SYMMETRIC FIBERED LINKS

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0. *Introduction*

The main points of this paper are a construction for fibered links, and a description of some interplay between major problems in the topology of 3-manifolds; these latter are, notably, the Smith problem (can a knot be the fixed point set of a periodic homeomorphism of S^3), the problem of which knots are determined by their complement in the 3-sphere, and whether a simply connected manifold is obtainable from S^3 by surgery on a knot.

There are three sections. In the first, symmetry of links is defined, and a method for constructing fibered links is presented. It is shown how this method can sometimes be used to recognize that a symmetric link is fibered; then it reveals all information pertaining to the fibration, such as the genus of the fiber and the monodromy. By way of illustration, an analysis is made of the figure-8 knot and the Boromean rings, which, it turns out, are symmetric and fibered, and related to each other in an interesting way.

In Section II it is explained how to pass back and forth between different ways of presenting 3-manifolds.

Finally, the material developed in the first two sections is used to establish the interconnections referred to earlier. It is proved that completely symmetric fibered links which have repeated symmetries of order 2 (e.g., the figure-8 knot) are characterized by their complement in the 3-sphere.

I would like to thank Louis Kauffman and John W. Milnor for conversations.

I. Symmetric fibered links

§1. *Links with rotational symmetry*

By a *rotation of* S^3 we mean an orientation preserving homeomorphism of S^3 onto itself which has an unknotted simple closed curve A for fixed point set, called the *axis* of the rotation. If the rotation has finite period n, then the orbit space of its action on S^3 is again the 3-sphere, and the projection map $p: S^3 \to S^3$ to the orbit space is the n-fold cyclic branched cover of S^3 along $p(A)$.

An oriented link $L \subset S^3$ has a *symmetry of order* n if there is a rotation of S^3 with period n and axis A, where $A \cap L = \phi$, which leaves L invariant. We will sometimes refer to the rotation as the symmetry, and to its axis as the axis of symmetry of L.

The oriented link $L \subset S^3$ *is said to be completely symmetric relative to an oriented link* L_0, if there exists a sequence of oriented links $L_0, L_1, \cdots, L_n = L$ beginning with L_0 and ending with $L_n = L$, such that for each $i \neq 0$, the link L_i has a symmetry of order $n_i > 1$ with axis of symmetry A_i and projection $p_i: S^3 \to S^3$ to the orbit space of the symmetry, and $L_{i-1} \simeq p_i(L_i)$. If L_0 is the trivial knot, then L is called a *completely symmetric link.* The number n is the *complexity* of the sequence. Abusing this terminology, we will sometimes refer to a completely symmetric link L of complexity n (relative to L_0) to indicate the existence of such a sequence of complexity n.

Figure 1 depicts a completely symmetric link L of complexity 3, having a symmetry of order 3.

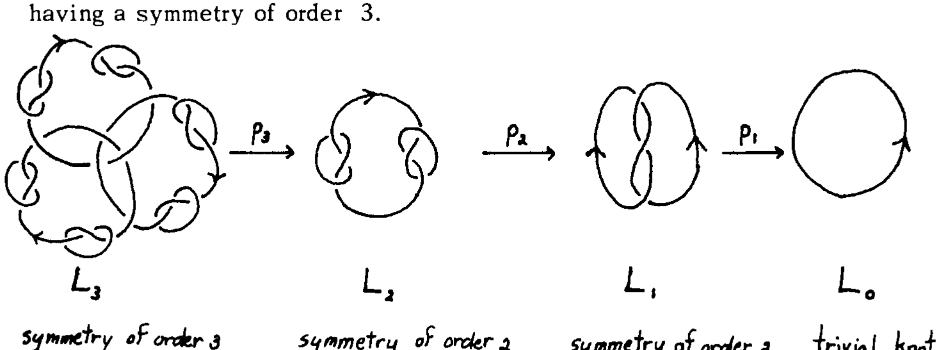

Fig. 1.

§2. *Symmetric fibered links*

An oriented link $L \subset S^3$ is *fibered* if the complement $S^3 - L$ is a surface bundle over the circle whose fiber F over $1 \in S^1$ is the interior of a compact, oriented surface F with $\partial F = L$.

Such a link L is a *generalized axis* for a link $L' \subset S^3 - L$ if L' intersects each fiber of the bundle $S^3 - L$ transversely in n points. In the classical case (which this generalizes) a link $L' \subset \mathbf{R}^3$ is said to have the z-axis for an axis if each component L'_i has a parametrization $L'_i(\theta)$ by which, for each angle θ_0, the point $L'_i(\theta_0)$ lies inside the half-plane $\theta = \theta_0$ given by its equation in polar coordinates for $\mathbf{R}^3$. We will define L to be an *axis* for $L' \subset S^3$ if L is a generalized axis for L' and L is an unknotted simple closed curve.

We wish to investigate sufficient conditions under which symmetric links are fibered.

LEMMA 1 (A construction). *Let* L' *be a fibered link in the* 3*-sphere and suppose* $p: S^3 \to S^3$ *is a branched covering of* S^3 *by* S^3*, whose branch set is a link* $B \subset S^3 - L'$*. If* L' *is a generalized axis for* B*, then* $L = p^{-1}(L')$ *is a fibered link.*

Proof. The complement $S^3 - L'$ fibers over the circle with fibers $\mathring{F}_s$, $s \in S^1$, the interior of compact, oriented surfaces F_s such that $\partial F_s = L'$. Let $\hat{F}_s = p^{-1}(F_s)$ be the inverse image of the surface F_s under the branched covering projection. Then $\partial \hat{F}_s = L$ and $\hat{F}_s - L$, $s \in S^1$, is a locally trivial bundle over S^1 by virtue of the homotopy lifting property of the covering space $p: S^3 - (L \cup p^{-1}(B)) \to S^3 - (L' \cup B)$. Thus $S^3 - L$ fibers over S^1 with fiber, the interior of the surface $\hat{F}_1$.

