JOURNAL OF ALGEBRA 120, 361–391 (1989)

Chain Homotopy Projections

WOLFGANG LÜCK

Mathematisches Institut, Georg-August Universität, Bunsenstr. 3–5, 34 Göttingen, Bundesrepublik Deutschland

AND

ANDREW RANICKI

Mathematics Department, Edinburgh University, Edinburgh, Scotland, United Kingdom

Communicated by Richard G. Swan

Received May 1, 1987

INTRODUCTION

The Wall finiteness obstruction of a finitely dominated CW complex X is an element $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ of the reduced projective class group such that $[X]=0$ if and only if X is homotopy equivalent to a finite CW complex. The finiteness obstruction of a pointed homotopy idempotent $p \simeq p^2$: $Y \to Y$ of a finite CW complex Y is an element $[Y, p] \in \tilde{K}_0(\mathbb{Z}[\pi_1(Y)])$ such that $[Y, p]=0$ if and only if (Y, p) is split by a finite CW complex. The explicit formula of Ranicki [18] for a f.g. (finitely generated) projective $\mathbb{Z}[\pi_1(X)]$-module representing $[X]$ is here generalized to an explicit formula for a f.g. projective $\mathbb{Z}[\pi_1(Y)]$-module representing $[Y, p]$.

Let A be an associative ring with 1. The image of a projection of a f.g. free A-module $p=p^2$: $A^r \to A^r$ is a f.g. projective A-module $P = \operatorname{im}(p: A^r \to A^r)$ with a projective class $[P] \in K_0(A)$. The projective class of a finite chain complex C of f.g. projective A-modules is defined in the usual way by

$$[C] = \sum_{r=0}^{\infty} (-)^r [C_r] \in K_0(A).$$

The reduced projective class $[C] \in \tilde{K}_0(A) = \operatorname{coker}(K_0(\mathbb{Z}) \to K_0(A))$ is such that $[C]=0$ if and only if C is chain equivalent to a finite complex of f.g. free A-modules.

A chain homotopy projection (D, p) is a chain complex D together with a chain map $p: D \to D$ for which there exists a chain homotopy $p^2 \simeq p$:

0021-8693/89 $3.00

$D \to D$. A splitting (C, f, g) of (D, p) is a chain complex C together with chain maps $f: C \to D$, $g: D \to C$ such that $gf \simeq 1: C \to C$, $fg \simeq p: D \to D$. Splittings always exist (Lück [13]). Given a chain homotopy projection (D, p) with D a finite chain complex of f.g. projective A-modules we define a projective class invariant

$$[D, p] = [C] \in K_0(A)$$

for any splitting (C, f, g). The reduced projective class $[D, p] \in \tilde{K}_0(A)$ is such that $[D, p] = 0$ if and only if (D, p) is split by a finite complex of f.g. free A-modules.

A near-projection of an A-module M is an endomorphism $p: M \to M$ such that $(p(1-p))^N = 0: M \to M$ for some integer $N \geqslant 0$, in which case standard algebra leads to a projection

$$p_\omega = (p^N + (1-p)^N)^{-1} p^N: M \to M.$$

In fact, p_ω is the unique projection of M such that $pp_\omega = p_\omega p$ and $p - p_\omega$ is nilpotent.

A finite domination (D, f, g) of an A-module chain complex C is a finite chain complex D of f.g. free A-modules together with chain maps $f: C \to D$, $g: D \to C$ such that there exists a chain homotopy $h: gf \simeq 1: C \to C$. An A-module chain complex is finitely dominated if and only if it is chain equivalent to a finite complex of f.g. projective A-modules. The instant finiteness obstruction of Ranicki [18] is an explicit formula in terms of (D, f, g) and a choice of $h: gf \simeq 1$ for a projection

$$X = X^2: D_\omega = \sum_r D_r \to D_\omega$$

such that the projective class of C is

$$[C] = [\mathrm{im}(X)] - [D_{\mathrm{odd}}] \in K_0(A),$$

with $D_{\mathrm{odd}} = D_1 \oplus D_3 \oplus D_5 \oplus \cdots$. (See Proposition 6.1 below for the actual formula.) Our main result is a generalization of the instant finiteness obstruction to chain homotopy projections:

THEOREM. *Given a chain homotopy projection (D, p) with D a finite chain complex of f.g. projective A-modules and a chain homotopy $q: p^2 \simeq p: D \to D$ there is defined a near-projection*

$$X = \begin{pmatrix} p & -d & 0 & \cdots \\ -q & 1-p & d & \cdots \\ 0 & q & p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : D_\omega = D_0 \oplus D_1 \oplus D_2 \oplus \cdots \to D_\omega$$
$$= D_0 \oplus D_1 \oplus D_2 \oplus \cdots$$

such that

$$[D, p] = [\mathrm{im}(X_\omega: D_\omega \to D_\omega)] - [D_{\mathrm{odd}}] \in K_0(A).$$

The projective class of a finitely dominated CW complex X is defined by

$$[X] = [C(\tilde{X})] \in K_0(\mathbb{Z}[\pi_1(X)]),$$

with $C(\tilde{X})$ the cellular chain complex of the universal cover $\tilde{X}$. The Wall finiteness obstruction is the reduced projective class $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$.

A homotopy idempotent (Y, p) is a space Y together with a map $p: Y \to Y$ such that there exists a homotopy $p^2 \simeq p: Y \to Y$. A splitting (X, f, g) of (Y, p) is a space X together with maps $f: X \to Y$, $g: Y \to X$ such that $gf \simeq 1: X \to X$, $fg \simeq p: Y \to Y$. Hastings and Heller [8] have shown that every homotopy idempotent (Y, p) with Y dominated by a finite CW complex admits a splitting (X, f, g), in which case X is also dominated by a finite CW complex. In Section 4 we define the projective class of a homotopy idempotent (Y, p) with Y finitely dominated

$$[Y, p] = f_*[X] \in K_0(\mathbb{Z}[\pi_1(Y)])$$

using any splitting (X, f, g) of (Y, p). The reduced projective class $[Y, p] \in \tilde{K}_0(\mathbb{Z}[\pi_1(Y)])$ is such that $[Y, p] = 0$ if and only if (Y, p) is split by a finite CW complex. In the case of a pointed homotopy idempotent (Y, p) we express the projective class in terms of the chain homotopy idempotent $(C(\bar{Y}), \bar{p})$ defined over $\mathbb{Z}[\mathrm{im}(p_*)]$, with $\bar{Y}$ the regular cover of Y associated to the normal subgroup $\ker(p_*: \pi_1(Y) \to \pi_1(Y)) \subseteq \pi_1(Y)$, such that the group of covering translations is $\mathrm{im}(p_*: \pi_1(Y) \to \pi_1(Y))$. The inclusion $\mathrm{im}(p_*) \to \pi_1(Y)$ is a split injection inducing a split injection $K_0(\mathbb{Z}[\mathrm{im}(p_*)]) \to K_0(\mathbb{Z}[\pi_1(Y)])$ sending $[C(\bar{Y}), \bar{p}]$ to $[Y, p]$. For any pointed splitting (X, f, g) of (Y, p) there are identifications

$$\bar{Y} = g^*\tilde{X}, \qquad \pi_1(X) = \mathrm{im}(p_*),$$
$$[X] = [C(\bar{Y}), \bar{p}] \in K_0(\mathbb{Z}[\pi_1(X)]) = K_0(\mathbb{Z}[\mathrm{im}(p_*)]),$$

with $\tilde{X}$ the universal cover of X.

By way of application of the Theorem consider a Hurewicz fibration $F \to E \to B$ with the base B and the fibre F dominated by finite CW complexes, in which case the total space E is also dominated by a finite CW complex. It was shown in Lück [14] that the homotopy action of the loop space ΩB on the covering $\tilde{F}$ of F pulled back from the universal cover $\tilde{E}$ of E determines a morphism of rings

$$U: \mathbb{Z}[\pi_1(B)] \to H_0(\mathrm{Hom}_{\mathbb{Z}[\pi_1(E)]}(C(\tilde{F}), C(\tilde{F})))^{op}$$

with the following property: if

$$[B]=[\mathrm{im}(p)]-[\mathbb{Z}[\pi_1(B)]^s]\in K_0(\mathbb{Z}[\pi_1(B)])$$

for a projection $p=p^2\colon \mathbb{Z}[\pi_1(B)]^r\to\mathbb{Z}[\pi_1(B)]^r$ $(r, s\geqslant 0)$ then the chain homotopy class of chain homotopy projections $U(p)\colon C(\tilde{F})^r\to C(\tilde{F})^r$ is such that

$$[E]=[C(\tilde{F})^r, U(p)]-[C(\tilde{F})^s]\in K_0(\mathbb{Z}[\pi_1(E)]).$$

The original instant finiteness obstruction can be used to construct a f.g. projective $\mathbb{Z}[\pi_1(E)]$-module representing $[C(\tilde{F})^s]$, and the Theorem can be used to do the same for $[C(\tilde{F})^r, U(p)]$.

Other applications require a greater algebraic generality, so that in the main body of the paper we shall be working with chain homotopy projections in any additive category. For example, lower K-theory in the treatment of Pedersen [17] works with chain complexes in the additive category $\mathscr{C}_i(A)$ of $\mathbb{Z}^i$-graded A-modules and bounded morphisms.

The second author would like to thank the Sonderforschungsbereich 170 "Geometrie und Analysis" at the Mathematics Institute of Göttingen University for the opportunity to spend several months there in 1986, during which this paper was written.

Contents.

1. Splitting Idempotents

We bring together general results on the idempotent completion $\hat{\mathscr{A}}$ of an additive category $\mathscr{A}$ and the splitting of idempotents.

A *projection* (D, p) in $\mathscr{A}$ is an endomorphism $p\colon D\to D$ of an object D in $\mathscr{A}$ such that $p^2=p\colon D\to D$, or equivalently $p(1-p)=0\colon D\to D$.

