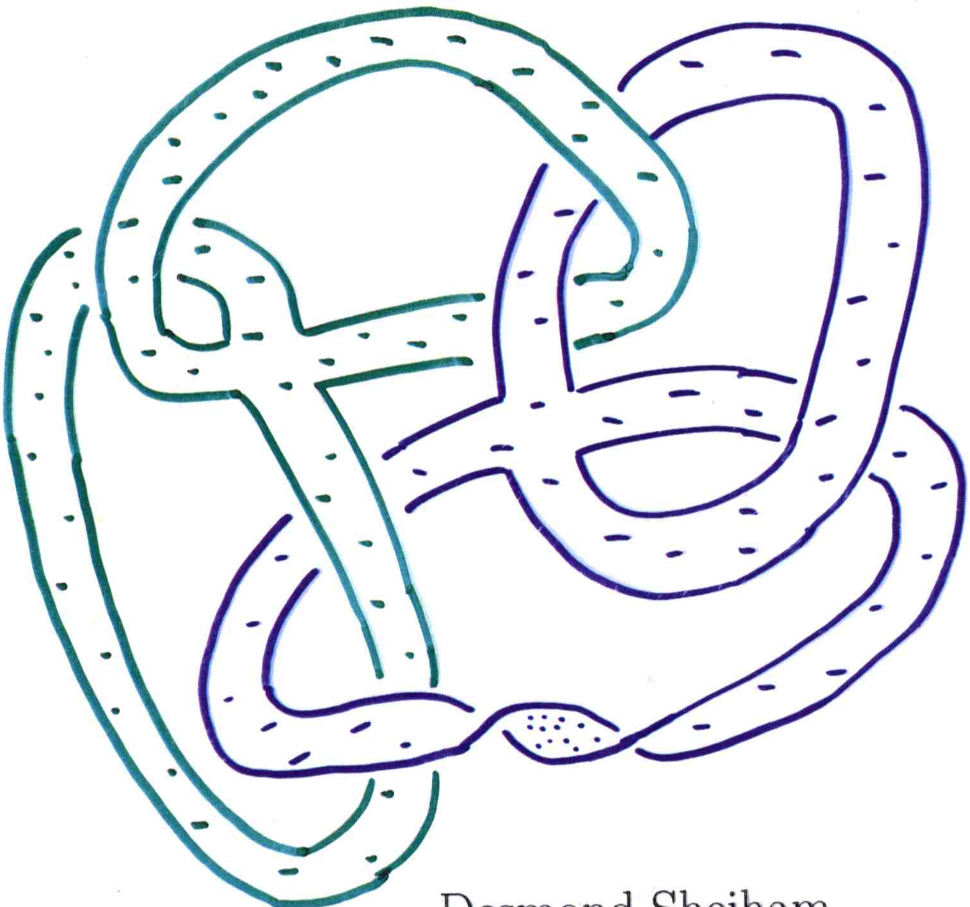
High dimensional knots and boundary links

Aberdeen 2004



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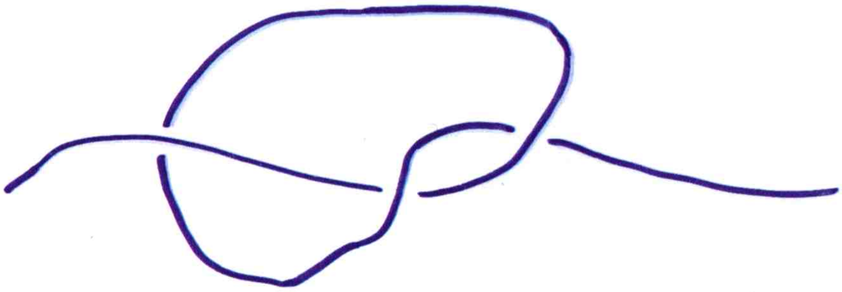
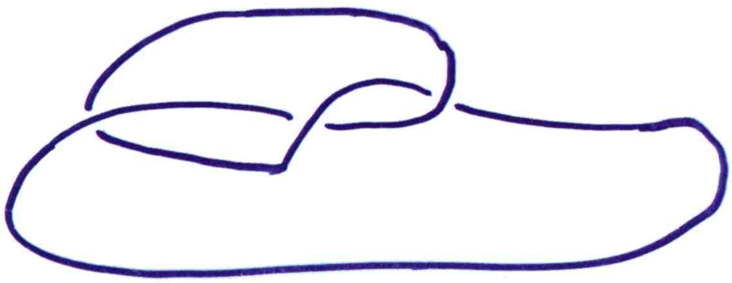
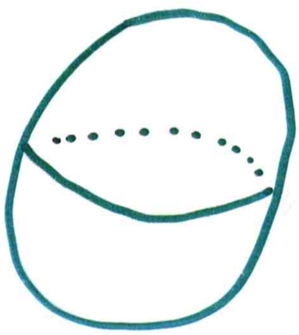


Join the ends of the string to make a knotted circle S^1.



