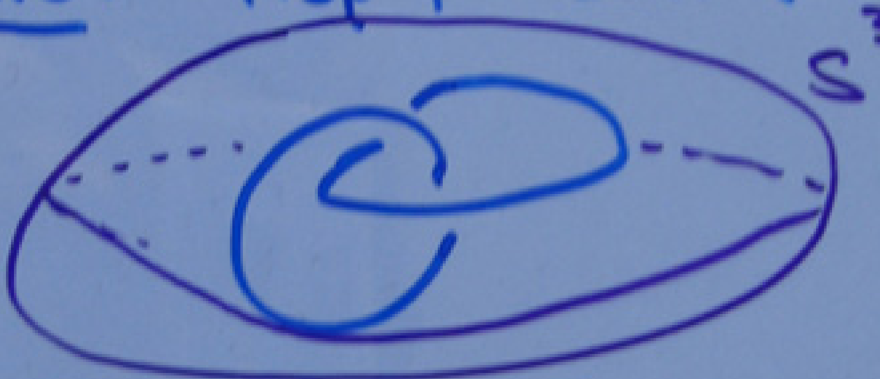
Links

$$L \cong \underbrace{S^n \sqcup \dots \sqcup S^n}_{\mu} \hookrightarrow S^{n+2}$$

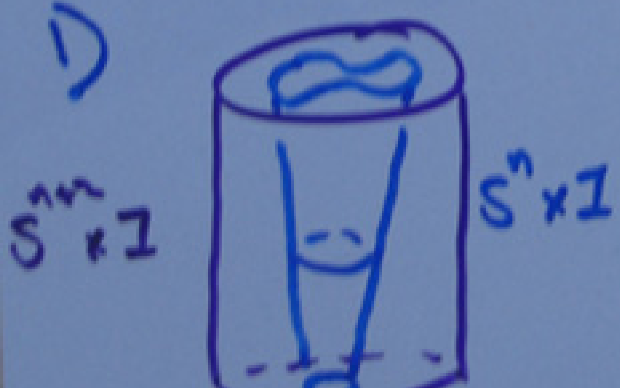
E.G. Hopf Link



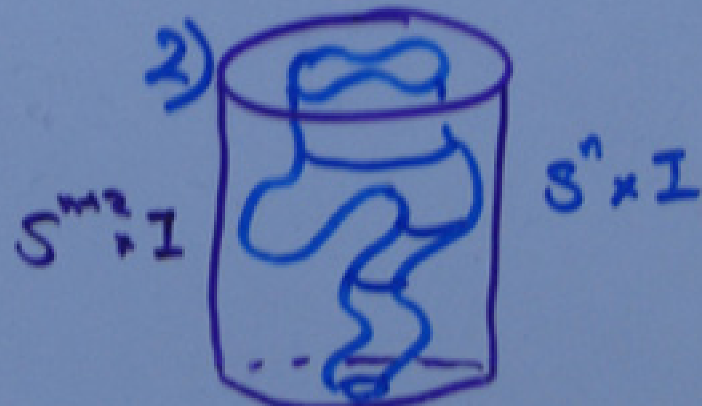
Relations: 1) Isotopy

2) Cobordism = concordance

1)



2)



Boundary Links

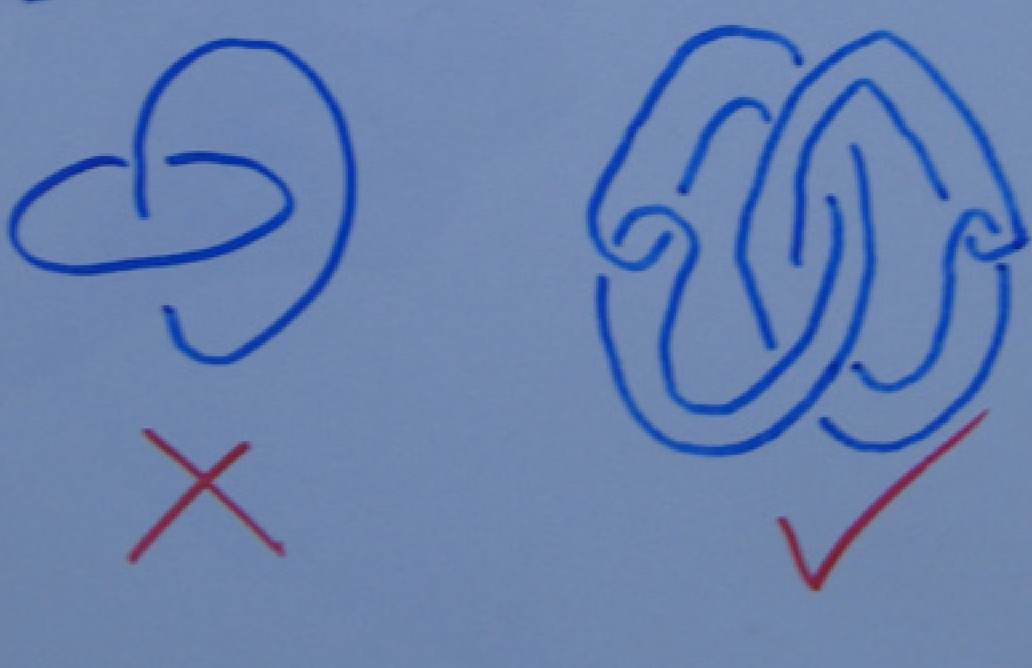
Defn: $L: \bigsqcup_{i=1}^{\mu} S_i^n \hookrightarrow S^{n+2}$