The *idempotent completion* $\hat{\mathscr{A}}$ of $\mathscr{A}$ is the additive category with objects projections $(D, p=p^2\colon D\to D)$ in $\mathscr{A}$ and morphisms

$$f\colon (D,p)\to(E,q)$$

defined by morphisms $f\colon D\to E$ in $\mathscr{A}$ with $qfp=f\colon D\to E$. The identity morphism of an object (D, p) in $\hat{\mathscr{A}}$ to itself is defined by

$$p\colon (D,p)\to(D,p).$$

The full embedding

$$\mathscr{A}\to\hat{\mathscr{A}};\qquad D\to(D,1)$$

is used to identify $\mathscr{A}$ with a full subcategory of $\hat{\mathscr{A}}$. Every object (D, p) in $\hat{\mathscr{A}}$ is a direct summand of the object $(D, 1)$ in $\mathscr{A}\subset\hat{\mathscr{A}}$, with inverse isomorphisms

$$(D,p)\oplus(D,1-p)\underset{\binom{p}{1-p}}{\overset{(p\ \ 1-p)}{\rightleftarrows}}(D,1).$$

Example 1.1. If $\mathscr{A}=\{$f.g. free A-modules$\}$ for a ring A there is defined an equivalence of additive categories

$$\hat{\mathscr{A}}\to\{\text{f.g. projective } A\text{-modules}\};$$
$$(A^m, p)\to\mathrm{im}(p\colon A^m\to A^m).$$

A *splitting* (C, f, g) of a projection (D, p) in $\mathscr{A}$ is an object C together with morphisms

$$f\colon C\to D,\qquad g\colon D\to C$$

such that $gf=1\colon C\to C$, $fg=p\colon D\to D$.

Proposition 1.2. (i) *A splitting (C, f, g) of a projection (D, p) in $\mathscr{A}$ is the same as a pair of inverse isomorphisms in $\hat{\mathscr{A}}$*

$$(D,p)\underset{f}{\overset{g}{\rightleftarrows}}(C,1).$$

(ii) *For any two splittings (C, f, g), (C', f', g') of a projection (D, p) in $\mathscr{A}$ there are defined inverse isomorphisms*

$$g'f\colon C\to C',\qquad gf'\colon C'\to C.$$

Proof. Trivial. ∎

An additive category $\mathscr{A}$ is *idempotent complete* if the embedding $\mathscr{A}\subset\hat{\mathscr{A}}$ is an equivalence of categories.

Proposition 1.3. *An additive category $\mathscr{A}$ is idempotent complete if and only if every projection (D, p) in $\mathscr{A}$ splits.*

Proof. Trivial. ∎

In particular, the idempotent completion $\hat{\mathscr{A}}$ is idempotent complete, since every projection $q\colon (D,p)\to(D,p)$ in $\hat{\mathscr{A}}$ has a splitting $((D,q), q, q)$ in $\hat{\mathscr{A}}$.

The additive category $\mathscr{A}$ is *countable* if for every object D in $\mathscr{A}$ the countable direct sum $\sum_0^\infty D$ is defined in $\mathscr{A}$.

Given a countable category $\mathcal{A}$ and a projection (D, p) in $\mathcal{A}$ define the *Eilenberg–Freyd direct sum system* in $\hat{\mathcal{A}}$

$$\left(\sum_0^\infty D, 1\right) \underset{j}{\overset{i}{\rightleftarrows}} \left(\sum_0^\infty D, 1\right) \underset{f}{\overset{g}{\rightleftarrows}} (D, p)$$

by

$$f = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} : D \to D \oplus D \oplus D \cdots,$$

$$g = (p \quad 0 \quad 0 \quad \cdots) : D \oplus D \oplus D \oplus \cdots \to D,$$

$$i = \begin{pmatrix} 1-p & 0 & 0 & \cdots \\ p & 1-p & 0 & \cdots \\ 0 & p & 1-p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : D \oplus D \oplus D \oplus \cdots \to D \oplus D \oplus D \oplus \cdots,$$

$$j = \begin{pmatrix} 1-p & p & 0 & \cdots \\ 0 & 1-p & p & \cdots \\ 0 & 0 & 1-p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : D \oplus D \oplus D \oplus \cdots \to D \oplus D \oplus D \oplus \cdots.$$

An additive category $\mathcal{A}$ is *complementary* if for every projection (D, p) in $\mathcal{A}$ which admits a splitting (C, f, g) there exists a splitting (B, i, j) of the projection $(D, 1-p)$. Equivalently, any morphisms $f: C \to D$, $g: D \to C$ in $\mathcal{A}$ such that $gf = 1: C \to C$ can be extended a direct sum system

$$B \underset{j}{\overset{i}{\rightleftarrows}} D \underset{f}{\overset{g}{\rightleftarrows}} C.$$

PROPOSITION 1.4. *A countable complementary additive category $\mathcal{A}$ is idempotent complete.*

Proof. Given a projection (D, p) in $\mathcal{A}$ let f, g, i, j be the maps in the Eilenberg–Freyd direct sum system, so that in particular the projection $(\sum_0^\infty D, 1 - fg = ij)$ is split by $(\sum_0^\infty D, i, j)$. Since $\mathcal{A}$ is complementary the projection

$$\left(\sum_0^\infty D, fg = 1 - ij = (1-p) \oplus \sum_1^\infty 0\right)$$

splits in $\mathcal{A}$. A splitting (B, h, k) of $(\sum_0^\infty D, 1 - fg)$ determines the splitting (B, fh, kg) of (D, p). ∎

Proposition 1.4 is a reformulation of the result of Freyd [6] on the splitting of idempotents in a countable complementary category, such as the stable homotopy category.

EXAMPLE 1.5. The idempotent complete additive category $\mathcal{A} = \{$projective A-modules$\}$ defined for any ring A is countable and complementary. The splitting of a projection (F, p) in $\mathcal{A}$ given by Proposition 1.4 corresponds to the A-module isomorphism of the Eilenberg swindle

$$P \oplus \sum_0^\infty F \to \sum_0^\infty F;$$

$$(x, y_0, y_1, \ldots) \to (x + (1-p)(y_0), p(y_0) + (1-p)(y_1), \ldots).$$

2. CHAIN HOMOTOPY PROJECTIONS

Given an additive category $\mathcal{A}$ let $\mathcal{D}(\mathcal{A})$ denote the *derived category*, the additive category with objects chain complexes in $\mathcal{A}$

$$C: \cdots \longrightarrow C_r \xrightarrow{d} C_{r-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d} C_0,$$

and morphisms the chain homotopy classes of chain maps.

In dealing with chain complexes and chain maps we adopt the sign convention that a chain homotopy

$$g: f \simeq f': C \to D$$

is such that

$$f' - f = d_D g + g d_C: C_r \to D_r.$$

The sign convention for the algebraic mapping cone $C(f)$ of a chain map $f: C \to D$ is

$$d_{C(f)} = \begin{pmatrix} d_D & f \\ 0 & -d_C \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \to C(f)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

PROPOSITION 2.1. *The derived category $\mathcal{D}(\mathcal{A})$ is complementary.*

Proof. Let (D, p) be a projection in $\mathcal{D}(\mathcal{A})$ with a splitting (C, f, g). For any representative chain maps in $\mathcal{A}$

$$f: C \to D, \qquad g: D \to C$$

there exist chain homotopies

$$h\colon gf \simeq 1\colon C \to C, \qquad k\colon fg \simeq p\colon D \to D.$$

The algebraic mapping cone

$$B = C(f\colon C \to D)$$

and the chain maps

$$i = (1 - fg \quad fh)\colon B \to D, \qquad j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\colon D \to B$$

are such that there are defined chain homotopies

$$\begin{pmatrix} 0 & 0 \\ g & -h \end{pmatrix}\colon ji = \begin{pmatrix} 1-fg & fh \\ 0 & 0 \end{pmatrix} \simeq 1\colon B \to B,$$

$$-k\colon ij = 1 - fg \simeq 1 - p\colon D \to D,$$

and hence a splitting (B, i, j) of $(D, 1-p)$ in $\mathcal{A}$. Now apply Proposition 1.4. ▮

Proposition 2.2. *If $\mathcal{A}$ is countable then $\mathscr{D}(\mathcal{A})$ is idempotent complete.*

Proof. $\mathscr{D}(\mathcal{A})$ is complementary (by Proposition 2.1) and countable, and hence idempotent complete by Proposition 1.2. ▮

A *domination* (D, f, g) of a chain complex C in $\mathcal{A}$ is a chain complex D together with chain maps

$$f\colon C \to D, \qquad g\colon D \to C$$

such that there exists a chain homotopy $h\colon gf \simeq 1\colon C \to C$.

A *chain homotopy projection* (D, p) is a chain complex D in $\mathcal{A}$ together with a self chain map $p\colon D \to D$ such that there exists a chain homotopy $q\colon p^2 \simeq p\colon D \to D$.

Example 2.3. Given a chain complex C in $\mathcal{A}$ and a domination (D, f, g) of C there is defined a chain homotopy projection $(D, p = fg\colon D \to D)$, with a chain homotopy

$$fhg\colon p^2 \simeq p\colon D \to D.$$

A *splitting* (C, f, g) of a chain homotopy projection (D, p) is a chain complex C with a domination (D, f, g) such that there exists a chain homotopy $fg \simeq p\colon D \to D$. This is just a splitting in $\mathscr{D}(\mathcal{A})$ together with a choice of representative chain maps.

