b	Theorems2.1	Theorem 2.1
75	2b	We use ' '	We use '$\sim$'
85	3t	cubmanifold	submanifold

A