REMARK. An exact calculation of genus $(\hat{F}_1)$ follows easily from the equation $\chi(\hat{F}_1 - p^{-1}(B)) = n\chi(F_1 - B)$ for the Euler characteristic of the covering space $\hat{F}_1 - p^{-1}(B) \to F_1 - B$. For example, if $p: S^3 \to S^3$ is a regular branched covering, L has only one component and k is the number of points in the intersection $B \cap F_1$ of B with the surface F_1, we can derive the inequality: genus $(\hat{F}_1) \geq n$ genus $(F_1) + \frac{1}{2} + \frac{n(k-2)}{4}$. From this it follows that if $k > 1$, or genus $(F_1) > 0$, then genus $(\hat{F}_1) > 0$ and L is knotted.

Recall that a completely symmetric link $L \subset S^3$ (relative to L_0) is given by a sequence of links $L_0, L_1, \cdots, L_n = L$ such that for each $i \neq 0$, the link L_i has a symmetry of order n_i with axis of symmetry A_i, and such that $p_i : S^3 \to S^3$ is the projection to the orbit space of the symmetry.

THEOREM 1. *Let* $L \subset S^3$ *be a completely symmetric link relative to the fibered link* L_0, *defined by the sequence of links* $L_0, L_1, \cdots, L_n = L$. *If for each* $i \neq 0$, *the projection* $p_i(L_i)$ *of the link* L_i *is a generalized axis for the projection* $p_i(A_i)$ *of its axis of symmetry, then* L *is a non-trivial fibered link.*

Proof. Apply Lemma 1 repeatedly to the branched coverings $p_i : S^3 \to S^3$ branched alorg the trivial knot $p_i(A_i)$ having $p_i(L_i) \simeq L_{i-1}$ for generalized axis.

The completely symmetric links L which are obtained from a sequence $L_0, L_1, \cdots, L_n = L$ satisfying the conditions of the theorem, where L_0 is the trivial knot, are called *completely symmetric fibered links*.

EXAMPLES. In Figure 2 we see a proof that the figure-8 knot L is a completely symmetric fibered knot of complexity 1, with a symmetry of order 2. It is fibered because $p(A)$ is the braid $\sigma_2^{-1}\sigma_1$ closed about the axis $p(L)$. The shaded disk F with $\partial F = L_0$ intersects $p(A)$ in three points; hence the shaded surface $\hat{F} = p^{-1}(F)$, which is the closed fiber of the fibration of $S^3 - L$ over S^1, is the 2-fold cyclic branched cover of the disk F branching along the points $F \cap p(A)$, and has genus 1.

In Figure 3, it is shown that the Boromean rings L is a completely symmetric fibered link of complexity 1, with a symmetry of order 3.

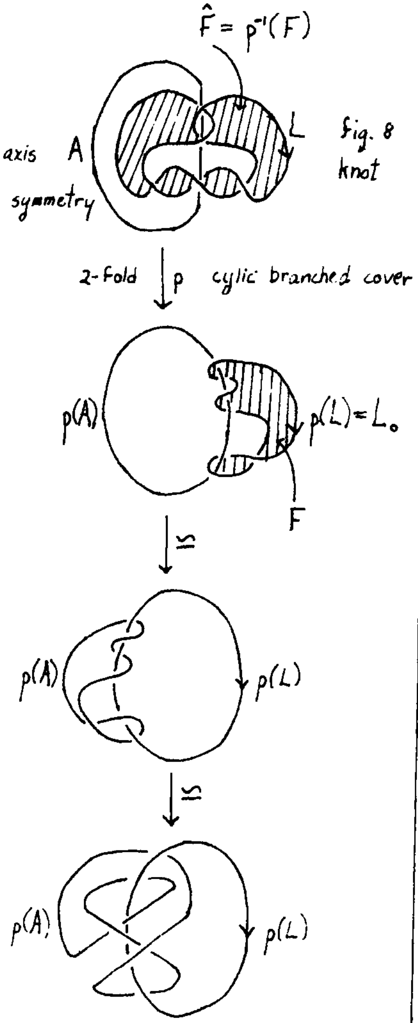

Fig. 2.

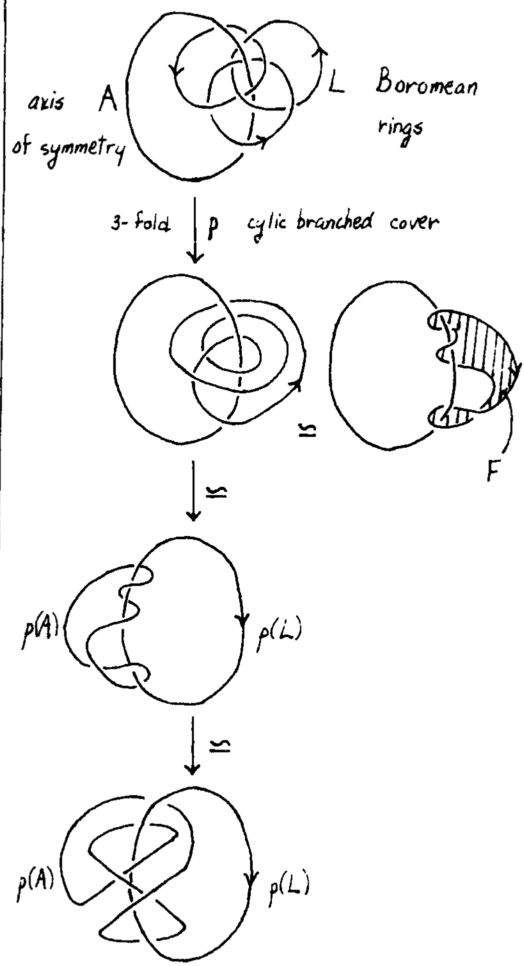

Fig. 3.