Proposition 2.4. *If $\mathcal{A}$ is countable every chain homotopy projection (D, p) in $\mathcal{A}$ admits a splitting (C, f, g).*

Proof. Immediate from Propositions 1.3, 2.2. ▮

Remark 2.5. The splitting of chain homotopy idempotents of Proposition 2.4 was first obtained in Lück [13] by the following explicit construction. Given a chain homotopy projection (D, p) in a countable additive category $\mathcal{A}$ define chain maps

$$f = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\colon D \to \sum_0^\infty D = D \oplus D \oplus D \oplus \cdots,$$

$$g = (p \quad 0 \quad 0 \quad \cdots)\colon \sum_0^\infty D = D \oplus D \oplus D \oplus \cdots \to D,$$

$$i = \begin{pmatrix} 1-p & 0 & 0 & \cdots \\ p & 1-p & 0 & \cdots \\ 0 & p & 1-p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\colon \sum_0^\infty D = D \oplus D \oplus D \oplus \cdots \to \sum_0^\infty D = D \oplus D \oplus D \oplus \cdots,$$

$$j = \begin{pmatrix} 1-p & p & 0 & \cdots \\ 0 & 1-p & p & \cdots \\ 0 & 0 & 1-p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\colon \sum_0^\infty D = D \oplus D \oplus D \oplus \cdots \to \sum_0^\infty D = D \oplus D \oplus D \oplus \cdots,$$

such that

$$gf = p\colon D \to D.$$

If $p\colon D \to D$ were an actual projection these chain maps would define the Eilenberg–Freyd direct sum system of Section 1

$$\left(\sum_0^\infty D, 1\right) \underset{j}{\overset{i}{\rightleftarrows}} \left(\sum_0^\infty D, 1\right) \underset{f}{\overset{g}{\rightleftarrows}} (D, p).$$

Choosing a chain homotopy

$$q\colon p^2 \simeq p\colon D \to D$$

let

$$s = pq - qp\colon D_r \to D_{r+1}.$$

Define chain homotopies

$$\alpha\colon gi \simeq 0\colon \sum_0^\infty D \to D, \qquad \beta\colon ji \simeq 1\colon \sum_0^\infty D \to \sum_0^\infty D,$$

$$\gamma\colon ij + fg \simeq 1\colon \sum_0^\infty D \to \sum_0^\infty D,$$

by

$$\alpha = (-q \quad 0 \quad 0 \quad \cdots)\colon \sum_0^\infty D_r = D_r \oplus D_r \oplus D_r \oplus \cdots \to D_{r+1},$$

$$\beta = \begin{pmatrix} 2q & -q & 0 & \cdots \\ -q & 2q & -q & \cdots \\ 0 & -q & 2q & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\colon \sum_0^\infty D_r = D_r \oplus D_r \oplus D_r \oplus \cdots \to \sum_0^\infty D_{r+1}$$

$$= D_{r+1} \oplus D_{r+1} \oplus D_{r+1} \oplus \cdots,$$

$$\gamma = \begin{pmatrix} q & -q & 0 & \cdots \\ -q & 2q & -q & \cdots \\ 0 & -q & 2q & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\colon \sum_0^\infty D_r = D_r \oplus D_r \oplus D_r \oplus \cdots \to \sum_0^\infty D_{r+1}$$

$$= D_{r+1} \oplus D_{r+1} \oplus D_{r+1} \oplus \cdots.$$

Define a splitting (C', f', g') of (D, p) by

$$C' = C\left(i\colon \sum_0^\infty D \to \sum_0^\infty D\right),$$

$$f' = (g - \alpha)\colon C' \to D, \qquad g' = \begin{pmatrix} f \\ 0 \end{pmatrix}\colon D \to C',$$

with

$$f'g' = gf = p\colon D \to D.$$

There is defined a chain homotopy

$$h' = \begin{pmatrix} 1 & f\alpha + i\beta - \gamma i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ j & -\beta \end{pmatrix}\colon g'f' \simeq 1\colon C' \to C',$$

which can be expressed as

$$h' = \begin{pmatrix} \gamma + \phi & \theta \\ j & -\beta \end{pmatrix}\colon C'_r = \sum_0^\infty D_r \oplus \sum_0^\infty D_{r-1} \to C'_{r+1} = \sum_0^\infty D_{r+1} \oplus \sum_0^\infty D_r,$$

with

$$\theta = -(f\alpha + i\beta - \gamma i)\beta = \begin{pmatrix} 5qs & -4qs & qs & \cdots \\ -9qs & 10qs & -5qs & \cdots \\ 5qs & -10qs & 10qs & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\colon \sum_0^\infty D_{r-1}$$

$$= D_{r-1} \oplus D_{r-1} \oplus D_{r-1} \oplus \cdots \to \sum_0^\infty D_{r+1} = D_{r+1} \oplus D_{r+1} \oplus D_{r+1} \oplus \cdots,$$

$$\phi = (f\alpha + i\beta - \gamma i)\, j = \begin{pmatrix} 2ps - 2s & -3ps + s & \cdots \\ -3ps + s & 6ps - 3s & \cdots \\ \vdots & \vdots & \end{pmatrix}\colon \sum_0^\infty D_r$$

$$= D_r \oplus D_r \oplus \cdots \to \sum_0^\infty D_{r+1} = D_{r+1} \oplus D_{r+1} \oplus \cdots.$$

3. Projective Class

In dealing with a pair of additive categories $(\mathscr{B}, \mathscr{A})$ we assume that $\mathscr{A}$ is a full subcategory of $\mathscr{B}$, and that $\mathscr{B}$ is countable (= has countable direct sums). We recall the $\mathscr{A}$-finiteness obstruction theory for $\mathscr{A}$-finitely dominated chain complexes in $\mathscr{B}$, and interpret it in terms of chain homotopy projections.

A chain complex C is *finite* if there exists an integer $n \geqslant 0$ such that $C_r = 0$ for $r > n$.

The *class group* $K_0(\mathscr{A})$ of an additive category $\mathscr{A}$ is the abelian group with one generator $[A]$ for each isomorphism class of objects A in $\mathscr{A}$, subject to the relations

$$[A \oplus A'] = [A] + [A'] \in K_0(\mathscr{A}).$$

The *class* of a finite chain complex C in $\mathscr{A}$ is the chain homotopy invariant defined as usual by

$$[C] = \sum_{r=0}^\infty (-)^r [C_r] \in K_0(\mathscr{A}).$$

A domination (D, f, g) of a chain complex C in $\mathscr{B}$ is *$\mathscr{A}$-finite* if D is a finite chain complex in $\mathscr{A}$.

PROPOSITION 3.1. (i) *A chain complex C in $\mathscr{B}$ is $\mathscr{A}$-finitely dominated if and only if the chain complex $(C, 1)$ in $\hat{\mathscr{B}}$ is chain equivalent to a finite chain complex (E, p) in $\hat{\mathscr{A}}$.*

(ii) *The* projective class *of an $\mathscr{A}$-finitely dominated chain complex C in $\mathscr{B}$ is defined by*

$$[C]=[E, p] \in K_0(\hat{\mathscr{A}})$$

for any finite chain complex (E, p) in $\hat{\mathscr{A}}$ chain equivalent to $(C, 1)$. The projective class is such that $[C] \in \operatorname{im}(K_0(\mathscr{A}) \to K_0(\hat{\mathscr{A}}))$ if and only if C is chain equivalent to a finite chain complex in $\mathscr{A}$.

Proof. See Ranicki [18]. ∎

A chain homotopy projection (D, p) in $\mathscr{B}$ is *$\mathscr{A}$-finite* (resp. *$\mathscr{A}$-finitely dominated*) if the chain complex D is $\mathscr{A}$-finite (resp. $\mathscr{A}$-finitely dominated).

PROPOSITION 3.2. *Every $\mathscr{A}$-finitely dominated chain homotopy projection (D, p) in $\mathscr{B}$ admits a splitting (C, f, g) with C an $\mathscr{A}$-finitely dominated chain complex in $\mathscr{B}$, and for any two splittings (C, f, g), (C', f', g') there are defined inverse chain equivalences $g'f\colon C \to C'$, $gf'\colon C' \to C$.*

Proof. Immediate from Proposition 2.4. ∎

DEFINITION 3.3. The *projective class* of an $\mathscr{A}$-finitely dominated chain homotopy projection (D, p) in $\mathscr{B}$ is

$$[D, p]=[C] \in K_0(\hat{\mathscr{A}}),$$

for any splitting (C, f, g) of (D, p) in $\mathscr{B}$.

The projective class has the following properties:

PROPOSITION 3.4. (i) *$[D, p] \in \operatorname{im}(K_0(\mathscr{A}) \to K_0(\hat{\mathscr{A}}))$ if and only if (D, p) admits a splitting (C, f, g) with C $\mathscr{A}$-finite.*

(ii) *If (B, i, j, k) is a domination of D with B $\mathscr{A}$-finitely dominated*

$$[D, p]=[B, ipj] \in K_0(\hat{\mathscr{A}}).$$

(iii) *If $p=p^2\colon D \to D$ is a projection of a finite chain complex D in $\mathscr{A}$*

$$[D, p]=\sum_{r=0}^{\infty}(-)^r[D_r, p] \in K_0(\hat{\mathscr{A}}).$$

Proof. (i) By Proposition 3.1, C is chain homotopy $\mathscr{A}$-finite if and only if $[C] \in \operatorname{im}(K_0(\mathscr{A}) \to K_0(\hat{\mathscr{A}}))$.

(ii) A splitting (C, f, g) of (D, p) determines a splitting (C, if, gj) of (B, ipj), so that

$$[D, p]=[C]=[B, ipj] \in K_0(\hat{\mathscr{A}}).$$

(iii) For any splitting (C, f, g) of (D, p) in $\mathscr{B}$ the chain complex $(C, 1)$ is chain equivalent in $\hat{\mathscr{B}}$ to the chain complex (D, p) in $\hat{\mathscr{A}}$, so that

$$[C]=[D, p]=\sum_{r=0}^{\infty}(-)^r[D_r, p] \in K_0(\hat{\mathscr{A}}). \quad ∎$$

EXAMPLE 3.5. Given a ring A let

$$(\mathscr{B}, \mathscr{A})=(\{\text{free } A\text{-modules}\}, \{\text{f.g. free } A\text{-modules}\}),$$

abbreviating $\mathscr{A}$-finite to finite. An A-module chain complex C is finitely dominated if and only if it is chain equivalent to a finite complex P of f.g. projective A-modules, in which case

$$[C]=\sum_{r=0}^{\infty}(-)^r[P_r] \in K_0(\hat{\mathscr{A}})=K_0(A).$$

The projective class $[D, p] \in K_0(A)$ of a finitely dominated chain homotopy projection (D, p) in $\mathscr{B}$ is such that $[D, p] \in \operatorname{im}(K_0(\mathbb{Z}) \to K_0(A))$ if and only if (D, p) admits a splitting by a finite complex of f.g. free A-modules.

