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THE ALGEBRAIC THEORY OF FINITENESS OBSTRUCTION

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Introduction.

The finiteness obstruction $[X] \in \tilde{K}_0(\mathsf{Z}[\pi_1(X)])$ of Wall [13], [14] is an algebraic K-theory invariant of a finitely dominated CW complex X such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW complex. We develop here an algebraic theory of finiteness obstruction for chain complexes in an additive category. The theory helps to clarify the passage $X \to [X]$ from topology to algebra. Such a clarification may be of interest in its own right, but in any case the new theory is necessary for some recent generalizations of the original obstruction theory to more complicated topological finiteness problems.

Let Λ be a ring. An endomorphism $p: F \to F$ of a Λ-module F is a *projection* if it is idempotent

$$p^2 = p: F \to F,$$

so that $1-p: F \to F$ is also a projection and F has a direct sum decomposition

$$F = \operatorname{im}(p) \oplus \operatorname{im}(1-p).$$

A Λ-module P is *projective* if it is isomorphic to $\operatorname{im}(p)$ for a projection $p: F \to F$ of a free Λ-module F, or equivalently if P is a direct summand of a free Λ-module F. A projective Λ-module P is f.g. (= finitely generated) if and only if F can be chosen to be a f.g. free Λ-module. The *projective class group of* Λ, $K_0(\Lambda)$ is the abelian group with one generator $[P]$ for each isomorphism class of f.g. projective Λ-modules P, and relations

$$[P] + [Q] = [P \oplus Q] \in K_0(\Lambda).$$

The *reduced projective class group of* Λ, $\tilde{K}_0(\Lambda)$ is the quotient of $K_0(\Lambda)$ defined by

$$\tilde{K}_0(\Lambda) = K_0(\Lambda)/\{[F] \mid F \text{ f.g. free}\}.$$

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A f.g. projective Λ-module P is such that $[P] = 0 \in \tilde{K}_0(\Lambda)$ if and only if it is stably f.g. free, that is if there exist f.g. free Λ-modules F,G such that $P \oplus F$ is isomorphic to G. More generally, given a bounded positive chain complex of f.g. projective Λ-modules

$$P : \cdots \to 0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0$$

there is defined an invariant in the reduced projective class group

$$[P] = \sum_{i=0}^{\infty} (-)^i [P_i] \in \tilde{K}_0(\Lambda)$$

such that $[P] = 0$ if and only if P is chain equivalent to a bounded positive chain complex of f.g. free Λ-modules

$$F : \cdots \to 0 \to F_N \to F_{N-1} \to \cdots \to F_1 \to F_0 .$$

This invariant was originally defined by Swan [12], in a precursor of the general theory of Wall [13], [14]. A chain complex C is *finite* if it is a bounded positive complex of f.g. free modules; thus $[P] \in \tilde{K}_0(\Lambda)$ is the *finiteness obstruction* of P. A *finite domination* (D,f,g,h) of a Λ-module chain complex C is a finite Λ-module chain complex D together with chain maps

$$f : C \to D,\ \ g : D \to C$$

and a chain homotopy

$$h : gf \simeq 1 : C \to C,$$

so that C is a chain homotopy theoretic direct summand of D, just as a f.g. projective module is a direct summand of a f.g. free module. Our main algebraic result (Proposition 3.2) asserts that a Λ-module chain complex C is chain equivalent to a bounded positive f. g. projective Λ-module chain complex P if and only if it admits a finite domination (D,f,g,h), in which case there is defined a projection p of a f.g. free Λ-module F

$$p = \begin{bmatrix} fg & -d & 0 & \cdots \\ -fhg & 1-fg & d & \cdots \\ -fh^2g & fhg & fg & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} :$$

$$F = D_0 \oplus D_1 \oplus D_2 \oplus \ldots \to F = D_0 \oplus D_1 \oplus D_2 \oplus \ldots$$

such that the f.g. projective Λ-module $\operatorname{im}(p)$ represents the finiteness obstruction of C

$$[C] = [P] = [\mathrm{im}\,(p)] \in \tilde{K}_0(\Lambda).$$

The f.g. projective Λ-module $\mathrm{im}\,(p)$ is the *instant finiteness obstruction* of C determined by (D,f,g,h).

Let X be a connected CW complex. A *finite domination* (Y,f,g,h) of X is a finite CW complex Y together with maps

$$f: X \to Y,\ g: Y \to X$$

and a homotopy

$$h: gf \simeq 1: X \to X,$$

so that X is a homotopy-theoretic direct summand of Y. Let $\tilde{X}$ be the universal cover of X, and let $\tilde{Y} = g^*\tilde{X}$ be the cover of Y obtained from $\tilde{X}$ by pullback along $g: Y \to X$. Define a ring

$$\Lambda = \mathbf{Z}[\pi_1(X)].$$

The cellular Λ-module chain complexes $C(\tilde{X}), C(\tilde{Y})$ are such that $C(\tilde{Y})$ is finite and dominates $C(\tilde{X})$ by

$$(C(\tilde{Y}), \tilde{f}: C(\tilde{X}) \to C(\tilde{Y}), \tilde{g}: C(\tilde{Y}) \to C(\tilde{X}), \tilde{h}: \tilde{g}\tilde{f} \simeq 1: C(\tilde{X})).$$

The *finiteness obstruction* of X given by the theory of Wall [13], [14] is the finiteness obstruction of $C(\tilde{X})$

$$[X] = [C(\tilde{X})] \in \tilde{K}_0(\Lambda).$$

The above formula gives an instant finiteness obstruction

$$\mathrm{im}\left(p: F = \sum_{i=0}^{\infty} C(\tilde{Y})_i \to F = \sum_{i=0}^{\infty} C(\tilde{Y})_i\right)$$

which may be read off from the geometry. Thus X is homotopy equivalent to a finite CW complex if and only if $\mathrm{im}\,(p)$ is a stably f.g. free Λ-module; the finiteness obstruction of X is given by $[X] = [\mathrm{im}\,(p)] \in \tilde{K}_0(\Lambda)$.

Given a finite CW complex Y let $\chi_{\mathrm{even}}(Y)$ (respectively $\chi_{\mathrm{odd}}(Y)$) be the total number of even- (respectively odd-) dimensional cells in Y. The finiteness obstruction is a generalization of the Euler number $\chi(Y) = \chi_{\mathrm{even}}(Y) - \chi_{\mathrm{odd}}(Y) \in \mathbf{Z}$ of a finite CW complex Y. The rank of the f.g. free Λ-module F appearing above is $\chi_{\mathrm{even}}(Y) + \chi_{\mathrm{odd}}(Y)$ = the total number of cells in the dominating finite CW complex Y.