This link is fibered because p(A) is the braid $\sigma_2^{-1}\sigma_1$ closed about the axis p(L). The surface $\hat{F} = p^{-1}(F)$ which is the closed fiber of the fibration of $S^3 - L$ over S^1 is not shaded, but is precisely the surface obtained by Seifert's algorithm (see [12]). It is a particular 3-fold cyclic branched cover of the disk F (shaded) branching along the three points $F \cap p(A)$, and has genus 1.

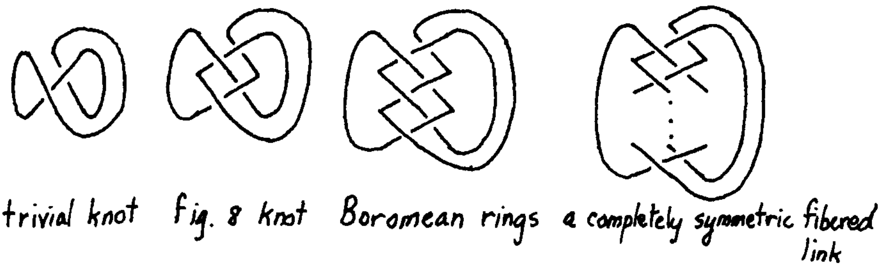

Fig. 4.

Finally, we see from Figure 4 that these two examples are special cases of a class of completely symmetric fibered links of complexity 1 with a symmetry of order n, obtained by closing the braid b^n, where $b = \sigma_2^{-1}\sigma_1$.

II. Presentations of 3-manifolds

There are three well-known constructions for a 3-manifold M: M may be obtained from a Heegaard diagram, or as the result of branched covering or performing "surgery" on another 3-manifold. A specific construction may be called a *presentation*; and just as group presentations determine the group, but not vice-versa, so M has many Heegaard, branched covering and surgery presentations which determine it up to homeomorphism.

Insight is gained by changing from one to another of the three types of presentations for M, and methods for doing this have been evolved by various people; in particular, given a Heegaard diagram for M, it is known how to derive a surgery presentation ([9]) and in some cases, how to present M as a double branched cover of S^3 along a link ([2]). This section deals with the remaining case, that of relating surgery and branched covering constructions.

§1. *The operation of surgery*

Let C be a closed, oriented 1-dimensional submanifold of the oriented 3-manifold M, consisting of the oriented simple closed curves $c_1, \cdots, c_k$. An oriented 3-manifold N is said to be obtained from M by surgery on C if N is the result of removing the interior of disjoint, closed tubular neighborhoods T_i of the c_i's and regluing the closed neighborhoods by orientation preserving self-homeomorphisms $\phi_i : \partial T_i \to \partial T_i$ of their boundary. It is not hard to see that N is determined up to homeomorphism by the homology classes of the image curves $\phi_i(m_i)$ in $H_1(\partial T_i; Z)$, where m_i is a meridian on ∂T_i (i.e., m_i is an oriented simple closed curve on ∂T_i which spans a disk in T_i and links c_i with linking number +1 in T_i). If γ_i is the homology class in $H_1(\partial T_i; Z)$ represented by $\phi_i(m_i)$, then let $M(C; \gamma_1, \cdots, \gamma_k)$ denote the manifold N obtained according to the above surgery procedure.

When it is possible to find a longitude ℓ_i on ∂T_i (i.e., an oriented simple closed curve on ∂T_i which is homologous to c_i in T_i and links c_i with linking number zero in M), then γ_i will usually be expressed as a linear combination $rm_i + s\ell_i$, $r, s \in Z$, of these two generators for $H_1(\partial T_i; Z)$, where the symbols m_i and ℓ_i serve dually to denote both the simple closed curve and its homology class. An easy fact is that for a knot C in the homology 3-sphere M, $M(C; rm + s\ell)$ is again a homology sphere exactly when $r = \pm 1$.

§2. *Surgery on the trivial knot in* S^3

An important feature of the trivial knot $C \subset S^3$ is that any 3-manifold $S^3(C; m + k\ell)$, $k \in Z$, obtained from S^3 by surgery on C is again S^3. To see this, decompose S^3 into two solid tori sharing a common boundary, the tubular neighborhood T_1 of C, and the complementary solid torus T_2. Let $\hat{\phi} : T_2 \to T_2$ be a homeomorphism which carries m to the curve $m + k\ell$; then $\hat{\phi}$ extends to a homeomorphism $\phi : S^3 \to S^3(C; , + k\ell)$.

Now suppose $B \subset S^3$ is some link disjoint from C. The link $B \subset S^3(C; m + k\ell)$ is generally different from the link $B \subset S^3$. Specifically, B is transformed by the surgery to its inverse image $\phi^{-1}(B)$ under the identification $\phi : S^3 \to S^3(C; m + k\ell)$. The alteration may be described in the following way:

Let B be transverse to some cross-sectional disk of T_2 having ℓ for boundary. Cut S^3 and B open along this disk, and label the two copies the negative side and the positive side of the disk, according as the meridian m enters that side or leaves it. Now twist the negative side k full rotations in the direction of $-\ell$, and reglue it to the positive side. The resulting link is $\phi^{-1}(B)$.

For example, if B is the n-stringed braid $b \in B_n$ closed about the axis C, where B_n is the braid group on n-strings, and if c is an appropriate generator of center (B_n), then $B \subset S^3(C; m+k\ell)$ is the closed braid $\overline{b \cdot c^k}$. Figure 5 illustrates this phenomenon. In Figure 6 it is shown how to change a crossing of a link B by doing surgery on an unknotted simple closed curve C in the complement of B.

Fig. 5a.

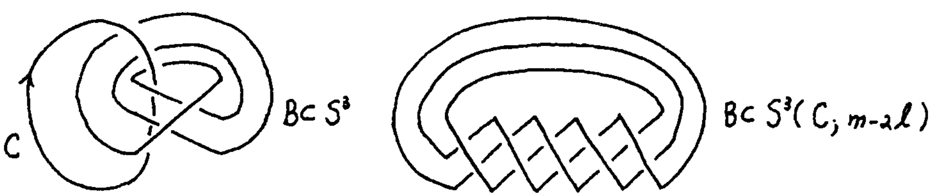

Fig. 5b.