4. GEOMETRIC HOMOTOPY PROJECTIONS

We now connect the algebra and the topology.

A *domination* (Y, f, g) of a space X is a space Y together with maps

$$f\colon X \to Y, \qquad g\colon Y \to X$$

such that there exists a homotopy $h\colon gf \simeq 1\colon X \to X$.

A *homotopy idempotent* (Y, p) is a space Y together with a map $p\colon Y \to Y$ such that there exists a homotopy $p^2 \simeq p\colon Y \to Y$. For example, a domination (Y, f, g) determines a homotopy idempotent (Y, fg). A *splitting* (X, f, g) of a homotopy idempotent (Y, p) is a space X together with a domination (Y, f, g) such that there exists a homotopy $fg \simeq p\colon Y \to Y$. A homotopy idempotent (Y, p) is *pointed* if the space Y is connected and pointed, and there exists a pointed homotopy $p^2 \simeq p\colon Y \to Y$. Similarly for splittings.

PROPOSITION 4.1 (Hastings and Heller [8]). *Every pointed homotopy idempotent (Y, p) with Y a CW complex admits a pointed splitting (X, f, g).*

Proof. Define the mapping telescope of $p: Y \to Y$, the identification space

$$X = (Y \times [0, 1] \times \mathbb{N}) / \{(p(y), 0, n) = (y, 1, n+1) \mid y \in Y, n \in \mathbb{N}\},$$

with $\mathbb{N} = \{0, 1, 2, ...\}$. Given a pointed homotopy

$$q: p^2 \simeq p: Y \to Y$$

with $q(y, 0) = p^2(y)$, $q(y, 1) = p(y)$ $(y \in Y)$ there are defined pointed maps

$$f: X \to Y; \quad (y, t, n) \to q(y, t),$$
$$g: Y \to X; \quad y \to (y, 0, 0)$$

such that

$$fg = p^2 \simeq p: Y \to Y,$$

and such that $gf: X \to X$ is a pointed homotopy equivalence by Whitehead's theorem, since it induces isomorphisms of fundamental groups and the homology groups of the universal covers. It follows that there exists a pointed homotopy $gf \simeq 1: X \to X$, since

$$gf \simeq (gf)^{-1} (gf)(gf)(gf)(gf)^{-1} = (gf)^{-1} g(fg)(fg) f(gf)^{-1}$$
$$\simeq (gf)^{-1} g(fg) f(gf)^{-1} \simeq 1: X \to X,$$

so that there exists a pointed splitting (X, f, g). ∎

Remark 4.2. The use of the infinite mapping telescope in the proof of Proposition 4.1 corresponds to the countable direct sum in the Eilenberg–Freyd direct sum system.

Remark 4.3. Dydak [3] and Freyd and Heller [7] have constructed an unpointed homotopy idempotent of an infinite-dimensional *CW* complex which does not split. Hastings and Heller [9] have shown that unpointed homotopy idempotents of finite-dimensional *CW* complexes split. Every finitely dominated *CW* complex is homotopy equivalent to a finite-dimensional (but not necessarily finite) *CW* complex. Thus for finitely dominated Y every unpointed homotopy idempotent (Y, p) splits.

Let X be a connected *CW* complex, and let $\tilde{X}$ be a regular cover of X with group of covering translations π. The cellular chain complex $C(\tilde{X})$ is a free $\mathbb{Z}[\pi]$-module chain complex which is finite (resp. finitely dominated) if

X is finite (resp. finitely dominated). The projective class of a finitely dominated X is defined as usual by

$$[X] = [C(\tilde{X})] \in K_0(\mathbb{Z}[\pi_1(X)]),$$

with $\tilde{X}$ the universal cover of X. The reduced projective class $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ is the Wall finiteness obstruction, such that $[X] = 0$ if and only if X is homotopy equivalent to a finite *CW* complex. See Ferry and Ranicki [5] for an exposition of finiteness obstruction theory in terms of homotopy idempotents on finite *CW* complexes.

Let Groups be the category of groups. Two morphisms $f, f': \pi \to \rho$ in Groups are *homotopic* if there exist inner automorphisms $\alpha: \pi \to \pi$, $\beta: \rho \to \rho$ such that $f' = \beta f \alpha$, which we denote by

$$f \simeq f': \pi \to \rho.$$

Following Freyd and Heller [7] and Hastings and Heller [8] let Ho(Groups) be the *homotopy category of groups*, with objects groups and morphisms the homotopy classes of morphisms in Groups. Pointed maps of pointed connected spaces which are unpointed homotopic induce homotopic maps on fundamental groups. An unpointed homotopy idempotent (Y, p) with Y a pointed connected space induces a projection $p_*: \pi_1(Y) \to \pi_1(Y)$ in Ho(Groups), with a homotopy

$$(p_*)^2 \simeq p_*: \pi_1(Y) \to \pi_1(Y).$$

Inner automorphisms induce the identity in the projective class group, so that there is defined a functor

$$K_0: \text{Ho(Groups)} \to \text{Abelian Groups}; \quad \pi \to K_0(\mathbb{Z}[\pi]).$$

Given an unpointed homotopy idempotent (Y, p) with Y a pointed connected finitely dominated *CW* complex there exists by Remark 4.3 an unpointed splitting (X, f, g) with X a pointed connected finitely dominated *CW* complex, and with the maps $f: X \to Y$ and $g: Y \to X$ pointed. However, there may not exist a pointed homotopy $fg \simeq p: Y \to Y$. The projection $(\pi_1(Y), p_*)$ is split in Ho(Groups) by $(\pi_1(X), f_*, g_*)$, so that the projection $(K_0(\mathbb{Z}[\pi_1(Y)]), p_*)$ is split in Abelian Groups by $(K_0(\mathbb{Z}[\pi_1(X)]), f_*, g_*)$. Thus $f_*: K_0(\mathbb{Z}[\pi_1(X)]) \to K_0(\mathbb{Z}[\pi_1(Y)])$ is a split injection onto the image of the projection $p_* = (p_*)^2: K_0(\mathbb{Z}[\pi_1(Y)]) \to K_0(\mathbb{Z}[\pi_1(Y)])$.

DEFINITION 4.4. The *projective class* of an unpointed homotopy idempotent (Y, p) of a finitely dominated *CW* complex Y is defined for any splitting (X, f, g) by

$$[Y, p] = f_*[X] \in \text{im}(f_*: K_0(\mathbb{Z}[\pi_1(X)]) \to K_0(\mathbb{Z}[\pi_1(Y)]))$$
$$= \text{im}(p_* = (p_*)^2: K_0(\mathbb{Z}[\pi_1(Y)]) \to K_0(\mathbb{Z}[\pi_1(Y)])),$$

the image of the projective class $[X] \in K_0(\mathbb{Z}[\pi_1(X)])$ under the split injection f_*.

The projective class has the following properties:

PROPOSITION 4.5. *The projective class* $[Y, p] = f_*[X] \in K_0(\mathbb{Z}[\pi_1(Y)])$ *is independent of the choice of splitting* (X, f, g). *The reduced projective class* $[Y, p] = f_*[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(Y)])$ *is such that* $[Y, p] = 0$ *if and only if there exists a splitting* (X, f, g) *with* X *a finite CW complex.*

It should be possible to express the projective class $[Y, p]$ in terms of the $\mathbb{Z}$-module chain map $\tilde{p}: C(\tilde{Y}) \to C(\tilde{Y})$ induced by a lift $\tilde{p}: \tilde{Y} \to \tilde{Y}$ of $p: Y \to Y$ to the universal cover $\tilde{Y}$ of Y, for which

$$\tilde{p}(gy) = p_*(g)(\tilde{p}(y)) \in C(\tilde{Y}) \qquad (g \in \pi_1(Y),\ y \in C(\tilde{Y})),$$

with $p_*: \pi_1(Y) \to \pi_1(Y)$ the morphism in Groups induced by $p: Y \to Y$ and $\tilde{p}$ such that $\tilde{p}(\tilde{b}) = \tilde{b}$ for a lift $\tilde{b} \in \tilde{Y}$ of the basepoint $b \in Y$. If $h: p^2 \simeq p: Y \to Y$ is an unpointed homotopy and $w \in \pi_1(Y)$ is the track of $b \in Y$ then in Groups

$$(p_*)^2 = c(w)\, p_*: \pi_1(Y) \to \pi_1(Y),$$

with

$$c(w): \pi_1(Y) \to \pi_1(Y); \qquad g \to wgw^{-1}.$$

A lift $\tilde{h}: \tilde{Y} \times I \to \tilde{Y}$ of the homotopy $h: Y \times I \to Y$ determines a $\mathbb{Z}$-module chain homotopy

$$\tilde{h}: \tilde{p}^2 \simeq \lambda_w \tilde{p}: C(\tilde{Y}) \to C(\tilde{Y}),$$

with

$$\lambda_w: C(\tilde{Y}) \to C(\tilde{Y}); \qquad y \to wy,$$

$$\tilde{h}(gy) = (p_*)^2(g)\,\tilde{h}(y) \in C(\tilde{Y}) \qquad (g \in \pi_1(Y),\ y \in C(\tilde{Y})).$$

We have only been able to express the projective class $[Y, p]$ in terms of $\tilde{p}$ in the case of a pointed homotopy idempotent, as follows.