The instant finiteness obstruction is an analogue of the *instant torsion* $M \in \mathrm{GL}_N(\Lambda)$ used by Whitehead [16] to define the *torsion*

$\tau(f)\in \mathrm{Wh}(\pi_1(X))$ of a homotopy equivalence $f:X\to Y$ of finite CW complexes. Here, $\Lambda = \mathbf{Z}[\pi_1(X)]$ (as before),

$$N = \chi_{\mathrm{even}}(X) + \chi_{\mathrm{odd}}(Y) = \chi_{\mathrm{odd}}(X) + \chi_{\mathrm{even}}(Y),$$

and M is the matrix of the isomorphism of f.g. free Λ-modules of rank N determined by any chain contraction $\Gamma: 0\simeq 1: C\to C$ of the algebraic mapping cone $C = C(f: C(\tilde{X})\to C(\tilde{Y}))$

$$M = \begin{bmatrix} d & 0 & 0 & \cdots \\ \Gamma & d & 0 & \cdots \\ 0 & \Gamma & d & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}:$$

$$C_{\mathrm{odd}} = C_1\oplus C_3\oplus C_5\oplus\ldots \xrightarrow{\sim} C_{\mathrm{even}} = C_0\oplus C_2\oplus C_4\oplus\ldots$$

with respect to the geometrically determined bases. The torsion

$$\tau(f)\in \mathrm{Wh}(\pi_1(X)) = \mathrm{GL}(\Lambda)/(E(\Lambda)+\{\pm\pi_1(X)\})$$

is represented by the matrix $M\in \mathrm{GL}_N(\Lambda)\subset \mathrm{GL}(\Lambda)$, which may be read off from the geometry. See Ranicki [11] for the development of the algebraic theory of torsion parallel to the treatment here of the finiteness obstruction.

The algebraic theory of finiteness obstruction developed here is an analogue of the algebraic theory of surgery developed in Ranicki [9], [10]. The instant finiteness obstruction is the counterpart of the *instant surgery obstruction* of [10], which associates to a normal map $(f,b): M\to X$ of n-dimensional geometric Poincaré complexes an n-dimensional algebraic Poincaré complex representing the surgery obstruction $\sigma_*(f,b)\in L_n(\mathbf{Z}[\pi_1(X)])$ of Wall [15], without preliminary surgery below the middle dimension. Although this will not be done here it is possible to fuse together the two theories, so as to apply to the projective L-groups $L^p_*(\mathbf{Z}[\pi])$ of Ranicki [8] and the projective surgery obstruction of Pedersen and Ranicki [5].

In fact, the algebraic theory of finiteness obstruction arose from the extension to the projective L-groups of the algebraic S^1-bundle transfer map for the projective class groups K_0 obtained by Munkholm and Ranicki [3]. The latter was itself an extension of the works of Munkholm, Pedersen and myself on algebraic S^1-bundle transfer maps for the Whitehead group Wh and the L-groups L^h_*, L^s_* defined by free modules.

In the first version of this paper only chain complexes of modules were considered. The instant finiteness obstruction in this case was used by Pedersen [4] to define lower K-theory invariants of chain complexes of $\mathbf{Z}^i$-

graded modules. A study of this application and of the work of Pedersen and Weibel [6] on the lower K-theory of additive categories and their idempotent completions convinced me that the proper level of generality for the theory should be that of chain complexes in any additive category, not just an abelian category.

There are many connections between the theory here, the algebraic K- and L-theory of polynomial extensions, and non-compact manifolds. I hope to deal with these in further work.

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1. Finite chain complexes.

The standard notions of chain homotopy theory are developed for chain complexes in an additive category $\mathscr{A}$, in particular the connection between finite chain complexes and the isomorphism class group $K_0(\mathscr{A})$.

EXAMPLE. Given a ring Λ let $\mathscr{A}$ be the additive category of f.g. free Λ-modules.

Let then $\mathscr{A}$ be an additive category, with direct sum $\oplus$.

A *chain complex* C in $\mathscr{A}$ is a sequence of objects and morphisms

$$C:\ldots \to C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \to \ldots \to C_0$$

such that $d^2 = 0$.

A *chain map* of chain complexes in $\mathscr{A}$

$$f: C \to D$$

is a collection of morphisms $\{f: C_r \to D_r \,|\, r \geqq 0\}$ such that $d_D f = f d_C$, defining a commutative diagram

$$\begin{array}{ccccccccc} C:\ldots & \to & C_{r+1} & \xrightarrow{d_C} & C_r & \xrightarrow{d_C} & C_{r-1} & \to & \ldots \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ D:\ldots & \to & D_{r+1} & \xrightarrow{d_D} & D_r & \xrightarrow{d_D} & D_{r-1} & \to & \ldots \end{array} .$$

A *chain homotopy* of chain maps in $\mathscr{A}$

$$e: f \simeq f' : C \to D$$

is a collection of morphisms $\{e: C_r \to D_{r+1} \mid r \geqq 0\}$ such that

$$f' - f = d_D e + e d_C : C_r \to D_r.$$

A *chain equivalence* is a chain map $f: C \to D$ which admits a chain homotopy inverse, that is a chain map $g: D \to C$ and chain homotopies

$$h: gf \simeq 1: C \to C,\ k: fg \simeq 1: D \to D.$$

A chain complex C is *chain contractible* if it is chain equivalent to 0, that is if there exists a *chain contraction*

$$\Gamma: 0 \simeq 1: C \to C.$$

The *algebraic mapping cone* $C(f)$ of a chain map $f: C \to D$ is the chain complex defined by

$$d_{C(f)} = \begin{pmatrix} d_D & (-)^{r-1} f \\ 0 & d_C \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \to C(f)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

PROPOSITION 1.1. *A chain map $f: C \to D$ is a chain equivalence if and only if the algebraic mapping cone $C(f)$ is chain contractible.*

PROOF. Given a chain contraction $\Gamma: 0 \simeq 1: C(f) \to C(f)$ let g, h, k be the morphisms defined by

$$\Gamma = \begin{pmatrix} k & ? \\ (-)^r g & h \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \to C(f)_{r+1} = D_{r+1} \oplus C_r.$$