Q: Can we knot a sphere?



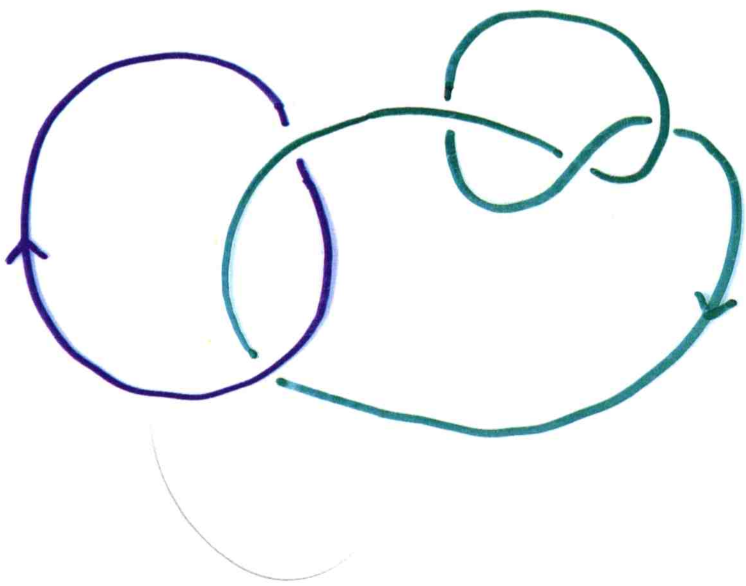
Yes! In 4-dimensional space.

A **knot** is an embedding of an n-dimensional sphere S^n in an $(n+2)$-dimensional sphere S^{n+2}.

A **link** is an embedding

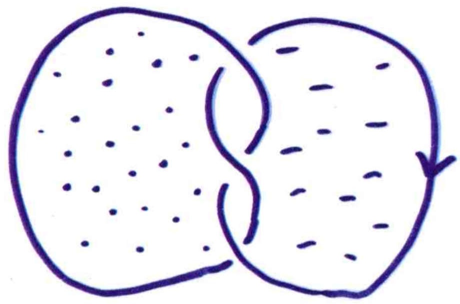
$$S^n \cup \cdots \cup S^n \subset S^{n+2}$$

of disjoint spheres S^n in S^{n+2}.



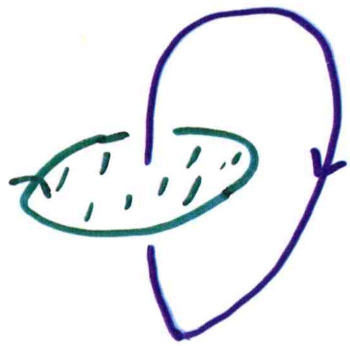
(n=1 in illustration)

Fact: Every knot $S^n \subset S^{n+2}$ bounds an (oriented) $(n+1)$-manifold.