Let (Y, p) be a pointed homotopy idempotent with Y a finitely dominated CW complex, so that

$$w = 1 \in \pi_1(Y), \qquad c(w) = id: \pi_1(Y) \to \pi_1(Y),$$
$$\lambda_w = 1: C(\tilde{Y}) \to C(\tilde{Y}).$$

By Proposition 4.1 there exists a pointed splitting (X, f, g). Thus

$(\pi_1(Y), p_*)$ is a projection in Groups which is split by $(\pi_1(X), f_*, g_*)$. Let $\tilde{X}$ be the universal cover of X, and let

$$\bar{Y} = g^*\tilde{X} = \tilde{Y}/\ker(g_*: \pi_1(Y) \to \pi_1(X))$$

be the induced cover of Y. The group of covering translations $\bar{\pi} = \pi_1(Y)/\ker(g_*)$ is isomorphic to $\pi_1(X)$ and also to $\mathrm{im}(p_*: \pi_1(Y) \to \pi_1(Y))$. Let $q: \pi_1(Y) \to \bar{\pi}$ be the projection, and let $i: \bar{\pi} \to \pi_1(Y)$ be the injection induced by p_*. Then $(\bar{\pi}, i, q)$ is a splitting in Groups of the projection $(\pi_1(Y), p_*)$ and there are defined inverse isomorphisms

$$\bar{\pi} \underset{qf_*}{\overset{g_*i}{\rightleftarrows}} \pi_1(X).$$

The $\mathbb{Z}[\bar{\pi}]$-module chain map

$$\bar{p} = 1 \otimes \tilde{p}: C(\bar{Y}) = \mathbb{Z}[\bar{\pi}] \otimes_{\mathbb{Z}[\pi_1(Y)]} C(\tilde{Y}) \to C(\bar{Y})$$

defines a finitely dominated chain homotopy projection $(C(\bar{Y}), \bar{p})$ over $\mathbb{Z}[\bar{\pi}]$, with a chain homotopy

$$\bar{h} = 1 \otimes \tilde{h}: \bar{p}^2 \simeq \bar{p}: C(\bar{Y}) \to C(\bar{Y}).$$

The following result expresses the projective class of a pointed homotopy idempotent in terms of the projective class of a chain homotopy projection.

PROPOSITION 4.6. *The projective class of a finitely dominated pointed homotopy idempotent* (Y, p) *is such that*

$$[Y, p] = i_*[C(\bar{Y}), \bar{p}] \in K_0(\mathbb{Z}[\pi_1(Y)]),$$

with $[C(\bar{Y}), \bar{p}] \in K_0(\mathbb{Z}[\bar{\pi}])$ *the projective class of the finitely dominated chain homotopy projection* $(C(\bar{Y}), \bar{p})$ *over* $\mathbb{Z}[\bar{\pi}]$ *and* $i_*: K_0(\mathbb{Z}[\bar{\pi}]) \to K_0(\mathbb{Z}[\pi_1(Y)])$ *the split injection of projective class groups induced by the split injection of groups* $i: \bar{\pi} \to \pi_1(Y)$.

Proof. By Definition 4.4, $[Y, p] = f_*[X]$ for any geometric splitting (X, f, g) of (Y, p). By Definition 3.3, $[C(\bar{Y}), \bar{p}] = [D]$ for any algebraic splitting (D, h, k) of $(C(\bar{Y}), \bar{p})$. A pointed geometric splitting (X, f, g) of (Y, p) induces a splitting $q_* f_*(C(\tilde{X}), \tilde{f}, \tilde{g})$ of $(C(\bar{Y}), \bar{p})$ over $\mathbb{Z}[\bar{\pi}]$, so that

$$[Y, p] = f_*[X] = i_* q_* f_*[X] = i_*[C(\bar{Y}), \bar{p}] \in K_0(\mathbb{Z}[\pi_1(Y)]). \quad ∎$$

EXAMPLE 4.7. Let π be a finitely presented group, and let Q be a f.g. projective $\mathbb{Z}[\pi]$-module, with

$$Q = \mathrm{im}(q = q^2: \mathbb{Z}[\pi]^m \to \mathbb{Z}[\pi]^m).$$

For any finite CW complex K such that $\pi_1(K)=\pi$ and any integer $r \geqslant 2$ there exists by the Hurewicz theorem a pointed homotopy idempotent

$$p\colon Y=K\vee\bigvee_m S^r \to Y$$

extending $1\colon K\to K$, and inducing

$$p_* = q\colon H_r(\tilde{Y},\tilde{K}) = \mathbb{Z}[\pi]^m \to \mathbb{Z}[\pi]^m$$

with $\tilde{Y}$, $\tilde{K}$ the universal covers of Y, K, respectively. The projective class of (Y, p) is

$$[Y, p]=[K]+(-)^r[Q]\in K_0(\mathbb{Z}[\pi]),$$

and the reduced projective class is

$$[Y, p]=(-)^r[Q]\in \tilde{K}_0(\mathbb{Z}[\pi]).$$

Remark 4.8. Let (Y, p) be an unpointed homotopy idempotent with Y finitely dominated. The homology $H_*(Y)$ is finitely generated, so that the Lefschetz number of $p\colon Y\to Y$ is defined

$$\Lambda(p)=\sum_{r=0}^{\infty}(-)^r \operatorname{tr}(p_*\colon H_r(Y)\to H_r(Y))\in\mathbb{Z}.$$

For any splitting (X, f, g) of (Y, p) the homology $H_*(X)$ is finitely generated, so that the Euler characteristic of X is defined

$$\chi(X)=\sum_{r=0}^{\infty}(-)^r \operatorname{rank}(H_r(X))\in\mathbb{Z}.$$

The trace is such that $\operatorname{tr}(\alpha\beta)=\operatorname{tr}(\beta\alpha)$ for any morphisms $\alpha\colon M\to N$, $\beta\colon N\to M$ of f.g. abelian groups, so that

$$\Lambda(p)=\Lambda(fg)=\Lambda(gf)=\Lambda(1_X)=\chi(X)\in\mathbb{Z}.$$

Thus the augmentation map

$$K_0(\mathbb{Z}[\pi_1(Y)])\to K_0(\mathbb{Z})=\mathbb{Z};\qquad [P]\to \operatorname{rank}(\mathbb{Z}\otimes_{\mathbb{Z}[\pi_1(Y)]}P)$$

sends the projective class $[Y, p]$ to $\Lambda(p)=\chi(X)\in\mathbb{Z}$.

5. Lifting Idempotents

A near-projection is an endomorphism which fails to be a projection only up to nilpotence. The approximation of near-projections by projections ("lifting of idempotents") is a classical procedure (Jacobson [10, Sect. III.8]), which is also used in the perturbation theory of linear operators (Kato [11]). The origin of the technique appears to be the use of the binomial theorem by Wedderburn [23, II.2.13] to construct the square root of a matrix.

Let $\mathcal{A}$ be an additive category. An endomorphism $e\colon A\to A$ in $\mathcal{A}$ is *nilpotent* if for some integer $N\geqslant 1$

$$e^N=0\colon A\to A,$$

in which case $1-e\colon A\to A$ is an automorphism with inverse

$$(1-e)^{-1}=1+e+e^2+\cdots+e^{N-1}\colon A\to A.$$

Any such integer $N\geqslant 1$ is an *exponent* of e.

Proposition 5.1. *Projections (A, p), (A, p') with $p'-p\colon A\to A$ nilpotent are isomorphic in the idempotent completion $\hat{\mathcal{A}}$.*

Proof. Define inverse isomorphisms in $\hat{\mathcal{A}}$ by

$$p'p\colon (A, p)\to (A, p'),$$
$$(1+p'-p)^{-1}p'=p(1-p'+p)^{-1}\colon (A, p')\to (A, p). \quad ∎$$

Remark 5.2. If the objects (A, p), (B, q) in $\hat{\mathcal{A}}$ are such that there exists an isomorphism $h\colon A\to B$ with $qh=hp\colon A\to B$ then there are defined inverse isomorphisms in $\hat{\mathcal{A}}$

$$(A, p)\underset{h^{-1}q}{\overset{hp}{\rightleftarrows}}(B, q).$$

Given (A, p), (A, p') in $\hat{\mathcal{A}}$ as in Proposition 5.1 there exists an automorphism in $\mathcal{A}$

$$h=(1-(p-p')^2)^{-1/2}(p'p+(1-p')(1-p))\colon A\to A$$

(Kato [11, Sect. 4.6]) such that

$$p'h=hp\colon A\to A,$$

giving a different isomorphism $hp\colon (A, p)\to (A, p')$ in $\hat{\mathcal{A}}$.

A *near-projection* (A, p) in an additive category $\mathcal{A}$ is an object A together with an endomorphism $p\colon A\to A$ such that the *defect* endomorphism

$$q=p(1-p)\colon A\to A$$

is nilpotent.

EXAMPLE 5.3. A projection $(A, p=p^2)$ is a near-projection with defect of exponent 1, or equivalently with defect 0.