Then $g: D \to C$ is a chain homotopy inverse for $f: C \to D$, with chain homotopies

$$h: gf \simeq 1: C \to C,\ k: fg \simeq 1: D \to D.$$

Conversely, suppose that $f: C \to D$ is a chain equivalence, with chain homotopy inverse $g: D \to C$ and chain homotopies

$$h: gf \simeq 1: C \to C,\ k: fg \simeq 1: D \to D.$$

Define morphisms $\beta: C(f)_r \to C(f)_{r+1}$, $\alpha: C(f)_r \to C(f)_r$ $(r \geqq 0)$ by

$$\beta = \begin{pmatrix} k & 0 \\ (-)^r g & h \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \to C(f)_{r+1} = D_{r+1} \oplus C_r,$$

$$\alpha = d_{C(f)}\beta + \beta d_{C(f)} = \begin{pmatrix} 1 & (-)^r(fh-kf) \\ 0 & 1 \end{pmatrix}:$$

$$C(f)_r = D_r \oplus C_{r-1} \to C(f)_r = D_r \oplus C_{r-1}.$$

Then $\alpha: C(f) \to C(f)$ is an automorphism of $C(f)$, and the morphisms

$$\Gamma = \alpha^{-1}\beta: C(f)_r \to C(f)_{r+1}$$

define a chain contraction

$$\Gamma: 0 \simeq 1: C(f) \to C(f).$$

The *isomorphism class group of* $\mathscr{A}$, $K_0(\mathscr{A})$, is the abelian group with one generator $[A]$ for each isomorphism class of objects A in $\mathscr{A}$, with relations

$$[A] + [B] = [A \oplus B] \in K_0(\mathscr{A}).$$

This is just the Grothendieck group of $\mathscr{A}$, as defined by Bass [1, p. 346]. A typical element of $K_0(\mathscr{A})$ is a formal difference $[A] - [B]$, with

$$[A] - [B] = [A'] - [B']$$

if and only if there exists an isomorphism in $\mathscr{A}$

$$A \oplus B' \oplus C \xrightarrow{\sim} A' \oplus B \oplus C$$

for some object C in $\mathscr{A}$.

EXAMPLE. The projective class group $K_0(\Lambda)$ of a ring Λ is the isomorphism class group $K_0(\mathscr{P})$ of the additive category $\mathscr{P}$ of f.g. projective (left) Λ-modules

$$K_0(\Lambda) = K_0(\mathscr{P}).$$

A chain complex C in $\mathscr{A}$ is *finite* if there exists an integer $n \geqq 0$ such that $C_i = 0$ for $i > n$.

Define the *class* of a finite chain complex C in $\mathscr{A}$ by

$$[C] = \sum_{i=0}^{\infty} (-)^i [C_i] \in K_0(\mathscr{A}),$$

where the sum is only formally infinite.

EXAMPLE. If Λ is a ring such that f.g. free Λ-modules have a well-defined rank, such as a group ring $\mathsf{Z}[\pi]$, and $\mathscr{A}$ is the additive category of f.g. free Λ-modules then rank defines an isomorphism of abelian groups

$$K_0(\mathscr{A}) \xrightarrow{\sim} \mathsf{Z};\ [F]-[G] \to \mathrm{rank}(F)-\mathrm{rank}(G).$$

The class of a finite chain complex C of f.g. free Λ-modules is just the Euler number of C

$$[C] = \sum_{i=0}^{\infty} (-)^i \mathrm{rank}(C_i) = \chi(C) \in K_0(\mathscr{A}) \cong \mathsf{Z}.$$

If $\Lambda = \mathsf{Z}$ and $C = C(X)$ is the cellular Z-module chain complex of a finite CW complex X, then $\chi(C) = \chi(X) \in \mathsf{Z}$ is the Euler number of X.

PROPOSITION 1.2. *The class of a finite chain complex is a chain homotopy invariant, with*

$$[C] = [D] \in K_0(\mathscr{A})$$

for chain equivalent finite complexes C, D in $\mathscr{A}$.

PROOF. In the first instance, we show that $[C] = 0 \in K_0(\mathscr{A})$ for a contractible finite chain complex C in $\mathscr{A}$.

Given a chain contraction $\Gamma : 0 \simeq 1 : C \to C$ define morphisms f, g in $\mathscr{A}$ by

$$f = \begin{bmatrix} d & 0 & 0 & \cdots \\ \Gamma & d & 0 & \cdots \\ 0 & \Gamma & d & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} :$$

$$C_{\mathrm{odd}} = C_1 \oplus C_3 \oplus C_5 \oplus \ldots \to C_{\mathrm{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \ldots,$$

$$g = \begin{bmatrix} \Gamma & d & 0 & \cdots \\ 0 & \Gamma & d & \cdots \\ 0 & 0 & \Gamma & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} :$$

$$C_{\mathrm{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \ldots \to C_{\mathrm{odd}} = C_1 \oplus C_3 \oplus C_5 \oplus \ldots$$

(cf. the instant Whitehead torsion quoted in the introduction). Both the composites

$$fg = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \Gamma^2 & 1 & 0 & \cdots \\ 0 & \Gamma^2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} :$$

$$C_{\text{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \ldots \to C_{\text{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \ldots,$$

$$gf = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \Gamma^2 & 1 & 0 & \cdots \\ 0 & \Gamma^2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}:$$

$$C_{\text{odd}} = C_1 \oplus C_3 \oplus C_5 \oplus \ldots \to C_{\text{odd}} = C_1 \oplus C_3 \oplus C_5 \oplus \ldots$$

are automorphisms. It follows that there is defined a two-sided inverse $f^{-1}: C_{\text{even}} \to C_{\text{odd}}$ for $f: C_{\text{odd}} \to C_{\text{even}}$,

$$f^{-1} = (gf)^{-1}g = g(fg)^{-1} : C_{\text{even}} \to C_{\text{odd}}.$$

Thus f is an isomorphism, and

$$[C] = [C_{\text{even}}] - [C_{\text{odd}}] = 0 \in K_0(\mathscr{A}).$$

The algebraic mapping cone $C(f)$ of any chain map $f: C \to D$ of finite chain complexes C,D in $\mathscr{A}$ has class

$$[C(f)] = [D] - [C] \in K_0(\mathscr{A}).$$

If f is a chain equivalence then $C(f)$ is chain contractible (by Proposition 1.1), so that by the above

$$[D] - [C] = [C(f)] = 0 \in K_0(\mathscr{A}).$$

A chain complex C in $\mathscr{A}$ is *homotopy finite* if it is chain equivalent to a finite chain complex D. The *class* of a homotopy finite chain complex C is defined by

$$[C] = [D] \in K_0(\mathscr{A})$$

for any chain equivalent finite complex D. Proposition 1.2 shows that this definition is independent of the choice of D.

2. The idempotent completion.

The idempotent completion $\mathscr{P}$ of the additive category $\mathscr{A}$ is used to define a class of infinite chain complexes C in $\mathscr{A}$ with an invariant $[C] \in K_0(\mathscr{P})$ such that C is chain equivalent to a finite chain complex in $\mathscr{A}$ if and only if $[C] \in \text{im}\,(K_0(\mathscr{A}) \to K_0(\mathscr{P}))$.

A morphism $p: A \to A$ of an object A in $\mathscr{A}$ to itself is a *projection* if it is idempotent, that is

$$p^2 = p: A \to A,$$

in which case $1 - p: A \to A$ is also a projection.

