TWO COMPLEXES WHICH ARE HOMEO MORPHIC BUT
COMBINATORIALLY DISTINCT

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Let $L_q$ denote the 3-dimensional lens manifold of type $(7, q)$, suitably
triangulated (see § 1), and let $\sigma^n$ denote an $n$-simplex. A finite simplicial
complex $X_q$ is obtained from the product $L_q \times \sigma^n$ by adjoining a cone over
the boundary $L_q \times \partial \sigma^n$. The dimension of $X_q$ is $n + 3$.

THEOREM 1. For $n + 3 \geq 6$ the complex $X_1$ is homeomorphic to $X_7$.

THEOREM 2. No finite cell subdivision of the simplicial complex $X_1$ is
isomorphic to a cell subdivision of $X_7$. In particular there is no piece-wise
linear homeomorphism from $X_1$ to $X_7$.

The proof of Theorem 1 will be based on a recent result of B. Mazur.
For the special case $n = 3$ (which is somewhat more difficult) the proof
will make use of theorems of A. Haefliger and J. Stallings.

The proof of Theorem 2 will be based on the concept of "torsion" as
defined by Reidemeister, Franz, and de Rham.

These two theorems show that the Hauptvermutung\(^2\) for simplicial com-
plexes of dimension $\geq 6$ is false. On the other hand Papakyriakopoulos
[10] has proved the Hauptvermutung for complexes of dimension $\leq 2$.

The Hauptvermutung for manifolds remains open. However Moise [8]
has proved the Hauptvermutung for manifolds of dimension $\leq 3$; and
Smale [13] has proved it for triangulations of the sphere $S^n$, $n \neq 4, 5, 7$,
which look locally like the usual triangulation. A weak form of the
Hauptvermutung for cells and spheres has been proved by Gluck [4].

As bi-products of the argument, two other curious phenomena appear.

The symbols 

$$S^{n-1} \subset D^n \subset R^n$$

will always denote the unit sphere bounding the unit disk in euclidean
$n$-space.

THEOREM 3. The manifold-with-boundary $L_1 \times D^6$ is not diffeomor-
phic to $L_7 \times D^6$. However the interiors of these two manifolds are
diffeomorphic.

\(^1\) The author wishes to thank the Sloan Foundation for its support.

\(^2\) See, for example, Alexandroff and Hopf [1, p. 152]. I do not know who originated the
term "Hauptvermutung". The problem was clearly formulated by Tietze [18, pp. 13-14] in
1908. See also Steinitz [15, p. 23].
Two closed manifolds \( M_1 \) and \( M_2 \) will be called \( h \)-cobordant (ignoring orientation) if their disjoint sum \( M_1 + M_2 \) bounds a compact differentiable manifold \( W \) such that both \( M_1 \) and \( M_2 \) are deformation retracts of \( W \). (The term "\( J \)-equivalent" has previously been used for this concept. Compare Thom [17], Smale [13].)

**Theorem 4.** The manifold \( L_1 \times S^1 \) is \( h \)-cobordant to \( L_2 \times S^1 \); but these two manifolds are not diffeomorphic.

1. **Mazur's theorem and lens manifolds**

Let \( M_1 \) and \( M_2 \) be two closed differentiable manifolds of dimension \( k \) which are parallelizable\(^8\) and have the same homotopy type.

**Theorem of Mazur [6].** If \( n > k \) then \( M_1 \times R^n \) is diffeomorphic to \( M_2 \times R^n \).

An outline of the proof is given in §2.

The *lens manifold* \( L = L(p, q) \) can be constructed as follows. Let \( p > q \) be relatively prime positive integers. Identify \( S^3 \) with the unit sphere in the complex plane, consisting of all \((z_1, z_2)\) with \( |z_1|^2 + |z_2|^2 = 1 \). Let \( \omega \) denote the complex number \( \exp(2\pi i/p) \). Then the cyclic group \( \Pi \) of order \( p \) acts differentiably on \( S^3 \) without fixed points by the rule

\[
T(z_1, z_2) = (\omega z_1, \omega^q z_2),
\]

where \( T \) denotes a generator of \( \Pi \). The quotient manifold \( S^3/\Pi \) is the required lens manifold.

This manifold \( L \) can be considered as a CW-complex with only four cells, namely the images \( \tilde{e}_m \) in \( L \) of:

1. The point \( e_0 = (1, 0) \),
2. The set \( e_1 \) of \( (e^{i\theta}, 0) \),
3. The set \( e_2 \) of \( (z_1, \sqrt{1 - |z_1|^2}) \), and
4. The set \( e_3 \) of \( (z_1, e^{i\theta} \sqrt{1 - |z_1|^2}) \);

where \( 0 < \theta < 2\pi/p \) and \( |z_1| < 1 \). (Compare de Rham [12].)