PROPOSITION 5.4. *For any near-projection (A, p) with defect $q=p(1-p)$ of exponent N there is defined a projection*

$$\begin{aligned} p_\omega &= (p^N+(1-p)^N)^{-1}\, p^N \\ &= p+(1/2)(2p-1)((1-4q)^{-1/2}-1) \\ &= p+(2p-1)(q+3q^2+10q^3+35q^4+126q^5+462q^6+1716q^7 \\ &\quad +6435q^8+24310q^9+92378q^{10}+352716q^{11} \\ &\quad +1352078q^{12}+\cdots)\colon A \to A. \end{aligned}$$

(A, p_ω) is the unique projection with $p_\omega - p$ nilpotent and $pp_\omega = p_\omega p$. In particular, for the near-projection $(A, 1-p)$

$$(1-p)_\omega = 1-p_\omega\colon A \to A.$$

Proof. For any integer $M \geqslant 1$ we have that the endomorphism

$$(p^M+(1-p)^M)-1 = q\left(\sum_{r=1}^{M-1}\left(\binom{M-1}{r}(-1)^r-1\right)p^{r-1}\right)\colon A \to A$$

is nilpotent, so that $p^M+(1-p)^M\colon A \to A$ is an automorphism. The endomorphism defined by

$$p_M = (p^M+(1-p)^M)^{-1}\, p^M\colon A \to A$$

is a near-projection with defect

$$p_M(1-p_M) = (p^M+(1-p)^M)^{-2}\,(p(1-p))^M\colon A \to A.$$

Thus for $M \geqslant N$ there is defined a projection

$$p_N = p_{N+1} = \cdots = p_\omega\colon A \to A.$$

The expression for p_ω in terms of the binomial expansion for $(1-4q)^{-1/2}$ follows from the identity

$$(1-4q)^{1/2} = (1-2p)(p^N+(1-p)^N)(-p^N+(1-p)^N)^{-1}\colon A \to A.$$

For uniqueness note that for any ring R and any indeterminates w, z over R the morphism of formal power series rings

$$R[[z]] \to R[[w]]; \qquad z \to w-w^2$$

sends the binomial expansion of $(1-4z)^{-1/2}$ to the binomial expansion of $(1-2w)^{-1}$, that is

$$(1-2w)^{-1} = 1+2w+4w^2+8w^3+\cdots.$$

Given a near-projection (A, p) in $\mathcal{A}$ and any expression of p as a sum $p=f+g$ of a projection $f=f^2\colon A \to A$ and a nilpotent map $g\colon A \to A$ such that $fg=gf$ let $R=\operatorname{Hom}_{\mathcal{A}}(A, A)$ be the endomorphism ring of A in $\mathcal{A}$, and define ring morphisms

$$\begin{aligned} R[[z]] \to R; &\qquad z \to q = p(1-p) = (1-2f)\,g-((1-2f)\,g)^2, \\ R[[w]] \to R; &\qquad w \to (1-2f)\,g \end{aligned}$$

compatible with $w \to z-z^2$. The binomial expansions of $(1-4z)^{-1/2}$ and $(1-2w)^{-1}$ are sent to the same element of R, namely

$$(1-4q)^{-1/2} = (1-2(1-2f)\,g)^{-1}\colon A \to A,$$

so that

$$p_\omega = p+(1/2)(2p-1)((1-4q)^{-1/2}-1) = f\colon A \to A. \quad \blacksquare$$

For any endomorphism $e\colon A \to A$ in $\mathcal{A}$ define an endomorphism in $\mathcal{B}$

$$i(e) = \begin{pmatrix} 1-e & 0 & 0 & \cdots \\ e & 1-e & 0 & \cdots \\ 0 & e & 1-e & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\colon \sum_0^\infty A = A \oplus A \oplus A \oplus \cdots \to \sum_0^\infty A$$
$$= A \oplus A \oplus A \oplus \cdots.$$

PROPOSITION 5.5. (i) *If e is nilpotent then $i(e)$ is an automorphism.*

(ii) *If $1-e$ is nilpotent there is defined a direct sum system in $\mathcal{B}$*

$$\sum_0^\infty A \underset{j}{\overset{i(e)}{\rightleftarrows}} \sum_0^\infty A \underset{f}{\overset{g}{\rightleftarrows}} A.$$

Proof. (i) $i(e)$ is an automorphism for nilpotent e since the difference

$$1-i(e)\colon \sum_0^\infty A \to \sum_0^\infty A$$

is nilpotent.

(ii) Define a morphism in $\mathcal{B}$

$$f = e \oplus 0 \oplus 0 \cdots\colon A \to \sum_0^\infty A = A \oplus A \oplus A \oplus \cdots$$

and an isomorphism in $\mathscr{B}$

$$h\colon \sum_0^\infty A \to A \oplus \sum_0^\infty A; \qquad (x_0, x_1, x_2, \ldots) \to (x_0, (x_1, x_2, \ldots)).$$

If $1-e$ is nilpotent then

$$(f i(e))\colon A \oplus \sum_0^\infty A \to \sum_0^\infty A$$

is an isomorphism, since the difference

$$1-(f i(e))h\colon \sum_0^\infty A \to \sum_0^\infty A$$

is nilpotent. ∎

For a projection (A, p), $i(p)$ is one of the maps in the Eilenberg–Freyd direct sum system of Section 1, which we shall now generalize to near-projections.

PROPOSITION 5.6. *For any near-projection (A, p) in $\mathscr{A}$ there is defined a direct sum system in $\hat{\mathscr{B}}$*

$$\left(\sum_0^\infty A, 1\right) \underset{j}{\overset{i(p)}{\rightleftarrows}} \left(\sum_0^\infty A, 1\right) \underset{f}{\overset{g}{\rightleftarrows}} (A, p_\omega).$$

Proof. The near-projection $p\colon A \to A$ splits in the idempotent completion $\hat{\mathscr{A}}$ as a direct sum

$$pp_\omega \oplus p(1-p_\omega)\colon (A, p_\omega) \oplus (A, 1-p_\omega) \to (A, p_\omega) \oplus (A, 1-p_\omega),$$

and similarly for $i(p) = i(pp_\omega) \oplus i(p(1-p_\omega))$. The endomorphisms

$$1_{(A, p_\omega)} - pp_\omega = p_\omega(p_\omega - p)\colon (A, p_\omega) \to (A, p_\omega),$$
$$p(1-p_\omega) = p(1-p) + p(p-p_\omega)\colon (A, 1-p_\omega) \to (A, 1-p_\omega)$$

are nilpotent. By Proposition 5.5(i), $i(p(1-p_\omega))$ is an automorphism, and by Proposition 5.5(ii) there is defined a direct sum system in $\hat{\mathscr{B}}$

$$\left(\sum_0^\infty A, 1\right) \underset{j}{\overset{i(p)}{\rightleftarrows}} \left(\sum_0^\infty A, 1\right) \underset{f}{\overset{g}{\rightleftarrows}} (A, p_\omega). \quad ∎$$

EXAMPLE 5.7 (Bass, Heller, and Swan [2]). Let A be a ring, with polynomial extension $A[z]$ and Laurent polynomial extension $A[z, z^{-1}]$.

Let $e\colon A^m \to A^m$ be an endomorphism of a f.g. free A-module. The endomorphism

$$1-e+ze\colon A[z]^m \to A[z]^m$$

of the induced f.g. free $A[z]$-module is an automorphism if and only if e is nilpotent, with inverse

$$(1-e+ze)^{-1} = 1-(z-1)e+(z-1)^2e^2+\cdots\colon A[z]^m \to A[z]^m.$$

The endomorphism of the induced $A[z, z^{-1}]$-module

$$1-e+ze\colon A[z, z^{-1}]^m \to A[z, z^{-1}]^m$$

is an automorphism if and only if e is a near-projection. If e is a near-projection the f.g. projective A-modules

$$P_+ = \operatorname{im}(1-e_\omega\colon A^m \to A^m), \qquad P_- = \operatorname{im}(e_\omega\colon A^m \to A^m)$$

are such that $P_+ \oplus P_- = A^m$ and

$$1-e+ze = \begin{pmatrix} 1-v_+ + zv_+ & 0 \\ 0 & v_- + z(1-v_-) \end{pmatrix}\colon A[z, z^{-1}]^m$$
$$= (P_+ \oplus P_-)[z, z^{-1}] \to A[z, z^{-1}]^m = (P_+ \oplus P_-)[z, z^{-1}]$$

with $v_+\colon P_+ \to P_+$, $v_-\colon P_- \to P_-$ nilpotent endomorphisms. Indeed, the proof of Proposition 5.6 above relies on the abstract version of this decomposition.

EXAMPLE 5.8 (Bass [1, III.2.10], Swan [22, 2.17]). Let A be a ring and let I be a nilpotent two-sided ideal of A, such that $I^N = 0$ for some integer $N \geqslant 1$. Given a f.g. projective A/I-module

$$P = \operatorname{im}(p = p^2\colon (A/I)^m \to (A/I)^m)$$

let $\tilde{p}\colon A^m \to A^m$ be a lift of the projection p. Then $(A^m, \tilde{p})$ is a near-projection with defect of exponent N, and there is defined a lift of p to a projection

$$\tilde{p}_\omega = (\tilde{p}^N + (1-\tilde{p})^N)^{-1}\tilde{p}^N\colon A^m \to A^m$$

such that

$$1 \otimes \tilde{p}_\omega = p\colon A/I \otimes_A A^m = (A/I)^m \to (A/I)^m.$$

The abelian group morphisms

$$K_0(A) \to K_0(A/I); \qquad [M] \to [A/I \otimes_A M] = [M/IM],$$
$$K_0(A/I) \to K_0(A); \qquad [\mathrm{im}(p)] \to [\mathrm{im}(\tilde{p}_\omega)]$$

are inverse isomorphisms.

EXAMPLE 5.9. For a near-projection (A, p) with defect $q = p(1-p)$ of exponent 2 the projection (A, p_ω) is given by

$$p_\omega = 3p^2 - 2p^3 \colon A \to A.$$

See Munkholm and Ranicki [16] for an application of this to the transfer map in the algebraic K_0-groups induced by an S^1-bundle.

EXAMPLE 5.10. See Lam, Ranicki, and Smith [12] for an application of near-projections to the Jordan normal form.

6. THE INSTANT PROJECTIVE CLASS

Given a finite chain homotopy projection (D, p) in $\mathcal{A}$ we obtain an explicit representative of the projective class $[D, p] \in K_0(\hat{\mathcal{A}})$, using the instant finiteness obstruction.