The *idempotent completion of* $\mathscr{A}$, $\mathscr{P}$, is the additive category with one object (A,p) for each projection $p: A \to A$ in $\mathscr{A}$, and morphisms

$$f:(A,p) \to (B,q)$$

the morphisms $f: A \to B$ in $\mathscr{A}$ such that

$$qfp = f: A \to B.$$

The identity morphism of (A,p) in $\mathscr{P}$ is defined by

$$1_{(A,p)} = p:(A,p) \to (A,p).$$

The embedding of additive categories

$$\mathscr{A} \hookrightarrow \mathscr{P};\ A \to (A,1)$$

is full and cofinal: the morphisms $f:(A,1) \to (B,1)$ in $\mathscr{P}$ are precisely the morphisms $f: A \to B$ in $\mathscr{A}$, and for every object (A,p) in $\mathscr{P}$ there are defined isomorphisms

$$(A,p) \oplus (A,1-p) \underset{\binom{p}{1-p}}{\overset{(p,1-p)}{\rightleftarrows}} (A,1)$$

expressing (A,p) as a direct summand of $(A,1)$.

The isomorphism class group $K_0(\mathscr{P})$ is the *projective class group of* $\mathscr{A}$. The cofinality of $\mathscr{A} \hookrightarrow \mathscr{P}$ implies that the natural map

$$K_0(\mathscr{A}) \to K_0(\mathscr{P});\ [A] \to [A,1]$$

is an injection, so that $K_0(\mathscr{A})$ can be regarded as a subgroup of $K_0(\mathscr{P})$. The *reduced projective class group of* $\mathscr{A}$ is the quotient group

$$\tilde{K}_0(\mathscr{P}) = K_0(\mathscr{P})/K_0(\mathscr{A}).$$

A typical element of $\tilde{K}_0(\mathscr{P})$ is of the form $[A,p]^\sim$ for some (A,p) in $\mathscr{P}$, with

$$[A,p]^\sim = [B,q]^\sim \in \tilde{K}_0(\mathscr{P})$$

if and only if there exists an isomorphism

$$(A,p) \oplus (F,1) \xrightarrow{\sim} (B,q) \oplus (G,1)$$

in $\mathscr{P}$ for some objects F,G in $\mathscr{A}$.

Note that the isomorphism $(A,p) \oplus (A,1-p) \xrightarrow{\sim} (A,1)$ (used above to prove that $\mathscr{A} \hookrightarrow \mathscr{P}$ is cofinal) implies that for any (A,p) in $\mathscr{P}$

$$-[A,p]^\sim = [A,1-p]^\sim \in \tilde{K}_0(\mathscr{P}).$$

EXAMPLE. The idempotent completion $\mathscr{P}$ of the additive category of f.g. free Λ-modules (for some ring Λ) is equivalent to the additive category of f.g. projective Λ-modules. The equivalence

$$\mathscr{P} \xrightarrow{\sim} \{\text{f.g. projective } \Lambda\text{-modules}\};\quad (A,p) \to \operatorname{im}(p: A \to A)$$

induces isomorphisms of abelian groups

$$K_0(\mathscr{P}) \xrightarrow{\sim} K_0(\Lambda); [A,p] \to [\operatorname{im}(p)]$$
$$\tilde{K}_0(\mathscr{P}) \xrightarrow{\sim} \tilde{K}_0(\Lambda); [A,p]^{\sim} \to [\operatorname{im}(p)]$$

with $K_0(\Lambda)$ (respectively $\tilde{K}_0(\Lambda)$) the usual (respectively reduced) projective class group of Λ.

We shall use the terminology (C,p) for a chain complex in $\mathscr{P}$

$$(C,p): \ldots \to (C_{i+1},p_{i+1}) \xrightarrow{d} (C_i,p_i) \xrightarrow{d} (C_{i-1},p_{i-1}) \to \ldots \to (C_0,p_0).$$

The class of a homotopy finite chain complex (C,p) in $\mathscr{P}$ is denoted by $[C,p] \in K_0(\mathscr{P})$. The *reduced class* of (C,p)

$$[C,p]^{\sim} \in \tilde{K}_0(\mathscr{P})$$

is the image of $[C,p]$ under the canonical projection $K_0(\mathscr{P}) \twoheadrightarrow \tilde{K}_0(\mathscr{P})$.

PROPOSITION 2.1. *A homotopy finite chain complex (C,p) in $\mathscr{P}$ is chain equivalent to $(D,1)$ for a finite chain complex D in $\mathscr{A}$ if and only if $[C,p]^{\sim} = 0 \in \tilde{K}_0(\mathscr{P})$.*

PROOF. By the chain homotopy invariance of $[C,p]^{\sim} \in \tilde{K}_0(\mathscr{P})$ (given by Proposition 1.2) there is no loss of generality in assuming that (C,p) is a finite chain complex in $\mathscr{P}$.

If (C,p) is chain equivalent to $(D,1)$ for a finite chain complex D in $\mathscr{A}$ then

$$[C,p]^{\sim} = [D,1]^{\sim} = 0 \in \tilde{K}_0(\mathscr{P}).$$

For the converse it is convenient to have the following notion: given an object (A,p) in $\mathscr{P}$ and an integer $i \geqq 0$ define the *elementary* chain complex in $\mathscr{P}$

$$(A,p)[i,i+1]: \ldots \to 0 \to (A,p) \xrightarrow{p} (A,p) \to 0 \to \ldots$$

with the non-zero entries in degrees $i,i+1$, which is clearly finite and chain contractible.

LEMMA. *Every finite chain complex (C,p) in $\mathscr{P}$ is chain equivalent to a finite chain complex (D,q) such that $q_i = 1: D_i \to D_i$ for $i \geqq 1$, and*

$$\begin{aligned}[C,p] &= [D_0,q_0] + \sum_{i=1}^{\infty} (-)^i [D_i,1] \in K_0(\mathscr{P}),\\ [C,p]^{\sim} &= [D_0,q_0]^{\sim} \in \tilde{K}_0(\mathscr{P}).\end{aligned}$$

PROOF. Assume inductively on j that (C,p) is such that $p_i = 1: C_i \to C_i$ for $i > j$, with $j \geqq 1$. Now (C,p) is chain equivalent to $(C,p) \oplus (C_j, 1-p_j)[j-1,j]$, and replacing $(C_j,p_j) \oplus (C_j,1-p_j)$ with the isomorphic object $(C_j,1)$ in $\mathscr{P}$ there is obtained a chain equivalent finite complex (C',p') with $p'_i = 1: C'_i \to C'_i$ for $i \geqq j$.