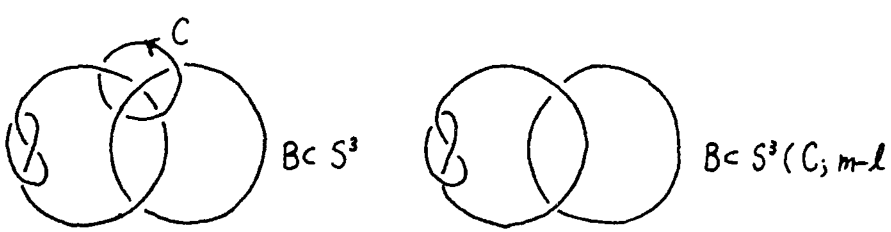

Fig. 6.

§3. *The branched covering operation*

For our purposes, a map $f: N \to M$ between the 3-manifolds N and M is a branched covering map with branch set $B \subset M$, if there are triangulations of N and M for which f is a simplicial map where no simplex is mapped degenerately by f, and if B is a pure 1-dimensional subcomplex of M such that the restriction

$$f \mid N - f^{-1}(B) : N - f^{-1}(B) \to M - B$$

is a covering (see [5]). The foldedness of the branched covering f is defined to be the index of the covering $f \mid N - f^{-1}(B)$.

We will only consider the case where the branch set $B \subset M$ is a 1-dimensional submanifold, and the foldedness of f is a finite number, n. Then f and N are determined by a representation $\pi_1(M-B) \to S(n)$ of the fundamental group of the complement of B in M to the symmetric group on n numbers (see [4]). Given this representation, the manifold N is constructed by forming the covering space $f': N' \to M - B$ corresponding to the subgroup of $\pi_1(M-B)$ represented onto permutations which fix 1, and then completing to $f: N \to M$ by filling in the tubular neighborhood of B and extending f' to f.

A regular branched covering is one for which $f': N' \to M - B$ is a regular covering, or in other words, one for which the subgroup of $\pi_1(M-B)$ in question is normal. Among these are the cyclic branched coverings, given by representations $\pi_1(M-B) \to Z_n$ onto the cyclic group of order n, such that the projection $f: N \to M$ is one-to-one over the branch set. Since Z_n is abelian, these all factor through the first homology group

$$\pi_1(M-B) \to H_1(M-B; Z) \to Z_n .$$

Does there always exist an n-fold cyclic branched covering $N \to M$ with a given branch set $B \subset M$? The simplest case to consider is the one in which M is a homology 3-sphere. Here $H_1(M-B; Z) \simeq Z \oplus Z \oplus \cdots \oplus Z$ is generated by meridians lying on tubes about each of the components of the branch set. Clearly all representations of $H_1(M-B; \mathbf{Z})$ onto $\mathbf{Z}_n$ which come from cyclic branched coverings are obtained by linearly extending arbitrary assignments of these meridians to ± 1. This guarantees the existence of many n-fold cyclic branched coverings of M branched along B, except in the case $n = 2$, or in case B has one component, when there is only one.

Should M not be a homology sphere, an n-fold cyclic covering with branch set B will exist if each component of B belongs to the n-torsion of $H_1(M; \mathbf{Z})$, but this condition is not always necessary.

§4. *Commuting the two operations*

If one has in hand a branched covering space, and a surgery to be performed on the base manifold, one may ask whether the surgery can be lifted to the covering manifold in such a way that the surgered manifold upstairs naturally branched covers the surgered manifold downstairs. The answer to this is very interesting, because it shows one how to change the order in which the two operations are performed, without changing the resulting 3-manifold.

Let $f: N \to M$ be an n-fold branched covering of the oriented 3-manifold M along $B \subset M$ given by a representation $\phi: \pi_1(M-B) \to S(n)$, and let $M(C; \gamma_1, \cdots, \gamma_k)$ be obtained from M by surgery on $C \subset M$, where $C \cap B = \phi$. Note that the manifold $N - f^{-1}(C)$ is a branched covering space of $M - C$ branched along $B \subset M - C$, and is given by the representation

$$\tilde{\phi} = \phi \circ i : \pi_1(M - [C \cup B]) \to S(n) ,$$

where $i: \pi_1(M - [C \cup B]) \to \pi_1(M-B)$ is induced by inclusion. Now let the components of $f^{-1}(T_i)$ be the solid tori $\hat{T}_{ij}$, $j = 1, \cdots, n_i$, $i = 1, \cdots, k$; on the boundary of each tube choose a single oriented, simple closed curve in the inverse image of a representative of γ_i, and denote its homology class in $H_1(\partial \hat{T}_{ij}; \mathbf{Z})$ by $\hat{\gamma}_{ij}$.

THEOREM 2. *Suppose* $\gamma_{i_1},\cdots,\gamma_{i_r}$ *are precisely the classes among* $\gamma_1,\cdots,\gamma_k$ *which have a representative all of whose lifts are closed curves; let* $B' = C - \bigcup_{j=1}^{r} c_{i_j}$. *Then* $f: N \to M$ *induces a branched covering*

$$f': N(f^{-1}(C); \gamma_{ij}\ j=1,\cdots,n_i,\ i=1,\cdots,k) \to M(C;\gamma_1,\cdots,\gamma_k)$$

of the surgered manifolds, branched along $B \cup B' \subset M(C;\gamma_1,\cdots,\gamma_k)$. *The associated representation is* $\phi': \pi_1(M(C;\gamma_1,\cdots,\gamma_k) - [B \cup B']) \to S(n)$, *defined by the commutative diagram*

$$\begin{tikzcd} \pi_1(M-[C\cup B]) \arrow[r, "i"] \arrow[rr, bend right, "\tilde{\phi}"'] & \pi_1(M(C;\gamma_1,\cdots,\gamma_k)-[B\cup B']) \arrow[r, "\phi'"] & S(n) \end{tikzcd}$$

and off of a tubular neighborhood $f^{-1}(\cup T_i)$ *of the surgered set, the maps* f *and* f' *agree.*

Proof. One need only observe that the representation $\tilde{\phi}$ does indeed factor through $\pi_1(M(C;\gamma_1,\cdots,\gamma_k) - [B \cup B'])$ because of the hypothesis that there exist representatives of $\gamma_{i_1},\cdots,\gamma_{i_r}$ all of whose lifts are closed curves.