PROPOSITION 6.1 (Ranicki [18]). *An $\mathcal{A}$-finite domination (D, f, g) of a chain complex C in $\mathcal{B}$ and a choice of chain homotopy $h\colon gf \simeq 1\colon C \to C$ determine the* instant finiteness obstruction *projection*

$$X = \begin{pmatrix} fg & -d & 0 & \cdots \\ -fhg & 1-fg & d & \cdots \\ -fh^2g & fhg & fg & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \colon D_\omega = D_0 \oplus D_1 \oplus D_2 \oplus \cdots \to D_\omega$$
$$= D_0 \oplus D_1 \oplus D_2 \oplus \cdots$$

such that

$$[C] = [D_\omega, X] - [D_{\text{odd}}] \in K_0(\hat{\mathcal{A}}),$$

with $D_{\text{odd}} = D_1 \oplus D_3 \oplus D_5 \oplus \cdots$.

Given an $\mathcal{A}$-finite chain homotopy projection (D, p) and a chain homotopy $q\colon p^2 \simeq p\colon D \to D$ there is defined a particular splitting (C, f, g) of (D, p) as in Remark 2.5, so that the projective class $[D, p] \in K_0(\hat{\mathcal{A}})$ is represented by the instant finiteness obstruction obtained from the domination (D, f, g) of C, as in Proposition 6.1. This procedure is somewhat cumbersome, and can be shortened by means of the theory of Section 5, as follows.

PROPOSITION 6.2. *The projective class of an $\mathcal{A}$-finite chain homotopy projection (D, p) is such that*

$$[D, p] = [D_\omega, X_\omega] - [D_{\text{odd}}] \in K_0(\hat{\mathcal{A}}),$$

with

$$X_\omega = (X^N + (1-X)^N)^{-1} X^N \colon D_\omega \to D_\omega$$

the projection defined for any exponent N of the instant near-projection *$X\colon D_\omega \to D_\omega$, given for any chain homotopy $q\colon p^2 \simeq p\colon D \to D$ by*

$$X = \begin{pmatrix} p & -d & 0 & \cdots \\ -q & 1-p & d & \cdots \\ 0 & q & p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \colon D_\omega = D_0 \oplus D_1 \oplus D_2 \oplus \cdots \to D_\omega$$
$$= D_0 \oplus D_1 \oplus D_2 \oplus \cdots.$$

Proof. Define

$$s = pq - qp \colon D_r \to D_{r+1}.$$

The defect of the near-projection X

$$Y = X(1-X) = \begin{pmatrix} p & -d & 0 & \cdots \\ -q & 1-p & d & \cdots \\ 0 & q & p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} 1-p & d & 0 & \cdots \\ q & p & -d & \cdots \\ 0 & -q & 1-p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & \cdots \\ -s & 0 & 0 & \cdots \\ q^2 & -s & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \colon D_\omega = D_0 \oplus D_1 \oplus D_2 \oplus \cdots \to D_\omega$$
$$= D_0 \oplus D_1 \oplus D_2 \oplus \cdots$$

is nilpotent because the matrix is lower triangular, with exponent N such that $D_r = 0$ for $r \geqslant N$. The projection given by Proposition 5.4

$$X_\omega = (X^N + (1-X)^N)^{-1} X^N \colon D_\omega \to D_\omega$$

is such that

$$X_\omega - X = (1-2X)(Y+3Y^2+10Y^3+\cdots)$$
$$= \begin{pmatrix} 1-2p & 2d & 0 & \cdots \\ 2q & 2p-1 & -2d & \cdots \\ 0 & -2q & 1-2p & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots \\ -s & 0 & 0 & \cdots \\ q^2+3s^2 & -s & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$
$$= \begin{pmatrix} -2ds & 0 & 0 & \cdots \\ * & 2ds & 0 & \cdots \\ * & * & -2ds & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : D_\omega = D_0 \oplus D_1 \oplus D_2 \oplus \cdots \to D_\omega$$
$$= D_0 \oplus D_1 \oplus D_2 \oplus \cdots.$$

The diagonal entries are nilpotent of exponent 2 since $d^2 = 0$ and

$$ds + sd = p(qd+dq) - (qd+dq)p$$
$$= p(p-p^2) - (p-p^2)p = 0: D_r \to D_r.$$

By Remark 2.5 there exists a splitting (C, f, g) of (D, p) such that

$$fg = p: D \to D.$$

Choose a chain homotopy $h: gf \simeq 1: C \to C$. The instant finiteness obstruction projection

$$X' = \begin{pmatrix} fg & -d & 0 & \cdots \\ -fhg & 1-fg & d & \cdots \\ -fh^2g & fhg & fg & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : D_\omega = D_0 \oplus D_1 \oplus D_2 \oplus \cdots \to D_\omega$$
$$= D_0 \oplus D_1 \oplus D_2 \oplus \cdots$$

is such that

$$X - X' = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ * & 0 & 0 & \cdots \\ * & * & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : D_\omega = D_0 \oplus D_1 \oplus D_2 \oplus \cdots \to D_\omega$$
$$= D_0 \oplus D_1 \oplus D_2 \oplus \cdots.$$

The difference of the projections $X_\omega, X': D_\omega \to D_\omega$

$$X_\omega - X' = (X_\omega - X) + (X - X')$$
$$= \begin{pmatrix} -2ds & 0 & 0 & \cdots \\ * & 2ds & 0 & \cdots \\ * & * & -2ds & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} : D_\omega = D_0 \oplus D_1 \oplus D_2 \oplus \cdots \to D_\omega$$
$$= D_0 \oplus D_1 \oplus D_2 \oplus \cdots$$

is nilpotent and by Propositions 5.1, 6.1

$$[D, p] = [C] = [D_\omega, X'] - [D_{\text{odd}}] = [D_\omega, X_\omega] - [D_{\text{odd}}] \in K_0(\hat{\mathcal{A}}). \quad \blacksquare$$

EXAMPLE 6.3. Let (D, p) be a chain homotopy projection of a finite-dimensional f.g. projective A-module chain complex D, for some ring A, and let $q: p^2 \simeq p: D \to D$. The instant near-projection $X: D_\omega \to D_\omega$ and the associated projection $X_\omega: D_\omega \to D_\omega$ are such that

$$[D, p] = [\text{im}(X_\omega: D_\omega \to D_\omega)] - [D_{\text{odd}}] \in K_0(A),$$

by a direct application of Proposition 6.2 in the case

$$(\mathcal{B}, \mathcal{A}) = (\{\text{projective } A\text{-modules}\}, \{\text{f.g. projective } A\text{-modules}\}).$$

7. TORSION

We express the projective class of a chain homotopy projection as the torsion of a chain equivalence.

Given an additive category $\mathcal{A}$ define the *Laurent extension* $\mathcal{A}[z, z^{-1}]$ to be the additive category with one object $A[z, z^{-1}]$ for each object A in $\mathcal{A}$, and with morphisms

$$f = \sum_{j=-\infty}^{\infty} z^j f_j : A[z, z^{-1}] \to B[z, z^{-1}]$$

defined by Laurent polynomials with coefficients morphisms $f_j: A \to B$ in $\mathcal{A}$ such that $\{j \in \mathbb{Z} \mid f_j \neq 0\}$ is finite. The embedding

$$\mathcal{A} \to \mathcal{A}[z, z^{-1}]; \qquad A \to A[z, z^{-1}], \qquad f \to f_0 = f$$

identifies $\mathcal{A}$ with a subcategory of $\mathcal{A}[z, z^{-1}]$.

EXAMPLE 7.1. The Laurent extension $\mathcal{A}[z, z^{-1}]$ of the additive category $\mathcal{A} = \{\text{based f.g. free } A\text{-modules}\}$ for a ring A is such that there is defined a natural isomorphism of additive categories

$$\mathcal{A}[z, z^{-1}] \to \{\text{based f.g. free } A[z, z^{-1}]\text{-modules}\},$$

with

$$K_1(\mathcal{A}) = K_1(A), \qquad K_0(\hat{\mathcal{A}}) = K_0(A),$$
$$K_1(\mathcal{A}[z, z^{-1}]) = K_1(A[z, z^{-1}]).$$

The natural direct sum decomposition of Bass [1] for the torsion group of the Laurent polynomial extension $A[z, z^{-1}]$ of any ring A

$$K_1(A[z, z^{-1}]) = K_1(A) \oplus K_0(A) \oplus Nil_0(A) \oplus Nil_0(A)$$

involved the split injection

$$\bar{B}: K_0(A) \to K_1(A[z, z^{-1}]);$$
$$[P] \to \tau(z: P[z, z^{-1}] \to P[z, z^{-1}]).$$

In Ranicki [19] it was pointed out that the natural direct sum decomposition involving the split injection

$$\bar{B}': K_0(A) \to K_1(A[z, z^{-1}]);$$
$$[P] \to \tau(-z: P[z, z^{-1}] \to P[z, z^{-1}])$$

was more gemetrically significant. If $P = \mathrm{im}(p = p^2: A^r \to A^r)$ then

$$\bar{B}([P]) = \tau(1 - p + pz: A[z, z^{-1}]^r \to A[z, z^{-1}]^r) \in K_1(A[z, z^{-1}]),$$
$$\bar{B}'([P]) = \tau(1 - p - pz: A[z, z^{-1}]^r \to A[z, z^{-1}]^r) \in K_1(A[z, z^{-1}]).$$

In Ranicki [21] it is shown that the torsion group of the Laurent extension $\mathcal{A}[z, z^{-1}]$ of any additive category $\mathcal{A}$ is such that there are defined two natural direct sum decompositions of the type

$$K_1(\mathcal{A}[z, z^{-1}]) = K_1(\mathcal{A}) \oplus K_0(\hat{\mathcal{A}}) \oplus Nil_0(\mathcal{A}) \oplus Nil_0(\mathcal{A}),$$

involving the split injections

$$\bar{B}: K_0(\hat{\mathcal{A}}) \to K_1(\mathcal{A}[z, z^{-1}]);$$
$$[A, p] \to \tau(1 - p + pz: A[z, z^{-1}] \to A[z, z^{-1}])$$
$$\bar{B}': K_0(\hat{\mathcal{A}}) \to K_1(\mathcal{A}[z, z^{-1}]);$$
$$[A, p] \to \tau(1 - p - pz: A[z, z^{-1}] \to A[z, z^{-1}]),$$

using the split exact structures on $\mathcal{A}$ and $\mathcal{A}[z, z^{-1}]$ to define the torsion groups.