Alternatively \( L \) can be considered as a simplicial complex. Here is an example of a triangulation of \( L \) which is compatible both with the above cell subdivision and with the differentiable structure. Consider the convex polyhedron \( P \) spanned by the \( 2p \) points \((\omega^i, 0)\) and \((0, \omega^x)\) in the complex plane. The boundary \( \partial P \) is a simplicial complex which is homeomorphic

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\(^8\)Instead of parallelizability, it suffices to assume that the stable tangent bundles of \( M_1 \) and \( M_2 \) are compatible under some homotopy equivalence \( M_1 \to M_2 \).
to $S^3$ under radial projection from the origin. Taking two successive barycentric subdivisions of $\partial P$, and then collapsing under the action of $\Pi$, we obtain the required simplicial complex.

These complexes were discovered by Tietze [18, p. 110] in 1908. Tietze computed the fundamental group

$$\pi_1(L) \cong \Pi$$

and the homology of $L$. In particular he showed that the integer $p$ is a topological invariant of $L = L(p, q)$.

In 1935 Reidemeister [11] classified the lens manifolds combinatorially. He showed that $L(p, q)$ is combinatorially equivalent to $L(p, q')$ if and only if either

$$q' \equiv \pm q \text{ or } \pm qq' \equiv 1 \pmod{p}.$$

(According to Moise [8] or Brody [2] two lens manifolds are homeomorphic if and only if they are combinatorially equivalent. This fact will not be needed in the present paper.)

In 1941 J. H. C. Whitehead [20] classified the lens manifolds up to homotopy type. (For a more recent version see Olum [9].) Whitehead showed that $L(p, q)$ has the homotopy type of $L(p, q')$ if and only if $\pm qq'$ is a quadratic residue modulo $p$. As an example, for $p = 7$, we obtain two distinct combinatorial manifolds $L(7, 1)$ and $L(7, 2)$; but only one homotopy type; since $1 \cdot 2 \equiv 3^2$ is a quadratic residue modulo 7.

All lens manifolds are parallelizable. This follows from the theorem of Stiefel [16] and Whitney that all orientable 3-manifolds are parallelizable. (For $p$ odd the proof is quite easy since the obstructions to parallelizability lie in groups $H^n(L; \pi_m - (SO_3))$ which are zero.)

Hence we can apply Mazur's theorem and conclude that:

**Lemma 1.** If $\pm qq'$ is a quadratic residue modulo $p$, and if $n > 3$, then $L(p, q) \times R^n$ is diffeomorphic to $L(p, q') \times R^n$.

**Proof of Theorem 1** for $n > 3$. Recall the definition:

$$X_q = L_q \times \sigma^n \cup \text{Cone } (L_q \times \partial \sigma^n),$$

where $L_q = L(7, q)$. Let $x_0$ denote the vertex of the cone. The complement $X_q - x_0$ is homeomorphic to the product $L_q \times R^n$. In fact a specific homeomorphism $f: X_q - x_0 \to L_q \times R^n$ can be given as follows. Let $h: \sigma^n \to D^n$ be a homeomorphism, and define

$$f(y, z) = (y, h(z))$$

$$f(t(y, z') + (1 - t)x_0) = (y, h(z')/t)$$

for $y \in L_q$, $z \in \sigma^n$, $z' \in \partial \sigma^n$, and $0 < t \leq 1$. 

Therefore $X_q$ is homeomorphic to the single point compactification of $L_q \times \mathbb{R}^n$. Using Lemma 1, this implies that $X_1$ is homeomorphic to $X_2$; which completes the proof of Theorem 1 for $n > 3$.

2. $h$-cobordism

First let me outline a proof of Mazur’s theorem. Given a homotopy equivalence $f : M_1 \to M_2$, choose a differentiable imbedding:

$$f' : M_1 \to \text{Interior } (M_2 \times D^n)$$

which approximates the function $x \to (f(x), 0)$. This is certainly possible if $n$ is greater than the dimension $k$ of $M_k$. Since both $M_1$ and $M_3$ are parallelizable, it follows that the normal bundle of $f'(M_1)$ is trivial providing that $n > k$. (See for example Milnor [7, Lemma 5].) Thus a tubular neighborhood of $f'(M_1)$ in $\text{Interior } (M_2 \times D^n)$ is diffeomorphic to $M_1 \times D^n$.

This gives an imbedding $i : M_1 \times D^n \to M_2 \times D^n$. Similarly, using a homotopy inverse to $f$, one obtains an imbedding $j : M_2 \times D^n \to M_1 \times D^n$. The main step in the proof is now the following.

**Lemma 1.** If $n > k > 1$ then any imbedding

$$h : M_1 \times D^n \to \text{Interior } M_1 \times D^n$$

which is homotopic to the identity can be extended to a diffeomorphism of the pair $(M_1 \times 2D^n, M_1 \times D^n)$ onto the pair $(M_1 \times D^n, h(M_1 \times D^n))$. In particular this applies to the imbedding $h = ji$.

Here $2D^n$ denotes the disk of radius 2. The key step in the proof is to show that $h$ restricted to $M_1 \times 0$ is differentiably isotopic to the standard inclusion map $M_1 \times 0 \to M_1 \times D^n$. For $n > k + 1$, this follows from a well known theorem of Whitney [23]. For the case $n = k + 1 > 2$, it follows from a recent theorem of A. Haefliger [5].