The meaning of this theorem should be made apparent by what follows.

EXAMPLE. It is known that the dodecahedral space is obtained from S^3 by surgery on the trefoil knot K; in fact, it is the manifold $S^3(K; m-\ell)$. We will use this to conclude that it is also the 3-fold cyclic branched cover of S^3 along the $(2,5)$ torus knot, as well as the 2-fold cyclic branched cover of S^3 along the $(3,5)$ torus knot (see [6]). These presentations are probably familiar to those who like to think of this homology sphere as the intersection of the algebraic variety $\{x \in \mathbb{C}^3 : x_1^2 + x_2^3 + x_3^5 = 0\}$ with the 3-sphere $\{x \in \mathbb{C}^3 : |x| = 1\}$.

According to Figure 7, the trefoil knot K is the inverse image of the circle C under the 3-fold cyclic branched cover of S^3 along the trivial knot B. By Theorem 2, $S^3(K; m-\ell)$ is the 3-fold cyclic branched cover of $S^3(C; m-3\ell)$ branched along $B \subset S^3(C; m-3\ell)$. Since C is the trivial knot, $S^3(C; m-3\ell)$ is the 3-sphere, and $B \subset S^3(C; m-3\ell)$ is the (2,5) torus knot, as in Figure 5a. We deduce that the dodecahedral space is the 3-fold cyclic branched cover of S^3 along the (2,5) torus knot.

A similar argument is applied to Figure 8, in which the trefoil knot is depicted as the inverse image of a circle C under the double branched cover of S^3 along the trivial knot B. By Theorem 2, the space $S^3(K; m-\ell)$ is then the 2-fold cyclic branched cover of $S^3(C; m-2\ell)$ along $B \subset S^3(C; m-2\ell)$, which according to Figure 5b is the (3,5) torus

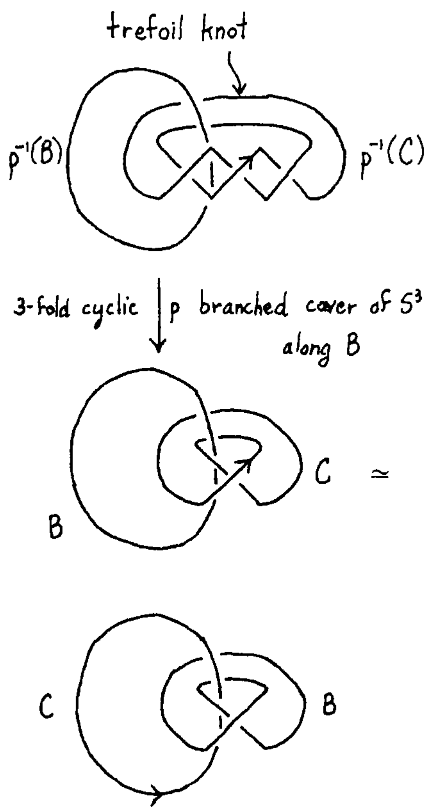

Fig. 7.

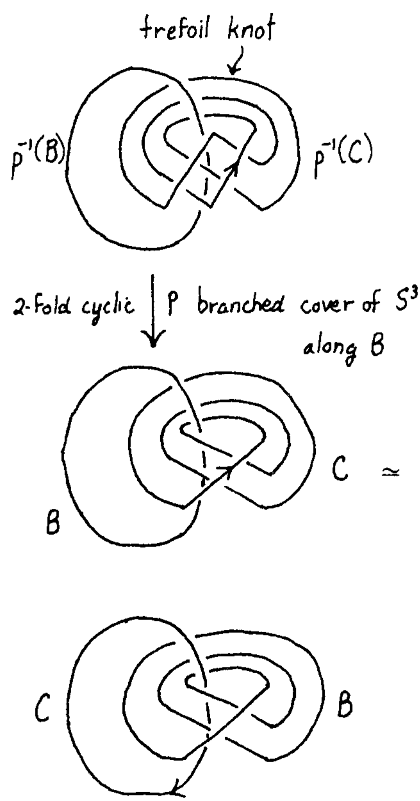

Fig. 8.

knot. Hence the dodecahedral space is the 2-fold cyclic branched cover of S^3 along the $(3,5)$ torus knot.

The following definition seems natural at this point:

DEFINITION. Let L be a link in a 3-manifold M which is left invariant by the action of a group G on M. Then any surgery $M(L;\gamma_1,\cdots,\gamma_k)$ in which the collection $\{\gamma_1,\cdots,\gamma_k\}$ of homology classes is left invariant by G, is said to be equivariant with respect to G.

The manifold obtained by equivariant surgery naturally inherits the action of the group G.

THEOREM 3 (An algorithm). *Every n-fold cyclic branched cover of* S^3 *branched along a knot* K *may be obtained from* S^3 *by equivariant surgery on a link* L *with a symmetry of order* n.

Proof. The algorithm proceeds as follows.

Step 1. Choose a knot projection for K. In the projection encircle the crossings which, if simultaneously reversed, cause K to become the trivial knot K'.

Step 2. Lift these disjoint circles into the complement $S^3 - K$ of the knot, so that each one has linking number zero with K.