is a **boundary link** if

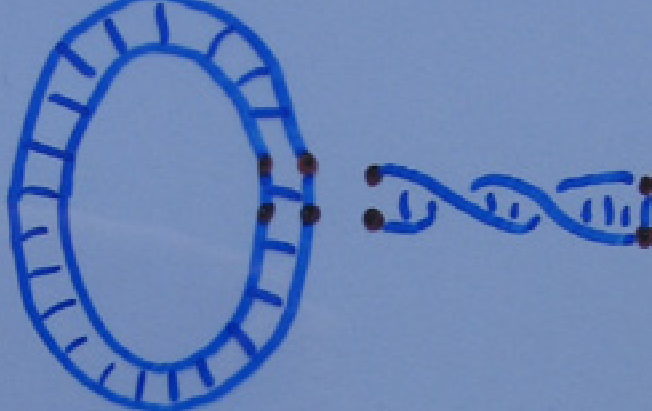
$$L = \partial\left(\bigcup_{i=1}^{\mu} F_i^{n+1} \hookrightarrow S^{n+2}\right).$$

F_i^{n+1} is a Seifert surface for S_i^n.

E.g.



Seifert surface:



(x2)

Proposition: (Gutierrez, Smythe)

A link L is a boundary link $\Longleftrightarrow$ $\exists$ a group homom.

$$\Theta : \pi_1(S^{n+2} - L) \longrightarrow F_\mu$$
$$m_i \longmapsto z_i$$

$\{z_1, \dots, z_\mu\}$ a basis for F_μ

$\{m_1, \dots, m_\mu\}$ a choice of meridian

Defn: (Cappell & Shaneson)

A pair (L, Θ) as above is called an F_μ-link.

- $C(n,\mu) := \{\mu\text{-links}\}/\text{conc.}$ group $n \geq 1$

- $B(n,\mu) := \dfrac{\{\mu\text{-boundary links}\}}{\text{boundary-conc.}}$ group $n \geq 1$

- $C_n(F_\mu) := \dfrac{\{F_\mu\text{-links}\}}{\text{boundary-conc.}}$ group $\forall n$.

$$C_n(F_\mu) \longrightarrow B(n,\mu) \longrightarrow C(n,\mu)$$

$$C_n(F_1) = B(n,1) = C(n,1)$$

= knot concordance gp.

History:

- Fox & Milnor (1966)

  Defined: $C(1,1)$, $|C(1,1)| = \infty$

- Kervaire (1965)

  $C(2q,1) = 0$

  ambient surgery on Seifert surface.

- Levine (1969)

  $C(2q+1,1) \cong \mathbb{Z}^{\infty} \oplus \mathbb{Z}/4^{\infty} \oplus \mathbb{Z}/2^{\infty}$

  $(q \geq 3)$.

  Seifert surface $\Rightarrow$ Seifert form.

$\Rightarrow C_{2q+1}(F_\mu)$ looks tractable.

History II

- Cappell + Shaneson:

$$C(2q,\mu) = 0$$

- $\prod_{\mu} C(2q+1) \not\twoheadrightarrow B(2q+1,\mu)$ (1980)

"Not all boundary links are conc. to split links."

- $$C_{2q+1}(F_\mu) \cong \Gamma \begin{pmatrix} \mathbb{Z}\pi & \longrightarrow & \mathbb{Z}\pi \\ \downarrow & & \downarrow \\ \mathbb{Z}\pi & \longrightarrow & 1 \end{pmatrix}$$

Here for any unital map of rings with involution $R \to S$

$\Rightarrow \Gamma(R \to S)$.

History III

- Cochran + Orr (1993)

$$B(2q+1,\mu) \xrightarrow{I} \!\!\!\!\!\!/ \;\; C(2q+1,\mu)$$

"Not all links are conc. to boundary links."

- Still open:

A) What is $C_{2q+1}(F\mu)$?

B) Does I have a kernel?

Problem 9 of Levine-Orr (1996)

Des begins Ph.D $\bar{c}$ A. Ranicki.

Des' Thesis

Theorem A:

$\mu \geq 2.$

$$C_{2q-1}(F_\mu) \cong \mathbb{Z}^\infty \oplus \mathbb{Z}/4^\infty \oplus \mathbb{Z}/8^\infty$$

- 30 year old problem
- Completely solved
- "Could only be solved by a student" A.R.
- "Regarded as intractable" K.O.

Open Problem:

Injectivity of I:

$$\begin{tikzcd} C_{2g+1}(F_\mu) \arrow[d] \arrow[dr, "I'"] & \\ B(2g+1,\mu) \arrow[r, "I"] & C(2g+1,\mu) \end{tikzcd}$$

$$\mathrm{Ker}(I') \cong \mathrm{Ker}\begin{pmatrix} \Gamma(\mathbb{Z}F_\mu \to \mathbb{Z}) \\ \downarrow \\ \Gamma(\mathbb{Z}\hat{F}_\mu \to \mathbb{Z}) \end{pmatrix}$$

Vogel

Let $f: F_\mu \to F_\mu$ s.t. $H_1(f)$ an $\cong$m.

$$\begin{tikzcd} \Gamma(f): \Gamma(\mathbb{Z}F_\mu \to \mathbb{Z}) \arrow[rr] \arrow[dr] & & \Gamma(\mathbb{Z}F_\mu \to \mathbb{Z}) \arrow[dl] \\ & \Gamma(\mathbb{Z}\hat{F}_\mu \to \mathbb{Z}) & \end{tikzcd}$$

Lillian Morrison Against ∞.

Poet as Mathematician

Having percieved the connections, he seeks
the proof, the clean revelation in its

simplest form, never doubting that
somewhere waiting in the chaos
is the unique

elegance, the precise, airy structure
defined, swift-lined,
and indestructible. $\blacksquare$

Remember: you have been touched
by Des' presence.
Ivan Sheiham.