CLASSIFICATION OF SIMPLICIAL TRIANGULATIONS
OF TOPOLOGICAL MANIFOLDS

BY DAVID E. GALEWSKI\textsuperscript{1} AND RONALD J. STERN\textsuperscript{2}

Communicated by S. Eilenberg, April 8, 1976

In this note we announce theorems which classify simplicial (not necessarily combinatorial) triangulations of a given topological \( n \)-manifold \( M, n \geq 7 \) (\( \geq 6 \) if \( \partial M = \emptyset \)), in terms of homotopy classes of lifts of the classifying map \( r: M \to BTOP \) for the stable topological tangent bundle of \( M \) to a classifying space \( BTRI_\text{top} \) which we introduce below. The (homotopic) fiber of the natural map \( j: BTRI_\text{top} \to BTOP \) is described in terms of certain groups of \( PL \) homology 3-spheres. We also give necessary and sufficient conditions for a closed topological \( n \)-manifold \( M, n \geq 6 \), to possess a simplicial triangulation.

The proofs of these results incorporate recent geometric results of F. Ancel and J. Cannon \cite{1}, J. Cannon \cite{2}, R. D. Edwards \cite{4}, and D. Galewski and R. Stern \cite{5}.

In \cite{8}, R. Kirby and L. Siebenmann show that in each dimension greater than four there exist closed topological manifolds which admit no piecewise linear manifold structure and hence cannot be triangulated as a combinatorial manifold. Also, R. D. Edwards \cite{3} has recently shown that the double suspension of the Mazer homology 3-sphere is homeomorphic to \( S^5 \), thus showing that a simplicial triangulation of a topological manifold need not be combinatorial. But it is still unknown whether or not every topological manifold can be triangulated as a simplicial complex.

Our classification theorems for simplicial triangulations on a given topological manifold take the following forms:

Let \( BTOP \) denote the classifying space for stable topological block bundles.

**Theorem 1.** There is a space \( BTRI_\text{top} \) and a natural map \( BTRI_\text{top} \to BTOP \) such that if \( M \) is a topological \( n \)-manifold, \( n \geq 7 \) (\( \geq 6 \) if \( \partial M = \emptyset \)) and \( r: M \to BTOP \) classifies the stable topological tangent bundle of \( M \), then there is a one-to-one correspondence between the set of concordance classes of simplicial triangulations of \( M \) and the set of vertical homotopy classes of lifts of \( r \) to \( BTRI_\text{top} \).

\textsuperscript{1}This research was supported in part by National Science Foundation grant GP29585—A4
\textsuperscript{2}This research was supported in part by a Faculty Research Grant at the University of Utah and by National Science Foundation grant MCS76—06393.
The obvious relative versions of Theorem 1 also hold true.

**Theorem 2.** The fiber $\text{TOP}/\text{TRI}_n$ of $B\text{TRI}_n \to B\text{TOP}$ has only two non-zero homotopy groups, namely $\pi_3$ and $\pi_4$, and the following sequence is exact.

$$0 \to \pi_4 \to \ker(\alpha: \theta^H_3 \to Z_2) \to \theta^{\text{TRI}_n}_3 \to \pi_3 \to 0.$$  

Here $\theta^H_3$ denotes the group of PL homology 3-spheres, modulo those which bound acyclic PL 4-manifolds, under the operation of connected sum; $\alpha: \theta^H_3 \to Z_2$ is the Kervaire-Milnor-Rochlin map $\alpha(H^3) = I(H^3)/8 \mod 2$, where $I(H^3)$ is the index of a parallelizable PL 4-manifold that $H^3$ bounds; and $\theta^{\text{TRI}_n}_3$ is the group of PL homology 3-spheres modulo those which bound acyclic homology 4-manifolds $W$ with $W \times R^{n-4}$ a topological manifold, under the operation of connected sum. Note that if $\Sigma^{n-3} H^3$ is homeomorphic to $S^n$, then $H^3$ represents the zero element of $\theta^{\text{TRI}_n}_3$.

**Theorem 3.** (i) $\pi_3(\text{TOP}/\text{TRI}_n) \subseteq Z_2$,

(ii) $\pi_3(\text{TOP}/\text{TRI}_n) = 0$ if and only if there exists a PL homology 3-sphere $H^3$ with $\alpha(H^3) = 1$ and the $(n-3)$-suspension of $H^3$, $\Sigma^{n-3} H^3$, is homeomorphic to $S^n$.

(iii) $\pi_4(\text{TOP}/\text{TRI}_n) = 0$ if and only if every PL homology 3-sphere $H^3$ with $\alpha(H^3) = 0$ and which bounds an acyclic homology 4-manifold $W$ with $W \times R^{n-4}$ a topological manifold, bounds an acyclic PL 4-manifold.

**Theorem 4.** There exists a PL homology 3-sphere $H^3$ such that

(i) $\alpha(H^3) = 1$,

(ii) $H^3 \# H^3$ bounds an acyclic PL 4-manifold, and

(iii) $\Sigma^{n-3} H^3$ is homeomorphic to $S^n$.

If and only if every closed topological $n$-manifold, $n \geq 6$, can be triangulated as a simplicial complex.

**Remark.** For $M = 5$ and $M^n$ oriented, Siebenmann [10] has shown under conditions (i) and (iii) that $M$ is simplicially triangulable. M. Scharlemann has pointed out that if $M^5$ is unoriented, then (i), (iii) and the fact that $H^3 \# H^3$ bounds a contractible PL 4-manifold implies the result. For $6 \leq n \leq 8$, Theorem 4 was proven by M. Scharlemann [9], where in place of (ii) he has the orientability condition that the integral Bockstein of the Kirby-Siebenmann obstruction to putting a PL structure on $M$ is zero. T. Matumoto has claimed a version of Theorem 4 under the stronger hypothesis that (iii) be replaced by the condition that $\Sigma^{n-4} H^3$ is homeomorphic to $S^{n-1}$.

We also investigate the question of whether a given topological $n$-manifold, $n \geq 6$, can be triangulated as a simplicial homotopy manifold. For example;

**Proposition 5.** Suppose that every PL homotopy 3-sphere bounds a contractible PL 4-manifold. Then there is a one-to-one correspondence between the set of concordance classes of simplicial homotopy manifold triangulations of
a topological n-manifold $M$, $n \geq 6$, and concordance classes of PL manifold structures on $M$.

**Proposition 6.** Suppose there exists a bad counterexample to the 3 dimensional Poincaré conjecture; namely suppose there exists a PL homotopy 3-sphere $H^3$, with

(i) $\alpha(H^3) = 1$, and

(ii) $H^3 \# H^3$ bounds a contractible PL 4-manifold.

Then every topological n-manifold, $n \geq 6$, can be triangulated as a simplicial homotopy manifold.

Details of these and related results will appear in [6] and [7].

**References**