Now consider the infinite direct sequence

$$M_1 \times D^n \xrightarrow{i} M_2 \times D^n \xrightarrow{j} M_1 \times D^n \xrightarrow{i} \cdots .$$

The “limit” or “union” of this sequence is non-compact manifold $V$. Using the lemma it is seen that $V$ is diffeomorphic to the union $M_1 \times \mathbb{R}^n$ of

$$M_1 \times D^n \subset M_1 \times 2D^n \subset M_1 \times 4D^n \subset \cdots .$$

But a similar proof shows that $V$ is diffeomorphic to $M_2 \times \mathbb{R}^n$. Hence $M_1 \times \mathbb{R}^n$ is diffeomorphic to $M_2 \times \mathbb{R}^n$. For details the reader is referred to Mazur's paper.

Now consider the region

$$W = M_2 \times D^n - \text{Interior } i(M_1 \times D^n) .$$
This is a compact differentiable manifold bounded by \( M_2 \times S^{n-1} \) and \( i(M_1 \times S^{n-1}) \).

**Lemma 2.** If \( n \geq 3 \) then both \( M_2 \times S^{n-1} \) and \( i(M_1 \times S^{n-1}) \) are deformation retracts of \( W \).

**Proof.** It will be convenient to denote the boundaries of \( W \) by \( W_2 \) and \( W_1 \) respectively. By a dimensional argument, any map of a 2-dimensional complex into \( M_2 \times D^n \) can be deformed off \( f'(M_1) \), and hence can be pushed into \( W \).

This implies that

\[
\pi_1(W) \cong \pi_1(M_2 \times D^n)
\]

and hence that

\[
\pi_1(W_q) \cong \pi_1(W) \quad \text{for } q = 1, 2.
\]

Given any system \( S \) of local coefficients on \( M_2 \times D^n \) we have

\[
H_*(W, W_1; S) \cong H_*(M_2 \times D^n, i(M_1 \times D^n); S)
\]

by excision. But \( i \) is a homotopy equivalence, hence these groups are zero. Using Whitehead [21, Theorem 3] it follows that \( W_1 \) is a deformation retract of \( W \).

The group \( H_*(W, W_2; S) \) is isomorphic by Poincaré duality to \( H^{n+k-*}(W, W_1; S) \), and therefore is zero. This implies that \( W_2 \) is a deformation retract of \( W \), which completes the proof of Lemma 2.

Thus: if \( M_1 \) and \( M_2 \) are closed parallelizable \( k \)-manifolds with the same homotopy type, and if \( n > k > 1 \), then \( M_1 \times S^{n-1} \) is \( h \)-cobordant to \( M_2 \times S^{n-1} \).

In particular this shows that \( L_1 \times S^4 \) is \( h \)-cobordant to \( L_2 \times S^4 \); which proves half of Theorem 4.

Next we will see that most of the above arguments still work for the case \( n = k = 3 \). According to Haefliger [5], any homotopy equivalence

\[
L_1 \to \text{Interior} \, (L_2 \times D^3)
\]

is homotopic to an imbedding \( f' \). The normal bundle of \( f'(L_i) \) will be trivial, since the obstructions to triviality lie in groups

\[
H^m(L_i; \pi_{m-1}(SO_3))
\]

which are zero. Hence, according to Lemma 2, both \( L_2 \times S^2 \) and \( i(L_1 \times S^2) \) are deformation retracts of the region

\[
W = L_2 \times D^3 - \text{Interior} \, i(L_1 \times D^3).
\]
Thus $L_1 \times S^2$ is h-cobordant to $L_2 \times S^2$.

According to Stallings [14, Theorem 7.4] the space $W$, with the boundary $L_2 \times S^2$ removed, is homeomorphic to $i(L_1 \times S^2) \times [0, \infty)$. Filling in the region $i(L_1 \times D^3)$ it follows that $(L_2 \times D^3) - (L_2 \times S^2)$ is homeomorphic to

$$(L_1 \times D^3) \cup (L_1 \times S^2 \times [0, \infty))$$

where the two sets are matched along the boundary $L_1 \times S^2$. Therefore $L_n \times R^3$ is homeomorphic to $L_1 \times R^3$.

It follows that $X_n$ is homeomorphic to $X_1$ for $n \geq 3$. This completes the proof of Theorem 1.

3. Torsion

This section will describe the torsion invariant of Reidemeister [11], Franz [3] and de Rham [12]. The presentation will be close to that of de Rham.

Let $\Pi$ be a discrete group which acts freely on a CW-complex $K$, and let

$$h : \Pi \rightarrow P$$

be a multiplicative homomorphism from $\Pi$ to a commutative ring $P$. If

1. the quotient complex $K/\Pi$ has only finitely many cells, and
2. the equivariant homology groups $H_i(P \otimes_\Pi C_*(K; Z))$ are all zero; then the torsion $\Delta_h(K)$, will be defined. The torsion is a unit of $P$ which is well defined up to multiplication by elements of the form $\pm h(\pi)$. We will use the notation

$$\Delta = \Delta_h(K) \in U/\pm h(\Pi),$$

where $U \subset P$ denotes the group of units. This element $\Delta$ is invariant under equivariant subdivision of $K$.