Let now (C,p) be a finite chain complex in $\mathscr{P}$ such that $[C,p]^{\sim} = 0 \in \tilde{K}_0(\mathscr{P})$. By the lemma (C,p) is chain equivalent to a finite complex (also denoted (C,p)) with $p_i = 1: C_i \to C_i$ for $i \geqq 1$, and

$$[C,p]^{\sim} = [C_0,p_0]^{\sim} = 0 \in \tilde{K}_0(\mathscr{P}).$$

Thus there exist objects A,B in $\mathscr{A}$ and an isomorphism in $\mathscr{P}$

$$f: (A,1) \xrightarrow{\sim} (B,1) \oplus (C_0,p_0).$$

Now (C,p) is chain equivalent to $(B,1)[0,1] \oplus (C,p)$, and using f to replace $(B,1) \oplus (C_0,p_0)$ by $(A,1)$ there is obtained a chain equivalent finite complex $(D,1)$.

A chain complex C in $\mathscr{A}$ is *homotopy $\mathscr{P}$-finite* if $(C,1)$ is homotopy finite in $\mathscr{P}$, that is if $(C,1)$ is chain equivalent to a finite complex (D,p) in $\mathscr{P}$. To avoid confusion we shall call homotopy finite complexes in $\mathscr{A}$ (as defined at the end of Section 1) *homotopy $\mathscr{A}$-finite*.

The *$\mathscr{P}$-class* of a homotopy $\mathscr{P}$-finite chain complex C in $\mathscr{A}$ is defined by

$$[C] = [C,1] \in K_0(\mathscr{P}),$$

so that for any finite complex (D,p) in $\mathscr{P}$ chain equivalent to $(C,1)$

$$\begin{aligned}[C] &= [D,p]\\ &= \sum_{i=0}^{\infty} (-)^i [D_i,p_i] \in K_0(\mathscr{P}).\end{aligned}$$

The *reduced $\mathscr{P}$-class* of a homotopy $\mathscr{P}$-finite complex C in $\mathscr{A}$ is the image of $[C] \in K_0(\mathscr{P})$ under the canonical projection $K_0(\mathscr{P}) \twoheadrightarrow \tilde{K}_0(\mathscr{P})$,

$$[\tilde{C}] = [C,1]^{\sim} \in \tilde{K}_0(\mathscr{P}).$$

PROPOSITION 2.2. (i) *A homotopy $\mathscr{P}$-finite chain complex C in $\mathscr{A}$ is homotopy $\mathscr{A}$-finite if and only if* $[\tilde{C}] = 0 \in \tilde{K}_0(\mathscr{P})$.

Thus the reduced $\mathscr{P}$*-class* $[\tilde{C}]\in\tilde{K}_0(\mathscr{P})$ *is the* ($\mathscr{A}$-) *finiteness obstruction of* C.

(ii) *Every element of* $\tilde{K}_0(\mathscr{P})$ *is the finiteness obstruction* $[\tilde{C}]$ *of a homotopy* $\mathscr{P}$*-finite complex* C *in* $\mathscr{A}$.

PROOF. (i) If C is homotopy $\mathscr{A}$-finite there exists a finite chain complex D in $\mathscr{A}$ chain equivalent to C, and

$$[C]=[D,1]\in \operatorname{im}(K_0(\mathscr{A})\to K_0(\mathscr{P}))\subseteqq K_0(\mathscr{P}),$$

so that $[\tilde{C}]=0\in\tilde{K}_0(\mathscr{P})$.

Conversely, if C is homotopy $\mathscr{P}$-finite and $[\tilde{C}]=0\in\tilde{K}_0(\mathscr{P})$ then by Proposition 2.1, $(C,1)$ is chain equivalent to $(D,1)$ for some finite chain complex D in $\mathscr{A}$. The embedding $\mathscr{A}\hookrightarrow\mathscr{P}$ is full, so that C is chain equivalent to D in $\mathscr{A}$, and C is homotopy $\mathscr{A}$-finite.

(ii) Given an object (A,p) in $\mathscr{P}$ define a chain complex C in $\mathscr{A}$ by

$$d=\begin{cases}1-p: C_{2i+1}=A\to C_{2i}=A\\ p: \quad C_{2i+2}=A\to C_{2i+1}=A\end{cases}\quad(i\geqq 0).$$

Define a finite chain complex (D,q) in $\mathscr{P}$ by

$$q_0=p:D_0=A\to D_0=A,\ D_i=0\ (i\neq 0).$$

The chain maps in $\mathscr{P}$

$$f:(C,1)\to(D,q),\ g:(D,q)\to(C,1)$$

defined by

$$f=p:C_0=A\to D_0=A$$
$$g=p:D_0=A\to C_0=A$$

are inverse chain equivalences, since

$$fg=q:(D,q)\to(D,q)$$

and there is defined a chain homotopy

$$h:gf\simeq 1:(C,1)\to(C,1),$$

with

$$h=1:C_i=A\to C_{i+1}=A\ \ (i\geqq 0).$$

Thus C is homotopy $\mathscr{P}$-finite, with $\mathscr{P}$-class

$$[C]=[D,q]=[A,p]\in K_0(\mathscr{P}).$$

3. Finitely dominated chain complexes.

A *domination* (D,f,g,h) of a chain complex C in $\mathscr{A}$ is a chain complex D in $\mathscr{A}$ together with chain maps

$$f:C\to D,\ D\to C$$

and a chain homotopy

$$h:gf\simeq 1:C\to C.$$

The domination is *finite* if D is finite.

We wish to make precise the sense in which a chain complex C with finite domination (D,f,g,h) is "the image of the projection $fg:D\to D$". The main difficulty is that fg is only a projection up to the chain homotopy

$$fhg:(fg)^2\simeq fg:D\to D,$$

so that it does not directly define a chain complex "(D,fg)" in the idempotent completion $\mathscr{P}$. This difficulty is overcome in the proof of Proposition 3.1 below by manufacturing from (D,f,g,h) a finite chain complex (E,q) in $\mathscr{P}$ chain equivalent to $(C,1)$.

PROPOSITION 3.1. *A chain complex C in $\mathscr{A}$ admits a finite domination (D,f,g,h) if and only if C is homotopy $\mathscr{P}$-finite. Any such finite domination determines an object of $\mathscr{P}$*

$$\Big(F=\sum_{i=0}^{\infty}D_i,p=\begin{bmatrix} fg & -d & 0 & \cdots \\ -fhg & 1-fg & d & \cdots \\ -fh^2g & fhg & fg & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}:$$

$$F=D_0\oplus D_1\oplus D_2\oplus\ldots\to F=D_0\oplus D_1\oplus D_2\oplus\ldots\Big),$$

the instant finiteness obstruction of C determined by (D,f,g,h), such that

$$[C]=[F,p]-[D_{\mathrm{odd}},1]\in K_0(\mathscr{P}),$$
$$[\tilde{C}]=[F,p]^{\sim}\in\tilde{K}_0(\mathscr{P}).$$

PROOF. If C is homotopy $\mathscr{P}$-finite, let (D,r) be a finite chain complex in $\mathscr{P}$ chain equivalent to $(C,1)$, via inverse chain equivalences

$$f:(C,1)\to(D,r),\ g:(D,r)\to(C,1)$$

with chain homotopies

$$h:gf\simeq 1:(C,1)\to(C,1),\ k:fg\simeq r:(D,r)\to(D,r).$$

In particular, there is defined a finite domination (D,f,g,h) of C in $\mathscr{A}$.