Step 3. Reverse the encircled crossings. Then orient each curve c_i so that the result of the surgery $S^3(c_i; m \mp \ell)$ is to reverse that crossing back to its original position (see Figure 6).

Step 4. Let $C = \bigcup_{i=1}^{k} c_i$ be the union of the oriented circles in $S^3 - K'$, and let $p: S^3 \to S^3$ be the n-fold cyclic branched cover of S^3 along the trivial knot K'. Then if $L = p^{-1}(C)$, it follows from Theorem 2 that the n-fold cyclic branched cover of S^3 along K is the manifold $S^3(L; r_1 m_1 \mp \ell_1, \cdots, r_k m_k \mp \ell_k)$ obtained from S^3 by equivariant surgery on the link L, which has a symmetry of order n.

EXAMPLE (Another presentation of the dodecahedral space). In Figure 9, let $p: S^3 \to S^3$ be the 5-fold cyclic branched cover of S^3 along the

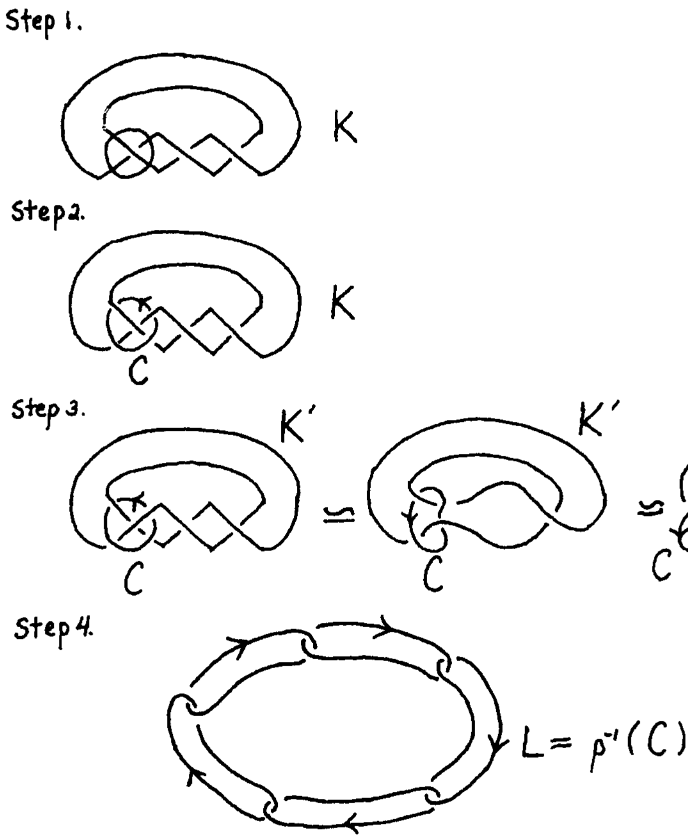

Fig. 9.

trivial knot K′. Then if $L = p^{-1}(C)$ as in step 4 of Figure 9, the 5-fold cyclic branched cover of S^3 along the (2,3) torus knot K is the manifold $S^3(L; m_1 - \ell_1, \cdots, m_5 - \ell_5)$ obtained from S^3 by equivariant surgery on the link L.

III. Applications

We will now derive properties of the special knots constructed in Section I. Recall that a knot K is characterized by its complement if no surgery $S^3(K; m + k\ell)$, $k \in Z$ and $k \neq 0$, is again S^3. A knot K is said to have property P if and only if no surgery $S^3(K; m + k\ell)$, $k \in Z$ and

$k \neq 0$, is a simply connected manifold. A fake 3-sphere is a homotopy 3-sphere which is not homeomorphic to S^3.

THEOREM 4. *Let* K *be a completely symmetric fibered knot defined by the sequence of knots* $K_0, K_1, \cdots, K_n = K$, *such that each* K_i, $i \neq 0$, *is symmetric of order* $n_i = 2$. *Then* K *is characterized by its complement.*

THEOREM 5. *Let* K *be a completely symmetric fibered knot of complexity* 1, *defined by the sequence* $K_0, K_1 = K$, *where* K *is symmetric of order* $n_1 = n$. *If* K *is not characterized by its complement, then there is a transformation of* S^3 *which is periodic of period* n, *having knotted fixed point set. If a fake 3-sphere is obtained from* S^3 *by surgery on* K, *then there is a periodic transformation of this homotopy sphere of period* n, *having knotted fixed point set.*

THEOREM 6. *Let* K *be a completely symmetric fibered knot. Then if* K *does not have property* P, *there exists a non-trivial knot* $K' \subset S^3$ *such that for some* $n > 1$, *the* n*-fold cyclic branched cover of* S^3 *branched along* K' *is simply connected.*

It should be pointed out that the property of a knot being characterized by its complement is considerably weaker than property P. For example, it is immediate from Theorem 4 that the figure-8 knot is characterized by its complement, while the proof that it has property P is known to be difficult (see [7]).

The following lemmas will be used to prove Theorems 4-6.

LEMMA 2. *The special genus of the torus link of type* (n, nk), $k \neq 0$, *is bounded below by*

$$\frac{n^2|k|-4}{4} \qquad \textit{if } n \textit{ even}$$

$$\frac{|k|(n^2-1)}{4} \qquad \textit{if } n \textit{ odd, } k \textit{ even}$$

$$\frac{(n-1)(|k|(n+1)-2)}{4} \qquad \textit{if } n \textit{ odd, } k \textit{ odd} .$$

Proof. The special genus of an oriented link L is defined here to be the infimum of all geni of connected, oriented surfaces F locally flatly embedded in D^4, whose oriented boundary ∂F is the link $L \subset \partial D^4$. This special genus, which will be denoted $g^*(L)$, satisfies an inequality

$$|\sigma(L)| \leq 2g^*(L) + \mu(L) - \eta(L)$$

where $\sigma(L)$ is the signature, $\mu(L)$ is the number of components and $\eta(L)$ is the nullity of the link L (see [8] or [10]). The lemma will be proved by calculating $\sigma(L)$, $\mu(L)$ and $\eta(L)$, where L is the torus link of type (n, nk), $k > 0$ (see [6]); then the result will automatically follow for torus links of type (n, nk), $k < 0$, since these are mirror images of the above.