In practice $K$ is taken to be the universal covering space of a finite cell complex, and $\Pi = \pi_1(K)$ the group of covering transformations. In particular, letting $K = \tilde{L}(p, q)$ be the universal covering space of a lens manifold, and letting $P$ be the field of complex numbers, the $\Delta_h L(p, q)$ were used by Reidemeister to give the complete combinatorial classification of the lens manifolds.

The proof of Theorem 2 will be based on a more general concept of torsion in which the CW-complex $K$ is replaced by a cw-pair $(K, L)$. The group $\Pi$ must act cellularly on $K$ and freely on $K - L$. The resulting torsion
\[ \Delta_h(K, L) \in U/\pm h(\Pi) \]

is still a combinatorial invariant. That is:

**Theorem A.** If the cw-pair \((K', L')\) is a \(\Pi\)-equivariant subdivision of \((K, L)\), and if \(\Delta_h(K, L)\) is defined, then

\[ \Delta_h(K', L') = \Delta_h(K, L). \]

The proof will be given in § 4.

In this generality, the torsion is definitely not a topological invariant: it depends on the cell structure of \((K, L)\). However in the classical case, with \(L\) vacuous, it is not known whether or not \(\Delta_h(K)\) really depends on the cell structure of \(K\). (If \(K/\Pi\) is a compact differentiable manifold then \(\Delta_h(K)\) can also be defined. See [12], [19].)

For the definition of torsion, it will be convenient to assume that \(P\) is a principal ideal domain. The more general case is considered in the appendix.

**Definition.** Let \(F\) be a free \(P\)-module of finite rank \(q\). A volume \(v\) in \(F\) will mean a generator for the \(q\)th exterior power \(\wedge^q F\). If \(q > 0\), then any volume can be written in the form \(b_1 \wedge \cdots \wedge b_q\) where \(b_1, \ldots, b_q\) form a basis for \(F\). If \(q = 0\) then a volume is defined to be a unit of \(P\).

Now let \(0 \to F' \to F \to F'' \to 0\) be a short exact sequence of free, finitely generated modules. Let \(v' = b'_1 \wedge \cdots \wedge b'_r\) and \(v'' = b''_1 \wedge \cdots \wedge b''_r\) be volumes in \(F'\) and \(F''\) respectively. Then each basis element \(b''_i\) can be lifted to an element \(b'_i\) of \(F\). Thus we obtain a well defined volume

\[ v = b_1 \wedge \cdots \wedge b_r \wedge b'_1 \wedge \cdots \wedge b'_r \]

in \(F\). It is clear that any two of the volumes \(v', v, v''\) determine the third uniquely. In particular we will write

\[ v'' = v/v' \]

to indicate the dependence of \(v''\) on \(v\) and \(v'\). If \(F'\) or \(F''\) is zero then this notation, suitably interpreted, still makes sense. For example if

\[ 0 \to F' \to F \to F'' \to 0 \]

then \(v'\) and \(v\) can be considered as generators of the same module. Their ratio \(v/v'\) is a unit of \(P\).

Now consider an exact sequence

\[ 0 \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \to C_1 \xrightarrow{\partial} C_0 \to 0 \]

of free \(P\)-modules, and suppose that a volume \(v_i\) is given in each \(C_i\). Since \(P\) is assumed to be a principal ideal domain, it follows that each
submodule \( \partial C_i \subset C_{i-1} \) is free. Using the exact sequence
\[
0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \partial C_{n-1} \rightarrow 0 ,
\]
the volumes \( v_n \) and \( v_{n-1} \) give rise to a volume \( v_{n-1}/v_n \) in \( \partial C_{n-1} \). Now using the sequence
\[
0 \rightarrow \partial C_{n-1} \rightarrow C_{n-2} \rightarrow \partial C_{n-2} \rightarrow 0 ,
\]
the volumes \( v_{n-1}/v_n \) and \( v_{n-2} \) give rise to a volume
\[
v_{n-2}/(v_{n-1}/v_n)
\]
in \( \partial C_{n-2} \). Continuing by induction we obtain a volume
\[
v_1/(v_2/\cdots/(v_{n-1}/v_n))\cdots)
\]
in \( \partial C_1 = C_0 \). Comparing this with the given volume \( v_0 \) in \( C_0 \) the ratio
\[
v_0/(v_1/(v_2/\cdots/(v_{n-1}/v_n))\cdots))
\]
is a well defined unit of \( P \). The unit obtained in this way will be denoted briefly by
\[
[v_0v_1^{-1}v_2^{-1}\cdots v_n^{\pm 1}] \in U \subset P .
\]

Now consider a \( CW \)-complex \( K \) on which the group \( \Pi \) operates.