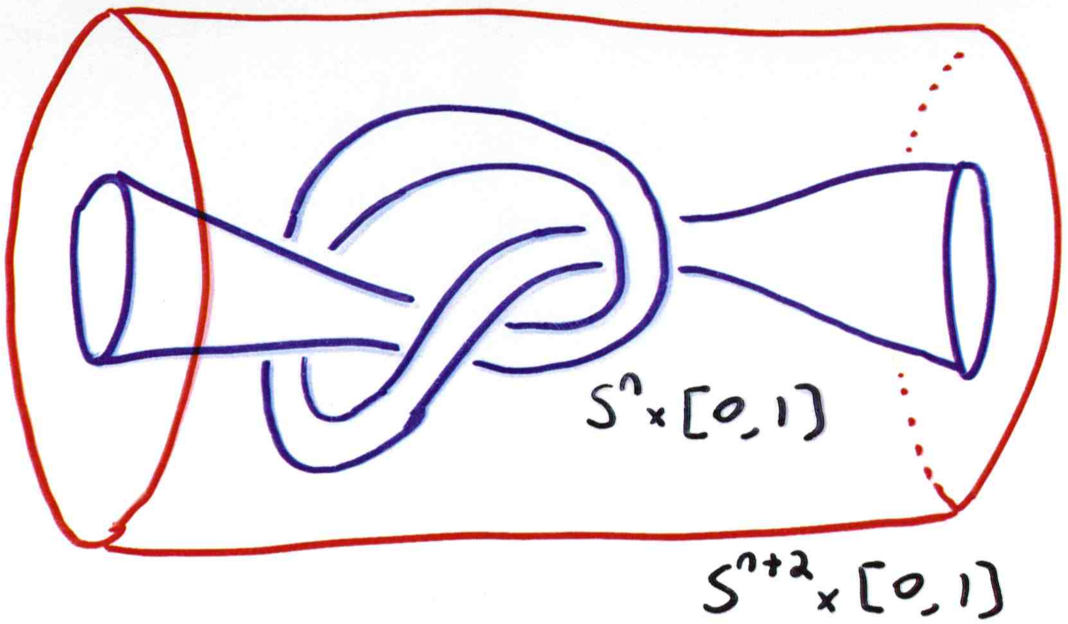


Do the components of a link bound *disjoint* oriented manifolds?

In general: No.



If Yes, the link is called a **boundary link**.

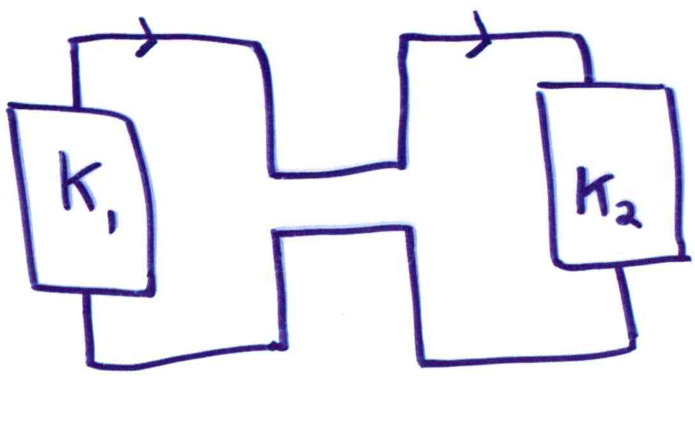


Two knots are **cobordant** (=**concordant**) if they can be joined by a cylinder $S^n \times [0,1]$ in $S^{n+2} \times [0,1]$. (Fox and Milnor)

Similarly for links, boundary links etc.

If n is even, there is a cobordism between *every* pair of knots. (Kervaire)

If n is odd the cobordism classes are the elements of an infinitely generated abelian group.



Theorem. (J.Levine) If n is odd and ≥ 3 then the knot cobordism classes correspond to elements of

$$\mathbb{Z}^\infty \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^\infty \oplus \left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^\infty .$$

Theorem. (S) If n is odd and ≥ 3 then the cobordism classes of boundary links with $\mu \geq 2$ components correspond to elements of

$$\mathbb{Z}^\infty \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^\infty \oplus \left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^\infty \oplus \left(\frac{\mathbb{Z}}{8\mathbb{Z}}\right)^\infty .$$

- Complete invariants.
- Torsion-free invariants (signatures) are indexed by "integers" on a family of real algebraic varieties.
- Algorithm to decide whether two boundary links are cobordant.

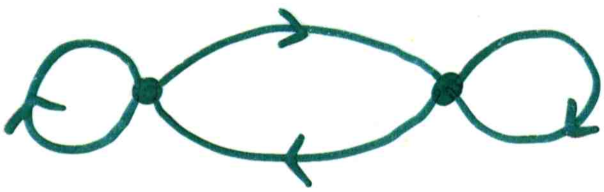
1) **Seifert surface** methods.

Advantages:

- Convenient in explicit calculations.
- Obtain algorithms.

Involves:

Finite-dimensional representations of certain quivers. For 2-component boundary links:



Symmetric and hermitian bilinear forms.

2) **Topology of Link complement.**

Advantages:

- Intrinsic; complete invariants without making a choice of Seifert surface.
- Fits naturally in the "Homology Surgery Theory" of Cappell and Shaneson.
- More amenable to potential generalization.

Involves:

Infinite-dimensional "torsion" representations of the free group.

Universal localization of rings.

(cf P.M.Cohn)

A Project in Progress:

To understand better the relationship between the categories of representations $\mathcal{C}_1$ and $\mathcal{C}_2$ which arise by the two approaches.

Theorem. (Ranicki, S) $\mathcal{C}_2$ is a universal localization of $\mathcal{C}_1$.

Theorem. (Ranicki, S) The kernel of the functor $\mathcal{C}_1 \longrightarrow \mathcal{C}_2$ consists of "near-projections".

Some Future Projects:

- Cobordism of arbitrary links. Requires *Vogel localization of spaces* and a corresponding *group localization.*
- Knot cobordism for $S^1 \subset S^3$ (cf recent work of Cochran, Orr and Teichner).