Conversely, suppose given a finite domination (D,f,g,h) of C in $\mathscr{A}$. Define an infinite chain complex C' in $\mathscr{A}$ chain equivalent to C by

$$d' = \sum_{j=0}^{i} \sum_{k=0}^{i-1} d'_{ijk} : C'_i = \sum_{j=0}^{i} D_j \to C'_{i-1} = \sum_{k=0}^{i-1} D_k \quad (i \geqq 1)$$

with $d'_{ijk} : D_j \to D_k$ given by

$$d'_{ijk} = \begin{cases} 0 & \text{if } j \geqq k+2 \\ (-)^{i+k} d & \text{if } j = k+1 \\ 1 - fg & \text{if } j = k,\ j \equiv i \,(\bmod 2) \\ fg & \text{if } j = k,\ j \equiv i+1 \,(\bmod 2) \\ (-)^{i+k+1} f h^{k-j} g & \text{if } j \leqq k-1 \end{cases}$$

In matrix terms

$$d' = \begin{bmatrix} fg & -d & 0 & \cdots \\ -fhg & 1-fg & d & \cdots \\ -fh^2g & fhg & fg & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} :$$

$$C'_{2i} = D_0 \oplus D_1 \oplus D_2 \oplus \ldots \oplus D_{2i} \to C'_{2i-1} = D_0 \oplus D_1 \oplus D_2 \oplus \ldots \oplus D_{2i-1},$$

$$d' = \begin{bmatrix} 1-fg & d & 0 & \cdots \\ fhg & fg & -d & \cdots \\ fh^2g & -fhg & 1-fg & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} :$$

$$C'_{2i+1} = D_0 \oplus D_1 \oplus D_2 \oplus \ldots \oplus D_{2i+1} \to C'_{2i} = D_0 \oplus D_1 \oplus D_2 \oplus \ldots \oplus D_{2i}.$$

The chain maps in $\mathscr{A}$

$$f' : C \to C',\ g' : C' \to C$$

defined by

$$f' = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix} : C_i \to C'_i = D_0 \oplus D_1 \oplus \ldots \oplus D_{i-1} \oplus D_i,$$

$$g' = (h^i g \ h^{i-1}g \ldots hg \ g) : C'_i = D_0 \oplus D_1 \oplus \ldots \oplus D_{i-1} \oplus D_i \to C_i$$

are inverse chain equivalences, as there are defined chain homotopies

$$h : g'f' = gf \simeq 1 : C \to C,$$
$$k' : f'g' \simeq 1 : C' \to C'$$

with

$$k' = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1 & 0 \\ & & & 0 & 1 \\ & & & 0 & 0 \end{bmatrix} :$$

$$C'_i = D_0 \oplus D_1 \oplus \ldots \oplus D_{i-1} \oplus D_i \to C'_{i+1} = D_0 \oplus D_1 \oplus \ldots \oplus D_i \oplus D_{i+1}.$$

Let $n \geqq 0$ be an integer such that $D_i = 0$ for $i > n$. The instant finiteness obstruction

$$(F,p) = (D_0 \oplus D_1 \oplus \ldots \oplus D_n, \begin{pmatrix} fg & -d & \cdots \\ -fhg & 1-fg & \cdots \\ \vdots & \vdots & \end{pmatrix})$$

is an object of $\mathscr{P}$ such that for $i \geqq n$

$$d' = \begin{cases} p \\ 1-p \end{cases} : C'_i = F \to C'_{i-1} = F \quad \begin{matrix} \text{if } i \equiv 0 \pmod 2 \\ \text{if } i \equiv 1 \pmod 2 \end{matrix}.$$

Let E be the finite chain complex in $\mathscr{A}$ defined by the n-skeleton of C'

$$E : \ldots \to 0 \to C'_n \xrightarrow{d'} C'_{n-1} \to \ldots \to C'_1 \xrightarrow{d'} C'_0,$$

and let $q = q^2 : E \to E$ be the projection of E defined by

$$q_i = 1 : E_i = C'_i \to E_i = C'_i \qquad (0 \leqq i \leqq n-1)$$

$$q_n = \begin{cases} p \\ 1-p \end{cases} : E_n = F \to E_n = F \quad \begin{matrix} \text{if } n \equiv 0 \pmod 2 \\ \text{if } n \equiv 1 \pmod 2 \end{matrix}$$

Now (E,q) is a finite chain complex in $\mathscr{P}$, and the chain maps in $\mathscr{P}$

$$I' : (C',1) \to (E,q), \ J' : (E,q) \to (C',1)$$

defined by

$$I' = \begin{cases} 1 : C'_i \to E_i = C'_i & \text{if } 0 \leqq i \leqq n-1 \\ q_n : C'_i = F \to E_i = F & \text{if } i = n \end{cases}$$

$$J' = \begin{cases} 1 : E_i = C'_i \to C'_i & \text{if } 0 \leqq i \leqq n-1 \\ q_n : E_i = F \to C'_i = F & \text{if } i = n \end{cases}$$

are inverse chain equivalences, with

$$I'J' = q : (E,q) \to (E,q),$$
$$K' : J'I' \simeq 1 : (C',1),$$

the chain homotopy K' being defined by

$$K' = \begin{cases} 0 : C'_i \to C'_{i+1} & \text{if } 0 \leqq i \leqq n-1 \\ 1 : C'_i = F \to C'_{i+1} = F & \text{if } i \geqq n \end{cases}$$

The composite chain maps in $\mathscr{P}$

$$I = I'f' : (C,1) \xrightarrow{f'} (C',1) \xrightarrow{I'} (E,q)$$
$$J = g'J' : (E,q) \xrightarrow{J'} (C',1) \xrightarrow{g'} (C,1)$$

are thus also inverse chain equivalences, with chain homotopies

$$h + g'K'f : JI \simeq 1 : (C,1) \to (C,1),$$
$$I'k'J' : IJ \simeq q : (E,q) \to (E,q).$$

By the chain homotopy invariance of class (Proposition 1.2)

$$\begin{aligned} [C] = [C,1] = [E,q] &= \sum_{i=0}^{n} (-)^i [E_i, q_i] \\ &= (-)^n [F, q_n] + \sum_{i=0}^{n-1} (-)^i \left(\sum_{j=0}^{i} [D_j, 1] \right) \\ &= (-)^n [F, q_n] + \sum_{j=0}^{n-1} \left(\sum_{i=j}^{n-1} (-)^i \right) [D_j, 1] \\ &= (-)^n [F, q_n] + (-)^{n-1} \left[\sum_{n-j \text{ odd}} D_j, 1 \right] \\ &= \begin{cases} [F,p] - [D_{\text{odd}}, 1] & \text{if } n \text{ is even} \\ -[F, 1-p] + [D_{\text{even}}, 1] & \text{if } n \text{ is odd} \end{cases} \\ &= [F,p] - [D_{\text{odd}}, 1] \in K_0(\mathscr{P}), \end{aligned}$$

where

$$D_{\text{even}} = \sum_{i \text{ even}} D_i, D_{\text{odd}} = \sum_{i \text{ odd}} D_i \quad (\text{so that } F = D_{\text{even}} \oplus D_{\text{odd}}).$$

It follows that the reduced class of C is represented by the object (F,p) of $\mathscr{P}$

$$[\tilde{C}] = [F,p] \in \tilde{K}_0(\mathscr{P}),$$

so that the instant finiteness obstruction (F,p) does indeed represent the finiteness obstruction of C.