**Hypothesis 1.** \( \Pi \) permutes the cells of \( K \) freely. The quotient complex \( K/\Pi \) has only finitely many cells.

Thus the integral chain groups \( C_*(K; Z) \) can be considered as free modules of finite rank over the integral group ring \( Z\Pi \). In fact each \( i \)-cell of \( K/\Pi \) gives rise to a basis element of \( C_*(K; Z) \) which is well defined up to sign, and up to multiplication, by elements of \( \Pi \).

Using the homomorphism \( h : \Pi \rightarrow U \subset P \) we can form the chain complex
\[
C_* = P \otimes_{\Pi} C_*(K; Z)
\]
where the subscript \( \Pi \) indicates that
\[
\rho h(\pi) \otimes c - \rho \otimes \pi_*(c)
\]
is set equal to zero for each \( \rho \in P, \pi \in \Pi \), and \( c \in C_*(K; Z) \). Thus \( C_i \) is a free \( P \)-module of finite rank with one basis element for each \( i \)-cell of \( K/\Pi \). Taking the exterior product of these basis elements, we obtain a volume \( v_i \) in \( C_i \) which is well defined up to multiplication by elements of the form \( \pm h(\pi) \).

**Hypothesis 2.** The homology groups \( H_*(P \otimes_{\Pi} C_*(K; Z)) \) are all zero, so that the sequence
is exact.

Then the torsion $\Delta_h(K)$ can be defined as the residue class of
\[ [v_0v_1^{-1}v_2v_3^{-1} \cdots v_n^{\pm 1}] \in U \]
modulo the multiplicative subgroup $\pm h(\Pi)$.

The definition of torsion for a cw-pair $(K, L)$ is similar. In this case one assumes that $\Pi$ is a group of automorphisms of the pair; that $\Pi$ operates freely on the cells of $K - L$; that $(K - L)/\Pi$ has only finitely many cells; and that
\[ H_i(P \otimes_n C_\ast(K, L; Z)) = 0 \quad \text{for all } i. \]
(The group $\Pi$ is definitely allowed to have fixed points in $L$.) The torsion
\[ \Delta_h(K, L) \in U/\pm h(\Pi) \]
is defined just as above; using the chain complex $C_\ast = P \otimes_n C_\ast(K, L; Z)$.

As an example let $K$ be the 3-sphere considered as the universal covering space $\tilde{L}$ of $L(p, q)$ and let $\Pi$ be the cyclic group of covering transformations. As described in §1, $\tilde{L}$ has a $\Pi$-equivariant cell structure with $4p$ cells; so that $L(p, q) = \tilde{L}/\Pi$ has only 4 cells. Thus $C_\ast(\tilde{L}; Z)$ is a free $\mathbb{Z}_p\Pi$-module with 4 generators: $e_0, e_1, e_2$ and $e_3$. The boundary relations are easily seen to be as follows:
\[ \partial e_1 = (T - 1)e_0 \]
\[ \partial e_2 = (1 + T + T^2 + \cdots + T^{p-1})e_1 \]
\[ \partial e_3 = (T^r - 1)e_2 , \]
where $r$ is determined by the congruence $qr = 1 \pmod{p}$.

A homomorphism $h$ from $\Pi$ to the complex numbers $P$ takes the generator $T$ into some $p^{th}$ root of unity $\tau$. If $\tau \neq 1$ then
\[ 1 + \tau + \tau^2 + \cdots + \tau^{p-1} = 0 ; \]
so that the boundary relations in
\[ C_\ast = P \otimes_\Pi C_\ast(\tilde{L}; Z) \]
become
\[ \partial e_1 = (\tau - 1)e_0 , \quad \partial e_2 = 0 , \quad \partial e_3 = (\tau^r - 1)e_2 . \]
Clearly the chain complex $C_\ast$ is acyclic. The torsion
\[ \Delta_h(\tilde{L}) = [e_0e_1^{-1}e_3e_3^{-1}] \in U/\pm h(\Pi) \]
is defined; and is equal to $(\tau - 1)^{-1}(\tau^r - 1)^{-1}$. This complex number is
well defined up to multiplication by numbers of form \( \pm \tau^k \). Taking the absolute value of \( \Delta_h(\tilde{L}) \) we obtain a well defined real number \( |\Delta| \).

Applying this construction to \( L(7, 1) \) we obtain \( |\Delta| = 1.33 \) or 0.41 or 0.26 (to two decimal places) depending on the choice of \( h \). On the other hand for \( L(7, 2) \) we obtain \( |\Delta| = 0.74 \) or 0.59 or 0.33. Thus the torsion invariant distinguishes \( L(7, 1) \) from \( L(7, 2) \). Together with Theorem A, it follows that no cw-subdivision of \( L(7, 1) \) is isomorphic to a cw-subdivision of \( L(7, 2) \).