In what follows, assume $k > 0$.

(i)
$$\sigma(L) = \begin{cases} \dfrac{2-n^2k}{2} & \text{if } n \text{ even} \\[2ex] \dfrac{k(1-n^2)}{2} & \text{if } n \text{ odd} . \end{cases}$$

The signature $\sigma(L)$ is the signature of any 4-manifold which is the double branched cover of D^4 along a spanning surface F of L having the properties described above (see [8]). The intersection of the algebraic variety $\{x \in C^3 : x_1^n + x_2^{nk} + x_3^2 = \delta\}$, for small δ, with the 4-ball $\{x \in C^3 : |x| \leq 1\}$ is such a 4-manifold. Its signature is calculated by Hirzebruch ([3]) to be $\sigma^+ - \sigma^-$, where

$$\sigma^+ = \#\{(i_1, i_2): 0 < i_1 < n,\ 0 < i_2 < nk\} \text{ such that } \quad 0 < \frac{i_1}{n} + \frac{i_2}{nk} + \frac{1}{2} < 1 \quad (\text{mod } 2)$$

$$\sigma^- = \#\{(i_1, i_2): 0 < i_1 < n,\ 0 < i_2 < nk\} \text{ such that } -1 < \frac{i_1}{n} + \frac{i_2}{nk} + \frac{1}{2} < 0 \quad (\text{mod } 2).$$

In other words, if we consider the lattice points $\left\{\left(\frac{i_1}{n}, \frac{i_2}{nk}\right): 0 < i_1 < n, 0 < i_2 < nk\right\}$ in the interior of the unit square of the xy-plane, and divide the unit square into positive and negative regions as in Figure 10, then σ^+ is the total number of points interior to the positive regions, σ^- is the number of points interior to the negative region, and their difference $\sigma^+ - \sigma^-$ is given by the formulae in (i).

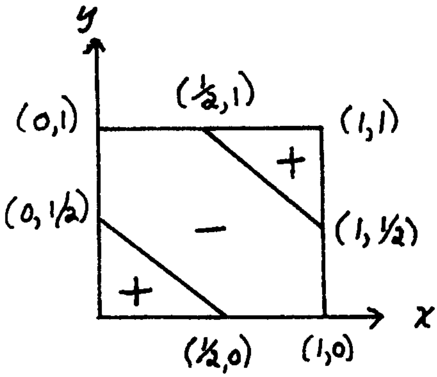

Fig. 10.

(ii) $\quad \eta(L) = n-1 \quad$ if n even

$\qquad\qquad n \quad$ if n odd, k even

$\qquad\qquad 1 \quad$ if n odd, k odd .

The nullity of a link L is defined to be one more than the rank of the first homology group $H_1(M; \mathbb{R})$ of the double branched cover M of S^3 branched along L; it follows that $\eta(L)$ is independent of the orientation of L. The result in (ii) can be easily obtained from any of the known methods for calculating nullities (see [11]).

(iii) $\quad \mu(L) = n$.

Substituting these quantities into the inequality gives the desired lower bounds for $g^*(L)$. Note that except for the $(2, \pm 2)$ torus links, none of the non-trivial torus links of type (n,nk) has special genus 0.

In the next few paragraphs, B_n denotes the braid group on n strings; a single letter will be used to signify both an equivalence class of braids, and a representative of that equivalence class; and the notation $\overline{b}$ will stand for the closure of the braid b (i.e., the link obtained by identifying the endpoints of b).

LEMMA 3. *If* $b \in B_n (n \geq 3)$ *is a braid with* n *strings which closes to the trivial knot, and* $c \in B_n$ *is a generator of the center of the braid group* B_n, *then the braid* $b \cdot c^k$, $k \in \mathbf{Z}$ *and* $k \neq 0$, *closes to a non-trivial knot.*

Proof. First observe that if b_1 and b_2 are n-stranded braids which have identical permutations and which close to a simple closed curve such that $g^*(\overline{b}_1) = g_1$ and $g^*(\overline{b}_2) = g_2$, then the closed braid $\overline{b_1^{-1} \cdot b_2}$ is a link of n components whose special genus $g^*(\overline{b_1^{-1} \cdot b_2}) \leq g_1 + g_2$. This is illustrated schematically by Figure 11. Imagine that the two abutting cubes are 4-dimensional cubes I_1^4 and I_2^4, that their boundaries are S^3, and that the closed braid $\overline{b}_i (i=1,2)$ is positioned in I_i^4 as shown, with the intersection $\overline{b}_1 \cap \overline{b}_2$ consisting of n arcs. Span each closed braid $\overline{b}_i$ by a connected, oriented, locally flatly embedded surface of genus g_i in the cube I_i^4. The union of the two surfaces is then a surface in $I^4 = I_1^4 \cup I_2^4$, whose boundary in the 3-sphere ∂I^4 is the closed braid $\overline{b_1^{-1} \cdot b_2}$

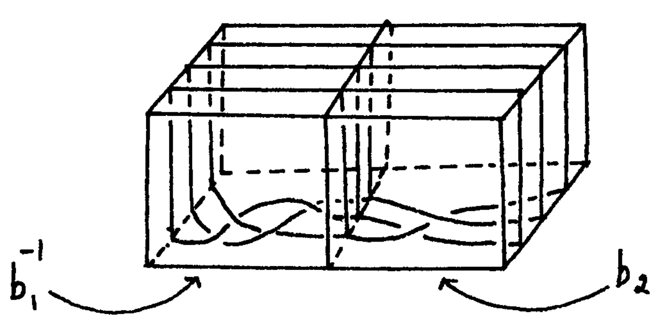

Fig. 11.