For the application of the theory to topology it is necessary to widen somewhat the context, from a single additive category $\mathscr{A}$ to a pair $(\mathscr{B}, \mathscr{A} \subset \mathscr{B})$.

Let then $\mathscr{A} \hookrightarrow \mathscr{B}$ be a full embedding of additive categories. Let $\mathscr{P}$ (respectively $\mathscr{Q}$) be the idempotent completion of $\mathscr{A}$ (respectively $\mathscr{B}$), so that there is defined a commutative square of additive categories and full embeddings

$$\begin{array}{ccc} \mathscr{A} & \hookrightarrow & \mathscr{B} \\ \downarrow & & \downarrow \\ \mathscr{P} & \hookrightarrow & \mathscr{Q} \end{array}$$

We shall now show that a chain complex C in $\mathscr{B}$ is dominated by a finite chain complex in $\mathscr{A}$ if and only if $(C,1)$ is chain equivalent in $\mathscr{Q}$ to a finite chain complex in $\mathscr{P}$, in which case there are defined a class $[C] \in K_0(\mathscr{P})$ and a reduced class $[\tilde{C}] \in \tilde{K}_0(\mathscr{P})$. The main result is that such an $\mathscr{A}$-finitely dominated C is homotopy $\mathscr{A}$-finite if and only if $[\tilde{C}] = 0 \in \tilde{K}_0(\mathscr{P})$, and that there is defined an instant $\mathscr{A}$-finiteness obstruction (F,p) in $\mathscr{P}$.

Example. Given a ring Λ let

$$\mathscr{A} = \{\text{f.g. free } \Lambda\text{-modules}\} \hookrightarrow \mathscr{B} = \{\Lambda\text{-modules}\}.$$

Then $\mathscr{P}$ is equivalent to $\{$f.g. projective Λ-modules$\}$, and $\mathscr{B} \hookrightarrow \mathscr{Q}$ is an equivalence with inverse

$$\mathscr{Q} \xrightarrow{\sim} \mathscr{B};\ (B,q) \to \operatorname{im}(q: B \to B).$$

A chain complex C in $\mathscr{B}$ is *homotopy $\mathscr{A}$-finite* (respectively *$\mathscr{P}$-finite*) if C (respectively $(C,1)$) is chain equivalent in $\mathscr{B}$ (respectively $\mathscr{Q}$) to a finite chain complex D in $\mathscr{A}$ (respectively (D,r) in $\mathscr{P}$) in which case the *$\mathscr{A}$-class* (respectively *$\mathscr{P}$-class*) of C

$$\begin{cases} [C] = [D] \in K_0(\mathscr{A}) \\ [C] = [D,r] \in K_0(\mathscr{P}) \end{cases}$$

is defined.

A domination (D,f,g,h) of a chain complex C in $\mathscr{B}$ is *$\mathscr{A}$-finite* if D is a finite chain complex in $\mathscr{A}$.

PROPOSITION 3.2. (i) *A homotopy $\mathscr{P}$-finite chain complex C in $\mathscr{B}$ is homotopy $\mathscr{A}$-finite if and only if $[\tilde{C}] = 0 \in \tilde{K}_0(\mathscr{P})$. Thus the reduced $\mathscr{P}$-class $[\tilde{C}] \in \tilde{K}_0(\mathscr{P})$ is the* $\mathscr{A}$-finiteness obstruction *of C.*

(ii) *A chain complex C in $\mathscr{B}$ admits an $\mathscr{A}$-finite domination (D,f,g,h) if and only id C is homotopy $\mathscr{P}$-finite. Any such $\mathscr{A}$-finite domination determines an object (F,p) of $\mathscr{P}$, the* instant $\mathscr{A}$-finiteness obstruction *of C, such that*

$$[C] = [F,p] - [D_{\text{odd}},1] \in K_0(\mathscr{P}),$$
$$[\tilde{C}] = [F,p]^{\sim} \in \tilde{K}_0(\mathscr{P}).$$

PROOF. The definitions and proofs of the previous case $\mathscr{A} = \mathscr{B}$ carry over verbatim to the general case $\mathscr{A} \subset \mathscr{B}$.

EXAMPLE. As before, given a ring Λ let

$$\mathscr{A} = \{\text{f.g. free } \Lambda\text{-modules}\} \hookrightarrow \mathscr{B} = \{\Lambda\text{-modules}\}.$$

If C is a Λ-module chain complex with a domination (D,f,g,h) by a finite n-dimensional f.g. free Λ-module chain complex D the instant finiteness obstruction (F,p) is the projection of a f.g. free Λ-module

$$p = \begin{bmatrix} fg & -d & 0 & \dots \\ -fhg & 1-fg & d & \dots \\ -fh^2g & fhg & fg & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}:$$

$$F = D_0 \oplus D_1 \oplus D_2 \oplus \dots \oplus D_n \to F = D_0 \oplus D_1 \oplus D_2 \oplus \dots \oplus D_n,$$

such that $P = \operatorname{im}(p : F \to F)$ is a f.g. projective Λ-module with

$$[C] = [P] - [D_{\text{odd}}] \in K_0(\Lambda), \ [\tilde{C}] = [P] \in \tilde{K}_0(\Lambda).$$

The proof of Proposition 3.1 gives an n-dimensional f.g. projective Λ-module chain complex $B = \operatorname{im}(q : E \to E)$ chain equivalent to C, with

$$B_i = D_0 \oplus D_1 \oplus \dots \oplus D_i \quad (0 \leqq i \leqq n-1)$$
$$B_n = \begin{cases} \operatorname{im}(p : F \to F) = P & \text{if } n \text{ is even} \\ \operatorname{im}(1-p : F \to F) & \text{if } n \text{ is odd.} \end{cases}$$