PROPOSITION 7.2. *An $\mathcal{A}$-finitely dominated chain homotopy projection (D, p) in $\mathcal{B}$ determines self chain equivalences $1 - p \pm zp: D[z, z^{-1}] \to D[z, z^{-1}]$ of the $\mathcal{A}[z, z^{-1}]$-finitely dominated chain complex $D[z, z^{-1}]$ in $\mathcal{B}[z, z^{-1}]$ such that*

$$\bar{B}([D, p]) = \tau(1 - p + zp: D[z, z^{-1}] \to D[z, z^{-1}]) \in K_1(\mathcal{A}[z, z^{-1}]),$$
$$\bar{B}'([D, p]) = \tau(1 - p - zp: D[z, z^{-1}] \to D[z, z^{-1}]) \in K_1(\mathcal{A}[z, z^{-1}]).$$

Proof. Given a domination (D', f, g) of D by an $\mathcal{A}$-finite chain complex D' define an $\mathcal{A}$-finite chain homotopy projection $(D', p' = fpg)$ such that

$$[D, p] = [D', p'] \in K_0(\hat{\mathcal{A}}),$$
$$\tau(1 - p \pm zp: D[z, z^{-1}] \to D[z, z^{-1}])$$
$$= \tau(1 - p' \pm zp': D'[z, z^{-1}] \to D'[z, z^{-1}]) \in K_1(\mathcal{A}[z, z^{-1}]).$$

As in Ranicki [18] it is possible to replace (D', p') by a chain equivalent $\mathcal{A}$-finite (D'', p'') with $p''^2 = p'': D'' \to D''$ an actual projection, such that

$$[D', p'] = [D'', p''] = \sum_{r=0}^{\infty} (-)^r [D''_r, p''] \in K_0(\hat{\mathcal{A}}),$$

$$\tau(1 - p' \pm zp': D'[z, z^{-1}] \to D'[z, z^{-1}])$$
$$= \tau(1 - p'' \pm zp'': D''[z, z^{-1}] \to D''[z, z^{-1}])$$
$$= \sum_{r=0}^{\infty} (-)^r \tau(1 - p'' \pm zp'': D''_r[z, z^{-1}] \to D''_r[z, z^{-1}]) \in K_1(\mathcal{A}[z, z^{-1}]).$$

∎

Remark 7.3. A linear automorphism $f = f_0 + zf_1: A[z, z^{-1}] \to A[z, z^{-1}]$ in $\mathcal{A}[z, z^{-1}]$ determines a near-projection

$$p = (f_0 + f_1)^{-1} f_1: A \to A$$

(cf. Example 5.7) and hence a projection $p_\omega: A \to A$. The torsions $\tau(f) \in K_1(\mathcal{A}[z, z^{-1}])$ of linear automorphisms $f = f_0 + zf_1$ generate $K_1(\mathcal{A}[z, z^{-1}])$, by the Higman linearization trick. Both the injections $\bar{B}, \bar{B}': K_0(\hat{\mathcal{A}}) \to K_1(\mathcal{A}[z, z^{-1}])$ are split by the projection

$$B: K_1(\mathcal{A}[z, z^{-1}]) \to K_0(\hat{\mathcal{A}});$$
$$\tau(f_0 + zf_1: A[z, z^{-1}] \to A[z, z^{-1}]) \to [A, p_\omega].$$

See Ranicki [21] for the details and the generalization to chain complexes.

Define the *mapping torus* of a map $f: X \to X$ in the usual way by

$$T(f) = X \times [0, 1]/\{(x, 0) = (f(x), 1) \mid x \in X\}.$$

Ferry [4] defined a geometric split injection

$$\tilde{K}_0(\mathbb{Z}[\pi]) \to Wh(\pi \times \mathbb{Z});$$

$$[X] \to \phi_* \tau(\phi^{-1}(1_X \times -1)\phi: T(fg) \to T(fg))$$

by sending the Wall finiteness obstruction

$$[X] = [C(\tilde{X})] \in \tilde{K}_0(\mathbb{Z}[\pi])$$

of a space X with $\pi_1(X) = \pi$ and with a domination (Y, f, g) by a finite CW complex Y to the Whitehead torsion of the self homotopy equivalence of a finite CW complex

$$\phi^{-1}(1_X \times -1)\phi: T(fg) \xrightarrow{\phi} X \times S^1 \xrightarrow{1_X \times -1} X \times S^1 \xrightarrow{\phi^{-1}} T(fg),$$

with $\phi: T(fg) \to T(gf) \simeq T(1_X) = X \times S^1$ the homotopy equivalence of Mather [15]. In Ranicki [19, 20] this was identified with the geometrically significant split injection

$$\bar{B}': \tilde{K}_0(\mathbb{Z}[\pi]) \to Wh(\pi \times \mathbb{Z});$$

$$[P] \to \tau(-z: P[z, z^{-1}] \to P[z, z^{-1}]).$$

Propositions 4.6, 7.2 show that the geometrically significant split injection sends the finiteness obstruction $[Y, p] \in \tilde{K}_0(\mathbb{Z}[\pi_1(Y)])$ of a finitely dominated pointed homotopy idempotent (Y, p) to the Whitehead torsion

$$\bar{B}'([Y, p]) = (f \times 1_{S^1})_* \phi_* \tau(\phi^{-1}(1_X \times -1)\phi: T(p) \to T(p))$$
$$\in Wh(\pi_1(Y \times S^1)) = Wh(\pi_1(Y) \times \mathbb{Z})$$

for any splitting (X, f, g) of (Y, p), with $\phi: T(p) \simeq T(fg) \to T(gf) \simeq X \times S^1$ the homotopy equivalence of Mather [15].

References

1. H. Bass, "Algebraic K-theory," Benjamin, New York, 1968.
2. H. Bass A. Heller, and R. Swan, The Whitehead group of a polynomial extension, *Publ. Math. I.H.E.S.* **22** (1964), 61–80.
3. J. Dydak, A simple proof that pointed, connected FANR-spaces are regular fundamental retracts of ANR's, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys.* **25** (1977), 55–62.

4. S. Ferry, A simple-homotopy approach to the finiteness obstruction, *in* "Proceedings, 1981 Dubrovnik Conf. on Shape Theory," pp. 73–81, Lecture Notes in Mathematics, Vol. 870, Springer-Verlag, New York/Berlin, 1981.
5. S. Ferry and A. Ranicki, in preparation.
6. P. Freyd, Splitting homotopy idempotents, *in* "Proceedings, 1965 La Jolla Conference on Categorical Algebra," pp. 173–176, Springer-Verlag, New York/Berlin, 1966.
7. P. Freyd and A. Heller, Splitting homotopy idempotents, to appear.
8. H. Hastings and A. Heller, Splitting homotopy idempotents, *in* "Proceedings, 1981 Dubrovnik Conf. on Shape Theory," pp. 23–36, Lecture Notes in Mathematics, Vol. 870, Springer-Verlag, New York/Berlin, 1981.
9. H. Hastings and A. Heller, Splitting homotopy idempotents of finite-dimensional CW complexes, *Proc. Amer. Math. Soc.* **85** (1982), 619–622.
10. N. Jacobson, "The Structure of Rings," Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence, RI, 1956.
11. T. Kato, "Perturbation Theory of Linear Operators," Springer, New York, 1966.
12. K. Y. Lam, A. Ranicki, and L. Smith, Jordan normal form projections, *Arch. Math.* **50** (1988), 113–117.
13. W. Lück, "Eine allgemeine algebraische Beschreibung des Transfers für Faserungen auf projektiven Klassengruppen und Whiteheadgruppen," Göttingen Ph.D. thesis, 1984.
14. W. Lück, The transfer maps induced in the algebraic K_0- and K_1-groups by a fibration, I, *Math. Scand.* **59** (1986), 93–121.
15. M. Mather, Counting homotopy types of manifolds, *Topology* **4** (1965), 93–94.
16. H. Munkholm and A. Ranicki, The projective class group transfer induced by an S^1-bundle, *in* "Proceedings, 1981 London, Ontario Topology Conference, Canadian Math. Conf. Proc. 2, part 2, 1982," pp. 461–484.
17. E. K. Pedersen, K_{-i}-invariants of chain complexes, *in* "Proceedings, 1982 Leningrad Topology Conference," pp. 174–186, Lecture Notes in Mathematics, Vol. 1060, Springer-Verlag, New York/Berlin, 1984.
18. A. Ranicki, The algebraic theory of finiteness obstruction, *Math. Scand.* **57** (1985), 105–126.
19. A. Ranicki, The algebraic and geometric splittings of the K- and L-groups of polynomial extensions, *in* "Proceedings, 1985 Poznań Conf. on Transformation Groups," pp. 321–364, Lecture Notes in Mathematics, Vol. 1217, Springer-Verlag, New York/Berlin, 1986.
20. A. Ranicki, The algebraic theory of torsion. II. Products, *K-theory* **1** (1987), 115–170.
21. A. Ranicki, Lower K- and L-theory, to appear.
22. R. G. Swan, "Algebraic K-theory," Lecture Notes in Mathematics, Vol. 76, Springer-Verlag, New York/Berlin, 1968.
23. J. H. M. Wedderburn, "Lectures on Matrices," Amer. Math. Soc. Colloq. Publ., Vol. XVII, Amer. Math. Soc., Providence, RI, 1934.