The boundary $\overline{b_1^{-1} \cdot b_2}$ has n components because the braid $b_1^{-1} \cdot b_2$ with n strings has the trivial permutation; hence attaching the two surfaces at n places along their boundaries does not increase the genus beyond the sum $g_1 + g_2$. The conclusion that $g^*(\overline{b_1^{-1} \cdot b_2}) \leq g_1 + g_2$ is immediate.

Now suppose the conclusion of the lemma is false; i.e., for some braid $b \in B_n$ and $k \in Z$, $k \neq 0$, both b and $b \cdot c^k$ close to a trivial knot. Applying the result with $b_1 = b$ and $b_2 = b \cdot c^k$, we reach a contradiction of Lemma 2, which is that $g^*(\overline{c^k}) \leq 0 + 0$, where $\overline{c^k}$ is the torus link of type (n, nk) $n \geq 3$. Therefore Lemma 3 must be true.

Now for the proofs of the theorems:

Proof of Theorem 4. Let $K' = p_n(K)$ and $B = p_n(A_n)$. Then K is the inverse image $p_n^{-1}(K')$ of the completely symmetric fibered knot K' under a 2-fold cyclic branched cover $p_n : S^3 \to S^3$ branched along the unknotted simple closed curve B having K' for generalized axis. The knot $K' \simeq K_{n-1}$ also has repeated symmetries of order 2, and its complexity is one less than that of K. Suppose K is not characterized by its complement. Then a 3-sphere $S^3(K; m + k\ell)$, $k \in Z$ and $k \neq 0$, may be obtained from S^3 by surgery on K. According to Theorem 2, this 3-sphere is the 2-fold cyclic branched cover of $S^3(K'; m + 2k\ell)$ branched along $B \subset S^3(K'; m + 2k\ell)$. By Waldhausen ([13]), $S^3(K'; m + 2k\ell)$ must be S^3 and $B \subset S^3(K; m + 2k\ell)$ must be unknotted.

We will proceed by induction on the complexity of K. If K has complexity 1, then B is some braid $b \in B_n$ closed about the axis K'. Since K' is unknotted, $S^3(K'; m + 2k\ell)$ is again S^3, and $B \subset S^3(K'; m + 2k\ell)$ is the closed braid $\overline{b \cdot c^{2k}}$ in S^3, for some generator c of center (B_n) (recall Section II, §2). This simple closed curve is knotted, by Lemma 3, which is a contradiction.

Next suppose that every knot of complexity $n < N$ meeting the requirements of the lemma is characterized by its complement, and let K have complexity N. From the induction hypothesis it follows that K' is characterized by its complement, and that $S^3(K'; m+2k\ell)$ cannot be S^3, which is a contradiction.

Hence K must have been characterized by its complement.

Proof of Theorem 5. Let $B = p_1(A_1)$ and $K' = p_1(K)$. Then $p_1 : S^3 \to S^3$ is an n-fold cyclic branched cover of S^3 along the trivial knot B, such that B is a braid $b \in B_n$ closed about the axis K', and $K = p^{-1}(K')$. If K is not characterized by its complement in S^3, then $S^3(K; m+k\ell)$ is the 3-sphere for some $k \in \mathbf{Z}$, $k \neq 0$. It follows from Theorem 2 that S^3 is the n-fold cyclic branched cover of $S^3(K; m+nk\ell)$ branched along $B \subset S^3(K'; m+nk\ell)$. Now since $K' \simeq K_0$ is unknotted, the manifold $S^3(K'; m+nk\ell)$ is S^3 and the simple closed curve $B \subset S^3(K'; m+nk\ell)$ is the closed braid $\overline{b \cdot c^{nk}}$, for some generator c of the center of the braid group B_n. This closed braid is knotted by Lemma 3!

Similarly, if a fake 3-sphere $S^3(K; m+k\ell)$ may be obtained from S^3 as the result of surgery on K, then this homotopy 3-sphere is the n-fold cyclic branched cover of the 3-sphere along the knot $\overline{b \cdot c^{nk}}$.

Proof of Theorem 6. The knot K is defined by a sequence $K_0, K_1, \cdots, K_j =$ Let $K'_{i-1} = p_i(K_i)$ and $B_{i-1} = p_i(A_i)$. Then there are n_i-fold cyclic branched coverings $p_i : S^3 \to S^3$ branched along the unknotted simple closed curves B_i having K'_i for generalized axis, $0 < i \leq j$, such that $K_i = p_i^{-1}(K'_{i-1})$. If K does not have property P, then a homotopy sphere $S^3(K; m+k\ell)$, $k \in \mathbf{Z}$ and $k \neq 0$, may be obtained from S^3 by surgery on K. This homotopy sphere is the n_j-fold cyclic branched cover of $S^3(K'_{j-1}; m+n_jk\ell)$ branched along $B_{j-1} \subset S^3(K'_{j-1}; m+n_jk\ell)$. It is easy to show that the manifold $S^3(K'_{j-1}; m+n_jk\ell) \simeq S^3(K_{j-1}; m+n_jk$ is simply connected, and so on, down to $S^3(K_1; m+n_j \cdots n_3n_2k\ell)$. Now $S^3(K_1; m+n_j \cdots n_2k\ell)$ is the n_1-fold cyclic branched cover of the manifold $S^3(K'_0; m+n_j \cdots n_2n_1k\ell)$ branched along $B_0 \subset S^3(K'_0; m+n_j \cdots n_2n_1k\ell)$. Let B_0 be the braid $b \in B_n$ closed about the axis K'_0. Then the homotopy sphere $S^3(K_1; m+n_j \cdots n_3 n_2 k\ell)$ is the n_1-fold cyclic branched cover of S^3 branched along the knot $\overline{b \cdot c^{n_j \cdots n_2 n_1 k}}$, where, as usual, c is some generator of center (B_n).

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