For $i < n$ the i-skeleton of B is a finite f.g. free Λ-module chain complex

$$B^{(i)} : \dots \to 0 \to B_i \xrightarrow{d_i} B_{i-1} \to \dots \to B_0$$

with

$$d_i = \begin{bmatrix} & \vdots & \vdots & \vdots \\ \cdots & d & 0 & 0 \\ \cdots & fg & -d & 0 \\ \cdots & -fhg & 1-fg & d \end{bmatrix}:$$

$$B_i = D_0 \oplus \ldots \oplus D_{i-2} \oplus D_{i-1} \oplus D_i \to B_{i-1} = D_0 \oplus \ldots \oplus D_{i-3} \oplus D_{i-2} \oplus D_{i-1},$$

such that there is defined a Λ-module chain map $\varphi^{(i)}: B^{(i)} \to C$ with $H_r(\varphi^{(i)}) = 0$ $(0 \leqq r \leqq i)$ and

$$H_{i+1}(\varphi^{(i)}) = \operatorname{coker}(d_{i+2}: B_{i+2} \to B_{i+1})$$

a f.g. Λ-module. The $(i+1)$-skeleton $B^{(i+1)}$ is obtained from $B^{(i)}$ by attaching a finite number of $(i+1)$-cells killing $H_{i+1}(\varphi^{(i)})$, by an algebraic mimicry of the geometric procedure of Wall [13], [14]. It is possible also to kill the f.g. projective Λ-module $H_n(\varphi^{(n-1)}) = B_n$ so as to obtain a finite complex chain equivalent to C if and only if B_n is a stably f.g. free Λ-module, that is if the reduced projective class

$$[\tilde{C}] = (-)^n[B_n] = [P] \in \tilde{K}_0(\Lambda)$$

vanishes. The purely algebraic procedure has the theoretical advantage that it is easier to keep track of the number of cell attachments required, which may be of relevance to the controlled finiteness obstruction theory of Quinn [7].

Given a CW complex X and a regular covering space $\tilde{X}$ with group of covering translations π let $\Lambda = \mathbb{Z}[\pi]$ be the group ring, and let $C(\tilde{X})$ be the cellular Λ-module chain complex defined as usual by

$$\begin{aligned} C(\tilde{X})_i &= H_i(\tilde{X}^{(i)}, \tilde{X}^{(i-1)}) \\ &= \text{the free } \Lambda\text{-module generated by the } i\text{-cells of } X, \end{aligned}$$

$d =$ the boundary map of the triple $(\tilde{X}^{(i)}, \tilde{X}^{(i-1)}, \tilde{X}^{(i-2)})$:

$$C(\tilde{X})_i = H_i(\tilde{X}^{(i)}, \tilde{X}^{(i-1)}) \to C(\tilde{X})_{i-1} = H_{i-1}(\tilde{X}^{(i-1)}, \tilde{X}^{(i-2)}).$$

Given a finite domination of X

$$(Y = \text{finite CW complex},\ f: X \to Y,\ g: Y \to X,\ h: gf \simeq 1: X \to X)$$

there is induced a finite domination of $C(\tilde{X})$

$(C(\tilde{Y}) =$ finite f.g. free Λ-module chain complex,

$$\tilde{f}: C(\tilde{X}) \to C(\tilde{Y}),\ \tilde{g}: C(\tilde{X}) \to C(\tilde{X}),\ \tilde{h}: \tilde{g}\tilde{f} \simeq 1: C(\tilde{X}) \to C(\tilde{X}))$$

with $\tilde{Y} = g^*\tilde{X}$ the regular covering of Y induced from $\tilde{X}$ by pullback along

$g: Y \to X$. (In practice Y is usually a finite subcomplex of X, with $g: Y \to X$ the inclusion, and both $\tilde{X}$ and $\tilde{Y}$ are the universal covers). Applying Proposition 3.2 in the case

$$\mathscr{A} = \{\text{f.g. free } \Lambda\text{-modules}\} \hookrightarrow \mathscr{B} = \{\text{free } \Lambda\text{-modules}\}$$

there is obtained an invariant in the reduced projective class group of the idempotent completion $\mathscr{P} \approx \{\text{f.g. projective } \Lambda\text{-modules}\}$ of $\mathscr{A}$, the reduced projective class group of Λ

$$[X] = [C(\tilde{X})] \in \tilde{K}_0(\mathscr{P}) = \tilde{K}_0(\Lambda),$$

such that the finitely dominated Λ-module chain complex $C(\tilde{X})$ is chain equivalent to a finite f.g. free Λ-module chain complex if and only if $[X] = 0$. Furthermore, the projection $p = p^2: F \to F$ of the f.g. free Λ-module

$$F = \sum_{i=0}^{n} C(\tilde{Y})_i \quad (n = \dim Y)$$

defined by

$$p = \begin{bmatrix} fg & -d & 0 & \cdots \\ -fhg & 1-fg & d & \cdots \\ -fh^2g & fhg & fg & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}:$$

$$F = C(\tilde{Y})_1 \oplus C(\tilde{Y})_1 \oplus C(\tilde{Y})_2 \oplus \ldots \oplus C(\tilde{Y})_n$$
$$\to F = C(\tilde{Y})_0 \oplus C(\tilde{Y})_1 \oplus C(\tilde{Y})_2 \oplus \ldots \oplus C(\tilde{Y})_n$$

is such that the f.g. projective Λ-module

$$P = \operatorname{im}(p: F \to F)$$

is an instant finiteness obstruction of $C(\tilde{X})$, with

$$[X] = [P] \in \tilde{K}_0(\Lambda).$$

For the universal cover $\tilde{X}$ of a connected finitely dominated CW complex X this invariant is the geometric finiteness obstruction of Wall [13], [14]

$$[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)]),$$

such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW complex.

REMARK. Lück [2] uses the instant finiteness obstruction to give an algebraic description of the transfer map

$$p^!: K_0(\mathbb{Z}[\pi_1(B)]) \to K_0(\mathbb{Z}[\pi_1(E)])$$

induced in the projective class groups by a fibration $F \to E \xrightarrow{p} B$ with finitely dominated fibre F. In the case when B (and hence E) is finitely dominated the transfer map sends the unreduced projective class $[B] \in K_0(\mathbb{Z}[\pi_1(B)])$ to $p^!([B]) = [E] \in K_0(\mathbb{Z}[\pi_1(E)])$. The algebraic effect of $p^!$ is to send the class of a f.g. projective $\mathbb{Z}[\pi_1(B)]$-module P which is a direct summand of the f.g. free $\mathbb{Z}[\pi_1(B)]$-module $\mathbb{Z}[\pi_1(B)]^n$ to the class of a finitely dominated $\mathbb{Z}[\pi_1(E)]$-module chain complex $P^!$ which is dominated by $C(\tilde{F})^n$, with $\tilde{F}$ the pullback to F of the universal cover $\tilde{E}$ of E.

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