Next consider the complexes \( X_1 \) and \( X_2 \) defined in the beginning of this paper. Each \( X_q \) is a manifold except at one exceptional point \( x_0 \). Removing this point we obtain a space \( X_q - x_0 \) which is homeomorphic to \( L(7, q) \times R^n \). The fundamental group \( \Pi \) of \( X_q - x_0 \) is cyclic of order 7.

Let \( K_q \) denote the single point compactification of the universal covering space of \( X_q - x_0 \). Thus the fundamental group \( \Pi \) of \( X_q - x_0 \) operates on \( K_q \) with a single fixed point. The quotient space \( K_q/\Pi \) is equal to \( X_q \). Any cell structure on the pair \( (X_q, x_0) \) gives rise to \( \Pi \)-equivariant cell structure on \( K_q \).

The simplest cell structure on \( X_q \) has five cells: namely the four cells \( \tilde{e}_i \times R^n \) of \( L(7, q) \times R^n \approx X_q - x_0 \); together with the vertex \( x_0 \). The corresponding cell structure on \( K_q \) has 28 cells of the form \( T^r e_i \times R^n \); together with one vertex which will be denoted by \( k_0 \).

Consider the chain complex \( C_\ast(K_q, k_0; Z) \). This complex is free over the group ring \( Z\Pi \) with 4 preferred generators \( e_i \times R^n \). It is isomorphic to the chain complex \( C_\ast(\tilde{L}(7, q); Z) \) except for a shift in dimension. Hence the torsion \( \Delta_h(K_q, k_0) \) is defined and is equal to \( \Delta_h(\tilde{L}(7, q))^{\pm 1} \). (The exponent is +1 or -1 according as \( n \) is even or odd.) Therefore the torsion invariant distinguishes \( (K_q, k_0; \Pi) \) from \( (K_2, k_0; \Pi) \). It follows that no cw-subdivision of the cw-complex \( X_q \) is isomorphic to a cw-subdivision of \( X_2 \). Since the simplicial structure on \( X_q \) defined in § 1 is a subdivision of the above cell structure, this completes the proof of Theorem 2; except for the verification that torsion is invariant under subdivision (Theorem A).

4. Invariance under subdivision

The proof of Theorem A will be based on three lemmas.

First consider a commutative diagram of short exact sequences.
The $F_{i,j}$ are to be free $P$-modules of finite rank.

**Lemma 3.** Given volumes $v_{i,j}$ in $F_{i,j}$ for $i, j \leq 2$ the identity

$$(v_{22}/v_{12})(v_{21}/v_{11}) = \pm (v_{22}/v_{21})(v_{12}/v_{11})$$

is satisfied.

**Proof.** Choose a basis $\{b_1, \ldots, b_{p}, \ldots, b_{q}, \ldots, b_r, \ldots, b_s\}$ for $F_{22}$ so that $\{b_1, \ldots, b_p\}$ forms a basis for $F_{11}$, so that $\{b_1, \ldots, b_q\}$ forms a basis for $F_{12}$ and so that $\{b_1, \ldots, b_p, b_{q+1}, \ldots, b_s\}$ forms a basis for $F_{21}$ (using the same symbol for corresponding elements in different groups). Set

$$v_{11} = u_{11}b_1 \wedge \cdots \wedge b_p \quad v_{12} = u_{12}b_1 \wedge \cdots \wedge b_q$$

$$v_{21} = u_{21}b_1 \wedge \cdots \wedge b_p \wedge b_{q+1} \wedge \cdots \wedge b_r \quad v_{22} = u_{22}b_1 \wedge \cdots \wedge b_s,$$

where the $u_{i,j}$ are units. Then it is easily verified that both $(v_{22}/v_{12})(v_{21}/v_{11})$ and $(v_{22}/v_{21})(v_{12}/v_{11})$ are equal to $\pm (u_{22}u_{12}^{-1}u_{21}^{-1}u_{11})b_{r+1} \wedge \cdots \wedge b_s$. This proves Lemma 3.

**Lemma 4.** Suppose that $\Pi$ operates cellularly on a CW-triple $(K, L, M)$. Then

$$\Delta_h(K, M) = \Delta_h(K, L)\Delta_h(L, M).$$

To be more precise: if two of these three invariants are defined, then the third is also defined and equality holds.

**Proof.** If two of the three invariants are defined, then certainly $\Pi$ permutes the cells of $K - M$ freely; and $(K - M)/\Pi$ has only finitely many cells. Let

$$C'_* = P \otimes_\pi C_* (L, M; Z)$$

$$C_* = P \otimes_\pi C_* (K, M; Z)$$

$$C''_* = P \otimes_\pi C_* (K, L; Z).$$

Then there is an exact sequence

$$0 \to C'_* \to C_* \to C''_* \to 0$$
of chain mappings. Since two of these three chain complexes are acyclic, it follows that the third is also. Let $v'_i$, $v_i$, $v''_i$ denote the preferred volumes in $C'_i$, $C_i$, $C''_i$ which are determined by the preferred bases. Each of these is well defined up to multiplication by elements of the form $\pm h(\pi)$. Furthermore it is clear that

$$v_i/v'_i = \pm h(\pi)v''_i$$

for some $\pi$. Applying Lemma 3 to each of the diagrams

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \partial C'_{i+1} & \partial C'_i \\
\downarrow & \downarrow & \downarrow \\
0 & C'_i & C_i \\
\downarrow & \downarrow & \downarrow \\
0 & \partial C'_i & \partial C_i \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

it follows by induction on $i$ that

$$\frac{(v_i/\ldots/v_n\ldots)}{(v'_i/\ldots/v'_n\ldots)} = \pm h(\pi)(v''_i/\ldots/v''_n\ldots)$$

for some $\pi_i$. This completes the proof of Lemma 4.

**Lemma 5.** If $\Pi$ permutes the components of $K - L$ freely, and if $H_*(K, L; Z) = 0$, then $\Delta_h(K, L) = 1$.

**Proof.** Let $K_0$ denote the union of $L$ with one component of $K - L$. Then the injection

$$P \otimes C_*(K_0, L; Z) \rightarrow P \otimes C_*(K, L; Z)$$

is an isomorphism. Thus the torsion

$$\Delta_h(K, L) \in U/\pm h(\Pi)$$

is the image in $U/\pm h(\Pi)$ of the torsion invariant

$$\Delta_t(K_0, L) \in U/\pm 1,$$

where the subscript 1 denotes the homomorphism from the trivial group to $U$. But this is in turn the image of a corresponding invariant with the ring $P$ replaced by the ring $Z$ of integers. Since the only units in $Z$ are $\pm 1$, it follows that $\Delta_t(K, L)$ is trivial.

**Proof of Theorem A** (following Whitehead [22]). Choose a sequence

$$L = K_0 \subset K_1 \subset \cdots \subset K_r = K$$
of subcomplexes of $K$ so that each $K_{i+1} - K_i$ consists a single cell, together with its translates under $\Pi$. Let $I$ denote the unit interval considered as cw-complex, with $\Pi$ acting trivially.

Given a subdivision $K'$ of $K$ let $(A, B)$ denote the CW-pair formed from $(K \times I, L \times I)$ by subdividing $K \times 1$ only. Let $A_i$ denote the subcomplex of $A$ formed from 

$$(K \times 0) \cup (K_i \times I)$$

by subdividing $K_i \times 1$.

The inclusion $C_*(K \times 0, L \times 0) \to C_*(A_0, B)$ is an excision isomorphism, and hence

$$\Delta_*(A_0, B) = \Delta_*(K, L).$$

Each pair $(A_{i+1}, A_i)$ clearly satisfies the conditions of Lemma 5. Hence by Lemma 4

$$\Delta_*(A_0, B) = \Delta_*(A_1, B) = \cdots = \Delta_*(A_r, B),$$

where $A_r = A$. Thus $\Delta_*(A, B)$ is equal to $\Delta_*(K, L)$.

Now let $\bar{A}_i$ denote the subcomplex of $A$ formed from $(K \times 1) \cup (K_i \times I)$ by subdividing $K \times 1$. Then by a similar argument

$$C_*(K' \times 1, L' \times 1) \xrightarrow{\cong} C_*(\bar{A}_0, B)$$

hence $\Delta_*(K', L') = \Delta_*(\bar{A}_0, B)$, and

$$\Delta_*(\bar{A}_0, B) = \Delta_*(\bar{A}_1, B) = \cdots = \Delta_*(\bar{A}_r, B)$$

where $\bar{A}_r = A$. Therefore

$$\Delta_*(K', L') = \Delta_*(A, B) = \Delta_*(K, L),$$

which completes the proof of Theorem A.

In conclusion, here is a theorem concerning the torsion of a product.

Let $A$ be a finite cw-complex with Euler characteristic $\chi(A)$. Assume that $\Pi$ acts trivially on $A$.

**Theorem B.** If $\Delta_*(K)$ is defined then $\Delta_*(K \times A)$ is defined and is equal to $\Delta_*(K)^{\chi(A)}$.

**Proof.** Choose subcomplexes $A_0 \subset A_1 \subset \cdots \subset A_r = A$ so that $A_0$ is vacuous and each $A_{i+1} - A_i$ consists of a single cell. The chain complex

$$C_*(K \times A_{i+1}, K \times A_i; Z)$$

is isomorphic to $C_*(K; Z)$ except for a shift in dimension; hence

$$\Delta_*(K \times A_{i+1}, K \times A_i) = \Delta_*(K)^{\pm 1}$$
where the exponent is exactly the difference $\chi(A_{i+1}) - \chi(A_i)$. Now by Lemma 4,
\[
\Delta_h(K \times A) = \Pi_{i=1}^{n-1} \Delta_h(K \times A_{i+1}, K \times A_i) = \Delta_h(K)^{\chi(A)};
\]
which completes the proof.

**Corollary 1.** For any $n$ the differentiable manifold $L_1 \times D^n$ is not diffeomorphic with $L_2 \times D^n$.

**Proof.** The triangulation of $L_q$ described in §1 is a $C^1$-triangulation in the sense of Whitehead [19]. Choosing any $C^1$-triangulation of $D^n$, consider the resulting product triangulation of $L_q \times D^n$. According to Theorem B
\[
\Delta_h(\tilde{L}_q \times D^n) = \Delta_h(\tilde{L}_q)^1
\]
hence $L_1 \times D^n$ (in this triangulation) is not combinatorially equivalent to $L_2 \times D^n$. But, according to Whitehead, if two manifolds are diffeomorphic then any $C^1$-triangulation of one is combinatorially equivalent to any $C^1$-triangulation of the other. Therefore $L_1 \times D^n$ is not diffeomorphic to $L_2 \times D^n$. This proves Corollary 1, and (together with Lemma 1) completes the proof of Theorem 3.

**Corollary 2.** For $n$ even the manifold $L_1 \times S^n$ is not diffeomorphic to $L_2 \times S^n$

(I do not know what happens for $n$ odd.) The proof is similar except that
\[
\Delta_h(\tilde{L}_q \times S^n) = \Delta_h(\tilde{L}_q)^2,
\]
since the Euler characteristic of an even dimensional sphere is $+2$. The absolute value of the torsion distinguishes $L_1$ from $L_2$, hence its square will also distinguish $L_1$ from $L_2$. This completes the proof of Corollary 2, and hence of Theorem 4.

**Appendix: Torsion and simple homotopy type**

The definition of torsion in §3 can be extended to the case where $P$ is an arbitrary commutative ring with unit as follows. Call a $P$-module $M$ quasi-free of rank $r$ if the direct sum of $M$ with a free module of rank $n$ is free of rank $r + n$ for large $n$. It follows easily that $\Lambda^r M$ is free on one generator, so that volumes can be defined as before. Furthermore, using the exact sequences
\[
0 \to \partial C_{i+1} \to C_i \to \partial C_i \to 0,
\]
it follows by induction on $i$ that each $\partial C_i$ is quasi-free. The definition of
torsion now proceeds as in § 3.

In his study of simple homotopy types, Whitehead has defined a sharper torsion invariant which makes sense even over a non-commutative ring. In this construction the group $U$ of units is replaced by an abelian group $W(P)$ which is defined as follows.

Let $G_n$ denote the group of all non-singular $n \times n$ matrices over $P$. Using the standard imbeddings

$$U = G_1 \subset G_2 \subset G_3 \subset \cdots,$$

one can form the union $G$: the infinite general linear group of $P$. Let $E$ denote the subgroup of $G$ generated by all elementary matrices (i.e., all matrices which coincide with the identity matrix except for one off-diagonal element). Whitehead shows that $E$ is exactly the commutator subgroup of $G$. Define the Whitehead group $W(P)$ to be the quotient $G/E$. Thus each non-singular matrix $A \in G_n$ determines an element of $W(P)$ which will be denoted by $w(A)$. (Note that $w(A)$ behaves very much like a determinant of $A$.)

**EXAMPLES.** If $P$ is an euclidean domain then $W(P) = U$; however I do not know whether or not this is true for a principal ideal domain. In general, if $P$ is a commutative ring, then $W(P)$ splits as the direct sum of $U$ and a second group $W_0(P)$. If $P$ is a skew-field, then $W(P)$ is the commutator quotient group of the multiplicative group $U$.

The definition of torsion using $W(P)$ in place of $U$ can be carried out as soon as one has a suitable concept of "volume". Let $M$ be a quasi-free left $P$-module of rank $r$ and let $F$ denote the free $P$-module generated by countably many elements $b_1, b_2, b_3, \cdots$. A quasi-basis for $M$ will mean an ordered basis $(m_1, m_2, m_3, \cdots)$ for the free module $M \oplus F$, which satisfies the condition $m_{r+i} = b_i$ for large $i$. An elementary transformation of such a quasi-basis will mean the operation of adding a left multiple of $m_i$ to $m_j$, $i \neq j$. Define a volume in $M$ to be an equivalence class of quasi-bases, where two quasi-bases are equivalent if and only if one can be obtained from the other by a finite sequence of elementary transformations. For the special case $M = 0$, note that a volume in $M$ can be considered as an element of the Whitehead group $W(P)$.

Proceeding just as in § 3 one can now define the torsion invariant

$$\Delta_s(K, L) \in W(P)/w(\pm h)\$.$

The hypotheses are the same as those of § 3 except that the ring $P$ need not be commutative.

As a case of particular interest suppose that II operates freely on the simply connected complexes $K \supset L$, and suppose that $H^s(K, L; Z) = 0$. 
Let $i: \Pi \to Z\Pi$ denote the inclusion homomorphism. Then the torsion

$$\Delta(K, L) \in W(Z\Pi)/w(\pm \Pi)$$

is defined. This invariant plays a fundamental role in Whitehead's theory. It vanishes if and only if the inclusion map

$$L/\Pi \to K/\Pi$$
is a simple homotopy equivalence.

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